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SOLVABILITY OF LINEAR BOUNDARY VALUE PROBLEMS FOR SUBDIFFUSION EQUATIONS WITH MEMORY

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ABSTRACT. For $\nu \in (0,1)$, the nonautonomous integro-differential equation

$$\mathbf{D}_t^\nu u - \mathcal{L}_1 u - \int_0^t \mathcal{K}_1(t-s)\mathcal{L}_2 u(\cdot,s)\,ds = f(x,t)$$

is considered here, where $\mathbf{D}_t^\nu$ is the Caputo fractional derivative and $\mathcal{L}_1$ and $\mathcal{L}_2$ are uniformly elliptic operators with smooth coefficients dependent on time. The global classical solvability of the associated initial-boundary value problems is addressed.

1. Introduction. Evolution equations with fractional derivatives in time are among the central objects of the contemporary theory of partial differential equations. On one hand, this is due to the multitude of applications in several sciences, such as physics [**3, 23, 34, 35**], biology [**44, 51**] and chemistry [**31, 50**]. On the other hand, the subject has a quite rich mathematical content that renders it of independent interest, see e.g., the monographs [**15, 25, 42**] and the references therein. In particular, fractional PDEs are apt to describe diffusive motions that cannot be modeled as the standard Brownian ones [**3, 35**]. The signature of an anomalous diffusion of this kind is that the mean square displacement of the diffusing species $\langle(\Delta\mathbf{x})^2\rangle$ scales as a nonlinear power law in time, i.e., $\langle(\Delta\mathbf{x})^2\rangle \sim t^\nu$, $\nu > 0$. When $\nu \in (0,1)$, this is referred to as subdiffusion.

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The present work is concerned with initial boundary-value problems describing heat flow in a rigid conductor with memory **[5, 9, 41]**. To this end, let $\Omega \subset \mathbb{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$, and, for an arbitrarily fixed time $T > 0$, denote

$$\Omega_T = \Omega \times (0, T) \quad \text{and} \quad \partial\Omega_T = \partial\Omega \times [0, T].$$

For a fixed $\nu \in (0, 1)$, we analyze the following, nonautonomous subdiffusion equation with a nonlocal term for the unknown function $u = u(x, t) : \Omega_T \to \mathbb{R}$,

$$\mathbf{D}_t^{\nu} u(x,t) - \mathcal{L}_1 u(x,t) - \int_0^t \mathcal{K}_1(t-s)\mathcal{L}_2 u(x,s)\, ds = f(x,t), \tag{1.1}$$

supplemented with the initial condition

$$u(x, 0) = u_0(x), \tag{1.2}$$

and subject either to the Dirichlet boundary condition (**DBC**)

$$u(x,t) = \psi(x,t) \quad \text{on } \partial\Omega_T, \tag{1.3}$$

or to the condition of the third kind (**III BC**)

$$\mathcal{M}_1 u(x,t) + \int_0^t \mathcal{K}_2(t-s)\mathcal{M}_2 u(x,s)\, ds = \psi_1(x,t) \quad \text{on } \partial\Omega_T. \tag{1.4}$$

Here, f represents an external force, and the so-called memory kernels $\mathcal{K}_1$ and $\mathcal{K}_2$ are supposed to be summable functions on $(0, T)$; ψ and ψ_1 are some given functions whose properties will be specified later. In regards to the operators involved, $\mathcal{L}_i$ are linear elliptic operators of the second order with time-dependent coefficients, namely,

$$\mathcal{L}_1 u(x,t) := \sum_{ij=1}^{n} a_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(x,t)\frac{\partial u}{\partial x_i} + a_0(x,t)u, \tag{1.5}$$

$$\mathcal{L}_2 u(x,t) := \sum_{ij=1}^{n} b_{ij}(x,t)\frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{n} b_i(x,t)\frac{\partial u}{\partial x_i} + b_0(x,t)u, \tag{1.6}$$

while the operators $\mathcal{M}_i$ read

$$\mathcal{M}_1 u(x,t) := \sum_{i=1}^{n} c_i(x,t)\frac{\partial u}{\partial x_i} + c_0(x,t)u, \tag{1.7}$$

$$\mathcal{M}_2 u(x,t) := \sum_{i=1}^{n} d_i(x,t)\frac{\partial u}{\partial x_i} + d_0(x,t)u. \tag{1.8}$$

Finally, the symbol $\mathbf{D}_t^\nu$ denotes the Caputo fractional derivative of order ν with respect to t (see, e.g., [**25,** (2.4.1)]), defined as

$$\mathbf{D}_t^\nu u(x,t) = \frac{1}{\Gamma(1-\nu)}\frac{\partial}{\partial t}\int_0^t \frac{u(x,s)-u(x,0)}{(t-s)^\nu}\,ds, \tag{1.9}$$

Γ being the Euler Gamma-function. A simple integration by parts shows that an equivalent definition is

$$\mathbf{D}_t^\nu u(x,t) = \frac{1}{\Gamma(1-\nu)}\int_0^t \frac{\partial_s u(x,s)}{(t-s)^\nu}\,ds.$$

In the limit cases $\nu \downarrow 0$ and $\nu \uparrow 1$, the Caputo fractional derivatives of $u(x,t)$ boil down to $u(x,t)$ and $\partial_t u(x,t)$, respectively. Accordingly, in what follows, we agree to set

$$\mathbf{D}_t^0 u(x,t) = u(x,t).$$

Existence, uniqueness, stability and longtime behavior of linear (as well as nonlinear) problems corresponding to (1.1)–(1.4) for $\nu = 1$ have been extensively studied by many authors in the past decades. With no claim of completeness, we merely recall several different approaches adopted in this context, such as abstract semigroup techniques, perturbation methods, compactness arguments, methods of continuity (see [**7, 8, 10, 11, 12, 13, 16, 19, 20, 33, 43, 52**] and the references therein). We also refer to the work of Staffans [**47**], who introduced special semigroups of operators in connection with finite and infinite delay (also see Desch and Miller [**14**]). Finally, we refer to the papers of Gripenberg, Londen and Prüss [**22**] and Cannarsa and Sforza [**4**], where the solvability of the Cauchy problem for equations of the form

$$u_t + \alpha \star u_t = Au + \beta \star u + \mathfrak{g}(t) + \mathfrak{F}(u)$$

was proven for a strictly negative (selfadjoint) linear operator A and summable functions α and β on $(0,\infty)$. For a particular choice of the kernels α and β, the above equation can be transformed into the evolution equation with fractional derivatives $\partial^\nu/\partial t^\nu$ in the Weyl sense:

$$u_t + \frac{\partial^\nu}{\partial t^\nu}u = Au + \frac{\partial^{\bar\nu}}{\partial t^{\bar\nu}}u + \mathfrak{g}(t) + \mathfrak{F}(u), \quad 0 \le \bar\nu < \nu < 1.$$

It is worth observing that some useful properties of the Weyl fractional derivative, such as

$$\frac{\partial^\nu}{\partial t^\nu}\frac{\partial^{\bar\nu}}{\partial t^{\bar\nu}}u = \frac{\partial^{\nu+\bar\nu}}{\partial t^{\nu+\bar\nu}}u, \qquad \int_{\mathbb{R}} u(t)\frac{\partial^\nu}{\partial t^\nu}v(t)\,dt = \int_{\mathbb{R}} v(t)\frac{\partial^\nu}{\partial t^\nu}u(t)\,dt,$$

for $\nu, \bar\nu > 0$ and u and v sufficiently smooth (see **[25]**), are no longer true in the case of the Caputo derivative.

At the same time, a number of works were published on the analysis of (1.1) with $\mathcal{K}_1 \equiv 0$ and $\nu \in (0,1)$ (i.e., for subdiffusion equations without the memory term), subject to several types of boundary conditions. Kochubei **[26, 27]** and Pskhu **[42]** constructed the fundamental solution in $\mathbb{R}^n$ and proved the maximum principle for the Cauchy problem. Gejji and Jafari **[18]** solved a nonhomogeneous fractional diffusion-wave equation in a one-dimensional bounded domain. Metzler and Klafter **[35]**, using the method of images and the Fourier-Laplace transform technique, obtained solutions to different boundary value problems for the linear homogenous subdiffusion equation in a half-space and in a box. Fujita **[17]** discussed an integrodifferential equation which interpolates the heat and wave equations in an unbounded domain. Mophou and Guérékata **[36]** and Sakamoto and Yamamoto **[45]** proved the one-valued solvability of the initial-boundary value problem for the linear fractional diffusion equation with time-independent coefficients subject to the homogenous Dirichlet boundary condition. Clement, Gripenberg and Londen **[6]** studied the evolution equation with the fractional Riemann-Liouville derivative

$$D^\nu_{0t}(u - u_0) + Bu = f$$

for a positive linear operator B on a Banach space. In **[28, 29]**, the global solvability of initial and initial-boundary value problems in Hölder spaces was obtained via suitable regularization techniques. As for quasilinear versions of (1.1) with $\mathcal{K}_1 \equiv 0$, the reader is referred to **[1, 7, 29, 37]**, where the local solvability in Hölder spaces and the global solvability in Hilbert spaces have been proved using different methods.

We also recall some investigations related to parabolic equations with evolution operator of the general form $\partial(\mathbf{k} \star (u - u_0))/\partial t$, where the kernel $\mathbf{k}$ is of $\mathcal{PC}$ type (see **[24, 49]** and the references therein). For the choice $\mathbf{k}(t) = t^{-\nu}/\Gamma(1-\nu)$, the operator $\partial(k \star u)/\partial t$ becomes the classical Riemann-Liouville fractional derivative $D^{\nu}_{0t} u$. In **[24, 49]**, the authors obtained weak global solvability results, as well as sharp estimates for the decay in time of solutions to equations like

$$\frac{\partial}{\partial t}(\mathbf{k} \star (u - u_0)) + Au = f.$$

Our work on the solvability and smoothness of (1.1)–(1.4) with $\nu \in (0,1)$ and $\mathcal{K}_1 \neq 0$ is motivated by nonlinear problems related to the generalized Oldroyd-B constitutive model **[46, 48]** and motion fluid through porous media with memory **[5]**. Indeed, equation (1.1) is the linear part of the nonlinear operators in the above-mentioned models.

Concerning (1.1) with the fractional derivative, and in the presence of the memory term $\mathcal{K}_1 \neq 0$, we point out the paper **[2]**, where the authors prove the existence and the regularity of the solutions in Sobolev spaces to the equation

$$D^{\nu}_{0t} u + \Delta u + \frac{1}{\Gamma(1-\nu)} \int_0^t (t-\tau)^{-\nu} \Delta u(x,\tau)\, d\tau = \mathcal{F}(x,t).$$

Lastly, we quote **[38, 39]** where the one-valued solvability in Hölder and L^p classes is established for the Cauchy problem with a Weyl fractional derivative

$$\frac{\partial^{\nu}}{\partial t^{\nu}} u(t) - Au - \int_{-\infty}^{t} \mathcal{K}_1(t-s) Au\, ds = f(t), \quad t \in \mathbb{R},$$

where A is a closed linear operator on a Banach space $\mathbf{X}$, and $\mathcal{K}_1$ is a summable kernel satisfying an additional restriction.

Nonetheless, to the best of our knowledge, we are unaware of works addressing the global classical solvability to the nonautonomous equation (1.1) with the Caputo fractional derivative and finite delay, subject to the nonhomogeneous conditions (1.2)–(1.4).

The goal of the present paper is the proof of the well-posedness and the regularity of solutions in smooth classes for any time, under the minimal requirement on the kernels $\mathcal{K}_1, \mathcal{K}_2 \in L_1(0,T)$. This will be obtained by adapting the technique of regularizers for parabolic equations **[30]** to the subdiffusion equation, so to establish the one-valued global classical solvability of (1.1)–(1.4). Once the linear case is fully understood, it is then possible to tackle the global classical solvability of boundary-value problems for nonlinear extensions of equation (1.1). This will be the subject of a future research.

1.1. Outline of the paper. In Section 2, we introduce the function spaces. The main results of the paper along with the general assumptions are stated in Section 3. Section 4 is devoted to some auxiliary results concerning the properties of solutions to subdiffusion equations, which will play a key role in the investigation. Finally, in Section 5, we provide the proofs of Theorems 3.1 and 3.2, combining some ideas from **[30]** with the coercive estimates of the solutions (see subsection 4.1). Moreover, in Remark 3.3, we show how results of Theorems 3.1 and 3.2 can be extended to the more general equation than (1.1).

2. Function spaces and notation. We will carry out our analysis in the framework of the fractional Hölder spaces. To this end, throughout the paper, let

$$\alpha, \nu \in (0,1)$$

be arbitrarily fixed. For any nonnegative integer l, we introduce the spaces

$$\mathcal{C}([0,T], \mathcal{C}^{l+\alpha}(\overline{\Omega})) \quad \text{and} \quad \mathcal{C}^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega}_T).$$

The first class has been used by several authors, and its definition and properties can be found, for instance, in **[32]**. Concerning the second class, we denote, for $\beta \in (0,1)$,

$$\langle v\rangle^{(\beta)}_{x,\Omega_T} = \sup_{\overline{\Omega}_T} \frac{|v(x,t)-v(\overline{x},t)|}{|x-\overline{x}|^{\beta}}, \quad x \neq \overline{x},$$

$$\langle v\rangle^{(\beta)}_{t,\Omega_T} = \sup_{\overline{\Omega}_T} \frac{|v(x,t)-v(x,\bar{t})|}{|t-\bar{t}|^{\beta}}, \quad t \neq \bar{t}.$$

Then, we have the following definition.

Definition 2.1. A function $v = v(x,t)$ belongs to the class $\mathcal{C}^{l+\alpha,(l+\alpha)\nu/2}$ $(\overline{\Omega}_T)$, for $l = 0,1,2$, if the following norms are finite:

$$\|v\|_{\mathcal{C}^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega}_T)} = \|v\|_{\mathcal{C}([0,T],\mathcal{C}^{l+\alpha}(\overline{\Omega}))} + \sum_{|j|=0}^{l} \langle D_x^j v\rangle_{t,\Omega_T}^{((l+\alpha-|j|)\nu/2)},$$

$l = 0, 1,$

$$\|v\|_{\mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}(\overline{\Omega}_T)} = \|v\|_{\mathcal{C}([0,T],\mathcal{C}^{2+\alpha}(\overline{\Omega}))} + \|\mathbf{D}_t^\nu v\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega}_T)} + \sum_{|j|=1}^{2} \langle D_x^j v\rangle_{t,\Omega_T}^{((2+\alpha-|j|)\nu/2)}.$$

In the limiting case $\nu = 1$, the class $\mathcal{C}^{l+\alpha,(l+\alpha)\nu/2}$ coincides with the usual parabolic Hölder space $H^{l+\alpha,(l+\alpha)/2}$. See, e.g., [**30,** (1.10)–(1.12)].

Definition 2.2. For $l = 0, 1, 2$, we define $\mathcal{C}_0^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega}_T)$ to be the space consisting of those functions $v \in \mathcal{C}^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega}_T)$, satisfying the zero initial conditions

$$v|_{t=0} = 0 \quad \text{and} \quad \mathbf{D}_t^{\nu m} v|_{t=0} = 0, \quad m = 0, \dots, \left\lfloor \frac{l}{2} \right\rfloor, \tag{2.1}$$

where $\lfloor\cdot\rfloor$ denotes the integer part.

In a similar manner, for $l = 0, 1, 2$, we introduce the spaces

$$\mathcal{C}^{l+\alpha,(l+\alpha)\nu/2}(\partial\Omega_T) \quad \text{and} \quad \mathcal{C}_0^{l+\alpha,(l+\alpha)\nu/2}(\partial\Omega_T).$$

Throughout the paper, we will also use the notation

$$(\mathcal{K} \star g)(\cdot, t) = \int_0^t \mathcal{K}(t-s) g(\cdot, s)\, ds,$$

sometimes omitting the argument $(\cdot, t)$. Finally, the symbol C will be used to denote a *generic* positive constant.

3. Statements of the results. We begin by stipulating the set of hypotheses.

H1. (Ellipticity conditions). There exist $0 < \mu_1 < \mu_2$ such that

$$\mu_1|\xi|^2 \leq \sum_{ij=1}^{n} a_{ij}(x,t)\xi_i\xi_j \leq \mu_2|\xi|^2, \tag{3.1}$$

for any $(x,t,\xi) \in \overline{\Omega}_T \times \mathbb{R}^n$; and there exists a $\mu_3 > 0$ such that

$$\left|\sum_{i=1}^{n} c_i(x,t)N_i(x)\right| \geq \mu_3 > 0, \tag{3.2}$$

for any $(x,t) \in \partial\Omega_T$, where $N = \{N_1(x), \dots, N_n(x)\}$ is the unit outward normal vector to Ω.

H2. (Conditions on the coefficients). For $i,\ j = 1, \dots, n$,

$$a_{ij}(x,t), a_i(x,t), a_0(x,t), b_{ij}(x,t), b_i(x,t), b_0(x,t) \in \mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega}_T), \tag{3.3}$$

and

$$c_i(x,t), c_0(x,t), d_i(x,t), d_0(x,t) \in \mathcal{C}^{l+\alpha,(l+\alpha)\nu/2}(\partial\Omega_T). \tag{3.4}$$

H3. (Conditions on the given functions)

$$\mathcal{K}_1(t), \mathcal{K}_2(t) \in L_1(0,T), \tag{3.5}$$

$$u_0(x) \in C^{2+\alpha}(\overline{\Omega}), \quad f(x,t) \in \mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega}_T), \tag{3.6}$$

$$\psi(x,t) \in \mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}(\partial\Omega_T), \tag{3.7}$$

$$\psi_1(x,t) \in \mathcal{C}^{1+\alpha,(1+\alpha)\nu/2}(\partial\Omega_T). \tag{3.8}$$

H4. (Compatibility conditions). When $x \in \partial\Omega$, the compatibility conditions at $t = 0$ hold

$$\psi(x,0) = u_0(x), \quad \mathbf{D}_t^\nu \psi(x,t)|_{t=0} = \mathcal{L}_1 u_0(x)|_{t=0} + f(x,0), \tag{3.9}$$

in the case of problem (1.1)–(1.3); and

$$\mathcal{M}_1 u_0(x)|_{t=0} = \psi_1(x,0), \tag{3.10}$$

for problem (1.1), (1.2), (1.4).

We observe that (3.2) can be equivalently written in the form

$$|\langle c, N\rangle| \geq \mu_3 > 0,$$

where $c = \{c_1(x,t), \dots, c_n(x,t)\}$, and $\langle\cdot,\cdot\rangle$ denotes the inner product. This means that the vector c does not lie in the tangent plane to $\partial\Omega$ at any point.

The compatibility conditions (3.9)–(3.10) do not contain the terms $(\mathcal{K}_1 \star \mathcal{L}_2 u)(x,0)$ and $(\mathcal{K}_2 \star \mathcal{M}_2 u)(x,0)$, since Lemma 4.1 together with assumptions **H2** and **H3** ensure the following equalities

$$(\mathcal{K}_1 \star \mathcal{L}_2 u)(x,0) = 0, \qquad (\mathcal{K}_2 \star \mathcal{M}_2 u)(x,0) = 0, \tag{3.11}$$

for any $x \in \partial\Omega$ and any function $u \in \mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}(\overline{\Omega}_T)$.

We are now ready to state our results on the global classical solvability of the problem.

Theorem 3.1. *Let $\partial\Omega \in C^{2+\alpha}$. For any $T > 0$, under conditions* (3.1)*,* (3.3)*,* (3.5)*–*(3.7) *and* (3.9)*, problem* (1.1)*–*(1.3) *admits a unique classical solution $u(x,t)$ on $[0,T]$. In addition, the following estimate holds*

$$\begin{aligned} \|u\|_{\mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}(\overline{\Omega}_T)} \\ \leq C_1\Big[\|\psi\|_{\mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}(\partial\Omega_T)} + \|f\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega}_T)} + \|u_0\|_{\mathcal{C}^{2+\alpha}(\overline{\Omega})}\Big], \end{aligned} \tag{3.12}$$

for some $C_1 > 0$ independent of the right-hand sides of (1.1)*–*(1.3).

Theorem 3.2. *Let $\partial\Omega \in C^{2+\alpha}$. For any $T > 0$, under conditions* **H1***,* **H2***,* (3.5)*,* (3.6)*,* (3.8) *and* (3.10)*, problem* (1.1)*,* (1.2) *and* (1.4) *admits a unique classical solution $u(x,t)$ on $[0,T]$. In addition, the following estimate holds*

$$\begin{aligned} \|u\|_{\mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}(\overline{\Omega}_T)} \\ \leq C_2\Big[\|\psi_1\|_{\mathcal{C}^{1+\alpha,(1+\alpha)\nu/2}(\partial\Omega_T)} + \|f\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega}_T)} + \|u_0\|_{\mathcal{C}^{2+\alpha}(\overline{\Omega})}\Big], \end{aligned} \tag{3.13}$$

for some $C_2 > 0$ independent of the right-hand sides of (1.1)*,* (1.2) *and* (1.4)*.*

Indeed, the positive constants C_1 and C_2 depend only on the Lebesgue measures of Ω and its boundary $\partial\Omega$, on the norm $\|\mathcal{K}_1\|_{L_1(0,T)}$, and on the norms of the coefficients of the operators $\mathcal{L}_i$ (as well as $\mathcal{M}_i$ and $\|\mathcal{K}_2\|_{L_1(0,T)}$ in the case of C_2).

Remark 3.3. Actually, with nonessential modifications in the proofs, the very same results hold for the more general equation

$$\mathbf{D}_t^{\nu} u(x,t) + \sum_{j=1}^{M} \overline{h}_j(x,t)\mathbf{D}_t^{\nu_j} u(x,t) - \mathcal{L}_1 u(x,t) - (\mathcal{K}_1 \star \mathcal{L}_2 u)(x,t)$$
$$- \sum_{i=1}^{N} \int_0^t k_i(t-s) h_i(x,s)\mathbf{D}_s^{\nu_i} u(x,s)\, ds = f(x,t), \quad \text{in } \Omega_T,$$

Here, for $i = 1, \dots, N$ and $j = 1, \dots, M$, the following assumptions are in place:

$$\begin{gathered} \nu_i \in (0, \nu], \\ \nu_j \in (0, \nu), \\ k_i(t) \in L_1(0,T), \\ h_i(x,t), \overline{h}_j(x,t) \in \mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega}_T), \end{gathered}$$

and, for $x \in \partial\Omega$,

$$\mathbf{D}_t^{\nu}\psi(x,t)|_{t=0} + \sum_{j=1}^{M} \overline{h}_j(x,0)\mathbf{D}_t^{\nu_j}\psi(x,t)|_{t=0} - \mathcal{L}_1 u_0(x,0) = f(x,0),$$

whenever the **DBC** (1.3) holds. The details are left to the interested reader.

4. Some technical lemmas.

4.1. Auxiliary estimates. We begin by stating some general properties of a kernel $\mathcal{K} \in L_1(0,T)$.

Lemma 4.1. *Let $\mathcal{K} \in L_1(0,T)$, and let $g_1 \in \mathcal{C}_0^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega}_T)$. Then, for $l = 0, 1$,*

$$(\mathcal{K} \star g_1)(x,t) \in \mathcal{C}_0^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega}_T),$$

and the estimate

$$\|\mathcal{K}\star g_1\|_{\mathcal{C}^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega}_T)} \le \|\mathcal{K}\|_{L_1(0,T)}\|g_1\|_{\mathcal{C}^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega}_T)} \tag{4.1}$$

holds.

Proof. Straightforward calculations lead to the estimate

$$\|\mathcal{K}\star g_1\|_{\mathcal{C}([0,T],\mathcal{C}^{l+\alpha}(\overline{\Omega}))} \le \|\mathcal{K}\|_{L_1(0,T)}\|g_1\|_{\mathcal{C}([0,T],\mathcal{C}^{l+\alpha}(\overline{\Omega}))}. \tag{4.2}$$

Thus, to prove (4.1), we shall evaluate the terms $\sum_{|j|=0}^{l}\langle D_x^j(\mathcal{K}\star g_1)\rangle_{t,\Omega_T}^{((l+\alpha-|j|)\nu/2)}$ (see Definition 2.1).

For $l=0$, $0\le t_1<t_2\le T$, we have

$$\begin{aligned} &\Big|\int_0^{t_2}\mathcal{K}(s)g_1(x,t_2-s)\,ds-\int_0^{t_1}\mathcal{K}(s)g_1(x,t_1-s)\,ds\Big| \\ &\le (t_2-t_1)^{(\alpha\nu/2)}\langle g_1\rangle_{t,\Omega_T}^{(\alpha\nu/2)}\int_0^{t_1}|\mathcal{K}(s)|\,ds \\ &\quad+(t_2-t_1)^{\alpha\nu/2}\int_{t_1}^{t_2}|\mathcal{K}(s)|\frac{|g_1(x,t_2-s)|}{[t_2-s]^{\alpha\nu/2}}\,ds. \end{aligned}\tag{4.3}$$

In order to obtain the last term on the right-hand side of (4.3), we used the simple inequality $0<t_2-s<t_2-t_1$. Due to $g_1(x,0)=0$, $x\in\overline{\Omega}$, we derive

$$\frac{|g_1(x,t_2-s)|}{[t_2-s]^{\alpha\nu/2}}\le\langle g_1\rangle_{t,\Omega_T}^{(\alpha\nu/2)}.$$

Thus, we can enhance estimate (4.3) in such a way as to get

$$\langle\mathcal{K}\star g_1\rangle_{t,\Omega_T}^{(\alpha\nu/2)}\le\|\mathcal{K}\|_{L_1(0,T)}\langle g_1\rangle_{t,\Omega_T}^{(\alpha\nu/2)}. \tag{4.4}$$

Finally, inequalities (4.2) together with (4.4) lead us to estimate (4.1) for $l=0$.

The case of $l=1$ is considered in the same manner: we obtain inequality (4.1), which guarantees that

$$\mathcal{K}\star g_1\in\mathcal{C}_0^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega}_T).$$

This completes the proof of Lemma 4.1. □

We next describe some properties of the solutions to initial and initial-boundary value problems for the subdiffusion equation, which will be important in constructing a regularizer to the linear problems (1.1)–(1.4) in Section 5. To this end, we denote

$$\mathbb{R}^n_+ = \{x : (x_1, \dots, x_{n-1}) \in \mathbb{R}^{n-1},\ x_n > 0\},$$
$$\mathbb{R}^n_{+T} = \mathbb{R}^n_+ \times (0, T).$$

Let the function $v_1(x,t)$ be the solution of the Cauchy problem:

$$\mathbf{D}_t^\nu v_1 - \Delta v_1 = F_0(x,t) \qquad \text{in } \mathbb{R}^n_T; \tag{4.5}$$

$$v_1(x,0) = v_{10}(x) \qquad \text{in } \mathbb{R}^n, \tag{4.6}$$

where F_0 and v_{10} are some given functions; and the functions $v_i(x,t)$, $i = 2, 3$, satisfy the following conditions:

$$\mathbf{D}_t^\nu v_i - \Delta v_i = 0 \quad \text{in } \mathbb{R}^n_{+T}, \tag{4.7}$$

$$v_i(x,0) = 0 \quad \text{in } \mathbb{R}^n_+; \qquad v_i(x,t) \to 0 \quad \text{if } |x| \to +\infty, \tag{4.8}$$

$$v_2(x,t)\big|_{x_n=0} = G_1(x,t) \qquad \text{on } \partial\mathbb{R}^n_{+T}, \tag{4.9}$$

$$\sum_{i=1}^{n} c_i \frac{\partial v_3}{\partial x_i}|_{x_n=0} = G_2(x,t) \quad \text{on } \partial\mathbb{R}^n_{+T}, \tag{4.10}$$

where G_1 and G_2 are given functions, and $c = \{c_1, \dots, c_n\}$ is a constant vector with $c_n \neq 0$.

The classical solvability of problems (4.5)–(4.10) has been studied in the one-dimensional case in **[29]**, and in the multi-dimensional case in **[28]**. The following results have been obtained.

Lemma 4.2. *Let* $c_n \neq 0$ *and* $v_{10} \in \mathcal{C}^{2+\alpha}(\mathbb{R}^n)$, *and let*

$$F_0 \in \mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\mathbb{R}}^n_T),$$
$$G_1 \in \mathcal{C}_0^{2+\alpha,(2+\alpha)\nu/2}(\partial\mathbb{R}^n_{+T}),$$
$$G_2 \in \mathcal{C}_0^{1+\alpha,(1+\alpha)\nu/2}(\partial\mathbb{R}^n_{+T}).$$

Finally, assume that there exists a positive number r_0 *such that*

$$v_{10}(x) = F_0(x,\cdot) = G_j(x,\cdot) = 0, \quad \text{if } |x| > r_0. \tag{4.11}$$

Then, there are unique classical solutions $v_i(x,t)$ *to problems* (4.5)–(4.10). *In addition, the following estimates hold*:

$$\|v_1\|_{\mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}(\overline{\mathbb{R}}^n_T)} \leq C[\|v_{10}\|_{\mathcal{C}^{2+\alpha}(\mathbb{R}^n)} + \|F_0\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\mathbb{R}}^n_T)}], \tag{4.12}$$

$$\|v_2\|_{\mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}(\overline{\mathbb{R}}^n_{+T})} \leq C\|G_1\|_{\mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}(\partial\mathbb{R}^n_{+T})}, \tag{4.13}$$

$$\|v_3\|_{\mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}(\overline{\mathbb{R}}^n_{+T})} \leq C\|G_2\|_{\mathcal{C}^{1+\alpha,(1+\alpha)\nu/2}(\partial\mathbb{R}^n_{+T})}. \tag{4.14}$$

Here, the generic constants C are independent of the right-hand sides in (4.5)–(4.10).

The next lemma allows us to reduce (1.1)–(1.4) to problems with homogenous initial data.

Lemma 4.3. *There exists a universal constant $C > 0$ with the following property: for any functions $W_0 \in \mathcal{C}^{2+\alpha}(\overline{\Omega})$ and $W_1 \in \mathcal{C}^{\alpha}(\overline{\Omega})$, there exists a function $W(x,t) \in \mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}(\overline{\Omega}_T)$ such that:*

$$W(x,0) = W_0(x), \qquad \mathbf{D}_t^{\nu} W(x,t)|_{t=0} = W_1(x),$$

and

$$\|W\|_{\mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}(\overline{\Omega}_T)} \leq C(\|W_0\|_{\mathcal{C}^{2+\alpha}(\overline{\Omega})} + \|W_1\|_{\mathcal{C}^{\alpha}(\overline{\Omega})}). \tag{4.15}$$

Proof. Denote $F(x) := W_1(x) - \Delta W_0(x)$. From [**30,** Theorem 4.1], there are extensions $\overline{W}_0$ and $\overline{F}$ of the functions W_0 and F on $\mathbb{R}^n$ such that:

$$\begin{aligned} &\|\overline{W}_0\|_{\mathcal{C}^{2+\alpha}(\mathbb{R}^n)} \leq C\|W_0\|_{\mathcal{C}^{2+\alpha}(\overline{\Omega})}, \\ &\qquad \|\overline{F}\|_{\mathcal{C}^{\alpha}(\mathbb{R}^n)} \leq C\|F\|_{\mathcal{C}^{\alpha}(\overline{\Omega})} \leq C(\|W_0\|_{\mathcal{C}^{2+\alpha}(\overline{\Omega})} + \|W_1\|_{\mathcal{C}^{\alpha}(\overline{\Omega})}), \end{aligned} \tag{4.16}$$

and the functions $\overline{W}_0$ and $\overline{F}$ have compact supports. Then, we define the function $W(x,t)$ to be the solution to the equation

$$\mathbf{D}_t^{\nu} W - \Delta W = \overline{F}(x) \quad \text{in } \mathbb{R}^n_T,$$

with initial datum

$$W(x,0) = \overline{W}_0(x).$$

Applying Lemma 4.2 to the above Cauchy problem, and taking into account (4.16), the claim is proven. □

4.2. Domains. Some auxiliary propositions. In order to prove Theorem 3.1, we will construct a regularizer (see [**30,** Section 4]). To this end, we need a special covering of the domain Ω. We take two collections of open sets $\{\varpi^m\}$ and $\{\Omega^m\}$, which consist of a finite number ϖ^m and Ω^m possessing the following properties for any small number $\lambda > 0$ and any point $x^m \in \overline{\Omega}$:

(i) Denoting by $B_r(x^m)$ the ball about x^m of radius r, we have that

$$\varpi^m = B_{\lambda/2}(x^m) \cap \overline{\Omega}, \qquad \Omega^m = B_\lambda(x^m) \cap \overline{\Omega},$$

and

$$\overline{\varpi^m} \subset \Omega^m \subset \overline{\Omega}, \qquad \bigcup_m \varpi^m = \bigcup_m \Omega^m = \overline{\Omega}.$$

(ii) There exists a number N_0, independent of λ, such that the intersection of any $N_0 + 1$ distinct Ω^m (and consequently any $N_0 + 1$ distinct ϖ^m) is empty.

The index m belongs to one of two sets, $\mathfrak{M}$ or $\mathfrak{N}$, where

$$m \in \mathfrak{M} \quad \text{if } \overline{\Omega^m} \cap \partial\Omega = \emptyset,$$
$$m \in \mathfrak{N} \quad \text{if } \overline{\varpi^m} \cap \partial\Omega \neq \emptyset.$$

Denote $\partial\Omega^m = \partial\Omega \cap B_\lambda(x^m)$. The covering $\{\varpi^m\}$ and $\{\Omega^m\}$ define a partition of unity for the domain Ω.

Let $\xi^m(x) : \Omega \to [0, 1]$ be a smooth function such that

$$\xi^m(x) = 1 \quad \text{if } x \in \overline{\varpi^m},$$
$$\xi^m(x) = 0 \quad \text{if } x \in \overline{\Omega} \backslash \overline{\Omega^m},$$
$$\xi^m(x) \in (0, 1) \quad \text{if } x \in \Omega^m \backslash \varpi^m,$$

and

$$|D_x^j \xi^m| \leq C\lambda^{-|j|}, \quad 1 \leq |j|, \ 1 \leq \sum_m (\xi^m)^2 \leq N_0.$$

Using ξ^m, we define the function

$$\eta^m = \frac{\xi^m}{\sum_j (\xi^j)^2}. \tag{4.17}$$

As it follows from the properties of the functions ξ^m, the functions η^m vanish for $x \in \overline{\Omega} \backslash \overline{\Omega^m}$, and, in addition, $|D_x^j \eta^m| \leq C\lambda^{-|j|}$. Thus, the product $\eta^m \xi^m$ defines the partition of unity by the formula

$$\sum_m \eta^m \xi^m = 1. \tag{4.18}$$

At this point, we define the local coordinate systems connected with each point x^m, $m \in \mathfrak{N}$. For each $m \in \mathfrak{N}$, we choose a point $x^m \in \varpi^m \cap \partial\Omega$, which will be the origin of a local coordinate system. Let $\partial\Omega$ be described by $y_n = \mathfrak{F}^m(y_1, \dots, y_{n-1})$ in a small neighborhood of every point x^m, $m \in \mathfrak{N}$, and

$$y = \mathfrak{B}^{(m)}(x - x^m), \qquad \left| \frac{\partial \mathfrak{F}^m}{\partial y_i} \right| \leq C\lambda, \quad i = 1, \dots, n-1, \tag{4.19}$$

where $\mathfrak{B}^{(m)} = (\gamma_{ij}^{(m)})_{i,j=1,\dots,n}$ is an orthogonal matrix with elements γ_{ij}^m, and $(\gamma_{ij}^m)^{-1}$ is an element of the inverse matrix to $\mathfrak{B}^{(m)}$. To obtain the local "flatness" of the boundary, we make the change of variables as

$$\begin{gathered} z_i = y_i, \ z_n = y_n - \mathfrak{F}^m(y_1, \dots, y_{n-1}), \\ i = 1, \dots, n-1, \ m \in \mathfrak{N}. \end{gathered} \tag{4.20}$$

Thus, we have constructed the mapping Z_m (see (4.19) and (4.20)) which connects the variables $(x_1, \dots, x_n)$ with $(z_1, \dots, z_n)$ in a neighborhood of every point x^m, $m \in \mathfrak{N}$:

$$x = z_m(z) \quad \text{and} \quad z = Z_m^{-1}(x).$$

We introduce the following norms in the spaces $\mathcal{C}_0^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega}_T)$, $l = 0, 1, 2$, which are associated with the covering $\{\Omega^m\}$:

$$\{v\}_{\mathcal{C}^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega}_T)} := \sup_m \|v\|_{\mathcal{C}^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega^m}_T)}.$$

Repeating the arguments of [**30,** Chapter 4], we can assert the following.

Proposition 4.4. *Let* $v \in \mathcal{C}_0^{2+\alpha,(2+\alpha)\nu/2}(\overline{\Omega}_T)$*. Then:*

$$\begin{aligned} &\|v\|_{\mathcal{C}([0,T],\mathcal{C}^l(\overline{\Omega}))} \\ &\qquad \leq CT^{(2+\alpha-l)\nu/2} \left[\langle \mathbf{D}_t^\nu v \rangle_{t,\Omega_T}^{(\alpha\nu/2)} + \sum_{|j|=1}^{2} \langle D_x^j v \rangle_{t,\Omega_T}^{((2+\alpha-|j|)\nu/2)} \right], \end{aligned} \tag{4.21}$$

$l = 0, 1, 2,$ *and*

(4.22) $$\|v\|_{\mathcal{C}^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega}_T)} \leq CT^{(2+\alpha-l)\nu/2}\Big[\langle \mathbf{D}_t^\nu v\rangle_{t,\Omega_T}^{(\alpha\nu/2)} + \sum_{|j|=1}^{2}\langle D_x^j v\rangle_{t,\Omega_T}^{((2+\alpha-|j|)\nu/2)}\Big],$$

$l=0,1$, and

(4.23) $$\|\mathbf{D}_t^\nu v\|_{\mathcal{C}([0,T],\mathcal{C}(\overline{\Omega}))} \leq CT^{\alpha\nu/2}\langle \mathbf{D}_t^\nu v\rangle_{t,\Omega_T}^{(\alpha\nu/2)},$$

where the positive constant C does not depend on T.

Proposition 4.5. *Let $\tau\in[0,T]$. For an arbitrarily given $0<\kappa<1$, define*

(4.24) $$\tau = \lambda^{2/\nu}\kappa.$$

Then, for any $v\in\mathcal{C}_0^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega}_\tau)$ with $l=0,1,2$, we have the following norm equivalence:

(4.25) $$\{v\}_{\mathcal{C}^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega}_\tau)} \leq \|v\|_{\mathcal{C}^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega}_\tau)} \leq C\{v\}_{\mathcal{C}^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega}_\tau)},$$

where the positive constant C is independent of λ and κ.

Proposition 4.6. *Suppose a function $\Phi_m(x)$ defined in Ω^m possesses the property*

(4.26) $$|D_x^j\Phi_m(x)| \leq C\lambda^{-|j|}, \quad 0\leq |j|\leq 2,$$

and the numbers τ and λ are related via (4.24). Then, for any function $v\in\mathcal{C}_0^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega}_\tau)$, $l=0,1,2$,

$$\|\Phi_m v\|_{\mathcal{C}^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega^m}_\tau)} \leq C\|v\|_{\mathcal{C}^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega^m}_\tau)},$$

where the positive constant C does not depend on λ and τ.

Proposition 4.7. *Let (4.24) hold, and set*

$$\widetilde{v}(x,t) = \sum_{m\in\mathfrak{M}\cup\mathfrak{N}} v^m(x,t),$$

where $v^m\in\mathcal{C}_0^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega^m}_\tau)$, $l=0,1,2$, and v^m vanishes outside Ω^m. Then:

$$\{\widetilde{v}\}_{\mathcal{C}^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega}_\tau)} \leq C\sup_{m\in\mathfrak{M}\cup\mathfrak{N}}\|v^m\|_{\mathcal{C}^{l+\alpha,(l+\alpha)\nu/2}(\overline{\Omega^m}_\tau)}.$$

5. Proofs of Theorems 3.1 and 3.2. We proceed with a detailed proof of Theorem 3.1. The proof of Theorem 3.2 is almost identical, and is left to the interested reader.

Using Lemma 4.3 with $W_0 := u_0$ and $W_1 := \mathcal{L}_1 u_0(x)|_{t=0} + f(x,0)$, $x \in \Omega$, we reduce problem (1.1)–(1.3) to the problem with homogenous initial data. Thus, we look for a solution to (1.1)–(1.3) in the form

$$u(x,t) = W(x,t) + w(x,t), \tag{5.1}$$

where the function $W(x,t)$ is constructed in Lemma 4.3, and the unknown function $w(x,t)$ satisfies

$$\mathbf{D}_t^\nu w - \mathcal{L}_1 w - (\mathcal{K}_1 \star \mathcal{L}_2 w) = \overline{f}(x,t) \quad \text{in } \Omega_T, \tag{5.2}$$

$$w(x,0) = 0, \quad x \in \overline{\Omega}, \tag{5.3}$$

$$w(x,t) = \overline{\psi}(x,t) \quad \text{on } \partial\Omega_T. \tag{5.4}$$

Here,

$$\overline{f}(x,t) = f(x,t) - \mathbf{D}_t^\nu W(x,t) + \mathcal{L}_1 W(x,t) + (\mathcal{K}_1 \star \mathcal{L}_2 W)(x,t), \tag{5.5}$$

and

$$\overline{\psi}(x,t) = \psi(x,t) - W(x,t)\big|_{\partial\Omega_T}. \tag{5.6}$$

The compatibility conditions (3.9) and **H2-H3**, together with Lemmas 4.1 and 4.3 and the representations (5.5) and (5.6), ensure that

$$\overline{f} \in \mathcal{C}_0^{\alpha,\alpha\nu/2}(\overline{\Omega}_T) \quad \text{and} \quad \overline{\psi} \in \mathcal{C}_0^{2+\alpha,(2+\alpha)\nu/2}(\partial\Omega_T). \tag{5.7}$$

For the sake of convenience, we rewrite problem (5.2)–(5.4) in the form

$$\mathcal{L} w = g, \quad g = (\overline{f}, \overline{\psi}), \tag{5.8}$$

where $\mathcal{L}$ is the linear operator defined by the left-hand side of (5.2)–(5.4), in other words, $\mathcal{L}w = \{\mathcal{A}w, \mathcal{A}_1 w|_{\partial\Omega_T}\}$, where $\mathcal{A}$ is defined by the left-hand side of (5.2) and $\mathcal{A}_1$ by the left-hand side of (5.4).

Denote

$$\begin{aligned} a_{ij}^m &:= a_{ij}(x^m, 0), \\ f_m(x,t) &:= \xi^m(x) \overline{f}(x,t), \end{aligned} \tag{5.9}$$

$m \in \mathfrak{M} \cup \mathfrak{N}$, and

$$\begin{aligned} \widetilde{f}_m(z,t) &= f_m(x,t)\big|_{x=Z_m(z)}, \\ \widetilde{\psi}_m(z,t) &= \xi^m(x)\overline{\psi}(x,t)\big|_{x=Z_m(z)}, \end{aligned} \tag{5.10}$$

$m \in \mathfrak{N}$, with ξ^m, $Z_m(z)$, $\mathfrak{M}$ and $\mathfrak{N}$ as in subsection 4.2.

Let $\tau \in [0,T]$, and let the functions $w_m(x,t)$, $m \in \mathfrak{M} \cup \mathfrak{N}$, be the solutions to the following problems. If $m \in \mathfrak{M}$, then

$$\begin{cases} \mathbf{D}_t^\nu w_m - \displaystyle\sum_{ij=1}^n a_{ij}^m \frac{\partial^2 w_m}{\partial x_i \partial x_j} = f_m(x,t) & \text{in } \mathbb{R}^n_\tau, \\ w_m(x,0) = 0 & \text{in } \mathbb{R}^n. \end{cases} \tag{5.11}$$

Instead, if $m \in \mathfrak{N}$,

$$w_m(x,t) = \widetilde{w}_m(z,t)\big|_{z=Z_m^{-1}(x)},$$

where $\widetilde{w}_m$ solves

$$\begin{cases} \mathbf{D}_t^\nu \widetilde{w}_m - \displaystyle\sum_{ij=1}^n a_{ij}^m \frac{\partial^2 \widetilde{w}_m}{\partial z_i \partial z_j} = \widetilde{f}_m(z,t) & \text{in } \mathbb{R}^n_{+\tau}, \\ \widetilde{w}_m(z,t) = \widetilde{\psi}_m(z,t) & \text{on } \partial\mathbb{R}^n_{+\tau}, \\ \widetilde{w}_m(z,0) = 0 & \text{in } \mathbb{R}^n_+. \end{cases} \tag{5.12}$$

Definition 5.1. Let $\tau \in (0,T]$. An operator $\mathfrak{R}$ is called a *regularizer*, on the time-interval $[0,\tau]$, if

$$\mathfrak{R} : \mathcal{C}_0^{\alpha,\alpha\nu/2}(\overline{\Omega}_\tau) \times \mathcal{C}_0^{2+\alpha,(2+\alpha)\nu/2}(\partial\Omega_\tau) \longrightarrow \mathcal{C}_0^{2+\alpha,(2+\alpha)\nu/2}(\overline{\Omega}_\tau),$$

and

$$\mathfrak{R}(\overline{f},\overline{\psi}) = \sum_{m \in \mathfrak{M}\cup\mathfrak{N}} \eta^m(x) w_m(x,t), \tag{5.13}$$

where the functions $\eta^m(x)$ and $w_m(x,t)$ are defined in (4.17), (5.11) and (5.12).

The operator $\mathfrak{R}$ enables us to construct an inverse operator to $\mathcal{L}$. First, we state the following key lemma.

Lemma 5.2. *Let* $\tau \in (0,T]$, *and assume* (4.24) *along with the hypotheses of Theorem* 3.1. *Then, setting*

$$\mathcal{H} = \mathcal{C}_0^{\alpha,\alpha\nu/2}(\overline{\Omega}_\tau) \times \mathcal{C}_0^{2+\alpha,(2+\alpha)\nu/2}(\partial\Omega_\tau),$$

for any

$$g \in \mathcal{H} \quad \text{and} \quad w \in \mathcal{C}_0^{2+\alpha,(2+\alpha)\nu/2}(\overline{\Omega}_\tau),$$

the following hold.

(i) $\mathfrak{R}$ *is a bounded operator*:

$$\|\mathfrak{R}g\|_{\mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}(\overline{\Omega}_\tau)} \leq C\|g\|_{\mathcal{H}}, \tag{5.14}$$

where the positive constant C *is independent of* λ *and* τ.

(ii) *There exists an operator* $\mathfrak{T}_1$ *such that*

$$\mathcal{L}\mathfrak{R}g = g + \mathfrak{T}_1 g, \qquad \|\mathfrak{T}_1 g\|_{\mathcal{H}} \leq \frac{1}{2}\|g\|_{\mathcal{H}}. \tag{5.15}$$

(iii) *There exists an operator* $\mathfrak{T}_2$ *such that*

$$\begin{gathered}\mathfrak{R}\mathcal{L}w = w + \mathfrak{T}_2 w,\\ \|\mathfrak{T}_2 w\|_{\mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}(\overline{\Omega}_\tau)} \leq \frac{1}{2}\|w\|_{\mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}(\overline{\Omega}_\tau)}.\end{gathered} \tag{5.16}$$

Proof. Simple linear changes of variables allow us to conclude that the results of Lemma 4.2 hold in problems (5.11) and (5.12). Then, Propositions 4.4–4.7 together with Lemma 4.2 lead to:

$$\begin{aligned}&\|\mathfrak{R}g\|_{\mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}(\overline{\Omega}_\tau)}\\ &\quad\leq C \sup_{m\in\mathfrak{M}\cup\mathfrak{N}} \|w_m\|_{\mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}(\overline{\Omega_\tau^m})}\\ &\quad\leq C\Big[\sup_{m\in\mathfrak{M}\cup\mathfrak{N}} \|\xi^m\overline{f}_0\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega_\tau^m})} + \sup_{m\in\mathfrak{N}} \|\xi^m\overline{\psi}\|_{\mathcal{C}^{1+\alpha,(1+\alpha)\nu/2}(\partial\Omega_\tau^m)}\Big]\\ &\quad\leq C\|g\|_{\mathcal{H}},\end{aligned} \tag{5.17}$$

where the constant C meets the requirement of the present lemma. Thus, the last inequality in (5.17) yields (5.14).

Let us verify point (ii). The definition of the operator $\mathcal{L}$ (see (5.8)) together with (4.18) and (5.9)–(5.12) allow us to conclude that

$$\mathcal{L}\mathfrak{R}g = \{\mathcal{A}\mathfrak{R}g, \mathcal{A}_1\mathfrak{R}g\big|_{\partial\Omega_\tau}\},$$

where

$$\begin{aligned}\mathcal{A}_1\mathfrak{R}g\big|_{\partial\Omega_\tau} &= \sum_{m\in\mathfrak{N}} \big(\eta^m(x)\widetilde{w}_m(z,t)\big|_{z=Z_m^{-1}(x)}\big)\big|_{\partial\Omega_\tau} \\ &= \sum_{m\in\mathfrak{N}} [\eta^m(x)\xi^m(x)\overline{\psi}(x,t)]\big|_{\partial\Omega_\tau} = \overline{\psi}(x,t)\big|_{\partial\Omega_\tau}\end{aligned} \tag{5.18}$$

and

$$\mathcal{A}\mathfrak{R}g = \mathcal{A}_0\mathfrak{R}g - \mathcal{K}_1 \star \mathcal{L}_2\mathfrak{R}g, \tag{5.19}$$

having denoted

$$\mathcal{A}_0 = \mathbf{D}_t^\nu - \mathcal{L}_1.$$

In **[28, 29]** it was shown that

$$\mathcal{A}_0\mathfrak{R}g = \overline{f} + \mathfrak{T}_1^1 g, \qquad \|\mathfrak{T}_1^1 g\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega}_\tau)} \leq \frac{1}{4}\|g\|_{\mathcal{H}}, \tag{5.20}$$

if λ and τ satisfy (4.24). Hence, if we are able to prove the estimate

$$\|\mathcal{K}_1 \star \mathcal{L}_2\mathfrak{R}g\|_{\mathcal{H}} \leq \frac{1}{4}\|g\|_{\mathcal{H}}, \tag{5.21}$$

then (ii) immediately follows from (5.18)–(5.21) and

$$\mathfrak{T}_1 g = \mathfrak{T}_1^1 g - \mathcal{K}_1 \star \mathcal{L}_2\mathfrak{R}g, \qquad \|\mathfrak{T}_1 g\|_{\mathcal{H}} \leq \frac{1}{2}\|g\|_{\mathcal{H}}. \tag{5.22}$$

In order to verify (5.21), we rewrite the term $\mathcal{K}_1 \star \mathcal{L}_2\mathfrak{R}g$ as

$$\mathcal{K}_1 \star \mathcal{L}_2\mathfrak{R}g = \mathfrak{T}_1^2 g + \mathfrak{T}_1^3 g, \tag{5.23}$$

where

$$\mathfrak{T}_1^2 g = \int_0^t \mathcal{K}_1(t-s)\Big[\sum_{i=1}^n b_i(x,s)\frac{\partial\mathfrak{R}g}{\partial x_i} + b_0(x,s)\mathfrak{R}g\Big]\,ds$$

and

$$\begin{aligned}\mathfrak{T}_1^3 g = \sum_m \sum_{ij=1}^n \int_0^t \mathcal{K}_1(t-s)b_{ij}(x,s)\Big[&\eta^m(x)\frac{\partial^2 w_m}{\partial x_i\partial x_j}(x,s) \\ &+ 2\frac{\partial\eta^m}{\partial x_j}\frac{\partial w_m}{\partial x_i}(x,s) + w_m(x,s)\frac{\partial^2\eta^m}{\partial x_i\partial x_j}\Big]\,ds.\end{aligned}$$

Propositions 4.4–4.7 along with (4.1), (4.24) and (5.14), where $l = 0$, lead to estimate

$$\begin{aligned}&\|\mathfrak{T}_1^2 g\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega}_\tau)}\\ &\le C\kappa\sum_{i=0}^{n}\|b_i\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega}_\tau)}\|\mathcal{K}_1\|_{L_1(0,\tau)}\|\mathfrak{R}g\|_{\mathcal{C}^{2+\alpha,(2+\alpha)\nu/2)}(\overline{\Omega}_\tau)}\\ &\le C\kappa\|\mathcal{K}_1\|_{L_1(0,\tau)}\|g\|_{\mathcal{H}},\end{aligned}\tag{5.24}$$

where the positive constant C depends only on $\|b_i\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega}_T)}$, $i = 0,\dots,n$.

In order to evaluate the term $\mathfrak{T}_1^3 g$, we apply the easily verified inequalities:

$$\left\|\eta^m\frac{\partial^2 w_m}{\partial x_i\partial x_j}\right\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega^m}_\tau)}\le C[1+\kappa^{\alpha\nu/2}]\|D_x^2 w_m\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega^m}_\tau)},\tag{5.25}$$

$$\begin{aligned}&\left\|\frac{\partial\eta^m}{\partial x_j}\frac{\partial w_m}{\partial x_i}\right\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega^m}_\tau)}\\ &\le C\left[(\kappa^{(1+\alpha)\nu/2}+\kappa^{\nu/2})\langle D_x w_m\rangle^{((1+\alpha)\nu/2)}_{t,\Omega^m_\tau}+\kappa^{\alpha\nu/2}\langle D_x^2 w_m\rangle^{(\alpha\nu/2)}_{t,\Omega^m_\tau}\right],\end{aligned}\tag{5.26}$$

$$\begin{aligned}&\left\|w_m\frac{\partial^2\eta^m}{\partial x_i\partial x_j}\right\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega^m}_\tau)}\\ &\le C\left[\kappa^{(1+\alpha)\nu/2}\langle D_x w_m\rangle^{((1+\alpha)\nu/2)}_{t,\Omega^m_\tau}+(1+\lambda^\alpha)\kappa^{(2+\alpha)\nu/2}\langle \mathbf{D}_t^\nu w_m\rangle^{(\alpha\nu/2)}_{t,\Omega^m_\tau}\right],\end{aligned}\tag{5.27}$$

where the positive constant C is independent of λ and τ. Therefore, taking into account Propositions 4.5–4.7, Lemmas 4.1 and 4.2 and estimates (5.25)–(5.27), we deduce

$$\begin{aligned}&\|\mathfrak{T}_1^3 g\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega}_\tau)}\le C[1+\kappa^{\alpha\nu/2}+\kappa^{(1+\alpha)\nu/2}+(1+\lambda^\alpha)\kappa^{(2+\alpha)\nu/2}]\\ &\quad\times\sum_{ij=1}^{n}\|b_{ij}\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega}_T)}\|\mathcal{K}_1\|_{L_1(0,\tau)}\{w_m\}_{\mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}(\overline{\Omega}_\tau)}\\ &\quad\le C\|\mathcal{K}_1\|_{L_1(0,\tau)}\|g\|_{\mathcal{H}},\end{aligned}\tag{5.28}$$

where the positive constant C depends only on $\|b_{ij}\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega}_T)}$, $i,j=1,\dots,n$.

Furthermore, keeping in mind inequalities (5.23), (5.24) and (5.28), we deduce from (4.24) that

$$\|\mathcal{K}_1\star\mathcal{L}_2\mathfrak{R}g\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega}_\tau)}\leq C^*(1+\kappa^{\alpha\nu/2})\|\mathcal{K}_1\|_{L_1(0,\kappa\lambda^{2/\nu})}\|g\|_{\mathcal{H}}, \tag{5.29}$$

where the positive constant $C^*=C^*(\|b_{ij}\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega}_T)},\|b_i\|_{\mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega}_T)})$ is independent of λ and τ. In light of (3.5), appealing on the absolute continuity of the Lebesgue integral, we choose λ small enough such that

$$\|\mathcal{K}_1\|_{L_1(0,\kappa\lambda^{2/\nu})}\leq\frac{1}{4C^*}. \tag{5.30}$$

Hence, inequalities (5.29) and (5.30) provide the estimate (5.21). This completes the proof of point (ii).

As for statement (iii), it is verified in the same manner using analogous arguments from [**30,** Chapter 4]. This completes the proof of Lemma 5.2. □

Relations (5.15) and (5.16) guarantee that the operators $\mathcal{I}+\mathfrak{T}_1$ and $\mathcal{I}+\mathfrak{T}_2$ (here, $\mathcal{I}$ is the identity) are invertible for a suitably small time τ, and $(\mathcal{I}+\mathfrak{T}_1)^{-1}$ and $(\mathcal{I}+\mathfrak{T}_2)^{-1}$ are bounded. Therefore, this yields the equalities

$$\mathcal{L}\mathfrak{R}(\mathcal{I}+\mathfrak{T}_1)^{-1}g=g,\qquad(\mathcal{I}+\mathfrak{T}_2)^{-1}\mathfrak{R}\mathcal{L}w=w,$$

implying that $\mathcal{L}$ has bounded right and left inverse operators such that

$$\mathfrak{R}(\mathcal{I}+\mathfrak{T}_1)^{-1}=(\mathcal{I}+\mathfrak{T}_2)^{-1}\mathfrak{R}=\mathcal{L}^{-1}. \tag{5.31}$$

Accordingly, the unique solution of (5.8) is given by

$$w=\mathcal{L}^{-1}(\overline{f},\overline{\psi}). \tag{5.32}$$

Returning to representation (5.1), we obtain a unique solution to the original problem (1.1)–(1.3) for $t\in[0,\tau]$:

$$u(x,t)=W(x,t)+\mathcal{L}^{-1}(\overline{f},\overline{\psi}).$$

The estimate of the norm of $\mathcal{L}^{-1}$ follows from (5.14)–(5.16). Moreover, inequality (3.12) follows from (5.1), (5.7) and (5.14). In summary, we have proved Theorem 3.1 for a small time interval $[0,\tau]$. In order to obtain the result in the general case, i.e., for $t\in[0,T]$, all we need is to extend the constructed solution (5.32) on the intervals $[\tau, 2\tau]$, $[2\tau, 3\tau]$, etc.

By virtue of [**30,** Theorem 4.1], we can extend the solution $w(x,t)$ from Ω to $\mathbb{R}^n$ for $t \in [0,\tau]$. We denote this extension by $\overline{w}(x,t)$. Note that $\overline{w}(x,t)$ satisfies

$$\overline{w} \in \mathcal{C}_0^{2+\alpha,(2+\alpha)\nu/2}(\overline{\mathbb{R}}^n_\tau),$$
$$\|\overline{w}\|_{\mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}(\overline{\mathbb{R}}^n_\tau)} \leq C\|w\|_{\mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}(\overline{\Omega}_\tau)}$$

and

$$\overline{w}(x,t) = w(x,t) \quad \text{if } (x,t) \in \overline{\Omega}_\tau. \tag{5.33}$$

In particular, (5.33) implies that

$$\mathbf{D}_t^\nu \overline{w}(x,t) = \mathbf{D}_t^\nu w(x,t) \quad \text{if } (x,t) \in \overline{\Omega}_\tau. \tag{5.34}$$

Then, we set

$$F_0(x,t) := \begin{cases} \mathbf{D}_t^\nu \overline{w}(x,t) - \Delta\overline{w}(x,t), & x \in \mathbb{R}^n,\ t \in [0,\tau], \\ \{\mathbf{D}_t^\nu \overline{w}(x,t) - \Delta\overline{w}(x,t)\}|_{t=\tau}, & x \in \mathbb{R}^n,\ t \in (\tau, 2\tau]. \end{cases}$$

It can easily be verified that $F_0 \in \mathcal{C}^{\alpha,\alpha\nu/2}(\overline{\Omega}_{2\tau})$.

Next, we define the function $\theta(x,t)$ as the solution to the Cauchy problem

$$\begin{cases} \mathbf{D}_t^\nu \theta(x,t) - \Delta\theta(x,t) = F_0(x,t), & (x,t) \in \mathbb{R}^n_{2\tau}, \\ \theta(x,0) = \overline{w}(x,0), & x \in \mathbb{R}^n. \end{cases} \tag{5.35}$$

Lemma 4.2 provides the existence of a unique classical solution $\theta(x,t)$ to problem (5.35) such that $\theta \in \mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}\,(\overline{\mathbb{R}}^n_{2\tau})$ and

$$\theta(x,t) = \overline{w}(x,t), \quad \text{if } t \in [0,\tau]. \tag{5.36}$$

Then, we seek the solution to (5.8) in the form

$$w(x,t) = V(x,t) + \theta(x,t),$$

where the unknown function $V(x,t)$ satisfies

$$\mathbf{D}_t^\nu V(x,t) - \mathcal{L}_1 V(x,t) - (\mathcal{K}_1 \star \mathcal{L}_2 V)(x,t) = \varphi(x,t) \quad \text{in } \Omega_{2\tau}, \tag{5.37}$$
$$V(x,0) = 0, \quad x \in \overline{\Omega}, \tag{5.38}$$
$$V(x,t) = \Psi(x,t), \quad (x,t) \in \partial\Omega_{2\tau}. \tag{5.39}$$

Here,

$$\varphi(x,t) := \overline{f}(x,t) - \mathbf{D}_t^{\nu}\theta(x,t) + \mathcal{L}_1\theta(x,t) + (\mathcal{K}_1 \star \mathcal{L}_2\theta)(x,t) \tag{5.40}$$

and

$$\Psi(x,t) := \overline{\psi}(x,t) - \theta(x,t). \tag{5.41}$$

Based on (5.33), (5.36), (5.40) and (5.41), we deduce that, for $t \in [0,\tau]$,

$$\begin{aligned} \varphi(x,t) &= 0, \quad x \in \overline{\Omega}, \\ \Psi(x,t) &= 0, \quad x \in \partial\Omega. \end{aligned} \tag{5.42}$$

Then, applying the above arguments (see Lemma 5.2, and so on) to problem (5.37)–(5.39), we can conclude that

$$V(x,t) = 0 \quad \text{for } x \in \overline{\Omega},\ t \in [0,\tau]. \tag{5.43}$$

Finally, we introduce a new variable $\sigma = t-\tau$, $\sigma \in [-\tau,\tau]$ in problem (5.37)–(5.39). Denote

$$\begin{aligned} \overline{V}(x,\sigma) &= V(x,\sigma+\tau), \\ \overline{\Psi}(x,\sigma) &= \Psi(x,\sigma+\tau), \\ \overline{\varphi}(x,\sigma) &= \varphi(x,\sigma+\tau), \end{aligned} \tag{5.44}$$

$$\overline{\mathcal{L}}_1 = \sum_{ij=1}^{n} \overline{a}_{ij}(x,\sigma)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \overline{a}_i(x,\sigma)\frac{\partial}{\partial x_i} + \overline{a}_0(x,\sigma),$$

$$\overline{\mathcal{L}}_2 = \sum_{ij=1}^{n} \bar{b}_{ij}(x,\sigma)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{n} \bar{b}_i(x,\sigma)\frac{\partial}{\partial x_i} + \bar{b}_0(x,\sigma),$$

where

$$\begin{aligned} \overline{a}_{ij}(x,\sigma) &:= a_{ij}(x,\sigma+\tau), \\ \overline{a}_i(x,\sigma) &:= a_i(x,\sigma+\tau), \\ \overline{a}_0(x,\sigma) &:= a_0(x,\sigma+\tau), \\ \bar{b}_{ij}(x,\sigma) &:= b_{ij}(x,\sigma+\tau), \\ \bar{b}_i(x,\sigma) &:= b_i(x,\sigma+\tau), \\ \bar{b}_0(x,\sigma) &:= b_0(x,\sigma+\tau). \end{aligned}$$

It is easy to see that the coefficients of the operators $\overline{\mathcal{L}}_k$, $k = 1, 2$, and the functions $\overline{h}$, $\overline{\Psi}$ and $\overline{\varphi}$ meet the requirements of Theorem 3.1. Moreover, equalities (5.42)–(5.43) ensure that

$$
\begin{gathered}
\overline{\varphi}(x,\sigma) = \overline{\Psi}(x,\sigma) = \overline{V}(x,\sigma) = 0 \quad \text{if } \sigma \in [-\tau, 0], \\
\mathbf{D}^{\nu}_{\sigma}\overline{V}(x,\sigma) = \mathbf{D}^{\nu}_{t}V(x,t) \quad \text{if } \sigma \in [-\tau,\tau],\ t \in [0, 2\tau].
\end{gathered}
\tag{5.45}
$$

The latter equality in (5.45) is analogously verified to the proof of formula (3.111) in **[29]**.

Next, we recalculate the term $(\mathcal{K}_1 \star \mathcal{L}_2 V)(x,t)$ in the new variable σ:

$$
\begin{aligned}
(\mathcal{K}_1 \star \mathcal{L}_2 V)(x,t) &= \int_0^{\sigma+\tau} \mathcal{K}_1(\sigma+\tau-s)\mathcal{L}_2 V(x,s)\,ds \\
&= \int_{-\tau}^{0} \mathcal{K}_1(\sigma - z)\mathcal{L}_2 V(x, z+\tau)\,dz + (\mathcal{K}_1 \star \overline{\mathcal{L}}_2\overline{V})(x,\sigma) \\
&= (\mathcal{K}_1 \star \overline{\mathcal{L}}_2\overline{V})(x,\sigma).
\end{aligned}
\tag{5.46}
$$

In order to obtain the last equality in (5.46), we used (5.43) and representations (5.44).

Thus, based on (5.43)–(5.46), we can rewrite problem (5.37)–(5.39) in the new variable as

$$
\begin{cases}
\mathbf{D}^{\nu}_{\sigma}\overline{V}(x,\sigma) - \overline{\mathcal{L}}_1\overline{V}(x,\sigma) - (\mathcal{K}_1 \star \overline{\mathcal{L}}_2\overline{V})(x,\sigma) = \overline{\varphi}(x,\sigma) & \text{in } \Omega_{\tau}, \\
\overline{V}(x,0) = 0, & x \in \overline{\Omega}, \\
\overline{V}(x,\sigma) = \overline{\Psi}(x,\sigma) & \text{on } \partial\Omega_{\tau}.
\end{cases}
\tag{5.47}
$$

Now, we can apply Lemma 5.2 to problem (5.47) and obtain the one-to-one classical solvability in $\mathcal{C}^{2+\alpha,(2+\alpha)\nu/2}$ for $\sigma \in [0,\tau]$. In other words, we have extended the solution $w(x,t)$ from $[0,\tau]$ to $[\tau, 2\tau]$. By the same token, we repeat the procedure to continue the constructed solution on the intervals $[i\tau, (i+1)\tau]$, $i = 2, 3, \ldots$, until the entire $[0,T]$ is exhausted. This allows us to obtain the classical solution $u(x,t)$ on $[0,T]$, which satisfies inequality (3.12). This completes the proof of Theorem 3.1.

As for Theorem 3.2, it is proven with the same arguments, by applying Lemma 4.1 for $l = 1$ and inequality (4.14) in place of (4.13).

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