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ON A BOHR-NEUGEBAUER PROPERTY FOR SOME ALMOST AUTOMORPHIC ABSTRACT DELAY EQUATIONS

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ABSTRACT. This paper is a continuation of the investigations done in the literature regarding the so called Bohr-Neugebauer property for almost periodic differential equations in Hilbert spaces. The aim of this work is to extend the investigation of this property to almost automorphic functional partial differential equations in Banach spaces. We use a compactness assumption which turns out to relax assumptions made in some earlier works for differential equations in Hilbert spaces. Two new integration theorems for almost automorphic functions are proven in the process. To illustrate our main results, we propose an application to a reaction-diffusion equation with continuous delay.

1. Introduction. The theory of almost periodic functions was initiated between 1924 and 1926 by Danish mathematician Harald Bohr [**8**]. In 1933, Bochner [**5**] published an important article devoted to the extension of the theory of almost periodic functions to vector-valued (abstract) functions with values in a Banach space.

From the earliest days, the theory of almost periodic functions has been developed in connection with problems of differential equations, stability theory and dynamical systems. In his fundamental paper [**22**], Favard studied almost periodic differential equations and connected the problem of existence of almost periodic solutions with some separation properties of the bounded solutions. This work of Favard was the starting point of many further investigations.

The circle of applications of the theory has been largely extended and includes not only ordinary differential equations [**23**], but wide classes of partial differential equations and equations in Banach spaces.

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In this process, an important role has been played starting in the 1950s with the investigations of the Italian school, Amerio, Biroli, Prouse and others [**2, 4**], which are directed at extending certain classical results of Favard, Bochner, von Neumann and Sobolev to differential equations in Banach spaces. Then in the spirit of the Italian school, Dafermos, Haraux, Ishii and others have given important contributions to the question of almost periodic solutions [**9, 10, 15, 25, 26, 27**].

In the literature, several books are devoted to almost periodic differential equations. For example, let us indicate the books of Amerio and Prouse [**2**], Corduneanu [**13**], Fink [**23**], Levitan and Zhikov [**29**] and Zaidman [**46**].

It is well known that some almost periodic systems do not carry necessarily almost periodic dynamics [**28, 35, 39**]. Although these systems may have bounded oscillating solutions, these oscillations belong to a class larger than the class of almost periodic functions; we are talking about *almost automorphic* functions.

Bochner introduced the concept of almost automorphy in the literature in [**7**] as a generalization of almost periodicity. This concept was then deeply investigated by Veech [**42**] and many other authors. The name "almost automorphic" was given by Bochner himself since he encountered this type of function first in his work on differential geometry. He also observed that almost automorphic functions can sometimes be used in obtaining simpler proofs of certain results concerning almost periodic functions by first proving these results for almost automorphic functions.

Other important contributions to the theory of almost automorphic functions include those from Zaki [**47**], N'Guérékata [**32**] and Shen and Yi [**39**].

Let us consider the following differential equation in $\mathbb{R}^n$:

$$x'(t) = G(t)x(t) + f(t) \quad \text{for } t \in \mathbb{R}, \tag{1.1}$$

where the matrix $G(t)$ and the vector $f(t)$ are both continuous and ω-periodic for some $\omega > 0$. In [**31**], Massera proved that the existence of a bounded solution of equation (1.1) on the positive real line is enough to obtain the existence of an ω-periodic solution. This result is known in the literature as the Massera theorem. Fixed point theory plays an important role in these kinds of results.

For almost periodic equations, the situation is more complicate since fixed point arguments cannot be used. Bohr and Neugebauer, see **[23]**, extended Massera's theorem for equation (1.1) to the almost periodic case when $G(t) = G$ is a constant matrix. In addition, they proved what it is known in the literature as the Bohr-Neugebauer theorem, namely, they showed that all bounded solutions of equation (1.1) on $\mathbb{R}$ are almost periodic. We note that this result (the Bohr-Neugebauer theorem) does not hold for the periodic case. In general, if this result holds for some differential equation, we say that this equation has the Bohr-Neugebauer property. For more results about differential equations having the Bohr-Neugebauer property, see **[11, 14, 24, 26, 37, 38, 43, 44, 45]**.

In **[11]**, Cooke proved a Bohr-Neugebauer type property for the following differential equation

$$y^{(n)}(t) + A_1 y^{(n-1)}(t) + \cdots A_n y(t) = f(t),$$

when $f : \mathbb{R} \to H$ is almost periodic and A_i, $i = 1, \dots, n$, are compact operators in a separable Hilbert space H, namely, he showed that all bounded solutions on $\mathbb{R}$ are almost periodic. In **[26]**, Haraux proved the same result for the following evolution equation

$$x'(t) + \widetilde{A}x(t) \ni f(t) \quad \text{for } t \in \mathbb{R}, \tag{1.2}$$

where $\widetilde{A}$ is a maximal monotone operator on $\mathbb{R}^2$. In **[43, 44]**, Zaidman proved that the following equation

$$\frac{d}{dt}x(t) = Ax(t) + f(t) \quad \text{for } t \in \mathbb{R}, \tag{1.3}$$

possesses the Bohr-Neugebauer property, when A is a self-adjoint operator in a Hilbert space.

In another paper, Zaidman **[45]** proved this result for equation (1.3) when A is a finite rank operator. In **[24]**, Goldstein extended the work of Zaidman by considering a more general "finite dimensionality assumption" when A is a closed linear operator in a Hilbert space.

While the above results investigated the Bohr-Neugebauer property in Hilbert spaces, it is interesting to know whether this property holds in Banach spaces. This is an important question since many partial differential equations can be formulated as abstract differential equations, such as (1.3) with solutions living in purely Banach spaces.

Motivated by this, we investigate in this paper the Bohr-Neugebauer property for the following partial functional differential equation

$$\frac{d}{dt}u(t) = Au(t) + L(u_t) + f(t) \quad \text{for } t \in \mathbb{R}, \tag{1.4}$$

where A is a linear operator on a Banach space X. Note that this equation is more general than equation (1.3) since it contains the additional term $L(u_t)$ which gives the possibility of considering delays in the equation. We assume that the domain $D(A)$ is not necessarily dense and that A satisfies the well known Hille-Yosida condition, namely, we suppose that:

(H0): there exist $\overline{M} \geq 1$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$ and

$$|R(\lambda, A)^n| \leq \frac{\overline{M}}{(\lambda - \omega)^n}, \quad \text{for } n \in \mathbb{N} \text{ and } \lambda > \omega,$$

where $\rho(A)$ is the resolvent set of A and $R(\lambda, A) := (\lambda I - A)^{-1}$ for $\lambda \in \rho(A)$.

Let $r > 0$ and $C := C([-r, 0], X)$ be the space of continuous functions from $[-r, 0]$ to X endowed with the uniform norm topology. For every $t \in \mathbb{R}$, the historical function $u_t \in C$ is defined by

$$u_t(\theta) := u(t + \theta), \quad \text{for } \theta \in [-r, 0].$$

L is a bounded linear operator from C to X.

Under a "finite dimensionality assumption," we prove that equation (1.4) has a Bohr-Neugebauer type property. More specifically, we prove that all bounded solutions of equation (1.4) on $\mathbb{R}$ are even compact almost automorphic if the input term f is only Stepanov almost automorphic, which is a weaker notion of almost automorphy. As a comparison to the results in **[11, 24, 45]**, the "finite dimensionality assumption" in our work takes the form of an immediate compactness condition on the C_0-semigroup generated by the part of the operator A. This condition finds its applications in several classes of partial differential equations. Our result is achieved by showing that the partial functional differential equation (1.4) can be partially reduced to a finite-dimensional ordinary differential equation, which makes sense given the compactness assumption we made. Two new integration theorems for almost automorphic functions are proven in the process.

This work is organized as follows. In Section 2, we recall the variation of constants formula and establish a variant of the reduction principle that will be used in this work. Section 3 is devoted to almost automorphic functions. In Section 4, we investigate the relative compactness of bounded solutions of equation (1.4). Moreover, we prove a Bohr-Neugebauer type theorem for some almost automorphic finite-dimensional ordinary differential equations. Two new results of integration of almost automorphic functions are proven in the process. Furthermore, we use a spectral decomposition (which is a consequence of the compactness assumption we made) to extend the Bohr-Neugebauer theorem to equation (1.4). In order to illustrate our approach, we propose an application to a reaction-diffusion equation with continuous delay.

2. Variation of constants formula and the reduction principle. To equation (1.4), we associate the following Cauchy problem:

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t) + L(u_t) + f(t) & \text{for } t \geq \sigma \\ u_\sigma = \varphi \in C. \end{cases} \tag{2.1}$$

Definition 2.1 ([**1**]). A continuous function $u : [-r+\sigma, \infty) \to X$ is called an integral solution of equation (2.1), if

(i) $\displaystyle\int_\sigma^t u(s)\,ds \in D(A)$ for $t \geq \sigma$,

(ii) $\displaystyle u(t) = \varphi(0) + A\int_\sigma^t u(s)\,ds + \int_\sigma^t [L(u_s) + f(s)]\,ds$ for $t \geq \sigma$,

(iii) $u_\sigma = \varphi$.

If u is an integral solution of equation (2.1), then, from the continuity of u, we have $u(t) \in \overline{D(A)}$, for all $t \geq \sigma$. In particular, $\varphi(0) \in \overline{D(A)}$.

We introduce the part A_0 of the operator A in $\overline{D(A)}$, defined by

$$\begin{cases} D(A_0) = \{x \in D(A) : Ax \in \overline{D(A)}\} \\ A_0 x = Ax & \text{for } x \in D(A_0). \end{cases}$$

Lemma 2.2 ([**41**]). *Assume that* **(H0)** *holds. Then,* A_0 *generates a strongly continuous semigroup* $(T_0(t))_{t\geq 0}$ *on* $\overline{D(A)}$.

For the existence of integral solutions, we have the following result:

Theorem 2.3 ([**1**]). *Assume that* **(H0)** *holds. Then, for all* $\varphi \in C$ *such that* $\varphi(0) \in \overline{D(A)}$, *equation* (2.1) *has a unique integral solution* u *on* $[-r+\sigma, \infty)$. *Moreover,* u *is given by*

$$\begin{cases} u(t) = T_0(t-\sigma)\varphi(0) \\ \qquad\qquad + \lim\limits_{\lambda\to\infty} \displaystyle\int_\sigma^t T_0(t-s)B_\lambda[L(u_s)+f(s)]\,ds \quad \text{for } t \geq \sigma, \\ u_\sigma = \varphi, \end{cases}$$

where $B_\lambda = \lambda R(\lambda, A)$ *for* $\lambda > \omega$.

From now on, integral solutions will be called solutions. $u(\cdot,\sigma,\varphi,f)$ denotes the solution of equation (2.1). The phase space C_0 of equation (2.1) is given by

$$C_0 = \{\varphi \in C : \varphi(0) \in \overline{D(A)}\}.$$

For each $t \geq 0$, we define the linear operator $U(t)$ on C_0 by

$$U(t)\varphi = u_t(\cdot, 0, \varphi, 0),$$

where $u(\cdot,0,\varphi,0)$ is the solution of the following homogeneous equation

$$\begin{cases} \dfrac{d}{dt}u(t) = Au(t) + L(v_t) \quad \text{for } t \geq 0, \\ u_0 = \varphi. \end{cases}$$

We have the following result:

Proposition 2.4 ([**1**]). *Assume that* **(H0)** *holds. Then,* $(U(t))_{t\geq 0}$ *is a strongly continuous semigroup of operators on* C_0. *Moreover, the operator* $\mathcal{A}_U$, *defined on* C_0 *by*

$$\begin{cases} D(\mathcal{A}_U) = \big\{\varphi \in C^1([-r,0],X) : \\ \quad \varphi(0) \in D(A), \varphi'(0) \in \overline{D(A)} \text{ and } \varphi'(0) = A\varphi(0) + L(\varphi)\big\} \\ \mathcal{A}_U\varphi = \varphi', \end{cases}$$

is the infinitesimal generator of $(U(t))_{t\geq 0}$ *on* C_0.

In order to give the variation of constants formula associated to equation (2.1), we need to recall some notation and results which are taken from [**1**]. Let $\langle X_0\rangle$ be the space defined by

$$\langle X_0\rangle = \{X_0 y : y \in X\},$$

where the function $X_0 y$ is given, for each $y \in X$, by

$$(X_0 y)(\theta) = \begin{cases} 0 & \text{if } \theta \in [-r, 0), \\ y & \text{if } \theta = 0. \end{cases}$$

The space $C_0 \oplus \langle X_0\rangle$, equipped with the norm $|\varphi + X_0 y| = |\varphi| + |y|$, for $(\varphi, y) \in C_0 \times X$, is a Banach space. Consider the extension $\widetilde{\mathcal{A}_U}$ of the operator $\mathcal{A}_U$ on $C_0 \oplus \langle X_0\rangle$, defined by

$$\begin{cases} D(\widetilde{\mathcal{A}_U}) = \{\varphi \in C^1([-r,0], X) : \varphi(0) \in D(A) \\ \qquad\qquad \text{and } \varphi'(0) \in \overline{D(A)}\}, \\ \widetilde{\mathcal{A}_U}\varphi = \varphi' + X_0(A\varphi(0) + L(\varphi) - \varphi'(0)). \end{cases}$$

Lemma 2.5 ([**1,** Theorem 13]). *Assume that* **(H0)** *holds. Then,* $\widetilde{\mathcal{A}_U}$ *satisfies the Hille-Yosida condition on* $C_0 \oplus \langle X_0\rangle$*, i.e., there exist* $\widetilde{M} \geq 0$ *and* $\widetilde{\omega} \in \mathbb{R}$ *such that* $(\widetilde{\omega}, \infty) \subset \rho(\widetilde{\mathcal{A}_U})$ *and*

$$\left|R(\lambda, \widetilde{\mathcal{A}_U})^n\right| \leq \frac{\widetilde{M}}{(\lambda - \widetilde{\omega})^n} \quad \text{for } n \in \mathbb{N} \text{ and } \lambda > \widetilde{\omega}.$$

Theorem 2.6 ([**1,** Theorem 16]). *Assume that* **(H0)** *holds. Then, for all* $\varphi \in C_0$*, the solution* $u(\cdot, \sigma, \varphi, f)$ *of equation* (2.1) *is given by the following variation of constants formula:*

$$u_t(\cdot, \sigma, \varphi, f) = U(t-\sigma)\varphi + \lim_{n\to\infty} \int_\sigma^t U(t-s)\widetilde{B}_n(X_0 f(s))\, ds \quad \text{for } t \geq \sigma,$$

where $\widetilde{B}_n := nR(n, \widetilde{\mathcal{A}_U})$ *for* $n > \widetilde{\omega}$*.*

The following assumption plays a crucial role in obtaining the reduction principle:

(H1): the operator $T_0(t)$ is compact on $\overline{D(A)}$ for every $t > 0$.

We have the following fundamental result on the semigroup $(U(t))_{t\geq 0}$:

Theorem 2.7 ([**1,** Lemma 10])**.** *Assume that* **(H0)** *and* **(H1)** *hold. Then, the operator* $U(t)$ *is compact for* $t > r$.

As a consequence of the compactness property of the operator $U(t)$, the spectrum $\sigma(A_U)$ is the point spectrum. Moreover, we have the following spectral decomposition result:

Theorem 2.8 ([**17**])**.** C_0 *is decomposed as follows*:

$$C_0 = S \oplus V, \tag{2.2}$$

where S *is* U*-invariant, and there are positive constants* α *and* N *such that*

$$|U(t)\varphi| \leq Ne^{-\alpha t}|\varphi| \quad \text{for each } \varphi \in S \text{ and } t \geq 0.$$

V *is a finite-dimensional space, and the restriction of* U *to* V *is a group.*

In the sequel, $U^s(t)$ and $U^v(t)$ will denote the restriction of $U(t)$, respectively, on S and V, which are given by the above decomposition. Let $d := \dim(V)$ with a vector basis $\Phi = \{\varphi_1, \ldots, \varphi_d\}$. Then, there exist d-elements $\{\psi_1, \ldots, \psi_d\}$ in C_0^* such that

$$\begin{cases} \langle \psi_i, \varphi_j \rangle = \delta_{ij}, \\ \langle \psi_i, \varphi \rangle = 0 \qquad \text{for all } \varphi \in S \text{ and } i \in \{1, \ldots, d\}, \end{cases} \tag{2.3}$$

where $\langle \cdot, \cdot \rangle$ denotes the duality pairing between C_0^* and C_0, and

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j. \end{cases}$$

Let $\Psi = \text{col}\{\psi_1, \ldots \psi_2\}$, $\langle \Psi, \Phi \rangle$ is a $(d \times d)$-matrix, where the (i, j)-component is $\langle \psi_i, \varphi_j \rangle$.

Denote by Π^s and Π^v the projections, respectively, on S and V. For each $\varphi \in C_0$, we have

$$\Pi^v \varphi = \Phi \langle \Psi, \varphi \rangle.$$

In fact, for $\varphi \in C_0$, we have $\varphi = \Pi^s \varphi + \Pi^v \varphi$ with $\Pi^v \varphi = \sum_{i=1}^d \alpha_i \varphi_i$ and $\alpha_i \in \mathbb{R}$. By (2.3), we conclude that

$$\alpha_i = \langle \psi_i, \varphi \rangle.$$

Hence,

$$\Pi^v\varphi = \sum_{i=1}^{d}\langle\psi_i,\varphi\rangle\varphi_i = \Phi\langle\Psi,\varphi\rangle.$$

Since $(U^v(t))_{t\geq 0}$ is a group on V, then there exists a $(d\times d)$-matrix G such that

$$U^v(t)\Phi = \Phi e^{tG} \quad \text{for } t\in\mathbb{R}.$$

For $n, n_0 \in \mathbb{N}$ such that $n \geq n_0 \geq \widetilde{\omega}$ and $i \in \{1,\dots,d\}$, we define the linear operator $x^*_{i,n}$ by

$$x^*_{i,n}(a) = \langle\psi_i, \widetilde{B}_n X_0 a\rangle \quad \text{for } a\in X.$$

Since $|\widetilde{B}_n| \leq n/(n-\widetilde{\omega})\widetilde{M}$ for any $n\geq n_0$, then $x^*_{i,n}$ is a bounded linear operator from X to $\mathbb{R}$ such that

$$|x^*_{i,n}| \leq \frac{n}{n-\widetilde{\omega}}\widetilde{M}|\psi_i| \quad \text{for any } n\geq n_0.$$

Define the d-column vector $x^*_n = \operatorname{col}(x^*_{1,n},\dots,x^*_{d,n})$. Then, it can be seen that

$$\langle x^*_n, a\rangle = \langle\Psi, \widetilde{B}_n X_0 a\rangle \quad \text{for } a\in X,$$

with

$$\langle x^*_n, a\rangle_i = \langle\psi_i, \widetilde{B}_n X_0 a\rangle \quad \text{for } i=1,\dots,d \text{ and } a\in X.$$

Consequently, we have

$$\sup_{n\geq n_0}|x^*_n| < \infty,$$

which implies that $(x^*_n)_{n\geq n_0}$ is a bounded sequence in $\mathcal{L}(X,\mathbb{R}^d)$. We recall the following important results:

Theorem 2.9 ([**21**]). *There exists an $x^*\in\mathcal{L}(X,\mathbb{R}^d)$ such that $(x^*_n)_{n\geq n_0}$ converges weakly to x^* in the sense that*

$$\langle x^*_n, x\rangle \longrightarrow \langle x^*, x\rangle \quad \textit{as } n\to\infty \textit{ for all } x\in X.$$

Theorem 2.10 ([**21**]). *Assume that* **(H0)** *and* **(H1)** *hold, f is continuous, and let u be a solution of equation* (1.4) *on $\mathbb{R}$. Then, the function z defined by $z(t) := \langle\Psi, u_t\rangle$ is a solution of the ordinary* *differential equation*

$$\frac{d}{dt}z(t) = Gz(t) + \langle x^*, f(t)\rangle \quad \text{for } t \in \mathbb{R}. \tag{2.4}$$

Conversely, if z *is a solution of equation* (2.4) *on* $\mathbb{R}$*, and if, in addition,* f *is bounded on* $\mathbb{R}$*, then the function* u *given by*

$$u(t) := \left[\Phi z(t) + \lim_{n\to\infty}\int_{-\infty}^{t} U^s(t-s)\Pi^s(\widetilde{B}_n X_0 f(s))\,ds\right](0) \quad \text{for } t \in \mathbb{R},$$

is a solution of equation (1.4) *on* $\mathbb{R}$.

Theorem 2.11. *Assume that* **(H0)** *and* **(H1)** *hold,* f *is locally integrable, and let* u *be a solution of equation* (1.4) *on* $\mathbb{R}$*. Then, the function* z *defined by* $z(t) := \langle \Psi, u_t\rangle$ *is given by*

$$z(t) = e^{tG}z(0) + \int_0^t e^{(t-s)G}\langle x^*, f(s)\rangle\,ds \quad \text{for } t \in \mathbb{R}. \tag{2.5}$$

Conversely, if z *satisfies equation* (2.5) *on* $\mathbb{R}$*, and if, in addition,* f *satisfies*

$$\sup_{t\in\mathbb{R}}\int_t^{t+1} |f(s)|\,ds < \infty,$$

then the function u *given by*

$$u(t) := \left[\Phi z(t) + \lim_{n\to\infty}\int_{-\infty}^{t} U^s(t-s)\Pi^s(\widetilde{B}_n X_0 f(s))\,ds\right](0) \quad \text{for } t \in \mathbb{R},$$

is a solution of equation (1.4) *on* $\mathbb{R}$.

Proof. The proof is similar to that of Theorem 2.10. We only must prove that the limit

$$\lim_{n\to\infty}\int_{-\infty}^{t} U^s(t-s)\Pi^s(\widetilde{B}_n X_0 f(s))\,ds$$

exists in C_0. For $t \in \mathbb{R}$ and for n sufficiently large, we have

$$\left|\int_{-\infty}^{t} U^s(t-s)\Pi^s(\widetilde{B}_n X_0 f(s))\,ds\right| \le K,$$

where

$$K = 2\widetilde{M}N\|\Pi^s| \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} |f(s)|\, ds \right) \frac{1}{1-e^{-\alpha}}.$$

Let

$$H(n,s,t) := U^s(t-s)\Pi^s(\widetilde{B}_n X_0 f(s)) \quad \text{for } n \in \mathbb{N} \text{ and } s \le t.$$

For n and m sufficiently large and $\sigma \le t$, we have

$$\left| \int_{-\infty}^t H(n,s,t)\, ds - \int_{-\infty}^t H(m,s,t)\, ds \right| \le 2Ke^{-\alpha(t-\sigma)} + \left| \int_\sigma^t H(n,s,t)\, ds - \int_\sigma^t H(m,s,t)\, ds \right|.$$

Since $\lim_{n\to\infty} \int_\sigma^t H(n,s,t)\, ds$ exists, it follows that

$$\limsup_{n,m\to\infty} \left| \int_{-\infty}^t H(n,s,t)\, ds - \int_{-\infty}^t H(m,s,t)\, ds \right| \le 2Ke^{-\alpha(t-\sigma)}.$$

By letting $\sigma \to -\infty$, we obtain

$$\limsup_{n,m\to\infty} \left| \int_{-\infty}^t H(n,s,t)\, ds - \int_{-\infty}^t H(m,s,t)\, ds \right| = 0.$$

Thus, by the completeness of the phase space C_0, we deduce that the limit $\lim_{n\to\infty} \int_{-\infty}^t H(n,s,t)\, ds$ exists in C_0. □

3. Almost automorphic functions. Let $(X, |\cdot|)$ be a Banach space and $BC(\mathbb{R}, X)$ the space of bounded continuous functions from $\mathbb{R}$ to X, equipped with the supremum norm

$$|f|_\infty := \sup_{t\in\mathbb{R}} |f(t)|. \tag{3.1}$$

Definition 3.1. **[23]** A continuous function $f : \mathbb{R} \to X$ is said to be *Bohr almost periodic* (or, simply, *almost periodic*) if, for every $\varepsilon > 0$, there exists a positive number l such that every interval of length l contains a number τ such that

$$|f(t+\tau) - f(t)| < \varepsilon \quad \text{for } t \in \mathbb{R}.$$

Theorem 3.2 (**[23]**). *Each almost periodic function is uniformly continuous.*

A useful characterization of almost periodic functions was given by Bochner.

Theorem 3.3 (**[6]**). *A continuous function $f : \mathbb{R} \to X$ is almost periodic if and only if, for every sequence of real numbers $(s_n)_n$, there exist a subsequence $(s'_n)_n \subset (s_n)_n$ and a function $\widetilde{f}$, such that*

$$f(t + s'_n) \longrightarrow \widetilde{f}(t)$$

uniformly on $\mathbb{R}$ as $n \to \infty$.

In **[7]**, Bochner introduced the concept of almost automorphy, which is a generalization of almost periodicity.

Definition 3.4 (**[7]**). A continuous function $f : \mathbb{R} \mapsto X$ is said to be *almost automorphic* if, for every sequence of real numbers $(s_n)_n$, there exist a subsequence $(s'_n)_n \subset (s_n)_n$ and a function $\widetilde{f}$ such that, for each $t \in \mathbb{R}$,

$$f(t + s'_n) \longrightarrow \widetilde{f}(t)$$

and

$$\widetilde{f}(t - s'_n) \longrightarrow f(t),$$

as $n \to \infty$. If the above limits hold uniformly in compact subsets of $\mathbb{R}$, then f is said to be *compact almost automorphic.*

Let $AA(\mathbb{R}, X)$ denote the space of almost automorphic X-valued functions.

Definition 3.5 (**[33]**). A weakly continuous function $f : \mathbb{R} \mapsto X$ is said to be *weakly almost automorphic* if, for every sequence of real numbers $(s_n)_n$, there exist a subsequence $(s'_n)_n \subset (s_n)_n$ and a function $\widetilde{f}$ such that, for each $t \in \mathbb{R}$,

$$f(t + s'_n) \longrightarrow \widetilde{f}(t)$$

and

$$\widetilde{f}(t - s'_n) \longrightarrow f(t),$$

as $n \to \infty$, where both of the above convergences hold in the weak sense.

The following concept is due to Bochner.

Definition 3.6 ([**2, 5, 36**])**.** The Bochner transform f^b of a function $f \in L^p_{\mathrm{loc}}(\mathbb{R}, X)$ is the function $f^b : \mathbb{R} \to L^p([0,1], X)$, defined for each $t \in \mathbb{R}$ by

$$(f^b(t))(s) = f(t+s) \quad \text{for } s \in [0,1].$$

Definition 3.7 ([**2, 5, 36**])**.** Let $p \geq 1$. The space $BS^p(\mathbb{R}, X)$ consists of all functions $f \in L^p_{\mathrm{loc}}(\mathbb{R}, X)$ such that $f^b : \mathbb{R} \to L^p([0,1], X)$ is bounded, that is, $\sup_{t\in\mathbb{R}}(\int_t^{t+1} |f(s)|^p ds)^{1/p} < \infty$. It is a normed space when equipped with the following norm

$$|f_{BS^p}| = \sup_{t\in\mathbb{R}} \left(\int_t^{t+1} |f(s)|^p ds \right)^{1/p}.$$

Remark. Note that the functions of $BS^p(\mathbb{R}, X)$ may not be bounded.

Definition 3.8 ([**16**])**.** A function $f \in L^p_{\mathrm{loc}}(\mathbb{R}, X)$ is said to be *Stepanov almost automorphic* for some $p \geq 1$ (or S^p*-almost automorphic*) if the function $f^b : \mathbb{R} \to L^p([0,1], X)$ is almost automorphic.

The following characterization of almost automorphy in the sense of Stepanov is essential for the remainder of this work.

Proposition 3.9 ([**20**])**.** *A function $f \in L^p_{\mathrm{loc}}(\mathbb{R}, X)$ is S^p-almost automorphic if and only if, for every sequence of real numbers $(s_n)_n$, there exist a subsequence $(s'_n)_n \subset (s_n)_n$ and a function $g \in L^p_{\mathrm{loc}}(\mathbb{R}, X)$, such that, for each $t \in \mathbb{R}$,*

$$\left(\int_t^{t+1} |f(s+s'_n) - g(s)|^p ds \right)^{1/p} \longrightarrow 0 \tag{3.2}$$

and

$$\left(\int_t^{t+1} |g(s-s'_n)-f(s)|^p ds\right)^{1/p} \longrightarrow 0, \tag{3.3}$$

as $n\to\infty$.

Let $SAA^p(\mathbb{R},X)$ denote the space of S^p-almost automorphic X-valued functions on $\mathbb{R}$. Then, for all $p\geq 1$, $AA(\mathbb{R},X)\subset SAA^p(\mathbb{R},X)$. Moreover, if $p\geq q$, then $SAA^p(\mathbb{R},X)\subset SAA^q(\mathbb{R},X)$. If $h\in AA(\mathbb{R},\mathbb{C})$ and $f\in SAA^p(\mathbb{R},\mathbb{C})$, then $h\cdot f\in SAA^p(\mathbb{R},\mathbb{C})$.

4. Behavior of the bounded solutions of equation (1.4). The goal of this section is to investigate the almost automorphic aspect of bounded solutions of equation (1.4). Since almost automorphic functions have a relatively compact range, it is appropriate to investigate first when a bounded solution of equation (1.4) has a relatively compact range for an input function $f\in BS^p(\mathbb{R},X)$. We distinguish two cases, $p>1$ and $p=1$.

4.1. $p>1$.

Lemma 4.1. *Assume that* **(H0)** *and* **(H1)** *hold and* $f\in BS^p(\mathbb{R},X)$ *with* $p>1$. *Then, every bounded solution of equation* (1.4) *on* $\mathbb{R}^+$ *has a relatively compact range.*

Proof. Let x be a solution of equation (1.4) which is bounded on $\mathbb{R}^+$ and $\overline{x}:=\sup_{t\geq 0}|x_t|$. Let $M_0\geq 1$ and $\omega_0\in\mathbb{R}$ be such that $|T(t)|\leq M_0e^{\omega_0 t}$ for all $t\geq 0$. For $0<\varepsilon\leq 1$ and $t\geq 1$, we have

$$x(t)=T_0(\varepsilon)x(t-\varepsilon)+\lim_{\lambda\to\infty}\int_{t-\varepsilon}^t T_0(t-s)B_\lambda[L(x_s)+f(s)]\,ds.$$

Since $T_0(\varepsilon)$ is compact, there exists a compact subset K_ε of X such that $T_0(\varepsilon)x(t-\varepsilon)\in K_\varepsilon$ for all $t\geq 1$. Moreover, for each $t\geq 1$ we have

$$\left|\lim_{\lambda\to\infty}\int_{t-\varepsilon}^t T_0(t-s)B_\lambda[L(x_s)+f(s)]\,ds\right|\leq\delta(\varepsilon),$$

where

$$\delta(\varepsilon) := M_0\overline{M}|L|\overline{x}\int_0^{\varepsilon} e^{\omega_0 s}ds + M_0\overline{M}|f|_{BS^p}\left(\int_0^{\varepsilon} e^{q\omega_0 s}ds\right)^{1/q}.$$

It follows that

$$\{x(t) : t \geq 1\} \subset K_\varepsilon + \overline{B}(0, \delta(\varepsilon)).$$

Taking Kuratowski's measure of noncompactness, we obtain that

$$\alpha(\{x(t) : t \geq 1\}) \leq 2\delta(\varepsilon).$$

The above inequality holds for all $0 < \varepsilon \leq 1$; thus, by letting ε go to 0, we deduce that

$$\alpha(\{x(t) : t \geq 1\}) = 0.$$

Thus, $\{x(t) : t \geq 1\}$ is relatively compact. Since $x(\cdot)$ is continuous, we conclude that $\{x(t) : t \geq 0\}$ is relatively compact. □

Lemma 4.2. *Assume that* **(H0)** *and* **(H1)** *hold and* $f \in BS^p(\mathbb{R}, X)$ *with* $p > 1$. *Then, every bounded solution of equation* (1.4) *on* $\mathbb{R}^+$ *is uniformly continuous.*

The next lemma is needed in the proof of Lemma 4.2.

Lemma 4.3 ([**3,** Theorem 4.1.2])**.** *Let* Y *be a normed space and* $(T_n)_n$ *a sequence of bounded linear operators on* Y *such that* $\sup_n |T_n| < \infty$. *If* D *is a dense subset of* Y *and if, for each* $y \in D$,

$$T_n y \longrightarrow Ty \quad \text{as } n \to \infty,$$

for some bounded linear operator T, *then, for every compact set* K *of* Y,

$$\sup_{y \in K} |T_n y - Ty| \longrightarrow 0 \quad \text{as } n \to \infty.$$

Proof of Lemma 4.2. Let x be a solution of equation (1.4) which is bounded on $\mathbb{R}^+$, and let $\overline{x} := \sup_{t \geq 0} |x_t|$. Let $M_0 \geq 1$ and $\omega_0 \in \mathbb{R}$ be such that $|T(t)| \leq M_0 e^{\omega_0 t}$ for all $t \geq 0$. For $0 \leq t' \leq t$, we have

$$x(t) = T_0(t - t')x(t') + \lim_{\lambda \to \infty} \int_{t'}^{t} T_0(t - s)B_\lambda[L(x_s) + f(s)]\, ds.$$

Let $K = \overline{\{x(t) : t \geq 0\}}$ which is a compact subset of X by Lemma 4.1. Then, for $|t - t'|$ small enough

$$\begin{aligned} |x(t) - x(t')| \leq & \sup_{y \in K} |T_0(t-t')y - y| \\ & + M_0 \overline{M} |L| \overline{x} \int_0^{t-t'} e^{\omega_0 s} ds \\ & + M_0 \overline{M} |f|_{BS^p} \left(\int_0^{t-t'} e^{q\omega_0 s} ds \right)^{1/q}. \end{aligned}$$

The Banach-Steinhaus theorem, together with Lemma 4.3, imply that $\sup_{y \in K} |T_0(t-t')y - y| \to 0$ as $|t - t'| \to 0$. Thus, x is uniformly continuous. □

Corollary 4.4. *Assume that* **(H0)** *and* **(H1)** *hold and* $f \in BS^p(\mathbb{R}, X)$ *with* $p > 1$. *If* $t \mapsto x(t)$ *is a bounded solution of equation* (1.4) *on* $\mathbb{R}^+$, *then the history function* $t \mapsto x_t$ *has a relatively compact range on* $\mathbb{R}^+$.

Proof. From Lemma 4.2, the solution $t \mapsto x(t)$ is uniformly continuous. Thus, the family of functions $\theta \mapsto x_t(\theta)$ indexed by t is equicontinuous. Since, for each $\theta \in [-r, 0]$, the set $\{x_t(\theta) : t \geq 0\}$ is relatively compact (Lemma 4.1), we obtain the compactness of $\overline{\{x_t : t \geq 0\}}$ in the space $C([-r, 0], X)$ by applying Arzelà-Ascoli's theorem. □

4.2. $p = 1$. In order to reproduce the above results for $f \in BS^p(\mathbb{R}, X)$ with $p = 1$, more than the boundedness in the Stepanov norm must be assumed.

In what follows, let $M_1 \geq 1$ and $\omega_1 \in \mathbb{R}$ be such that $|U(t)| \leq M_1 e^{\omega_1 t}$ for all $t \geq 0$.

Lemma 4.5. *Assume that* **(H0)** *and* **(H1)** *hold and* $f \in SAA^1(\mathbb{R}, X)$. *If* $t \mapsto x(t)$ *is a bounded solution of equation* (1.4) *on* $\mathbb{R}^+$, *then the history function* $t \mapsto x_t$ *has a relatively compact range on* $\mathbb{R}^+$.

Proof. Let x be a solution of equation (1.4) which is bounded on $\mathbb{R}^+$. Consider the following decomposition

$$\{x_t : t \geq 0\} = \{x_t : t \geq 2r\} \cup \{x_t : 0 \leq t < 2r\}. \tag{4.1}$$

Recall that the operator $U(t)$ is compact whenever $t > r$, where r is the delay (see Theorem 2.7). Let $(t_n)_n$ be a sequence of real numbers such that $t_n \geq 2r$ for all $n \in \mathbb{N}$. Then, we have

$$x_{t_n} = U(2r)x_{t_n-2r} + \lim_{m\to\infty}\int_0^{2r} U(s)\widetilde{B}_m(X_0 f(t_n - s))\, ds.$$

Since the operator $U(2r)$ is compact and $f \in SAA^1(\mathbb{R}, X)$, there exist a subsequence $(t'_n)_n \subset (t_n)_n$ and a function $\widetilde{f} \in L^1_{\rm loc}(\mathbb{R}, X)$ such that $U(2r)x_{t'_n-2r}$ converges to some $\varphi_1 \in C_0$ and, for all $t \in \mathbb{R}$,

$$\int_t^{t+1} \left|f(t'_n + s) - \widetilde{f}(s)\right| ds \longrightarrow 0 \quad \text{as } n \to \infty.$$

Let

$$\varphi_2 := \lim_{m\to\infty}\int_0^{2r} U(s)\widetilde{B}_m(X_0\widetilde{f}(-s))\, ds.$$

Then, $x_{t'_n} \to \varphi_1 + \varphi_2$ as $n \to \infty$. In fact,

$$\begin{aligned}&\left|\lim_{m\to\infty}\int_0^{2r} U(s)\widetilde{B}_m(X_0 f(t'_n - s))\, ds - \varphi_2\right| \\ &\qquad \leq M_1\widetilde{M}e^{|\omega_1|2r}\sum_{k=0}^{[2r]+1}\int_{-k-1}^{-k}\left|f(t'_n + s) - \widetilde{f}(s)\right| ds \longrightarrow 0,\end{aligned}$$

as $n \to \infty$. Therefore, the set $\{x_t : t \geq 2r\}$ is relatively compact. The relative compactness of $\{x_t : t \geq 0\}$ follows from the decomposition (4.1) and the continuity of the history function $t \mapsto x_t$. □

Lemma 4.6. *Assume that* **(H0)** *and* **(H1)** *hold and* $f \in SAA^1(\mathbb{R}, X)$. *If* $t \mapsto x(t)$ *is a bounded solution of equation* (1.4) *on* $\mathbb{R}^+$ (*respectively, on* $\mathbb{R}$), *then the history function* $t \mapsto x_t$ *is uniformly continuous on* $\mathbb{R}^+$ (*respectively, on* $\mathbb{R}$).

Proof. If $t \mapsto x_t$ is not uniformly continuous, then there exist $\varepsilon > 0$ and two real sequences $(s_n)_n$ and $(h_n)_n$ such that $\lim_{n\to\infty} h_n = 0$ and

$$|x_{s_n+h_n}| - x_{s_n} > \varepsilon \quad \text{for all } n \in \mathbb{N}. \tag{4.2}$$

Assume, without loss of generality, that $h_n \geq 0$ for all $n \in \mathbb{N}$. Thus, we have

$$x_{s_n+h_n} = U(h_n)x_{s_n} + \lim_{m\to\infty} \int_0^{h_n} U(h_n - s)(\widetilde{B}_m(X_0 f(s+s_n)))\, ds.$$

Let $K := \overline{\{x_t : t \in \mathbb{R}\}}$. Then, we have

(4.3)

$$|x_{s_n+h_n} - x_{s_n}| \leq \sup_{\phi\in K} |U(h_n)\phi - \phi| + \widetilde{M} M_1 e^{|\omega_1| h_n} \int_0^{h_n} |f(s+s_n)\, ds.$$

From Lemma 4.5, K is a compact subset of C. It follows by the Banach-Steinhaus theorem and Lemma 4.3 that

$$\sup_{\phi\in K} |U(h_n)\phi - \phi| \longrightarrow 0 \quad \text{as } n \to \infty.$$

On the other hand, from the S^1-almost automorphy of f, there exist a subsequence $(s'_n)_n \subset (s_n)_n$ and a function $\widehat{f} \in L^1_{\mathrm{loc}}(\mathbb{R}, X)$ such that, for each $t \in \mathbb{R}$,

$$\int_t^{t+1} |f(s+s'_n) - \widehat{f}(s)|\, ds \longrightarrow 0, \quad \text{as } n \to \infty. \tag{4.4}$$

Let $(h'_n)_n$ be the corresponding subsequence of $(h_n)_n$. We can assume that $|h_n| \leq 1$ for all $n \in \mathbb{N}$. Then, we have

$$\int_0^{h'_n} |f(s+s'_n)|\, ds \leq \int_0^1 |f(s+s'_n) - \widehat{f}(s)|\, ds + \int_0^{h'_n} |\widehat{f}(s)|\, ds \longrightarrow 0$$

as $n \to \infty$. Therefore, we deduce from (4.3) that

$$|x_{s'_n+h'_n} - x_{s'_n}| \longrightarrow 0 \quad \text{as } n \to \infty,$$

which contradicts (4.2). We conclude that $t \mapsto x_t$ must be uniformly continuous. □

The following result shows that, to get a bounded solution on $\mathbb{R}$, we only need a bounded solution on $\mathbb{R}^+$.

Theorem 4.7. *Assume that* **(H0)** *and* **(H1)** *hold and* $f \in SAA^1(\mathbb{R}, X)$. *Then, if equation* (1.4) *has a solution which is bounded on* $\mathbb{R}^+$, *it has a solution which is bounded on* $\mathbb{R}$.

Proof. Let x be a solution of equation (1.4) which is bounded on $\mathbb{R}^+$. From Lemma 4.5 and Lemma 4.6, x is compact and uniformly continuous. Let $(t_n)_n$ be a sequence of real numbers such that $\lim_{n\to\infty} t_n = \infty$. If $t \in [-1, 1]$, then, for sufficiently large n, the sequence of functions $x_n : t \mapsto x(t + t_n)$ is well defined and equicontinuous. It follows by Arzelà-Ascoli's theorem and the diagonal extraction procedure that there exists a subsequence $(t'_n)_n \subset (t_n)_n$ such that

$$x(t + t'_n) \longrightarrow y(t) \quad \text{as } n \to \infty,$$

uniformly on each compact subset of $\mathbb{R}$. Since $f \in SAA^1(\mathbb{R}, X)$, then there exist a subsequence $(t''_n)_n \subset (t'_n)_n$ and a function $\widetilde{f} \in L^1_{\text{loc}}(\mathbb{R}, X)$ such that

$$\int_t^{t+1} |f(t''_n + s) - \widetilde{f}(s)|\, ds \longrightarrow 0 \tag{4.5}$$

and

$$\int_t^{t+1} |\widetilde{f}(t''_n - s) - f(s)|\, ds* \longrightarrow 0, \tag{4.6}$$

as $n \to \infty$. For each $t \geq s$ and for $n \in \mathbb{N}$ sufficiently large, we have

$$x(t+t''_n) = T_0(t-s)x(s+t''_n) + \lim_{\lambda\to\infty} \int_s^t T_0(t-\sigma)B_\lambda[L(x_{\sigma+t''_n}) + f(\sigma+t''_n)]\, d\sigma. \tag{4.7}$$

By taking the limit as $n \to \infty$ in (4.7) using (4.5) and the fact that $x_{\sigma+t''_n} \to y_\sigma$, we obtain, for each $t \geq s$,

$$y(t) = T_0(t-s)y(s) + \lim_{\lambda\to\infty} \int_s^t T_0(t-\sigma)B_\lambda[L(y_\sigma) + \widetilde{f}(\sigma)]\, d\sigma.$$

We observe that y is also compact and uniformly continuous. Using (4.6) and applying the above argument to the returning sequence $(-t''_n)_n$, we obtain a solution z of equation (1.4) which is bounded on $\mathbb{R}$. □

4.3. Almost automorphy of bounded solutions. We first study the behavior of bounded integral solutions of the following finite dimensional differential equation:

$$x'(t) = Bx(t) + g(t) \quad \text{for } t \in \mathbb{R}, \tag{4.8}$$

where $g : \mathbb{R} \to \mathbb{C}^n$ is S^1-almost automorphic and $B : \mathbb{C}^n \to \mathbb{C}^n$ a matrix.

Since g is only locally integrable, we mean by an *integral solution* of equation (4.8) a locally integrable function $x : \mathbb{R} \to \mathbb{C}^n$ which satisfies the following integral equation

$$x(t) = x(0) + \int_0^t Bx(s)\,ds + \int_0^t g(s)\,ds \quad \text{for } t \in \mathbb{R}.$$

Using this convention, an integral solution of equation (4.8) is locally absolutely continuous and given by the following formula

$$x(t) = e^{tB}x(0) + \int_0^t e^{(t-s)B} g(s)\,ds \quad \text{for all } t \in \mathbb{R}.$$

Moreover, an integral solution of equation (4.8) satisfies (4.8) almost everywhere.

Next, we find the conditions under which the integral of a Stepanov almost automorphic function is almost automorphic. We first recall the following result.

Theorem 4.8 ([**33,** page 29, Theorem 2.4.6])**.** *Let X be a uniformly convex Banach space and $f : \mathbb{R} \to X$ an almost automorphic function. If $F(t) := \int_0^t f(s)\,ds$ is bounded on $\mathbb{R}$, then it is almost automorphic.*

The following result gives the same conclusion as in Theorem 4.8, but with weaker assumptions.

Theorem 4.9. *Let X be a uniformly convex Banach space and $f : \mathbb{R} \to X$ an S^p-almost automorphic function with $p > 1$. Let $F(t) := \int_0^t f(s)\,ds$. If $F \in BS^p(\mathbb{R}, X)$, then F is almost automorphic.*

The next lemmas are needed in the proof of Theorem 4.9.

Lemma 4.10 ([**30,** Lemma 3.2])**.** *Let X be a Banach space and $f : \mathbb{R} \to X$ an S^p-almost automorphic function with $p \geq 1$. Then, the function F defined on $\mathbb{R}$ by $F(t) := \int_0^t f(s)\,ds$ is uniformly continuous.*

Lemma 4.11 ([**30,** Lemma 3.1])**.** *Let X be a Banach space and $f : \mathbb{R} \to X$ an S^p-almost automorphic function with $p \geq 1$. If f is uniformly continuous, then f is almost automorphic.*

Proof of Theorem 4.9. We first observe that, for each $t \in \mathbb{R}$,

$$\int_0^t f^b(s)\,ds = F^b(t) - F^b(0),$$

where f^b denotes the Bochner transform of f (see Definition 3.6). Since $f \in SAA^p(\mathbb{R}, X)$, then $f^b \in AA(\mathbb{R}, L^p([0,1], X))$. F^b is bounded on $\mathbb{R}$ since $F \in BS^p(\mathbb{R}, X)$. The uniform convexity of $L^p([0,1], X)$ follows from the uniform convexity of X and the fact that $p > 1$. Now, from Theorem 4.8, we deduce that F^b is almost automorphic, that is, $F \in SAA^p(\mathbb{R}, X)$. From Lemma 4.10, F is uniformly continuous. It follows by Lemma 4.11 that F is almost automorphic. □

The proof of Theorem 4.9 relies on the fact that the space $L^p[(0,1), X]$ is uniformly convex for $p > 1$, when X is uniformly convex. For $p = 1$, the space $L^p((0,1), X)$ is no longer uniformly convex; thus, the approach in Theorem 4.9 cannot be used to give a similar result when $p = 1$.

Theorem 4.12. *Let X be a uniformly convex Banach space and $f : \mathbb{R} \to X$ an S^1-almost automorphic function. Let $F(t) := \int_0^t f(s)\,ds$. If F is bounded on $\mathbb{R}$, then it is almost automorphic.*

Proof. The proof is divided into two steps:

Step 1. In this step, we will prove that F is almost automorphic in the weak topology. Let $(s_n)_{n \in \mathbb{N}}$ be an arbitrary sequence. We have for, all $n \in \mathbb{N}$ and $t \in \mathbb{R}$,

$$F(t + s_n) = F(s_n) + \int_0^t f(s + s_n)\,ds.$$

Since the Banach space X is reflexive, F is bounded and f is S^1-almost automorphic, there exist a subsequence $(s'_n)_n \subset (s_n)_n$, a function

$\widetilde{f} \in L^1_{\text{loc}}(\mathbb{R}, X)$, and a constant $c \in X$ such that, for each $t \in \mathbb{R}$,

$$\begin{cases} \int_t^{t+1} |f(s+s'_n) - \widetilde{f}(s)|\, ds \longrightarrow 0, \\ \int_t^{t+1} |\widetilde{f}(s-s'_n) - f(s)|\, ds \longrightarrow 0, \end{cases} \tag{4.9}$$

and

$$F\left(s'_n\right) \longrightarrow c, \tag{4.10}$$

in the weak topology as $n \to \infty$. Consider the function $\widetilde{F}(t) := c + \int_0^t \widetilde{f}(s)\, ds$ defined for all $t \in \mathbb{R}$. Then, we can see that, for each $t \in \mathbb{R}$,

$$\int_0^t f(s+s'_n)\, ds \longrightarrow \int_0^t \widetilde{f}(s)\, ds \quad \text{as } n \to \infty. \tag{4.11}$$

From (4.10) and (4.11), we deduce that, for each $t \in \mathbb{R}$,

$$F(t+s'_n) \longrightarrow \widetilde{F}(t) \tag{4.12}$$

in the weak topology as $n \to \infty$. Using the fact that $|\widetilde{F}(t)| = \sup_{|\varphi| \leq 1,\, \varphi \in X^*} |\langle \varphi, \widetilde{F}(t) \rangle|$, it can be observed from (4.12) that

$$|\widetilde{F}|_\infty \leq |F|_\infty, \tag{4.13}$$

where $|\cdot|_\infty$ denotes the supremum norm, see (3.1).

On the other hand, for all $n \in \mathbb{N}$ and $t \in \mathbb{R}$, we have

$$\widetilde{F}(t-s'_n) = \widetilde{F}(-s'_n) + \int_0^t \widetilde{f}(s-s'_n)\, ds.$$

Since $\widetilde{F}$ is also bounded, then there exist a subsequence $(s''_n)_n \subset (s'_n)_n$ and a constant $d \in \mathbb{R}$ such that

$$\widetilde{F}(-s''_n) \longrightarrow d, \tag{4.14}$$

in the weak topology as $n \to \infty$. On the other hand, we have

$$\int_0^t \widetilde{f}(s-s''_n)\, ds \longrightarrow \int_0^t f(s)\, ds = F(t) \quad \text{as } n \to \infty. \tag{4.15}$$

From (4.14) and (4.15), we deduce that, for each $t \in \mathbb{R}$,

$$\widetilde{F}(t - s''_n) \longrightarrow d + F(t) \tag{4.16}$$

in the weak topology as $n \to \infty$. Using the same approach as in the proof of [**33,** page 27, Theorem 2.4.4] we can prove that $d = 0$. Therefore, we deduce that, for each $t \in \mathbb{R}$,

$$\widetilde{F}(t - s''_n) \longrightarrow F(t) \tag{4.17}$$

in the weak topology as $n \to \infty$. This implies that F is weakly almost automorphic. Note that we have not yet used the uniform convexity assumption. Thus, the result obtained in this step is valid for an arbitrary reflexive Banach space.

Step 2. The equation $F(t) = \int_0^t f(s)\, ds$ can be seen as an evolution equation of the form

$$\frac{dF}{dt}(t) = AF(t) + f(t) \quad \text{for } t \in \mathbb{R}.$$

Here, $A = 0$ generates the trivial, bounded C_0-group $T(t) = I$, where I is the identity operator on X. Since X is uniformly convex, by [**20,** Theorem 31], we deduce that F is almost automorphic. □

Consider the following scalar differential equation

$$x'(t) = \lambda x(t) + g(t) \quad \text{for } t \in \mathbb{R}, \tag{4.18}$$

where $g : \mathbb{R} \to \mathbb{C}$ is an S^1-almost automorphic function and $\lambda \in \mathbb{C}$.

Theorem 4.13. *Every bounded integral solution of equation* (4.18) *on* $\mathbb{R}$ *is almost automorphic.*

Proof. The proof is similar to [**34,** Theorem 2.1]. We only must use Theorem 4.12 instead of Theorem 4.8. □

We have the following Bohr-Neugebauer type theorem for equation (4.8). This result is similar to [**34,** Theorem 2.4]; however, we do not require the function g to be almost automorphic in the classical sense.

Theorem 4.14. *If g is S^1-almostautomorphic, then every bounded integral solution of equation* (4.8) *on* $\mathbb{R}$ *is almost automorphic.*

Proof. The proof is similar to [**34,** Theorem 2.4]. We only must use Theorem 4.13 instead of [**34,** Theorem 2.1]. □

Now, we use the reduction principle in Theorem 2.11 to extend the Bohr-Neugebauer property in Theorem 4.14 to the partial functional differential equation (1.4).

Theorem 4.15. *Assume that* **(H0)** *and* **(H1)** *hold and* $f : \mathbb{R} \to X$ *is* S^1*-almost automorphic. Then, every bounded solution of equation* (1.4) *on* $\mathbb{R}$ *is compact almost automorphic.*

The next lemma is needed in the proof of Theorem 4.15.

Lemma 4.16 ([**18**]). *A function* $f : \mathbb{R} \to X$ *is compact almost automorphic if and only if it is almost automorphic and uniformly continuous.*

Proof of Theorem 4.15. Let x be a bounded solution of equation (1.4) on $\mathbb{R}$. Using the spectral decomposition (2.2), we have, for each $t \in \mathbb{R}$,

$$x_t = \Pi^v x_t + \Pi^s x_t. \tag{4.19}$$

On one hand, we have for $t \geq \sigma$,

$$\Pi^s x_t = U^s(t-\sigma)\Pi^s x_\sigma + \lim_{n\to\infty} \int_\sigma^t U^s(t-s)\Pi^s(\widetilde{B}_n(X_0 f(s)))\, ds. \tag{4.20}$$

Since $t \mapsto x_t$ is bounded on $\mathbb{R}$ and $U(t)$ is exponentially stable in S, by letting $\sigma \to -\infty$ in (4.20), we obtain that, for all $t \in \mathbb{R}$,

$$\Pi^s x_t = \lim_{n\to\infty} \int_{-\infty}^t U^s(t-s)\Pi^s(\widetilde{B}_n(X_0 f(s)))\, ds.$$

In fact, for each fixed $t \in \mathbb{R}$, we have, for all $\sigma \leq t$,

$$|U^s(t-\sigma)\Pi^s x_\sigma| \leq e^{-\alpha(t-\sigma)}|\Pi^s| \sup_{s\in\mathbb{R}} |x_s| \longrightarrow 0 \quad \text{as } \sigma \to -\infty$$

and

$$\Bigg| \lim_{n\to\infty} \int_{\sigma}^{t} U^s(t-s)\Pi^s(\widetilde{B}_n(X_0 f(s)))\,ds - \lim_{n\to\infty} \int_{-\infty}^{t} U^s(t-s)\Pi^s(\widetilde{B}_n(X_0 f(s)))\,ds \Bigg| \le \widetilde{M} N |\Pi^s| |f|_{BS^1} \frac{e^{-\alpha(t-\sigma)}}{1-e^{-\alpha}},$$

which goes to 0 as $\sigma \to -\infty$.

Using the same approach as in [**19,** Theorem 30], it can be proven that the function

$$t \longmapsto \Pi^s x_t = \lim_{n\to\infty} \int_{-\infty}^{t} U^s(t-s)\Pi^s(\widetilde{B}_n(X_0 f(s)))\,ds$$

is almost automorphic.

On the other hand, for each $t \in \mathbb{R}$,

$$\Pi^v x_t = \Phi\langle \Psi, x_t\rangle = \sum_{i=1}^{d} \langle \psi_i, x_t\rangle \varphi_i. \tag{4.21}$$

From Theorem 2.11, the function $z(t) = \langle \Psi, x_t\rangle$ is an integral solution of the following differential equation

$$z'(t) = Gz(t) + \langle x^*, f(t)\rangle \quad \text{for } t \in \mathbb{R}.$$

Moreover, the function $t \mapsto \langle \Psi, x_t\rangle$ is bounded on $\mathbb{R}$, and the function $t \mapsto \langle x^*, f(t)\rangle$ is S^1-almost automorphic. It follows from Theorem 4.14 that $t \mapsto \langle \Psi, x_t\rangle$ is almost automorphic. We deduce from (4.21) that the function $t \mapsto \Pi^v x_t$ is almost automorphic. The almost automorphy of $t \mapsto x_t$ follows from (4.19). From Lemma 4.6, the solution $t \mapsto x_t$ is uniformly continuous. The compact almost automorphy of x follows from Lemma 4.16. □

We recall the following Massera-type theorem.

Theorem 4.17 ([**21**]). *Assume that* **(H0)** *and* **(H1)** *hold and* $f : \mathbb{R} \to X$ *is almost automorphic. If equation* (1.4) *has a bounded solution on* $\mathbb{R}^+$*, then it has an almost automorphic solution.*

The following result is a consequence of Theorem 4.7 and Theorem 4.15. It shows that Theorem 4.17 holds even if we assume that f is S^1-almost automorphic, which is a weaker condition than the almost automorphy condition. Moreover, this result yields more than almost automorphy.

Corollary 4.18. *Assume that* **(H0)** *and* **(H1)** *hold and* $f : \mathbb{R} \to X$ *is* S^1*-almost automorphic. If equation* (1.4) *has a bounded solution on* $\mathbb{R}^+$*, then it has a compact almost automorphic solution.*

5. Application. To apply our results, we consider the following reaction-diffusion equation with delay in a bounded open subset Ω of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$.

(5.1)

$$\begin{cases} \dfrac{\partial}{\partial t} v(t,x) = \Delta v(t,x) + av(t,x) \\ \qquad\qquad + \displaystyle\int_{t-r}^{t} h(s-t)v(s,x)\,ds + F(t)\psi(x) & \text{for } t \in \mathbb{R} \text{ and } x \in \Omega, \\ v(t,x) = 0 & \text{on } \mathbb{R} \times \partial\Omega, \end{cases}$$

where $a \in \mathbb{R}$, $h : [-r, 0] \to \mathbb{R}$ and $\psi : \overline{\Omega} \to \mathbb{R}$ are continuous functions. The function $F : \mathbb{R} \to \mathbb{R}$ is given by

$$F(t) = \sum_{n \geq 1} F_n(t),$$

where the F_n are defined for every integer $n \geq 1$ by

$$F_n(t) = \sum_{k \in P_n} H(n^2(t-k)),$$

with $P_n = 3^n(2\mathbb{Z}+1) = \{3^n(2k+1),\ k \in \mathbb{Z}\}$ and $H \in C_0^\infty(\mathbb{R}, \mathbb{R})$, with support in $(-1/2, 1/2)$ such that

$$H \geq 0, \qquad H(0) = 1 \quad \text{and} \quad \int_{-1/2}^{1/2} H(s)\,ds = 1.$$

The function F is not almost automorphic since it is not bounded. However, $F \in C^\infty(\mathbb{R}, \mathbb{R}) \cap SAA^1(\mathbb{R}, \mathbb{R})$, see **[40]**.

To rewrite equation (5.1) in the abstract form (1.4), we introduce the Banach space $X := C(\overline{\Omega})$ of continuous functions from $\overline{\Omega}$ to $\mathbb{R}$

endowed with the uniform norm topology, and we define the operator $A : D(A) \subset X \to X$ by

$$\begin{cases} D(A) = \{u \in C_0(\Omega) \cap H_0^1(\Omega) : \Delta u \in C(\overline{\Omega})\}, \\ Au = \Delta u, s \end{cases}$$

where Δ is the Laplacian operator. We denote by λ_1 the smallest eigenvalue of $-\Delta$ in $H_0^1(\Omega)$ ($\lambda_1 > 0$ since Ω is bounded and smooth). The operator A satisfies the Hille-Yosida condition **(H0)** on X and

$$\overline{D(A)} = \{u \in X : \, u|_{\partial\Omega} = 0\} \neq X.$$

Let A_0 be the part of the operator A in $\overline{D(A)}$. Then, A_0 is given by

$$\begin{cases} D(A_0) = \{u \in C_0(\Omega) \cap H_0^1(\Omega) : \Delta u \in C_0(\Omega)\}, \\ A_0 u = \Delta u. \end{cases}$$

Lemma 5.1 (**[10]**). *The linear operator A_0 generates a compact C_0-semigroup $(T_0(t))_{t\geq 0}$ on $\overline{D(A)}$ such that for each $t \geq 0$*

$$|T_0(t)| \leq \exp\left(\frac{\lambda_1 |\Omega|^{2/n}}{4\pi}\right) e^{-\lambda_1 t}. \tag{5.2}$$

Let $L : C \to X$ be the operator defined by

$$L(\varphi)(\xi) = a\varphi(0)(\xi) + \int_{-r}^{0} h(s)\varphi(s)(\xi)\, ds \quad \text{for } \xi \in \overline{\Omega} \text{ and } \varphi \in C,$$

and let $f : \mathbb{R} \to X$ be given by

$$f(t)(\xi) = F(t)\psi(\xi) \quad \text{for } \xi \in \overline{\Omega} \text{ and } t \in \mathbb{R}.$$

Then, L is a bounded linear operator from C to X, and $f \in SAA^1(\mathbb{R}, X)$. Equation (5.1) takes the following abstract form

$$\frac{d}{dt} u(t) = Au(t) + L(u_t) + f(t) \quad \text{for } t \in \mathbb{R}. \tag{5.3}$$

Theorem 5.2. *Assume that*

$$|a| + \int_{-r}^{0} |h(s)|\, ds < \lambda_1 \exp\left[-\lambda_1 \left(\frac{|\Omega|^{2/n}}{2\pi} + r \right) \right]. \tag{5.4}$$

Then, equation (5.3) *has a unique compact almost automorphic solution that is globally attractive.*

For the proof of Theorem 5.2, we need the following lemma.

Lemma 5.3 (**[12]**). *If*

$$x(t) \leq h(t) + \int_{t_0}^{t} k(s)x(s)\, ds \quad \text{for } t \in [t_0, \tau),$$

where all of the functions involved are continuous and nonnegative on $[t_0, \tau)$, $\tau \leq \infty$ *and* $k(t) \geq 0$, *then* x *satisfies*

$$x(t) \leq h(t) + \int_{t_0}^{t} h(s)k(s)e^{\int_s^t k(u)\, du} ds \quad \text{for } t \in [t_0, \tau).$$

Proof of Theorem 5.2. From Theorem 2.3, for any initial data $\varphi \in C_0$, equation (5.3) has a solution x, given by

$$x(t) = T_0(t)\varphi(0) + \lim_{\lambda \to \infty} \int_0^t T_0(t-s)B_\lambda[L(x_s) + f(s)]\, ds \quad \text{for } t \geq 0,$$

where $B_\lambda = \lambda R(\lambda, A)$. Let

$$M := \exp\left(\frac{\lambda_1 |\Omega|^{2/n}}{4\pi}\right).$$

Then, we have $(-\lambda_1, \infty) \subset \rho(A)$ and

$$|R(\lambda, A)| \leq \frac{M}{\lambda + \lambda_1} \quad \text{for } \lambda > -\lambda_1. \tag{5.5}$$

Therefore, by (5.5) and (5.2), we obtain

$$e^{\lambda_1 t}|x(t)| \leq M|\varphi| + M^2 \int_0^t e^{\lambda_1 s}[|L||x_s| + |f(s)|]\, ds \quad \text{for } t \geq 0. \tag{5.6}$$

Let $\theta \in [-r, 0]$ and $t \geq 0$. If $t + \theta < 0$, then

$$\begin{aligned} e^{\lambda_1 t}|x(t+\theta)| &= e^{\lambda_1 t}|\varphi(t+\theta)| \\ &\leq e^{\lambda_1 r}|\varphi| \leq M e^{\lambda_1 r}|\varphi|. \end{aligned}$$

If $t+\theta \geq 0$, then, from (5.6), and, since $-\theta \leq r$, we have

$$e^{\lambda_1 t}|x(t+\theta)| \leq Me^{\lambda_1 r}|\varphi| + M^2 e^{\lambda_1 r}\int_0^t e^{\lambda_1 s}|f(s)|\,ds + M^2|L|e^{\lambda_1 r}\int_0^t e^{\lambda_1 s}|x_s|\,ds.$$

Thus, for each $t \geq 0$,

$$\begin{aligned} e^{\lambda_1 t}|x_t| &= \sup_{-r\leq\theta\leq 0} e^{\lambda_1 t}|x(t+\theta)| \\ &\leq Me^{\lambda_1 r}|\varphi| + M_2(e^{\lambda_1(t+1)}-1) \\ &\quad + M^2|L|e^{\lambda_1 r}\int_0^t e^{\lambda_1 s}|x_s|\,ds, \end{aligned}$$

where

$$M_2 = \frac{M^2 e^{\lambda_1(r+1)}|f|_{BS^1}}{e^{\lambda_1}-1}.$$

From the generalized Gronwall inequality in Lemma 5.3, we obtain, for each $t \geq 0$,

$$\begin{aligned} e^{\lambda_1 t}|x_t| \leq{}& Me^{\lambda_1 r}|\varphi| + M_2(e^{\lambda_1(t+1)}-1) \\ &+ M^2|L|e^{\lambda_1 r}\int_0^t (Me^{\lambda_1 r}|\varphi| + M_2(e^{\lambda_1(s+1)}-1)) \\ &\cdot e^{(M^2|L|e^{\lambda_1 r})(t-s)}ds. \end{aligned}$$

However, from (5.4), we have

$$|L| \leq |a| + \int_{-r}^0 |h(s)|\,ds < \frac{\lambda_1}{M^2 e^{r\lambda_1}}.$$

Thus,

$$\lambda_1 - M^2|L|e^{r\lambda_1} > 0. \tag{5.7}$$

It follows that, for each $t \geq 0$,

$$|x_t| \leq M_2 e^{\lambda_1} + M|\varphi|e^{r\lambda_1} + \frac{M^2 M_2|L|e^{\lambda_1(r+1)}}{\lambda_1 - M^2|L|e^{r\lambda_1}}.$$

This shows that x is a bounded solution of equation (5.3) on $\mathbb{R}^+$. Using Corollary 4.18, we deduce that equation (5.3) has a compact almost automorphic solution y.

Let z be another solution. Then,

$$y(t) - z(t) = T_0(t)(y(0) - z(0)) + \lim_{\lambda \to \infty} \int_0^t T_0(t-s) B_\lambda [L(y_s - z_s)]\, ds$$

for $t \geq 0$. Using the same computations as above, we have, for $t \geq 0$,

$$e^{\lambda_1 t} |y_t - z_t| \leq M e^{\lambda_1 r} |y_0 - z_0| + M^2 \int_0^t |L| e^{\lambda_1 r} e^{\lambda_1 s} |y_s - z_s|\, ds.$$

Now, using the classical Gronwall's lemma, we obtain, for $t \geq 0$,

$$|y_t - z_t| \leq M e^{\lambda_1 r} |y_0 - z_0| e^{(M^2 |L| e^{\lambda_1 r} - \lambda_1) t}.$$

Thus, from (5.7), we deduce that $|y_t - z_t| \to 0$ as $t \to \infty$, that is, y is globally attractive.

We claim that y is the unique solution of (5.3) which is bounded on the entire real line. In fact, if w is another solution which is bounded on $\mathbb{R}$, then, for all $t, \sigma \in \mathbb{R}$ with $\sigma \leq t$, we can show that

$$|y_t - w_t| \leq M e^{\lambda_1 r} |y_\sigma - w_\sigma| e^{(M^2 |L| e^{\lambda_1 r} - \lambda_1)(t - \sigma)}.$$

By letting $\sigma \to -\infty$, we deduce that $y_t = w_t$ for all $t \in \mathbb{R}$. ☐

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