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ENERGY DECAY RATES FOR SOLUTIONS OF THE KIRCHHOFF TYPE WAVE EQUATION WITH BOUNDARY DAMPING AND SOURCE TERMS

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ABSTRACT. In this work, we are concerned with uniform stabilization for an initial-boundary value problem associated with the Kirchhoff type wave equation with feedback terms and memory condition at the boundary. We prove that the energy decays exponentially when the boundary damping term has a linear growth near zero and polynomially when the boundary damping term has a polynomial growth near zero. Furthermore, we study the decay rate of the energy without imposing any restrictive growth assumption on the damping term near zero.

1. Introduction. In this paper, we are concerned with asymptotic behavior of the Kirchhoff type wave equation with

$$(1.1)\qquad \begin{cases} u'' - \left(1 + \beta \displaystyle\int_\Omega |\nabla u|^2 dx\right)\Delta u = |u|^\rho u & \text{in } \Omega \times \mathbb{R}_+, \\ u = 0 & \text{on } \Gamma_0 \times \mathbb{R}_+, \\ \dfrac{\partial u}{\partial \nu} + \displaystyle\int_0^t k(t-s,x)u'(s)\, ds + a(x)g(u') = 0 & \text{on } \Gamma_1 \times \mathbb{R}_+, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x), & x \in \Omega. \end{cases}$$

Here, $\Omega \subset \mathbb{R}^n$ is an open bounded domain, $n \geq 1$, with a boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ of class C^2, where Γ_0 and Γ_1 are closed and disjoint. The map

$$a : \Gamma_1 \longrightarrow \mathbb{R}_+ \in L^\infty(\Gamma_1)$$

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is such that $a(x) \geq a_0 > 0$,

$$k : \Gamma_1 \times \mathbb{R}_+ \longrightarrow \mathbb{R}_+ \in C^2(\mathbb{R}_+, L^\infty(\Gamma_1)),$$

and

$$g : \mathbb{R} \longrightarrow \mathbb{R}$$

is a nondecreasing function. We shall denote by ν the unit outward normal vector to Γ. Δ and ∇ stand for the Laplacian and gradient with respect to the spatial variables, respectively, $'$ denotes the derivative with respect to time t, and $\mathbb{R}_+ = [0, \infty)$.

The integro-differential equation $(1.1)_1$ (the first equation of (1.1)) describes vertical vibrations of a flexible string, but does not portray the existence of a whirling out of plan motion. This phenomenon was observed by Harrison **[22]** and Nayfeh and Mook **[30]**. On the other hand, the boundary condition $(1.1)_3$ (the third equation of (1.1)) describes the reflection of sound at surfaces of some materials with memory of interest in engineering practice. It is quite general and covers a fairly large variety of physical configurations. Problems containing the boundary condition $(1.1)_3$ were studied by many authors (e.g., Aassila, et al., **[1]**, Alabau-Boussouira **[2]**, Nicaise and Pignotti **[31]**, Park and Ha **[35]**, Park, et al., **[37]**, Prüss **[38]**, and the references therein).

The problem of proving uniform decay rates for solutions to the wave equation with a boundary dissipation has attracted much attention in recent years. The linear problem has been treated by many authors, see for instance, **[25, 26, 41]**. The nonlinear boundary conditions were studied by **[1, 24, 45]**. Aassila, et al., **[1]** and Alabau-Boussouira, et al., **[2]** studied the uniform decay for the solutions of the wave equation (1.1) without the source terms $|u|^\rho u$ and $\beta = 0$. Park, et al., **[37]** studied (1.1) with $\beta = 0$ and the same boundary condition, and found that the function g does not have polynomial growth near zero. For comprehensive studies of nonlinear wave equations, the reader is referred to **[7, 11, 12, 13, 14, 15, 16, 17, 18, 19, 28, 34, 36, 43, 44]** and the references therein. In particular, Autuori and Pucci **[3, 4]**, Bae **[5]**, Bae and Nakao **[6]**, Gorain **[9, 10]**, Ha and Park **[20, 21]**, Lasiecka and Ong **[27]**, Park, et al., **[33]** and Santos, et al., **[40]** researched Kirchhoff type problems. For instance, the case of the n-dimensional quasilinear wave equation of Kirchhoff type with a suitable nonlinear boundary dissipation was treated in **[27]** (this case is equation (1.1) without source terms $|u|^\rho u$, $k(t,x) = 0$ and $a(x) = 1$). Global existence, uniqueness and uniform decay of solutions for such a problem were examined, subject to some restriction on the norms of the initial data.

There have been very few results when the function g has no polynomial growth near zero. Lasiecka and Tataru [**28**] studied the more general case of a semilinear wave equation damped with a nonlinear velocity feedback acting on Γ_1, under some very weak geometrical conditions on Γ_0 and Γ_1. Without the assumption that g has a polynomial behavior near zero, they proved that the energy decays as fast as the solution of an associated differential equation. More precisely, they generalized the method used to obtain uniform decay estimates when g has a polynomial behavior near zero. However, they did not obtain an explicit decay rate estimate for the energy. On the other hand, Martinez [**29**] studied the linear wave equation with a boundary damping term. He proved the explicit decay estimate of the energy even if the damping term g has no polynomial growth near zero. In order to obtain the explicit decay estimate, he used the construction of a special weight function and the generalization of a technique of partition of the boundary.

In this paper, we prove the uniform decay rates of solutions for the Kirchhoff type wave equation with linear or polynomial growth on the damping term near zero. Moreover, we study the energy decay rate without imposing any restrictive growth assumption on the damping term near zero. The goal of this paper is to extend the results of [**37**] by applying the method developed in [**29**].

This paper is organized as follows. In Section 2, we recall the hypotheses used to prove our results and introduce our main results. In Section 3, under hypothesis ($\mathbf{H_3}$), we prove the exponential or polynomial decay rate using Komornik's method and Lasiecka and Tataru's method. In Section 4, using Martinez's method, we prove the energy decay rate, which has a more general case of g on the hypothesis ($\overline{\mathbf{H_3}}$).

2. Hypotheses and main results. We begin this section by introducing some hypotheses and our main results. Throughout this paper, we use standard functional spaces and denote $||\cdot||_p$ as the $L^p(\Omega)$ norm. We consider the Hilbert space

$$H^1_{\Gamma_0}(\Omega) := \{u \in H^1(\Omega); u = 0 \text{ on } \Gamma_0\}.$$

($\mathbf{H_1}$) Hypotheses on Ω. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain, $n \geq 1$, with a boundary $\Gamma = \Gamma_0 \cup \Gamma_1$ of class C^2, where Γ_0 and Γ_1 are closed and disjoint satisfying the following conditions:

$$\Gamma_0 \neq \emptyset \quad \text{or} \quad \inf_{\Gamma_1 \times \mathbb{R}_+} k \neq 0, \tag{2.1}$$

$$\begin{aligned} m \cdot \nu \geq \delta > 0 \quad & \text{on } \Gamma_1, \\ m \cdot \nu \leq 0 \qquad & \text{on } \Gamma_0, \\ m := m(x) = x - x^0, \quad & \text{for any } x^0 \in \mathbb{R}^n. \end{aligned} \tag{2.2}$$

($\mathbf{H_2}$) Hypotheses on k. $k : \Gamma_1 \times \mathbb{R}_+ \to \mathbb{R}_+ \in C^2(\mathbb{R}_+, L^\infty(\Gamma_1))$, such that

$$k' \leq 0 \quad \text{on } \Gamma_1 \times \mathbb{R}_+, \tag{2.3}$$

$$k'' \geq -\alpha k' \quad \text{on } \Gamma_1 \times \mathbb{R}_+, \text{ for some } \alpha > 0, \tag{2.4}$$

$$\alpha \inf_{\Gamma_1} k(0) > -2 \inf_{\Gamma_1} k'(0). \tag{2.5}$$

($\mathbf{H_3}$) Hypotheses on g. Let $g : \mathbb{R} \to \mathbb{R}$ be a continuous, nondecreasing function such that

$$g(0) = 0, \qquad |g(x)| \leq 1 + C_1|x| \quad \text{and} \quad g(s)s > 0 \text{ for } s \neq 0, \tag{2.6}$$

$$C_2|x|^p \leq |g(x)| \leq C_3|x|^{1/p} \quad \text{if } |x| \leq 1, \tag{2.7}$$

$$C_4|x| \leq |g(x)| \leq C_5|x| \quad \text{if } |x| \geq 1, \tag{2.8}$$

where $p \geq 1$ and $C_i (1 \leq i \leq 5)$ are five positive constants.

($\mathbf{H_4}$) Hypotheses on $a(x)$**,** β**,** ρ. Suppose that $a : \Gamma_1 \to \mathbb{R}_+ \in L^\infty(\Gamma_1)$ is such that $a(x) \geq a_0 > 0$, $\beta > 0$ is a real number and

$$0 < \rho < \frac{2}{n-2} \quad \text{if } n \geq 3 \quad \text{and} \quad \rho > 0 \text{ if } n = 1, 2. \tag{2.9}$$

The next lemma is used to estimate the energy identity.

Lemma 2.1. *For $\psi, \varphi \in C^1([0,\infty) : \mathbb{R})$, we have*

$$\left(\int_0^t \psi(t-s)\varphi(s)\,ds\right)\varphi' = \frac{1}{2}\int_0^t \psi'(t-s)|\varphi(t)-\varphi(s)|^2 ds - \frac{1}{2}\psi(t)|\varphi|^2 - \frac{1}{2}\frac{d}{dt}\left[\int_0^t \psi(t-s)|\varphi(t)-\varphi(s)|^2 ds - \left(\int_0^t \psi(s)\,ds\right)|\varphi|^2\right].$$

Proof. Differentiating the term $\int_0^t \psi(t-s)|\varphi(t)-\varphi(s)|^2 ds$, we arrive at the above equality. □

We define the energy of the solution by the following formula

$$\begin{aligned} E(t) := {} & \frac{1}{2}\int_\Omega |u'(t,x)|^2 dx \\ & + \frac{1}{2}\int_\Omega |\nabla u(t,x)|^2 dx + \frac{\beta}{4}\left(\int_\Omega |\nabla u(t,x)|^2 dx\right)^2 \\ & - \frac{1}{2}\left(1+\beta\int_\Omega |\nabla u|^2 dx\right) \\ & \cdot \int_{\Gamma_1}\int_0^t k'(t-s,x)|u(t,x)-u(s,x)|^2 ds\, d\Gamma \\ & + \frac{1}{2}\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_{\Gamma_1} k(t,x)|u(t,x)-u_0(x)|^2 d\Gamma \\ & - \frac{1}{\rho+2}\int_\Omega |u(t,x)|^{\rho+2} dx. \end{aligned} \tag{2.10}$$

According to (2.9), we have the imbedding:

$$H^1_{\Gamma_0}(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega) \hookrightarrow L^{\rho+2}(\Omega).$$

Let $B_1 > 0$ be the optimal constant of Sobolev immersion which satisfies the inequality

$$||v||_{\rho+2} \le B_1||\nabla v||_2, \quad \text{for all } v \in H^1_{\Gamma_0}(\Omega).$$

From the above inequality, we have

$$K_0 := \sup_{v\in H^1_{\Gamma_0}(\Omega) v\neq 0}\left(\frac{(1/\rho+2)||v||^{\rho+2}_{\rho+2}}{||\nabla v||_2^{\rho+2}}\right) \le \frac{B_1^{\rho+2}}{\rho+2}.$$

Note that $K_0 > 0$, and

$$\frac{1}{\rho+2}||v||_{\rho+2}^{\rho+2} \le K_0||\nabla v||_2^{\rho+2}, \quad \text{for all } v \in H^1_{\Gamma_0}(\Omega). \tag{2.11}$$

We consider the functional

$$J(u) = \frac{1}{2}||\nabla u||_2^2 - \frac{1}{\rho+2}||u||_{\rho+2}^{\rho+2}, \quad u \in H^1_{\Gamma_0}(\Omega), \tag{2.12}$$

and define the positive number

$$d := \inf_{\substack{v \in H^1_{\Gamma_0}(\Omega) \\ v \neq 0}} \left\{ \sup_{\lambda > 0} J(\lambda v) \right\}.$$

Setting

$$f(\lambda) = \frac{1}{2}\lambda^2 - K_0\lambda^{\rho+2}, \quad \lambda > 0,$$

then

$$\lambda_1 = \left(\frac{1}{K_0(\rho+2)} \right)^{1/\rho}$$

is the absolute maximum point of f and $d = f(\lambda_1) > 0$.

It is well known that the number d is the Mountain Pass level associated to the elliptic problem

$$\begin{cases} -\Delta u = |u|^\rho u & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1. \end{cases}$$

In fact (see **[42]**),

$$d = \inf_{\gamma \in \Lambda} \sup_{t \in [0,1]} J(\gamma(t)),$$

where

$$\Lambda = \{\gamma \in C([0,1];\ H^1_{\Gamma_0}(\Omega));\ \gamma(0) = 0,\ J(\gamma(1)) < 0\}.$$

Furthermore, we obtain

$$d = f(\lambda_1) = \frac{\rho}{2(\rho+2)}\lambda_1^2.$$

The energy associated to problem (1.1) is given by

$$\begin{aligned} E(t) &= \frac{1}{2}\int_\Omega |u'(t,x)|^2 dx \\ &+ \frac{1}{2}\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_{\Gamma_1} k(t,x)|u(t,x)-u_0(x)|^2 d\Gamma \\ &- \frac{1}{2}\left(1+\beta\int_\Omega |\nabla u|^2 dx\right) \\ &\cdot \int_{\Gamma_1}\int_0^t k'(t-s,x)|u(t,x)-u(s,x)|^2 ds\, d\Gamma + J(u(t)), \end{aligned}$$

for $u \in H^1_{\Gamma_0}(\Omega)$. From (2.3) and (2.11), we have

$$E(t) \geq J(u(t)) \geq \frac{1}{2}||\nabla u||_2^2 - K_0||\nabla u||_2^{\rho+2} = f(||\nabla u(t)||_2). \tag{2.13}$$

Now, if one considers

$$||\nabla u(t)||_2 < \lambda_1, \tag{2.14}$$

then, from (2.13) and (2.14), we have

$$E(t) \geq J(u(t)) > ||\nabla u(t)||_2^2\left(\frac{1}{2} - \lambda_1^\rho K_0\right) = ||\nabla u(t)||_2^2\left(\frac{1}{2} - \frac{1}{\rho+2}\right). \tag{2.15}$$

Thus, if (2.14) is satisfied, from the above inequality, we deduce that

$$J(t) \geq 0 \quad \text{and} \quad ||\nabla u(t)||_2^2 \leq \frac{2(\rho+2)}{\rho}E(t). \tag{2.16}$$

In order to obtain global existence for regular solutions, the following assumption is made on the initial data:

$$E(0) < d \quad \text{and} \quad ||\nabla u_0||_2 < \lambda_1. \tag{2.17}$$

We now state the global existence result, which can be obtained by **[27, 37, 38, 40]**: let us consider whether the hypotheses (H_1)–(H_4) and (2.17) hold. If the initial data $\{u_0, u_1\}$ belong to $H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, then problem (1.1) possesses a unique weak solution in the class

$$u \in C(\mathbb{R}_+, H^1_{\Gamma_0}(\Omega)) \cap C^1(\mathbb{R}_+, L^2(\Omega)),$$

with $||\nabla u(t)||_2 < \lambda_1$ for all $t > 0$.

Furthermore, if the initial data $\{u_0, u_1\}$ belong to $H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega) \times H^1_{\Gamma_0}(\Omega)$ and g is globally Lipschitz continuous, then the solution has the following regularity (**[1, 5, 37]**):

$$\begin{aligned} u &\in L^\infty(\mathbb{R}_+, H^1_{\Gamma_0}(\Omega)), \\ u' &\in L^\infty_{\text{loc}}(\mathbb{R}_+, H^1_{\Gamma_0}(\Omega)), \\ u'' &\in L^\infty_{\text{loc}}(\mathbb{R}_+, L^2(\Omega)), \end{aligned}$$

with $||\nabla u(t)||_2 < \lambda_1$ for all $t > 0$.

Now, we are in a position to state our main results.

Theorem 2.2. *Assume that hypotheses* $(\mathbf{H_1})$–$(\mathbf{H_4})$ *and* (2.17) *hold. Then, we have that, if* $p = 1$, *there exist positive constants* C_6 *and* ω *such that*

$$E(t) \leq C_6 E(0) e^{-\omega t}. \tag{2.18}$$

If $p > 1$, *there exists a positive constant* C_7 *such that*

$$E(t) \leq \frac{C_7 E(0)}{(1+t)^{2/(p-1)}}. \tag{2.19}$$

We will now consider the more general case of g.

$(\overline{\mathbf{H_3}})$**.** Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing C^1 function such that $g(0) = 0$, and suppose that there exists a strictly increasing and odd function α of C^1 class on $[-1, 1]$ such that

$$\begin{aligned} |\alpha(s)| \leq |g(s)| &\leq |\alpha^{-1}(s)| \quad \text{if } |s| \leq 1, \\ C_8|s| \leq |g(s)| &\leq C_9|s| \qquad \text{if } |s| > 1, \end{aligned}$$

where α^{-1} denotes the inverse function of α and C_8 and C_9 are positive constants.

Theorem 2.3. *Assume that hypotheses* $(\mathbf{H_1})$, $(\mathbf{H_2})$, $(\mathbf{H_4})$, $(\overline{\mathbf{H_3}})$ *and* (2.17) *hold. Then, we have*

$$E(t) \leq C_{10} \left(F^{-1}\left(\frac{1}{t}\right)\right)^2, \tag{2.20}$$

where $F(s) := s\alpha(s)$, *and* C_{10} *is a positive constant. Moreover, if the function* $G(s) := \alpha(s)/s$ *is nondecreasing on* $[0, \eta]$ *for some* $\eta > 0$ *and*

$G(0) = 0$, then we have

$$E(t) \le C_{11}\left(\alpha^{-1}\left(\frac{1}{t}\right)\right)^2, \tag{2.21}$$

where C_{11} is a positive constant.

3. Proof of Theorem 2.1. In order to simplify the computations, and without lost of generality, we will transform the boundary condition into another more practical one by considering $u_0 = 0$ on Γ_1. Thus, problem (1.1) is now transformed into

$$\begin{cases} u'' - \left(1 + \beta \int_\Omega |\nabla u|^2 dx\right)\Delta u = |u|^\rho u & \text{in } \Omega \times \mathbb{R}_+, \\ u = 0 & \text{on } \Gamma_0 \times \mathbb{R}_+, \\ \dfrac{\partial u}{\partial \nu} + \displaystyle\int_0^t k'(t-s,x)u(s,x)\,ds & \\ \quad +k(0)u + a(x)g(u') = 0 & \text{on } \Gamma_1 \times \mathbb{R}_+, \\ u(x,0) = u_0(x), \quad u'(x,0) = u_1(x) & x \in \Omega. \end{cases} \tag{3.1}$$

In order to solve the energy decay of (3.1), we use the following lemmas.

Lemma 3.1 (cf., **[23]**)**.** *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function, and assume that there exist two constants $\alpha > 0$ and $T > 0$ such that*

$$\int_t^{+\infty} E^{\alpha+1}(s)\,ds \le TE(0)^\alpha E(t) \quad \textit{for all } t \in \mathbb{R}_+.$$

Then, we have

$$E(t) \le E(0)\left(\frac{T + \alpha t}{T + \alpha T}\right)^{-1/\alpha} \qquad \textit{for all } t \ge T.$$

In particular, assume that

$$\int_t^{+\infty} E(s)\,ds \le TE(t) \quad \textit{for all } t \in \mathbb{R}_+.$$

Then,

$$E(t) \le E(0)e^{1-t/T} \quad \textit{for all } t \ge T.$$

Lemma 3.2 (Gagliardo-Nirenberg inequality **[8, 32]**). *Let* $1 \leq r < p < \infty$, $1 \leq q \leq p$ *and* $0 \leq m$. *Then*:

$$||v||_{W^{l,q}} \leq C||v||_{W^{m,p}}^{\theta}||v||_{L^r}^{1-\theta}$$

for $v \in W^{m,p}(\Omega) \cap L^r(\Omega)$, $\Omega \subset \mathbb{R}^N$, *where* C *is a positive constant and*

$$\theta = \left(\frac{l}{N} + \frac{1}{r} - \frac{1}{q}\right)\left(\frac{m}{N} + \frac{1}{r} - \frac{1}{p}\right)^{-1}$$

provided that $0 < \theta \leq 1$.

Using (3.1), it follows that

$$\begin{aligned}\frac{d}{dt}E(t) = &-\left(1 + \beta\int_{\Omega}|\nabla u|^2 dx\right)\\ &\cdot\int_{\Gamma_1}\frac{1}{2}k''(t-s,x)|u(t,x)-u(s,x)|^2 d\Gamma\\ &+\left(1 + \beta\int_{\Omega}|\nabla u|^2 dx\right)\\ &\cdot\int_{\Gamma_1}\frac{1}{2}k'(t,x)|u|^2 - a(x)g(u')u' d\Gamma.\end{aligned} \tag{3.2}$$

Hence, the energy is nonincreasing. Now, we will prove several estimates of the energy (3.1).

In the following section, the symbol C indicates positive constants, which may be different.

Lemma 3.3. *Setting* $Mu := 2(m \cdot \nabla u) + (n-1)u$, *the following holds*:

$$\int_S^T E^{(p+1)/2}(t)\,dt \leq C\int_S^T E^{(p-1)/2}(t)\int_{\Omega}|u|^2 dx\,dt + CE(S).$$

Proof. We shall divide the proof into several steps: (1) In order to derive the result, we multiply the equation (3.1) by $E^{(p-1)/2}(t)(Mu)$; (2) We analyze terms obtained by (1); (3) During the process of the estimation of (2), a significant boundary term remains. Thus, we calculate this term in the third step; (4) Using (2.10), we deduce the result of Lemma 3.3.

Step (1). (Multiplying $E^{(p-1)/2}(t)(Mu)$.) We multiply equation (3.1) by $E^{(p-1)/2}(t)(Mu)$ and then integrate the result obtained over $\Omega \times [S,T]$. Then, we have

$$
\begin{aligned}
0 &= \int_S^T E^{(p-1)/2}(t) \int_\Omega (Mu)\left(u'' - \left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\Delta u - |u|^\rho u\right) dx\,dt \\
&= \int_S^T E^{(p-1)/2}(t) \int_\Omega \big(2(m\cdot\nabla u) + (n-1)u\big) \\
&\quad \times \left(u'' - \left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\Delta u - |u|^\rho u\right) dx\,dt \\
&= \int_S^T E^{(p-1)/2}(t) \int_\Omega 2u''(m\cdot\nabla u)\,dx\,dt \\
&\quad + \int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_\Omega 2\nabla u\cdot\nabla(m\cdot\nabla u)\,dx\,dt \\
&\quad - \int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_\Gamma 2\frac{\partial u}{\partial\nu}(m\cdot\nabla u)\,d\Gamma\,dt \\
&\quad - \int_S^T E^{(p-1)/2}(t)\int_\Omega 2|u|^\rho u(m\cdot\nabla u)\,dx\,dt \\
&\quad + (n-1)\int_S^T E^{(p-1)/2}(t)\int_\Omega uu''dx\,dt \\
&\quad + (n-1)\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_\Omega |\nabla u|^2 dx\,dt \\
&\quad - (n-1)\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_{\Gamma_1}\frac{\partial u}{\partial\nu}u\,d\Gamma\,dt \\
&\quad - (n-1)\int_S^T E^{(p-1)/2}(t)\int_\Omega |u|^{\rho+2}dx\,dt.
\end{aligned} \tag{3.3}
$$

Note that

$$
\begin{aligned}
&\int_S^T E^{(p-1)/2}(t)\int_\Omega 2u''(m\cdot\nabla u)\,dx\,dt \\
&\quad = \left[E^{(p-1)/2}(t)\int_\Omega 2u'(m\cdot\nabla u)dx\right]_S^T \\
&\qquad - \frac{p-1}{2}\int_S^T E^{(p-3)/2}(t)E'(t)\int_\Omega 2u'(m\cdot\nabla u)\,dx\,dt \\
&\qquad - \int_S^T E^{(p-1)/2}(t)\int_\Omega 2u'(m\cdot\nabla u')\,dx\,dt,
\end{aligned}
$$

$$\begin{aligned}
&\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_\Omega 2\nabla u\cdot\nabla(m\cdot\nabla u)\,dx\,dt\\
&\quad=(2-n)\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_\Omega |\nabla u|^2 dx\,dt\\
&\qquad+\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_\Gamma (m\cdot\nu)|\nabla u|^2 d\Gamma\,dt
\end{aligned}$$

and

$$\begin{aligned}
&(n-1)\int_S^T E^{(p-1)/2}(t)\int_\Omega uu''dx\,dt\\
&\quad=(n-1)\left[E^{(p-1)/2}(t)\int_\Omega u'udx\right]_S^T\\
&\qquad-\frac{(n-1)(p-1)}{2}\int_S^T E^{(p-3)/2}(t)E'(t)\int_\Omega u'u\,dx\,dt\\
&\qquad-(n-1)\int_S^T E^{(p-1)/2}(t)\int_\Omega |u'|^2 dx\,dt.
\end{aligned}$$

Replacing the above calculations in (3.3), we obtain

$$\begin{aligned}
0=&\left[E^{(p-1)/2}(t)\int_\Omega 2u'(m\cdot\nabla u)\,dx\right]_S^T\\
&-\frac{p-1}{2}\int_S^T E^{(p-3)/2}(t)E'(t)\int_\Omega 2u'(m\cdot\nabla u)\,dx\,dt\\
&-\int_S^T E^{(p-1)/2}(t)\int_\Omega 2u'(m\cdot\nabla u')\,dx\,dt\\
&+(2-n)\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_\Omega |\nabla u|^2 dx\,dt\\
&+\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_\Gamma (m\cdot\nu)|\nabla u|^2 d\Gamma\,dt\\
&-\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_\Gamma 2\frac{\partial u}{\partial\nu}(m\cdot\nabla u)\,d\Gamma\,dt\\
&-\int_S^T E^{(p-1)/2}(t)\int_\Omega 2|u|^\rho u(m\cdot\nabla u)\,dx\,dt
\end{aligned}$$

$$
\begin{aligned}
&+ (n-1)\left[E^{(p-1)/2}(t)\int_\Omega u'u\,dx\right]_S^T \\
&- \frac{(n-1)(p-1)}{2}\int_S^T E^{(p-3)/2}(t)E'(t)\int_\Omega u'u\,dx\,dt \\
&- (n-1)\int_S^T E^{(p-1)/2}(t)\int_\Omega |u'|^2dx\,dt \\
&+ (n-1)\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2dx\right)\int_\Omega |\nabla u|^2dx\,dt \\
&- (n-1)\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2dx\right)\int_{\Gamma_1} \frac{\partial u}{\partial \nu}u\,d\Gamma\,dt \\
&- (n-1)\int_S^T E^{(p-1)/2}(t)\int_\Omega |u|^{\rho+2}dx\,dt \\
= &\left[E^{(p-1)/2}(t)\int_\Omega u'(Mu)\,dx\right]_S^T \\
&- \frac{p-1}{2}\int_S^T E^{(p-3)/2}(t)E'(t)\int_\Omega u'(Mu)\,dx\,dt \\
&+ n\int_S^T E^{(p-1)/2}(t)\int_\Omega |u'|^2dx\,dt \\
&- \int_S^T E^{(p-1)/2}(t)\int_{\Gamma_1} (m\cdot\nu)|u'|^2d\Gamma\,dt \\
&+ \int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2dx\right)\int_\Omega |\nabla u|^2dx\,dt \\
&+ \int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2dx\right)\int_{\Gamma_0} (m\cdot\nu)|\nabla u|^2d\Gamma\,dt \\
&+ \int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2dx\right)\int_{\Gamma_1} (m\cdot\nu)|\nabla u|^2d\Gamma\,dt \\
&- \int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2dx\right)\int_{\Gamma_0} 2\frac{\partial u}{\partial \nu}(m\cdot\nabla u)\,d\Gamma\,dt \\
&- \int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2dx\right)\int_{\Gamma_1} \frac{\partial u}{\partial \nu}(Mu)\,d\Gamma\,dt \\
&- \int_S^T E^{(p-1)/2}(t)\int_\Omega 2|u|^\rho u(m\cdot\nabla u)\,dx\,dt
\end{aligned}
$$

$$
\begin{aligned}
&- (n-1)\int_S^T E^{(p-1)/2}(t)\int_\Omega |u'|^2 dx\, dt \\
&- (n-1)\int_S^T E^{(p-1)/2}(t)\int_\Omega |u|^{\rho+2} dx\, dt.
\end{aligned}
$$

Using $\nabla u \cdot \nu = \partial u/\partial \nu$ on Γ_0, it follows that

$$
\begin{aligned}
(3.4)\quad & \int_S^T E^{(p-1)/2}(t)\int_\Omega \left(|u'|^2 - \frac{2}{\rho+2}|u|^{\rho+2}\right) dx\, dt \\
&+ \int_S^T E^{(p-1)/2}(t)\left(1+\frac{\beta}{2}\int_\Omega |\nabla u|^2 dx\right)\int_\Omega |\nabla u|^2 dx\, dt \\
&= -\left[E^{(p-1)/2}(t)\int_\Omega u'(Mu) dx\right]_S^T \\
&+ \frac{p-1}{2}\int_S^T E^{(p-3)/2}(t)E'(t)\int_\Omega u'(Mu)\, dx\, dt \\
&+ 2\int_S^T E^{(p-1)/2}(t)\int_\Omega |u|^\rho u(m\cdot\nabla u)\, dx\, dt \\
&+ \left((n-1)-\frac{2}{\rho+2}\right)\int_S^T E^{(p-1)/2}(t)\int_\Omega |u|^{\rho+2} dx\, dt \\
&- \frac{\beta}{2}\int_S^T E^{(p-1)/2}(t)\int_\Omega |\nabla u|^2 dx\int_\Omega |\nabla u|^2 dx\, dt \\
&+ \int_S^T E^{(p-1)/2}(t)\left(1+\frac{\beta}{2}\int_\Omega |\nabla u|^2 dx\right)\int_{\Gamma_0} (m\cdot\nu)|\nabla u|^2 d\Gamma\, dt \\
&+ \int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_{\Gamma_1} \frac{\partial u}{\partial \nu}(Mu)\, d\Gamma\, dt \\
&+ \int_S^T E^{(p-1)/2}(t)\int_{\Gamma_1} (m\cdot\nu)|u'|^2 d\Gamma\, dt \\
&- \int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_{\Gamma_1} (m\cdot\nu)|\nabla u|^2 d\Gamma\, dt.
\end{aligned}
$$

Step (2). (Analysis of (3.4).) We will estimate terms of the right hand side of (3.4).

Estimates for $I_1 := -\left[E^{(p-1)/2}(t)\int_\Omega u'(Mu)\,dx\right]_S^T$: From Young's inequality and Poincare's inequality, we obtain

$$\left|\int_\Omega u'(Mu)\,dx\right| \le \int_\Omega |u'|^2 dx + 2R^2\int_\Omega |\nabla u|^2 dx + \frac{(n-1)^2}{2}\int_\Omega |u|^2 dx \le CE(t), \tag{3.5}$$

where $R = \max_{x\in\overline{\Omega}} |m(x)|$. Since $E(t)$ is nonincreasing, we have

$$I_1 \le -C\left[E^{(p-1)/2}(t)E(t)\right]_S^T \le CE(S). \tag{3.6}$$

Estimates for $I_2 := \frac{p-1}{2}\int_S^T E^{(p-3)/2}(t)E'(t)\int_\Omega u'(Mu)\,dx\,dt$: By using (3.5), we obtain

$$\left|E^{(p-3)/2}(t)E'(t)\int_\Omega u'(Mu)\,dx\right| \le -C\Big(E^{(p+1)/2}(t)\Big)'.$$

Hence,

$$I_2 \le CE(S). \tag{3.7}$$

Estimates for $I_3 := 2\int_S^T E^{(p-1)/2}(t)\int_\Omega |u|^\rho u(m\cdot\nabla u)\,dx\,dt$:

$$\begin{aligned} |I_3| &\le 2R\int_S^T E^{(p-1)/2}(t)\int_\Omega |\nabla u||u|^{\rho+1}dx\,dt \\ &\le 2R\int_S^T E^{(p-1)/2}(t)||\nabla u||_2||u||_{2(\rho+1)}^{\rho+1}\,dt. \end{aligned} \tag{3.8}$$

On the other hand, by Lemma 3.2, we obtain that

$$||u||_q \le C||u||_2^{1-\theta}||u||_p^\theta, \quad \theta = \frac{p(q-2)}{q(p-2)}. \tag{3.9}$$

Let us consider $0 < \rho < 2/(n-2)$, where, if $n \ge 3$, $0 < s < 2n/(n-2) - 2(\rho+1)$ and (3.9) holds with $q = 2(\rho+1)$ and $p = 2(\rho+1)+s$, then

$$||u(t)||_{2(\rho+1)} \le C||u(t)||_2^{1-\theta}||u(t)||_{2(\rho+1)+s}^{\theta} \quad \text{for all } t \ge 0, \tag{3.10}$$

where $\theta = [\rho(2(\rho+1)+s)]/[(\rho+1)(2\rho+s)]$. From (3.10), we have

$$||u||_{2(\rho+1)}^{\rho+1} \le C||u||_2^{(1-\theta)(\rho+1)}||u||_{2(\rho+1)+s}^{\theta(\rho+1)}.$$

Considering the choice of s, we have that $2(\rho+1)+s \le 2n/(n-2)$, which implies that $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^{2(\rho+1)+s}$. Thus, we obtain that

$$||u||_{2(\rho+1)}^{\rho+1} \le C||u||_2^{(1-\theta)(\rho+1)}||\nabla u||_2^{\theta(\rho+1)}. \tag{3.11}$$

From (3.8) and (3.11), we have

$$|I_3| \le 2RC\int_S^T E^{(p-1)/2}(t)||u||_2^{(1-\theta)(\rho+1)}||\nabla u||_2^{\theta(\rho+1)+1}dt. \tag{3.12}$$

Using Young's inequality with $p = 2/[(1-\theta)(\rho+1)]$ and $q = 2/[2-(1-\theta)(\rho+1)]$, we obtain

$$\begin{aligned}||u||_2^{(1-\theta)(\rho+1)}||\nabla u||_2^{\theta(\rho+1)+1} \\ \le C'(\epsilon)||u||_2^2 + \epsilon||\nabla u||_2^{[2\{\theta(\rho+1)+1\}]/[2-(1-\theta)(\rho+1)]},\end{aligned} \tag{3.13}$$

where $\epsilon > 0$ and $C'(\varepsilon)$ is for some positive constant. By (2.16), we can easily check that

$$||\nabla u||_2^2 \le \frac{2(\rho+2)}{\rho}E(0). \tag{3.14}$$

In order to calculate (3.12) with (3.13) and (3.14), we obtain that

$$I_3 \le C^*(\epsilon)\int_S^T E^{(p-1)/2}(t)\int_\Omega |u|^2 dx\, dt + \kappa\epsilon\int_S^T E^{(p+1)/2}(t)\, dt, \tag{3.15}$$

where

$$C^*(\epsilon) = 2RCC'(\epsilon), \qquad \kappa = 4RC\left\{\frac{2(\rho+2)}{\rho}E(0)\right\}^{\rho/[2-(1-\theta)(\rho+1)]}.$$

Estimates for $I_4 := \left(n - 1 - \frac{2}{\rho+2}\right) \int_S^T E^{(p-1)/2}(t) \int_\Omega |u|^{\rho+2} dx\, dt$: Using (3.9) with $q = \rho + 2$ and $p = 2(\rho + 1)$, we deduce that

$$||u||_{\rho+2} \le C||u||_2^{1/(\rho+2)} ||u||_{2(\rho+1)}^{(\rho+1)/(\rho+2)},$$

where $\theta = (\rho + 1)/(\rho + 2)$. In addition, using the imbedding $H^1_{\Gamma_0}(\Omega) \hookrightarrow L^{2(\rho+1)}(\Omega)$ and Cauchy's inequality,

$$||u||_{\rho+2}^{\rho+2} \le C||u||_2 ||\nabla u||_2^{\rho+1} \le C''(\epsilon)||u||_2^2 + C\epsilon E(t),$$

where $\epsilon > 0$, and $C''(\epsilon)$ stands for some positive constant. Hence,

$$I_4 \le C\epsilon \int_S^T E^{(p+1)/2}(t)\, dt + C^{**}(\epsilon) \int_S^T E^{(p-1)/2}(t) \int_\Omega |u|^2 dx\, dt, \tag{3.16}$$

where $C^{**}(\epsilon) = C''(\epsilon)\{(n - 1) - (2/\rho + 2)\}$.

Estimates for $I_5 := \int_S^T E^{(p-1)/2}(t) \int_{\Gamma_1} (m \cdot \nu)|u'|^2 d\Gamma\, dt$: By (2.7) and (2.8), we have

$$\begin{aligned} \int_{|u'|\le 1} (m \cdot \nu)|u'|^2 d\Gamma &\le C \int_{|u'|\le 1} (m \cdot \nu)(u' g(u'))^{2/(p+1)} d\Gamma \\ &\le C\left(\int_{\Gamma_1} (m \cdot \nu) u' g(u') d\Gamma\right)^{2/(p+1)} \\ &\le C(-E')^{2/(p+1)} \end{aligned}$$

and

$$\int_{|u'|\ge 1} (m \cdot \nu)|u'|^2 d\Gamma \le \int_{|u'|\ge 1} (m \cdot \nu) u' g(u')\, d\Gamma \le -CE'.$$

Hence,

$$\begin{aligned} \int_S^T E^{(p-1)/2}(t) \int_{|u'|\le 1} (m\cdot\nu)|u'|^2 d\Gamma\, dt &\le \int_S^T \epsilon E^{(p+1)/2}(t) - C(\epsilon)E'(t)\, dt \\ &\le \epsilon \int_S^T E^{(p+1)/2}(t)\, dt + C^{***}(\epsilon)E(S) \end{aligned}$$

and

$$\int_S^T E^{(p-1)/2}(t) \int_{|u'|\ge 1} (m \cdot \nu)|u'|^2 d\Gamma\, dt \le CE(S),$$

where $C^{***}(\epsilon)$ stands for some positive constant.

Replacing the above inequalities ((3.6), (3.7), (3.15) and (3.16)) in (3.4) and using (2.2) and Young's inequality, we arrive at

$$
\begin{aligned}
&\int_S^T E^{(p-1)/2}(t)\int_\Omega\left(|u'|^2-\frac{2}{\rho+2}|u|^{\rho+2}\right)dx\,dt \\
&\quad+\int_S^T E^{(p-1)/2}(t)\left(1+\frac{\beta}{2}\int_\Omega|\nabla u|^2dx\right)\int_\Omega|\nabla u|^2dx\,dt \\
&\le CE(S)+C(\epsilon)E(S) \\
&\quad+C(\epsilon)\int_S^T E^{(p-1)/2}(t)\int_\Omega|u|^2dx\,dt \\
&\quad+C\epsilon\int_S^T E^{(p+1)/2}(t)\,dt \\
&\quad+\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega|\nabla u|^2dx\right) \\
&\quad\cdot\int_{\Gamma_1}\left(\frac{R^2}{\delta}\left(\frac{\partial u}{\partial\nu}\right)^2+(n-1)u\frac{\partial u}{\partial\nu}\right)d\Gamma\,dt.
\end{aligned}
\tag{3.17}
$$

Step (3). (Significant boundary term.) Now we will estimate the last term of the right hand side of (3.17).

Estimates for $I_6:=\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega|\nabla u|^2dx\right)\int_{\Gamma_1}\left(\frac{R^2}{\delta}\left(\frac{\partial u}{\partial\nu}\right)^2\right.$ $\left.+(n-1)u\frac{\partial u}{\partial\nu}\right)d\Gamma\,dt$: From (3.1) and Young's inequality, we have

$$
\begin{aligned}
&\frac{R^2}{\delta}\left(\frac{\partial u}{\partial\nu}\right)^2+(n-1)u\frac{\partial u}{\partial\nu} \\
&\le\gamma\left(\frac{\partial u}{\partial\nu}+k(0)u\right)^2-\gamma k^2(0)|u|^2+(n-1-2\gamma k(0))u\frac{\partial u}{\partial\nu} \\
&\le 2\gamma a^2(x)g^2(u')+2\gamma\left(\int_0^t k'(t-s,x)u(s,x)\,ds\right)^2 \\
&\quad-\eta k(0)|u|^2+(n-1-2\gamma k(0))u\,\frac{\partial u}{\partial\nu},
\end{aligned}
$$

where $\gamma:=\gamma(x)=\eta/k(0)$ with $\eta>\max\{(n-1)/2,(R^2/\delta)||k(0)||_{L^\infty(\Gamma_1)}\}$. Therefore, we will obtain the desired estimate of I_6 if we can estimate the right hand side of the above inequality.

Firstly, by calculating I_5 in a similar manner, we have

$$\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_{|u'|\le 1} 2\gamma a^2(x)g^2(u')\,d\Gamma\,dt \le C(-E')^{2/(p+1)}$$

and

$$\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_{|u'|\ge 1} 2\gamma a^2(x)g^2(u')\,d\Gamma\,dt \le -CE'.$$

Hence, using Young's inequality with $[(p-1)/(p+1)] + (2/(p+1)) = 1$, we obtain that

(3.18)
$$\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_{|u'|\le 1} 2\gamma a^2(x)g^2(u')\,d\Gamma\,dt \le \epsilon\int_S^T E^{(p+1)/2}(t)\,dt + C(\epsilon)E(S)$$

and

(3.19)
$$\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_{|u'|\ge 1} 2\gamma a^2(x)g^2(u')\,d\Gamma\,dt \le CE(S).$$

Secondly, let $\xi > 0$ be satisfied such that

$$\xi \inf_{\Gamma_1} k'(0) + 1 > 0, \tag{3.20}$$

and define

$$j := j(x) = \frac{k(0)}{\alpha(1+\xi k'(0))} \quad \text{on } \Gamma_1,$$

from (3.20), $j \ge 0$ and $j \in L^\infty(\Gamma_1)$. On the other hand, we have

$$\begin{aligned}
&\left(\int_0^t k'(t-s,x)u(s,x)\,ds\right)^2 \\
&\quad - j\int_0^t k''(t-s,x)(u(t,x)-u(s,x))^2 ds + jk'u^2 \\
&\le \left(\int_0^t k'(t-s,x)\,ds\right)\left(\int_0^t k'(t-s,x)u^2(s,x)\,ds\right) \\
&\quad - j\int_0^t k''(t-s,x)u^2(s,x)\,ds
\end{aligned}$$

$$
\begin{aligned}
&+ 2ju \int_0^t k''(t-s,x)u(s,x)\,ds - jk'(0)u^2 - jk'u^2 + jk'u^2 \\
&\leq (k-k(0))\int_0^t k'(t-s,x)u^2(s,x)\,ds - j\int_0^t k''(t-s,x)u^2(s,x)\,ds \\
&\quad + j\Big(k'(0) + \frac{1}{\xi}\Big)u^2 + \xi j\Big(\int_0^t k''(t-s,x)u(s,x)\,ds\Big)^2 \\
&\leq \Big(\frac{1}{\alpha}k(0) - j(1+\xi k'(0))\Big)\int_0^t k''(t-s,x)u^2(s,x)\,ds \\
&\quad + j\Big(k'(0) + \frac{1}{\xi}\Big)|u|^2 \\
&\leq \frac{1}{\xi\alpha}k(0)|u|^2.
\end{aligned}
$$

Since $j \in L^\infty(\Gamma_1)$, it holds that

$$
\begin{aligned}
&\int_S^T E^{(p-1)/2}(t)\Big(1 + \beta\int_\Omega |\nabla u|^2 dx\Big) \\
&\quad \cdot \int_{\Gamma_1} 2\gamma\Big\{\Big(\int_0^t k'(t-s,x)u(s,x)\,ds\Big)^2 \\
&\quad - j\int_0^t k''(t-s,x)(u(t,x)-u(s,x))^2 ds + jk'u^2\Big\}\,d\Gamma\,dt \\
&\leq \frac{2\eta}{\xi\alpha}\int_S^T E^{(p-1)/2}(t)\Big(1 + \beta\int_\Omega |\nabla u|^2 dx\Big)\int_{\Gamma_1}|u|^2 d\Gamma\,dt.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
(3.21)\quad &2\int_S^T E^{(p-1)/2}(t)\Big(1 + \beta\int_\Omega |\nabla u|^2 dx\Big) \\
&\quad \cdot \int_{\Gamma_1}\gamma\Big(\int_0^t k'(t-s,x)u(s,x)\,ds\Big)^2 d\Gamma\,dt \\
&\leq CE(S) + \frac{2\eta}{\xi\alpha}\int_S^T E^{(p-1)/2}(t)\Big(1 + \beta\int_\Omega |\nabla u|^2 dx\Big)\int_{\Gamma_1}|u|^2 d\Gamma\,dt.
\end{aligned}
$$

Finally, let φ be a solution of $-\Delta\varphi = |u|^\rho u$ in Ω, $\varphi = u$ on Γ. By the classical results of the elliptic partial differential equations theory we have

$$\int_\Omega \nabla\varphi\cdot\nabla u dx - \int_\Omega |u|^{\rho+2}dx = \int_\Omega |\nabla\varphi|^2 dx - \int_\Omega |u|^\rho u\varphi\, dx,$$

and

$$\int_\Omega |\varphi|^2 dx \le C\int_\Gamma |u|^2 d\Gamma$$

$$\int_\Omega |\varphi'|^2 dx \le C\int_\Gamma |u'|^2 d\Gamma.$$

Multiplying (3.1) with $E^{(p-1)/2}(t)\varphi$ and integrating from S to T, we obtain

$$(3.22)\quad -\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_\Gamma \varphi\frac{\partial u}{\partial\nu}\,d\Gamma\,dt$$

$$= -\left[E^{(p-1)/2}\int_\Omega u'\varphi\,dx\right]_S^T$$

$$+\frac{p-1}{2}\int_S^T E^{(p-3)/2}(t)E'(t)\int_\Omega u'\varphi\,dx\,dt$$

$$+\int_S^T E^{(p-1)/2}(t)\int_\Omega u'\varphi' dx\,dt$$

$$-\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_\Omega \nabla u\cdot\nabla\varphi\,dx\,dt$$

$$+\int_S^T E^{(p-1)/2}(t)\int_\Omega |u|^\rho u\varphi\,dx\,dt$$

$$= -\left[E^{(p-1)/2}\int_\Omega u'\varphi\,dx\right]_S^T$$

$$+\frac{p-1}{2}\int_S^T E^{(p-3)/2}(t)E'(t)\int_\Omega u'\varphi\,dx\,dt$$

$$+\int_S^T E^{(p-1)/2}(t)\int_\Omega u'\varphi' dx\,dt$$

$$-\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_\Omega |\nabla\varphi|^2 dx\,dt$$

$$-\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_\Omega |u|^{\rho+2} dx\,dt$$

$$+\int_S^T E^{(p-1)/2}(t)\left(2+\beta\int_\Omega |\nabla u|^2 dx\right)\int_\Omega |u|^\rho u\varphi\,dx\,dt.$$

Note that

$$-\left[E^{(p-3)/2}\int_\Omega u'\varphi\,dx\right]_S^T \le CE(S),$$

$$\int_S^T E^{(p-3)/2}(t)E'(t)\int_\Omega u'\varphi\,dx\,dt \le CE(S)$$

and

$$\begin{aligned}\int_S^T E^{(p-1)/2}(t)\int_\Omega u'\varphi' dx\,dt &\le \epsilon\int_S^T E^{(p-1)/2}(t)\int_\Omega |u'|^2 dx\,dt\\ &\quad + C(\epsilon)\int_S^T E^{(p-1)/2}(t)\int_\Omega |\varphi'|^2 dx\,dt\\ &\le \epsilon\int_S^T E^{(p-1)/2}(t)\int_\Omega |u'|^2 dx\,dt\\ &\quad + C(\epsilon)\int_S^T E^{(p-1)/2}(t)\int_{\Gamma_1} |u'|^2 dx\,dt\\ &\le CE(S)+C(\epsilon)E(S)+3\epsilon\int_S^T E^{(p+1)/2}(t)\,dt.\end{aligned}$$

Using the imbedding $H^1_{\Gamma_0}(\Omega)\hookrightarrow L^{2(\rho+1)}(\Omega)$ and Poincare's inequality, we obtain

$$\begin{aligned}&\int_S^T E^{(p-1)/2}(t)\left(2+\beta\int_\Omega|\nabla u|^2dx\right)\int_\Omega |u|^\rho u\varphi\,dx\,dt\\ &\le \int_S^T E^{(p-1)/2}(t)\left(2+\beta\int_\Omega|\nabla u|^2dx\right)\int_\Omega \frac{1}{2}|u|^{2(\rho+1)}dx\,dt\\ &\quad+\int_S^T E^{(p-1)/2}(t)\left(2+\beta\int_\Omega|\nabla u|^2dx\right)\int_\Omega \frac{1}{2}|\varphi|^2dx\,dt\\ &\le C\int_S^T E^{(p+1)/2}(t)\,dt.\end{aligned}$$

Replacing the above inequality in (3.22), we obtain

$$\begin{aligned}(3.23)\quad -\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega|\nabla u|^2dx\right)\int_{\Gamma_1} u\,\frac{\partial u}{\partial\nu}\,d\Gamma\,dt&\\ \le CE(S)+C(\epsilon)E(S)+C\epsilon\int_S^T E^{(p+1)/2}(t)\,dt.&\end{aligned}$$

Hence, by replacing (3.18), (3.19), (3.21) and (3.23) in I_6, we arrive at

(3.24)
$$I_6 \le CE(S) + C(\epsilon)E(S) + C\epsilon \int_S^T E^{(p+1)/2}(t)\,dt + \int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_{\Gamma_1}\left(\frac{2\eta}{\xi\alpha} - \eta k(0)\right)|u|^2 d\Gamma\,dt.$$

Therefore, we replace (3.24) in (3.17) to obtain

(3.25)
$$\begin{aligned}\int_S^T E^{(p-1)/2}(t)\int_\Omega\left(|u'|^2 - \frac{2}{\rho+2}|u|^{\rho+2}\right)dx\,dt &+ \int_S^T E^{(p-1)/2}(t)\left(1+\frac{\beta}{2}\int_\Omega |\nabla u|^2 dx\right)\int_\Omega |\nabla u|^2 dx\,dt\\ \le CE(S) + C(\epsilon)E(S) &+ C(\epsilon)\int_S^T E^{(p-1)/2}(t)\int_\Omega |u|^2 dx\,dt\\ &+ C\epsilon\int_S^T E^{(p+1)/2}(t)\,dt + \int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega|\nabla u|^2 dx\right)\\ &\cdot\int_{\Gamma_1}\left(\frac{2\eta}{\xi\alpha}-\eta k(0)\right)|u|^2 d\Gamma\,dt.\end{aligned}$$

Step (4). (Deducing the result.) Multiplying (2.10) by $E^{(p-1)/2}(t)$ and then integrating from S to T using (3.25) and the hypotheses of k, we obtain

(3.26)
$$\begin{aligned}\int_S^T E^{(p+1)/2}(t)\,dt &\le C\int_S^T E^{(p-1)/2}(t)\int_\Omega |u|^2 dx\,dt + CE(S)\\ &+ C\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega|\nabla u|^2dx\right)\\ &\cdot\int_{\Gamma_1}\left(\frac{2\eta}{\xi\alpha}+(1-\eta)k(0)\right)|u|^2 d\Gamma\,dt.\end{aligned}$$

On the other hand, we can obtain $2\eta/(\xi\alpha) + (1-\eta)k(0) \le 0$. Indeed, the condition (2.5) implies that there exists a $\xi' > 0$ such that $\alpha \inf_{\Gamma_1} k(0) > -(2+\xi')\inf_{\Gamma_1} k'(0)$. We choose $\xi > 0$ such that

$-\xi \inf_{\Gamma_1} k'(0) = (\xi'+4)/(2(\xi'+2))$. Then, we get

$$\frac{2\eta}{\xi\alpha} + (1-\eta)k(0) \le \frac{\eta}{\xi\alpha}\Big(2 + \xi(2+\xi')\inf_{\Gamma_1} k'(0)\Big) + k(0) = -\frac{\xi'\eta}{2\xi\alpha} + k(0).$$

Therefore, if we choose

$$\eta = \max\left\{n-1, \left(\frac{R^2}{\delta} + \frac{2\xi\alpha}{\xi'}\right)||k(0)||_{L^\infty(\Gamma_1)}\right\},$$

then we have

$$\frac{2\eta}{\xi\alpha} + (1-\eta)k(0) \le 0. \tag{3.27}$$

Replacing (3.27) in (3.26), we obtain the result of Lemma 3.3. □

Lemma 3.4. *There exists a* $T_0 > 0$ *independent of* $u(t)$ *such that, if* $T > T_0$*, then the inequality*

$$\begin{aligned} \int_S^T E^{(p-1)/2}(t)\int_\Omega |u|^2 dx\, dt \le & \int_\Omega |\nabla u|^2 dx \\ & + C\int_S^T E^{(p-1)/2}(t)\Big(1+\beta\int_\Omega |\nabla u|^2 dx\Big)\int_{\Gamma_1} a^2(x)g^2(u') \\ & - \frac{1}{2}k'(t,x)|u|^2 d\Gamma\, dt \\ & + C\int_S^T E^{(p-1)/2}(t)\Big(1+\beta\int_\Omega |\nabla u|^2 dx\Big) \\ & \cdot \int_{\Gamma_1}\frac{1}{2}\int_0^t k''(t-s,x)|u(t,x)-u(s,x)|^2 ds\, d\Gamma\, dt \end{aligned} \tag{3.28}$$

holds for $0 \le S < T < +\infty$.

Proof. We will argue by contradiction. If (3.28) is false, then let $\{u_i(0), u_i'(0)\}$ be a sequence of initial data where the corresponding solutions $\{u_i\}$ of (3.1) with $E_i(0)$ uniformly bounded in i verifies

$$\lim_{i\to\infty} \frac{\int_S^T E_i^{(p-1)/2}(t)\int_\Omega |u_i|^2 dx\, dt}{A_i} = \infty, \tag{3.29}$$

where $E_i(t)$ is defined by $E(t)$ with u replaced by u_i and

$$\begin{aligned} A_i = & \int_\Omega |\nabla u_i|^2 dx \\ & + \int_S^T E_i^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u_i|^2 dx\right) \\ & \cdot \int_{\Gamma_1}\left(a^2(x)g^2(u_i') - \frac{1}{2}k'(t,x)|u_i|^2\right) d\Gamma\, dt \\ & + \int_S^T E_i^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u_i|^2 dx\right) \\ & \cdot \int_{\Gamma_1}\frac{1}{2}\int_0^t k''(t-s,x)|u_i(t,x)-u_i(s,x)|^2 ds\, d\Gamma\, dt. \end{aligned}$$

Since $E_i(0)$ is uniformly bounded in i, we have $E_i(t) \leq C$ for all $i \in \mathbb{N}$, for all $t \geq 0$. Then, we obtain a subsequence, still denoted by $\{u_i\}$, that satisfies the following properties:

$$\begin{aligned} & u_i \longrightarrow u \quad \text{weakly in } H^1(Q), \quad Q = \Omega \times (0,T), \\ & u_i \longrightarrow u \quad \text{weak star in } L^\infty(0,T; H^1_{\Gamma_0}(\Omega)), \\ & u_i' \longrightarrow u' \quad \text{weak star in } L^\infty(0,T; L^2(\Omega)), \\ & u_i \longrightarrow u \quad \text{weakly in } L^2(0,T; L^2(\Gamma)). \end{aligned}$$

By compactness results, we have that

$$u_i \longrightarrow u \quad \text{strongly in } L^2(0,T; L^2(\Omega)) \tag{3.30}$$

and

$$u_i \longrightarrow u \quad \text{strongly in } L^2(0,T; L^2(\Gamma)). \tag{3.31}$$

In what follows, we shall use the ideas contained in Lasiecka and Tataru [**28**], applied to our context. Assume that $u \neq 0$. According to (3.30), we obtain that

$$|u_i|^\rho u_i \longrightarrow |u|^\rho u \quad \text{almost everywhere in } Q.$$

From the above convergence and the sequence $\{|u_i|^\rho u_i\}$ which is bounded in $L^2(0,T; L^2(\Omega))$, we conclude by Lions's lemma that

$$|u_i|^\rho u_i \longrightarrow |u|^\rho u \quad \text{weakly in } L^2(0,T; L^2(\Omega)).$$

In addition, since $\int_S^T E_i^{(p-1)/2}(t) \int_\Omega |u_i|^2 dx\, dt$ is bounded,

$$
\begin{aligned}
(3.32)\quad & \int_\Omega |\nabla u_i|^2 dx + \int_S^T E_i^{(p-1)/2}(t)\Big(1 + \beta \int_\Omega |\nabla u_i|^2 dx\Big) \\
& \Big\{ \int_{\Gamma_1} a^2(x) g^2(u_i')\, d\Gamma - \int_{\Gamma_1} \frac{1}{2} k'(t,x)|u_i|^2 d\Gamma \\
& + \int_{\Gamma_1} \frac{1}{2} \int_0^t k''(t-s,x)|u_i(t,x) - u_i(s,x)|^2 ds\, d\Gamma \Big\} dt \longrightarrow 0,
\end{aligned}
$$

as $i \to \infty$. Since S is chosen in the interval $[0,T)$, and by assumptions of k', k'', $a(x)$ and $g(x)$, we can obtain that

$$
\begin{aligned}
u_i &\longrightarrow 0 \quad \text{strongly in } L^2(0,T; H^1_{\Gamma_0}(\Omega)), \\
a(x)g(u_i') &\longrightarrow 0 \quad \text{strongly in } L^2(0,T; L^2(\Gamma_1)), \\
u_i &\longrightarrow 0 \quad \text{strongly in } L^2(0,T; L^2(\Gamma_1)),
\end{aligned}
$$

$$
u(s,x) = u(t,x) \quad \text{almost everywhere in } \Gamma_1 \times (0,T).
$$

Passing to the limit in the equation, when $i \to \infty$, we obtain, for u,

$$
(3.33)\qquad \begin{cases} u'' - \Delta u = |u|^\rho u & \text{in } Q, \\ u = 0 & \text{on } \Gamma_0 \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = 0,\ u' = 0 & \text{on } \Gamma_1 \times (0,T), \end{cases}
$$

and, for $u' = v$,

$$
\begin{cases} v'' - \Delta v = (\rho+1)|u|^\rho v & \text{in } Q, \\ v = 0 & \text{on } \Gamma_0 \times (0,T), \\ \dfrac{\partial v}{\partial \nu} = 0,\ v = 0 & \text{on } \Gamma_1 \times (0,T). \end{cases}
$$

Since $u \in L^\infty(0,T; H^1_{\Gamma_0}(\Omega))$, $(\rho+1)|u|^\rho \in L^\infty(0,T; L^n(\Omega))$. Then, by the result of **[28]**, we conclude that $v = u' \equiv 0$ for sufficiently large T. Returning to (3.33), we obtain the following elliptic equation:

$$
\begin{cases} -\Delta u = |u|^\rho u & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_0, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_1. \end{cases}
$$

Multiplying the equation by u, we get

$$\int_\Omega |\nabla u|^2 dx - \int_\Omega |u|^{\rho+2} dx = 0;$$

hence,

$$J(u) = \frac{\rho}{2(\rho+2)} ||\nabla u||_2^2.$$

However, if $u \neq 0$, we obtain that

$$J(u) > \frac{\rho}{2(\rho+2)} ||\nabla u||_2^2,$$

due to (2.15). This is a contradiction.

Now, we assume that $u \equiv 0$. Setting

$$\chi_i^2 = \int_S^T E_i^{(p-1)/2}(t) \int_\Omega |u_i|^2 dx\, dt \quad \text{and} \quad \overline{u}_i(t) = \frac{u_i(t)}{\chi_i}.$$

Then, we obtain

$$\int_0^T E_i^{(p-1)/2}(t) \int_\Omega |\overline{u}_i|^2 dx\, dt = \frac{1}{\chi_i^2} \int_S^T E_i^{(p-1)/2}(t) \int_\Omega |u_i|^2 dx\, dt = 1. \tag{3.34}$$

In addition, since $u = 0$, we have that $\chi_i \to 0$ as $i \to \infty$. Since S is chosen in the interval $[0, T)$, from (3.29), we have

$$\begin{aligned} \int_\Omega |\nabla \overline{u}_i|^2 dx + \int_0^T E_i^{(p-1)/2}(t) \Big(1 + \beta \int_\Omega |\nabla u_i|^2 dx\Big) \Big\{ \int_{\Gamma_1} \Big(\frac{1}{\chi_i} a(x) g(u_i')\Big)^2 d\Gamma \\ - \int_{\Gamma_1} \frac{1}{2} k'(t,x) |\overline{u}_i|^2 d\Gamma \\ + \int_{\Gamma_1} \frac{1}{2} \int_0^t k''(t-s,x) |\overline{u}_i(t,x) - \overline{u}_i(s,x)|^2 ds\, d\Gamma \Big\} dt \longrightarrow 0 \end{aligned} \tag{3.35}$$

as $i \to \infty$.

On the other hand,

$$\begin{aligned} \overline{E}_i(t) := \frac{1}{2} \int_\Omega |\overline{u'}(t,x)|^2 dx + \frac{1}{2} \int_\Omega |\nabla \overline{u}(t,x)|^2 dx \\ + \frac{\beta}{4} \Big(\int_\Omega |\nabla \overline{u}(t,x)|^2 dx \Big)^2 - \frac{1}{2} \Big(1 + \beta \int_\Omega |\nabla u|^2 dx \Big) \end{aligned}$$

$$
\begin{aligned}
&\cdot \int_{\Gamma_1} \int_0^t k'(t-s,x)|\overline{u}(t,x) - \overline{u}(s,x)|^2 ds\, d\Gamma \\
&+ \frac{1}{2}\left(1 + \beta \int_\Omega |\nabla u|^2 dx\right) \int_{\Gamma_1} k(t,x)|\overline{u}(t,x)|^2 d\Gamma \\
&- \frac{1}{\rho+2} \int_\Omega |\overline{u}(t,x)|^{\rho+2} dx \\
\le\ & \frac{1}{2\chi_i^2}\left\{ \int_\Omega |u_i'(t)|^2 dx + \int_\Omega |\nabla u_i(t)|^2 dx + \frac{\beta}{2}\left(\int_\Omega |\nabla u(t,x)|^2 dx\right)^2 \right. \\
&+ \left(1 + \beta \int_\Omega |\nabla u|^2 dx\right) \int_{\Gamma_1} k(t,x)|u_i(t,x)|^2 d\Gamma \\
&- \left(1 + \beta \int_\Omega |\nabla u|^2 dx\right) \\
&\left. \cdot \int_{\Gamma_1} \int_0^t k'(t-s,x)|u_i(t,x) - u_i(s,x)|^2 ds\, d\Gamma \right\}.
\end{aligned}
$$

From (2.15), we have

$$
\frac{1}{2}||\nabla u_i(t)||_2^2 \le \frac{\rho+2}{\rho}\left(\frac{1}{2}||\nabla u_i(t)||_2^2 - \frac{1}{\rho+2}||u_i(t)||_{\rho+2}^{\rho+2}\right).
$$

Hence,

$$
\overline{E}_i(t) \le \frac{\rho+2}{\rho\chi_i^2} E_i(t),
$$

implying that $||\overline{u}_i'(t)||_2^2$ and $||\nabla \overline{u}_i(t)||_2^2$ are bounded. Then, in particular, for a subsequence $\{\overline{u}_i\}$, from the boundedness of $||\overline{u}_i'(t)||_2^2$ and $||\nabla \overline{u}_i(t)||_2^2$ and (3.35), we obtain

$$
\begin{aligned}
&\overline{u}_i \longrightarrow \overline{u} \quad \text{weak star in } L^\infty(0,T; H^1_{\Gamma_0}(\Omega)), \\
&\overline{u}_i' \longrightarrow \overline{u}' \quad \text{weak star in } L^\infty(0,T; L^2(\Omega)), \\
&\overline{u}_i \longrightarrow \overline{u} \quad \text{strongly in } L^2(0,T; L^2(\Omega)), \\
&\overline{u}_i \longrightarrow 0 \quad \text{strongly in } L^2(0,T; H^1_{\Gamma_0}(\Omega)),
\end{aligned}
$$

$$
\begin{gathered}
\frac{1}{\chi_i} a(x) g(u_i') \longrightarrow 0 \quad \text{strongly in } L^2(0,T; L^2(\Gamma_1)), \\
\overline{u}_i \longrightarrow 0 \quad \text{strongly in } L^2(0,T; L^2(\Gamma)), \\
\overline{u}(s,x) = \overline{u}(t,x) \quad \text{almost everywhere in } \Gamma_1 \times (0,T).
\end{gathered}
$$

In addition, $\overline{u}_i$ satisfies the equation

$$(3.36)\quad \begin{cases} \overline{u}_i'' - \left(1+\beta \int_\Omega |\nabla u_i|^2 dx\right)\Delta \overline{u}_i = |u_i|^\rho \overline{u}_i & \text{in } \Omega\times(0,T),\\ \overline{u}_i = 0 & \text{on } \Gamma_0\times(0,T),\\ \dfrac{\partial \overline{u}_i}{\partial \nu} + \int_0^t k'(t-s,x)\overline{u}_i(s,x)\,ds & \\ \quad +k(0)\overline{u}_i + \dfrac{1}{\chi_i}a(x)g(u_i') = 0 & \text{on } \Gamma_1\times(0,T). \end{cases}$$

We note that

$$(3.37)\quad \begin{aligned} \int_0^T \int_\Omega ||u_i|^\rho \overline{u}_i|^2 dx\,dt &= \int_Q |\overline{u}_i|^2 |u_i|^{2\rho} dx\,dt \\ &= \int_{|u_i|\le\epsilon} |\overline{u}_i|^2|u_i|^{2\rho}dx\,dt \\ &\quad + \int_{|u_i|>\epsilon} |\overline{u}_i|^2|u_i|^{2\rho}dx\,dt, \end{aligned}$$

where $Q = \Omega\times(0,T)$. According to the fact that the function $F(s)=|s|^\rho$ is continuous in $\mathbb{R}$ and $S_\epsilon = \sup_{|x|\le\epsilon}|F(x)|$ is well defined, from (3.37), we have

$$\int_0^T\int_\Omega ||u_i|^\rho\overline{u}_i|^2 dx\,dt \le S_\epsilon^2 ||\overline{u}_i||^2_{L^2(Q)} + \chi_i^{2\rho}||\overline{u}_i||^{2\rho+2}_{L^{2\rho+2}(Q)}.$$

Since $\{\overline{u}_i\}$ is bounded in $L^\infty(0,T;H^1_{\Gamma_0}(\Omega)) \hookrightarrow L^\infty(0,T;L^{2\rho+2}(\Omega))$, there exists a $C>0$ such that

$$\int_0^T\int_\Omega ||u_i|^\rho\overline{u}_i|^2 dx\,dt \le C(S_\epsilon^2 + \chi_i^{2\rho}).$$

Then, taking $\varepsilon\to 0$ and $i\to\infty$, we conclude that

$$(3.38)\quad |u_i|^\rho\overline{u}_i \to 0 \quad \text{in } L^2(0,T;L^2(\Omega)) \text{ as } i\to\infty.$$

Passing to the limit in (3.36) as $i\to\infty$, we obtain

$$(3.39)\quad \begin{cases} \overline{u}'' - \Delta\overline{u} = 0 & \text{in } \Omega\times(0,T),\\ \overline{u} = 0 & \text{on } \Gamma_0\times(0,T),\\ \dfrac{\partial\overline{u}}{\partial\nu} = 0 & \text{on } \Gamma_1\times(0,T). \end{cases}$$

Furthermore,

$$\int_0^T E^{(p-1)/2}(t)\int_\Omega |\overline{u}|^2 dx\,dt = 1. \tag{3.40}$$

Then, $v = \overline{u}'$ satisfies (differentiating problem (3.39) with respect to t)

$$\begin{cases} v'' - \Delta v = 0 & \text{in } \Omega \times (0,T), \\ v = 0 & \text{on } \Gamma_0 \times (0,T), \\ \dfrac{\partial v}{\partial \nu} = 0 & \text{on } \Gamma_1 \times (0,T). \end{cases}$$

Hence, we can obtain that $v = \overline{u}' = 0$ (see [**39**]). We are able to rewrite (3.39), as

$$\begin{cases} -\Delta \overline{u} = 0 & \text{in } \Omega, \\ \overline{u} = 0 & \text{on } \Gamma_0, \\ \dfrac{\partial \overline{u}}{\partial \nu} = 0 & \text{on } \Gamma_1. \end{cases}$$

Since $\overline{u} \in H^1_{\Gamma_0}(\Omega)$, we conclude that $\overline{u} \equiv 0$ in Ω. This is a contradiction to (3.40). Thus, the proof is complete.

By replacing (3.28) in Lemma 3.3, we have

$$\begin{aligned} \int_S^T E^{(p+1)/2}(t)\,dt \le{}& \int_\Omega |\nabla u|^2 dx \\ &+ C\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right)\int_{\Gamma_1} a^2(x)g^2(u') \\ &- \frac{1}{2}k'(t,x)|u|^2 d\Gamma\,dt \\ &+ C\int_S^T E^{(p-1)/2}(t)\left(1+\beta\int_\Omega |\nabla u|^2 dx\right) \\ &\cdot \int_{\Gamma_1}\frac{1}{2}\int_0^t k''(t-s,x)|u(t,x)-u(s,x)|^2 ds\,d\Gamma\,dt \\ &+ CE(S) \\ \le{}& C\int_S^T E^{(p-1)/2}(t)\int_{|u'|\le 1} a^2(x)g^2(u')\,d\Gamma\,dt \\ &+ C\int_S^T E^{(p-1)/2}(t)\int_{|u'|\ge 1} a^2(x)g^2(u')\,d\Gamma\,dt \end{aligned}$$

$$+ C \int_S^T E^{(p-1)/2}(t)\big(- E'(t)\big)\, dt + CE(S)$$
$$\leq C\epsilon \int_S^T E^{(p+1)/2}(t)\, dt + C(\epsilon)E(S) + CE(S).$$

Hence, we can choose ϵ satisfying the inequality

$$\int_S^T E^{(p+1)/2}(t)\, dt \leq CE(S).$$

Consequently, we arrive at the result of Theorem 2.1 using Lemma 3.1. □

4. Proof of Theorem 2.2. In Section 3, using a polynomial growth of function g near the origin, we proved the energy decay rates for the solution of (3.1). However, using hypotheses ($\overline{\mathbf{H_3}}$), we cannot prove the energy decay rates as in Section 3 since ($\overline{\mathbf{H_3}}$) means that g has no polynomial behavior near the origin. This implies that we cannot use Lemma 3.1 in this section. Therefore, we present two technical lemmas which will play an essential role in establishing the asymptotic behavior.

Lemma 4.1 ([**29**]). *Let $E : \mathbb{R}_+ \to \mathbb{R}_+$ be a nonincreasing function and $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ a strictly increasing function of class C^1 such that*

$$\phi(0) = 0 \quad \text{and} \quad \phi(t) \longrightarrow +\infty \quad \text{as } t \to +\infty.$$

Assume that there exist $\sigma > 0$, $\sigma' \geq 0$ and $C > 0$ such that

$$\int_S^{+\infty} E^{1+\sigma}(t)\phi'(t)\, dt \leq CE^{1+\sigma}(S) + \frac{C}{(1+\phi(S))^{\sigma'}} E^{\sigma}(0)E(S),$$

$0 \leq S < +\infty$. *Then, there exists a $C > 0$ such that*

$$E(t) \leq E(0)\frac{C}{(1+\phi(t))^{(1+\sigma')/\sigma}}, \quad \text{for all } t > 0.$$

Lemma 4.2 ([**29**]). *There exists a function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ of class C^2 increasing and such that ϕ is concave, $\phi(t) \to +\infty$ as $t \to +\infty$, $\phi'(t) \to 0$ as $t \to +\infty$ and*

$$\int_1^{+\infty} \phi'(t)(\alpha^{-1}(\phi'(t)))^2 dt < +\infty,$$

where α is defined on ($\overline{\mathbf{H_3}}$).

Proof. Now, we begin the proof of our result. In what follows, the symbol C indicates positive constants, which may be different. We multiply (3.1) by $E(t)\phi'(t)Mu$, where $\phi(t)$ is a function under the hypotheses of Lemmas 4.1 and 4.2. Then, we have

(4.1)
$$
\begin{aligned}
0 &= \int_S^T E(t)\phi'(t)\int_\Omega (Mu)\Big(u'' - \Big(1+\beta\int_\Omega |\nabla u|^2 dx\Big)\Delta u - |u|^\rho u\Big)\,dx\,dt \\
&= \int_S^T E(t)\phi'(t)\int_\Omega \big(2(m\cdot\nabla u) + (n-1)u\big) \\
&\quad \times \Big(u'' - \Big(1+\beta\int_\Omega |\nabla u|^2 dx\Big)\Delta u - |u|^\rho u\Big)\,dx\,dt \\
&= \int_S^T E(t)\phi'(t)\int_\Omega 2u''(m\cdot\nabla u)\,dx\,dt \\
&\quad + \int_S^T E(t)\phi'(t)\Big(1+\beta\int_\Omega |\nabla u|^2 dx\Big)\int_\Omega 2\nabla u\cdot\nabla(m\cdot\nabla u)\,dx\,dt \\
&\quad - \int_S^T E(t)\phi'(t)\Big(1+\beta\int_\Omega |\nabla u|^2 dx\Big)\int_\Gamma 2\frac{\partial u}{\partial\nu}(m\cdot\nabla u)\,d\Gamma\,dt \\
&\quad - \int_S^T E(t)\phi'(t)\int_\Omega 2|u|^\rho u(m\cdot\nabla u)\,dx\,dt \\
&\quad + (n-1)\int_S^T E(t)\phi'(t)\int_\Omega uu''dx\,dt \\
&\quad + (n-1)\int_S^T E(t)\phi'(t)\Big(1+\beta\int_\Omega |\nabla u|^2 dx\Big)\int_\Omega |\nabla u|^2 dx\,dt \\
&\quad - (n-1)\int_S^T E(t)\phi'(t)\Big(1+\beta\int_\Omega |\nabla u|^2 dx\Big)\int_{\Gamma_1} \frac{\partial u}{\partial\nu}u\,d\Gamma\,dt \\
&\quad - (n-1)\int_S^T E(t)\phi'(t)\int_\Omega |u|^{\rho+2}dx\,dt.
\end{aligned}
$$

Note that

$$
\begin{aligned}
&\int_S^T E(t)\phi'(t)\int_\Omega 2u''(m\cdot\nabla u)\,dx\,dt \\
&= \Big[E(t)\phi'(t)\int_\Omega 2u'(m\cdot\nabla u)\,dx\Big]_S^T
\end{aligned}
$$

$$
\begin{aligned}
&- \int_S^T \big(E'(t)\phi'(t) + E(t)\phi''(t)\big) \int_\Omega 2u'(m\cdot\nabla u)\, dx\, dt \\
&- \int_S^T E(t)\phi'(t) \int_\Omega 2u'(m\cdot\nabla u')\, dx\, dt,
\end{aligned}
$$

$$
\begin{aligned}
&\int_S^T E(t)\phi'(t)\Big(1+\beta\int_\Omega |\nabla u|^2 dx\Big)\int_\Omega 2\nabla u\cdot\nabla(m\cdot\nabla u)\, dx\, dt \\
&= (2-n)\int_S^T E(t)\phi'(t)\Big(1+\beta\int_\Omega |\nabla u|^2 dx\Big)\int_\Omega |\nabla u|^2 dx\, dt \\
&\quad + \int_S^T E(t)\phi'(t)\Big(1+\beta\int_\Omega |\nabla u|^2 dx\Big)\int_\Gamma (m\cdot\nu)|\nabla u|^2 d\Gamma\, dt
\end{aligned}
$$

and

$$
\begin{aligned}
&(n-1)\int_S^T E(t)\phi'(t)\int_\Omega uu'' dx\, dt \\
&= (n-1)\Big[E(t)\phi'(t)\int_\Omega u'u dx\Big]_S^T \\
&\quad - (n-1)\int_S^T \big(E'(t)\phi'(t)+E(t)\phi''(t)\big)\int_\Omega u'u\, dx\, dt \\
&\quad - (n-1)\int_S^T E(t)\phi'(t)\int_\Omega |u'|^2 dx\, dt.
\end{aligned}
$$

Replacing the above calculations in (4.1), we obtain

$$
\begin{aligned}
(4.2)\quad &\int_S^T E(t)\phi'(t)\int_\Omega \Big(|u'|^2 - \frac{2}{\rho+2}|u|^{\rho+2}\Big)\, dxdt \\
&\quad + \int_S^T E(t)\phi'(t)\Big(1+\frac{\beta}{2}\int_\Omega |\nabla u|^2 dx\Big)\int_\Omega |\nabla u|^2 dx\, dt \\
&= -\Big[E(t)\phi'(t)\int_\Omega u'(Mu)\, dx\Big]_S^T \\
&\quad + \int_S^T E'(t)\phi'(t)+E(t)\phi''(t)\int_\Omega u'(Mu)\, dx\, dt \\
&\quad + 2\int_S^T E(t)\phi'(t)\int_\Omega |u|^\rho u(m\cdot\nabla u)\, dx\, dt
\end{aligned}
$$

$$
\begin{aligned}
&+\left(n-1-\frac{2}{\rho+2}\right)\int_S^T E(t)\phi'(t)\int_\Omega |u|^{\rho+2}dx\,dt \\
&-\frac{\beta}{2}\int_S^T E(t)\phi'(t)\int_\Omega |\nabla u|^2dx\int_\Omega |\nabla u|^2dx\,dt \\
&+\int_S^T E(t)\phi'(t)\left(1+\frac{\beta}{2}\int_\Omega |\nabla u|^2dx\right)\int_{\Gamma_0}(m\cdot\nu)|\nabla u|^2d\Gamma\,dt \\
&+\int_S^T E(t)\phi'(t)\int_{\Gamma_1}(m\cdot\nu)|u'|^2d\Gamma\,dt \\
&+\int_S^T E(t)\phi'(t)\left(1+\beta\int_\Omega |\nabla u|^2dx\right)\int_{\Gamma_1}\frac{\partial u}{\partial\nu}(Mu)\,d\Gamma\,dt \\
&-\int_S^T E(t)\phi'(t)\left(1+\beta\int_\Omega |\nabla u|^2dx\right)\int_{\Gamma_1}(m\cdot\nu)|\nabla u|^2d\Gamma\,dt.
\end{aligned}
$$

By using same argument as in Section 3 and J_1-J_4 in **[37]**, we obtain

$$
\int_S^T E^2(t)\phi'(t)\,dt \le CE^2(S)+CE(S)\int_S^T \phi'(t)(\beta^{-1}(\phi'(t)))^2dt. \tag{4.3}
$$

Finally, we estimate the second term of the right hand side of (4.3). From the condition on ϕ, we can consider, without loss of generality, that $\phi(1)=1$. Considering the change of variables $s=\phi(t)$, we have

$$
\begin{aligned}
&\int_1^{+\infty}\phi'(t)(\alpha^{-1}(\phi'(t)))^2dt \\
&\quad=\int_1^{+\infty}(\alpha^{-1}(\phi'(\phi^{-1}(s))))^2ds=\int_1^{+\infty}\left(\alpha^{-1}\left(\frac{1}{(\phi^{-1})'(s)}\right)\right)^2ds.
\end{aligned}
$$

We consider the function ψ by

$$
\psi(t)=1+\int_1^t\frac{1}{\alpha(1/s)}\,ds,\quad t\ge 1.
$$

Then, ψ is a strictly increasing function of class C^2 that satisfies

$$
\begin{aligned}
\psi'(t)=\frac{1}{\alpha(1/t)}&\longrightarrow +\infty, \\
\psi(t)\longrightarrow +\infty&\quad \text{as } t\to+\infty
\end{aligned}
$$

and

$$\int_1^{+\infty} \left(\alpha^{-1}\left(\frac{1}{\psi'(s)}\right)\right)^2 ds = \int_1^{+\infty} \frac{1}{s^2}\, ds < +\infty.$$

Simple computation shows that $\psi'' \geq 0$, which implies that ψ' is nondecreasing and ψ is convex. Moreover, it is easy to verify that ψ^{-1} is concave.

Setting $\phi(t) = \psi^{-1}(t)$, we can rewrite (4.3) as

$$\begin{aligned}\int_S^T E^2(t)\phi'(t)\, dt &\leq CE^2(S) + CE(S)\int_S^{+\infty} \phi'(t)(\alpha^{-1}(\phi'(t)))^2 dt \\ &\leq CE^2(S) + CE(S)\int_{\phi(S)}^{+\infty} \left(\alpha^{-1}\left(\frac{1}{\psi'(s)}\right)\right)^2 ds \\ &\leq CE^2(S) + \frac{C}{\phi(S)}E(S),\end{aligned}$$

and, applying Lemma 4.1 with $\sigma = \sigma' = 1$, we deduce

$$E(t) \leq \frac{C}{\phi^2(t)} \quad \text{for all } t > 0. \tag{4.4}$$

Let s_0 be a number such that $\alpha(1/s_0) \leq 1$. Since α is nondecreasing, we have

$$\psi(s) \leq 1 + (s-1)\frac{1}{\beta(1/s)} \leq \frac{1}{F(1/s)}, \quad \text{for all } s \geq s_0.$$

Consequently, keeping in mind that $\phi = \psi^{-1}$, the last inequality yields

$$s \leq \phi\left(\frac{1}{F(1/s)}\right) = \phi(t) \quad \text{with } t = \frac{1}{F(1/s)}.$$

Then,

$$\frac{1}{\phi(t)} \leq F^{-1}\left(\frac{1}{t}\right). \tag{4.5}$$

Combining (4.4) and (4.5) proves (2.20).

It remains to prove (2.21). Assume that $G(0) = 0$, and G is nondecreasing on $[0, \eta]$ for some $\eta > 0$. Let T_1 be a number such that $G(1/t) \leq \eta$ for all $t \geq T_1$, and set $T_2 = \sup\{T_1, 1/\eta\}$. Assume that ϕ is an increasing and concave function such that, for all $t \geq T_2$, $\phi'(t) \leq \eta$ and $\phi'(t) \leq G(\eta)$ and $\phi'(t) \to 0$ as $t \to +\infty$. Then, due to

Martinez's idea [**29,** page 438], and by the same arguments as (4.3), we obtain

$$\int_S^T E^2(t)\phi'(t)\,dt \le CE^2(S) + CE(S)\int_S^T \phi'(t)(G^{-1}(\phi'(t)))^2 dt.$$

Define

$$\widetilde{\phi}^{-1}(t) = T_2 + \int_{T_2}^t \frac{1}{G(1/s)}\,ds, \quad \text{for all } t \ge T_2.$$

Then,

$$\widetilde{\phi}(t) \ge T_2 \ge \frac{1}{\eta},$$

so

$$\widetilde{\phi}'(t) = G\left(\frac{1}{\widetilde{\phi}(t)}\right) \le G(\eta)$$

and

$$\widetilde{\phi}'(t) \le \widetilde{\phi}'(T_2) = G\left(\frac{1}{\widetilde{\phi}(T_2)}\right) \le G\left(\frac{1}{T_2}\right) \le G\left(\frac{1}{T_1}\right) \le \eta \quad \text{for all } t \ge T_2.$$

Then, $\widetilde{\phi}$ satisfies all of the requisite properties. Thus, from the same argument as (4.4), we obtain

$$E(t) \le \frac{C}{\widetilde{\phi}^2(t)} \quad \text{for all } t > T_2. \tag{4.6}$$

Since, for all $s \in [0,1]$, $\alpha(s) \le s$, i.e., $G(s) \le 1$, we see that

$$\widetilde{\phi}^{-1}(t) \le T_2 + \frac{t - T_2}{G(1/t)} \le \frac{t}{G(1/t)} = \frac{1}{\alpha(1/t)}. \tag{4.7}$$

Combining (4.6) and (4.7), we can prove (2.21). Thus, the proof of Theorem 2.2 is complete. □

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