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GENERAL DECAY FOR A LAMINATED BEAM WITH STRUCTURAL DAMPING AND MEMORY: THE CASE OF NON-EQUAL WAVE SPEEDS

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ABSTRACT. In previous work **[12]**, Lo and Tatar studied the exponential decay for a laminated beam with viscoelastic damping acting on the effective rotation angle in the case of equal-speed wave propagations. In this paper, we continue consideration of the same problem in the case of non-equal wave speeds. In this case, the main difficulty is how to estimate the non-equal speed term. To overcome this difficulty, the second-order energy method introduced in Guesmia and Messaoudi **[6]** seems to be the best choice for our problem. For a wide class of relaxation functions, we establish the general decay result for the energy without any kind of internal or boundary control.

1. Introduction. In previous work **[12]**, Lo and Tatar considered the following laminated beam with structural damping and memory:

$$\begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0, \\ I_\rho(3w-\psi)_{tt} - G(\psi-\varphi_x) - D(3w-\psi)_{xx} \\ \qquad\qquad + \displaystyle\int_0^t g(t-s)(3w-\psi)_{xx}(s)\,\mathrm{d}s = 0, \\ I_\rho w_{tt} + G(\psi-\varphi_x) + \dfrac{4}{3}\gamma w + \dfrac{4}{3}\beta w_t - Dw_{xx} = 0, \end{cases} \tag{1.1}$$

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under initial conditions

$$
\begin{cases}
\varphi(x,0) = \varphi_0(x), \psi(x,0) = \psi_0(x), w(x,0) = w_0(x) \\
\varphi_t(x,0) = \varphi_1(x), \psi_t(x,0) = \psi_1(x), w_t(x,0) = w_1(x),
\end{cases} \tag{1.2}
$$

and boundary conditions

$$
\varphi_x(0,t) = \varphi(1,t) = \psi(0,t) = \psi_x(1,t) = w(0,t) = w_x(1,t) = 0, \tag{1.3}
$$

for $D = 1$, where $(x,t) \in (0,1) \times (0,+\infty)$, x is the space variable along the beam of length 1, t denotes the time variable, $\varphi(x,t)$ denotes the transverse displacement of the beam which departs from its equilibrium position, $\psi(x,t)$ represents the rotation angle, $w(x,t)$ is proportional to the amount of slip along the interface at time t and longitudinal spatial variable x, $3w - \psi$ denotes the effective rotation angle and the third equation of (1.1) describes the dynamics of the slip; the coefficients $\rho, G, I_\rho, D, \gamma, \beta > 0$ denote the density of the beams, the shear stiffness, the mass moment of inertia, the flexural rigidity, the adhesive stiffness of the beams and the adhesive damping parameter, respectively; $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-increasing function. In that paper, the authors established an exponential decay result under the equal-speed wave propagation case: $G/\rho = D/I_\rho$. As for the previous results and developments of the laminated beam, the authors have stated and summarized in great detail in **[12]**; thus, we just omit it here. For a better understanding of the present work, the reader is strongly recommended to **[12]** and the references therein.

It is easy to find that, if the slip w is assumed to be identically zero, then the first two equations of system (1.1)–(1.3) can be reduced exactly to the Timoshenko beam system. During the last few years, many people have been interested in the question of stability of Timoshenko systems with memory and different speeds of wave propagation. For example, in **[6]**, Guesmia and Messaoudi studied a one-dimensional Timoshenko system with only one control given by a viscoelastic term on the angular rotation equation of the form

$$
\begin{cases}
\rho_1 \varphi_{tt} - k_1(\varphi_x + \psi)_x = 0, \\
\rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1(\varphi_x + \psi) + \displaystyle\int_0^t g(s)\psi_{xx}(x, t-s)\,\mathrm{d}s = 0,
\end{cases} \tag{1.4}
$$

where $(x,t) \in (0,L) \times (0,+\infty)$, ρ_1, ρ_2, k_1 and k_2 are positive constants and g is a positive differentiable function satisfying: there exist a non-increasing differentiable function $\varsigma : \mathbb{R}^+ \to (0,+\infty)$ and a constant $p \geq 1$ such that

$$g'(t) \leq -\varsigma(t) g^p(t) \quad \text{for all } t \geq 0.$$

For sufficiently regular initial data, the authors established a general decay result for the energy of the solution in case of different wave speeds ($k_1/\rho_1 \neq k_2/\rho_2$). Guesmia, et al., **[8]** considered a vibrating system of Timoshenko type in a one-dimensional bounded domain with an infinite history acting in the equation of the rotation angle of the form

(1.5)

$$\begin{cases} \rho_1 \varphi_{tt} - k_1(\varphi_x + \psi)_x = 0, \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1(\varphi_x + \psi) + \displaystyle\int_0^{+\infty} g(s) \psi_{xx}(x, t-s)\, \mathrm{d}s = 0, \end{cases}$$

$(x,t) \in (0,L) \times (0,+\infty)$. The authors proved a general decay of the solution for the case of equal-speed wave propagation as well as for the nonequal-speed case. For more papers related to various systems with memory, we refer the reader to **[1, 3, 4, 5, 7, 9, 10, 11, 13, 15, 14, 16]**.

In this paper, for a wide class of relaxation functions, we intend to study the general decay rate of the solutions for problem (1.1)–(1.3) under the non-equal wave speed case: $G/\rho \neq D/I_\rho$. In this case, the additional difficulty arises in estimating the non-equal speed term (see (3.6) below). For our purposes, we shall introduce an additional functional and use the second-order energy method motivated by Guesmia and Messoudi's work **[6]**, in which a linear Timoshenko system with memory was studied.

The remaining part of this paper is organized as follows. In Section 2, we present some hypotheses needed for our work and state the main results. In Section 3, we prove the general decay result of problem (1.1)–(1.3).

2. Preliminaries and main results. In this section, we begin with some materials and known results for problem (1.1)–(1.3). We use c to denote a generic positive constant which does not depend upon the initial data. For the relaxation function g, we have the following assumptions:

(G1) $g:\mathbb{R}^+\to\mathbb{R}^+$ is a differentiable function such that

$$g(0)>0,\qquad D-\int_0^{+\infty} g(s)\mathrm{d}s=l>0.$$

(G2) There exists a non-increasing differentiable function $\xi:\mathbb{R}^+\to\mathbb{R}^+$, satisfying

$$g'(t)\le -\xi(t)g(t),\quad \text{for all } t\ge 0.$$

Next, let

$$U=(\varphi,3w-\psi,w,\varphi_t,3w_t-\psi_t,w_t)^T$$

and

$$U_0=(\varphi_0,3w_0-\psi_0,w_0,\varphi_1,3w_1-\psi_1,w_1)^T.$$

Then, we consider the following spaces:

$$\mathscr{H}_0=H^1_*(0,1)\times(\widetilde{H}^1_*(0,1))^2\times(L^2(0,1))^3,$$

$$\mathscr{H}_1=\Big\{U\in\mathscr{H}_0 \mid \varphi\in H^2_*(0,1),3w-\psi,w\in\widetilde{H}^2_*(0,1),\varphi_t\in H^1_*(0,1),$$
$$3w_t-\psi_t,w_t\in\widetilde{H}^1_*(0,1),\varphi_x(0,t)=0,\psi_x(1,t)=w_x(1,t)=0\Big\},$$

where

$$H^1_*(0,1)=\left\{\eta \;\middle|\; \eta\in H^1(0,1):\eta(1)=0\right\},$$
$$\widetilde{H}^1_*(0,1)=\left\{\eta \;\middle|\; \eta\in H^1(0,1):\eta(0)=0\right\},$$
$$H^2_*(0,1)=H^2(0,1)\cap H^1_*(0,1),$$
$$\widetilde{H}^2_*(0,1)=H^2(0,1)\cap\widetilde{H}^1_*(0,1).$$

The next lemma plays an important role in the proof our main result.

Lemma 2.1 ([**8**]). *The following inequalities hold*:

$$\left(\int_0^t g(t-s)((3w_x-\psi_x)(t)-(3w_x-\psi_x)(s))\,\mathrm{d}s\right)^2 \\ \leq g_0(t)\int_0^t g(t-s)((3w_x-\psi_x)(t)-(3w_x-\psi_x)(s))^2\mathrm{d}s, \tag{2.1}$$

$$\left(\int_0^t g'(t-s)((3w_x-\psi_x)(t)-(3w_x-\psi_x)(s))\,\mathrm{d}s\right)^2 \\ \leq -g(0)\int_0^t g'(t-s)((3w_x-\psi_x)(t)-(3w_x-\psi_x)(s))^2\mathrm{d}s, \tag{2.2}$$

where $g_0(t)=\int_0^t g(s)\,\mathrm{d}s$.

Now, we state the following well-posedness result, which can be proved by using the standard Galerkin method.

Theorem 2.2. *For any initial data* $U^0 \in \mathscr{H}_0$*, problem* (1.1)–(1.3) *has a unique weak solution*

$$\begin{aligned} \varphi &\in C(\mathbb{R}^+; H_*^1(0,1)) \cap C^1(\mathbb{R}^+; L^2(0,1)),\\ 3w-\psi,\ w &\in C(\mathbb{R}^+; \widetilde{H}_*^1(0,1)) \cap C^1(\mathbb{R}^+; L^2(0,1)). \end{aligned}$$

Moreover, if $U^0 \in \mathscr{H}_1$*, then the solution satisfies*

$$\begin{aligned} \varphi &\in C(\mathbb{R}^+; H_*^2(0,1)) \cap C^1(\mathbb{R}^+; H_*^1(0,1)) \\ &\qquad \cap C^2(\mathbb{R}^+; L^2(0,1)),\\ 3w-\psi,\ w &\in C(\mathbb{R}^+; \widetilde{H}_*^2(0,1)) \cap C^1(\mathbb{R}^+; \widetilde{H}_*^1(0,1)) \\ &\qquad \cap C^2(\mathbb{R}^+; L^2(0,1)). \end{aligned}$$

In order to state our decay result, we introduce the following energy functional:

$$
\begin{aligned}
E(t) = \frac{1}{2}\int_0^1 \Big(&\rho\varphi_t^2 + I_\rho(3w_t-\psi_t)^2 + 3I_\rho w_t^2 + G(\psi-\varphi_x)^2 \\
&+ \Big(D - \int_0^t g(s)\,\mathrm{d}s\Big)(3w_x-\psi_x)^2 + 3Dw_x^2 + 4\gamma w^2\Big)\,\mathrm{d}x \\
&+ \frac{1}{2} g\circ(3w_x-\psi_x),
\end{aligned} \tag{2.3}
$$

where, for all $v \in L^2(0,1)$,

$$
g\circ v = \int_0^1\int_0^t g(t-s)(v(t)-v(s))^2\mathrm{d}s\,\mathrm{d}x.
$$

In order to estimate the non-equal speed term, we define the second-order energy by

$$
\begin{aligned}
\widetilde{E}(t) = \frac{1}{2}\int_0^1 \Big(&\rho\varphi_{tt}^2 + I_\rho(3w_{tt}-\psi_{tt})^2 + 3I_\rho w_{tt}^2 + G(\psi_t-\varphi_{xt})^2 \\
&+ \Big(D - \int_0^t g(s)\,\mathrm{d}s\Big)(3w_{xt}-\psi_{xt})^2 + 3Dw_{xt}^2 + 4\gamma w_t^2\Big)\,\mathrm{d}x \\
&+ \frac{1}{2} g\circ(3w_{xt}-\psi_{xt}).
\end{aligned} \tag{2.4}
$$

Our main decay result reads as follows.

Theorem 2.3. *Assume that* (G1), (G2) *and* $G/\rho \neq D/I_\rho$ *hold. For any* $U^0 \in \mathscr{H}_1$, *there exists a positive constant* C *such that the energy* $E(t)$ *associated with problem* (1.1)–(1.3) *satisfies*

$$
E(t) \leq C\left(\frac{1+\int_0^t g(s)\,\mathrm{d}s}{\int_0^t \xi(s)\,\mathrm{d}s}\right) \quad \text{for all } t \geq 0, \tag{2.5}
$$

where

$$
C = c\left(E(0) + \widetilde{E}(0) + \int_0^1 (3w_{0xx}-\psi_{0xx})^2\mathrm{d}x\right).
$$

3. Proof of the main result. In this section, we prove the general decay result as stated in Theorem 2.3. For this purpose, we establish several lemmas.

Lemma 3.1. *The energy functional* $E(t)$ *defined by* (2.3) *satisfies*

$$\frac{d}{dt}E(t) = -4\beta\int_0^1 w_t^2 \mathrm{d}x - \frac{g(t)}{2}\int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x + \frac{1}{2}g' \circ (3w_x - \psi_x) \leq 0. \tag{3.1}$$

Proof. Multiplying the three equations in (1.1) by $\varphi_t, 3w_t - \psi_t$ and $3w_t$, respectively, integrating over $(0,1)$ and using the boundary conditions in (1.3), we obtain (3.1) (see [**12,** Lemma 2.3] for details). □

Next, we will prove several lemmas with the purpose of creating negative counterparts of the terms that appear in the energy $E(t)$.

As in [**12**], we first consider the following functionals:

$$I_1(t) = -\rho\int_0^1 \varphi\varphi_t \,\mathrm{d}x,$$
$$I_2(t) = I_\rho\int_0^1 (3w - \psi)(3w_t - \psi_t)\,\mathrm{d}x,$$
$$I_3(t) = I_\rho\int_0^1 w w_t \,\mathrm{d}x$$

and

$$I_4(t) = -I_\rho\int_0^1 (3w_t - \psi_t)\int_0^t g(t-s)[(3w-\psi)(t) - (3w-\psi)(s)]\,\mathrm{d}s\,\mathrm{d}x.$$

Then, the next result holds.

Lemma 3.2. *The functional* $I_1(t)$ *satisfies, for any* $\varepsilon_1 > 0$,

$$I_1'(t) \leq -\rho\int_0^1 \varphi_t^2 \mathrm{d}x + G(1+\varepsilon_1)\int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x + \frac{G}{2\varepsilon_1}\int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x + \frac{9G}{2\varepsilon_1}\int_0^1 w_x^2 \,\mathrm{d}x. \tag{3.2}$$

Proof. Differentiating $I_1(t)$ with respect to t, using the first equation in (1.1), integrating by parts and making use of the Young's inequality, we obtain (3.2) (see [**12,** Lemma 2.5] for details). □

Lemma 3.3. *The functional $I_2(t)$ satisfies, for any $\varepsilon_2 > 0$,*

(3.3)
$$I_2'(t) \le -(l - \varepsilon_2(G+1)) \int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x + I_\rho \int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x + \frac{G}{4\varepsilon_2} \int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x + \frac{D-l}{4\varepsilon_2} g \circ (3w_x - \psi_x).$$

Proof. Taking the derivative of $I_2(t)$ with respect to t, using the second equation in (1.1), integrating by parts, using Young's inequality and Lemma 2.1, we obtain

$$\begin{aligned} I_2'(t) \le & -D \int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x + I_\rho \int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x \\ & + \varepsilon_2 G \int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x + \frac{G}{4\varepsilon_2} \int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x \\ & + \varepsilon_2 \int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x + \frac{D-l}{4\varepsilon_2} g \circ (3w_x - \psi_x) \\ & + (D-l) \int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x \\ \le & -(l - \varepsilon_2(G+1)) \int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x + I_\rho \int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x \\ & + \frac{G}{4\varepsilon_2} \int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x + \frac{D-l}{4\varepsilon_2} g \circ (3w_x - \psi_x). \end{aligned}$$

Lemma 3.3 is proven. □

Lemma 3.4. *The functional $I_3(t)$ satisfies, for any $\varepsilon_3 > 0$,*

$$\begin{aligned} I_3'(t) \le & -\left(\frac{4\gamma}{3} - \varepsilon_3\left(G + \frac{4\beta}{3}\right)\right) \int_0^1 w^2 \mathrm{d}x - D \int_0^1 w_x^2 \,\mathrm{d}x \\ & + \left(I_\rho + \frac{\beta}{3\varepsilon_3}\right) \int_0^1 w_t^2 \,\mathrm{d}x + \frac{G}{4\varepsilon_3} \int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x. \end{aligned} \tag{3.4}$$

Proof. Differentiating $I_3(t)$ with respect to t, using the third equation in (1.1), integrating by parts and making use of Young's inequality, we arrive at

$$\begin{aligned}
I_3'(t) \leq & -\frac{4\gamma}{3}\int_0^1 w^2 \mathrm{d}x - D\int_0^1 w_x^2\,\mathrm{d}x + I_\rho \int_0^1 w_t^2 \mathrm{d}x + \varepsilon_3 G\int_0^1 w^2 \mathrm{d}x \\
& + \frac{G}{4\varepsilon_3}\int_0^1 (\psi-\varphi_x)^2 \mathrm{d}x + \frac{4\varepsilon_3\beta}{3}\int_0^1 w^2 \mathrm{d}x + \frac{\beta}{3\varepsilon_3}\int_0^1 w_t^2\,\mathrm{d}x \\
\leq & -\left(\frac{4\gamma}{3} - \varepsilon_3\left(G+\frac{4\beta}{3}\right)\right)\int_0^1 w^2 \mathrm{d}x - D\int_0^1 w_x^2 \mathrm{d}x \\
& + \left(I_\rho + \frac{\beta}{3\varepsilon_3}\right)\int_0^1 w_t^2\,\mathrm{d}x + \frac{G}{4\varepsilon_3}\int_0^1 (\psi-\varphi_x)^2 \mathrm{d}x,
\end{aligned}$$

which is exactly (3.4). □

Lemma 3.5. *The functional $I_4(t)$ satisfies, for any $\varepsilon_4 > 0$,*
(3.5)

$$\begin{aligned}
I_4'(t) \leq & -I_\rho(g_0(t)-\varepsilon_4)\int_0^1 (3w_t-\psi_t)^2 \mathrm{d}x + \varepsilon_4 G\int_0^1 (\psi-\varphi_x)^2 \mathrm{d}x \\
& + \varepsilon_4(2D-l)\int_0^1 (3w_x-\psi_x)^2 \mathrm{d}x - \frac{I_\rho g(0)}{4\varepsilon_4} g' \circ (3w_x-\psi_x) \\
& + (D-l)\left(1+\frac{G+2D-l}{4\varepsilon_4}\right) g\circ(3w_x-\psi_x).
\end{aligned}$$

Proof. Taking the derivative of $I_4(t)$ with respect to t, using the second equation in (1.1), integrating by parts, using Young's inequality and Lemma 2.1, we deduce

$$\begin{aligned}
I_4'(t) \leq\ & \varepsilon_4 G\int_0^1 (\psi-\varphi_x)^2 \mathrm{d}x + \frac{G(D-l)}{4\varepsilon_4} g\circ(3w_x-\psi_x) \\
& + \varepsilon_4 D\int_0^1 (3w_x-\psi_x)^2 \mathrm{d}x + \frac{D(D-l)}{4\varepsilon_4} g\circ(3w_x-\psi_x) \\
& + \varepsilon_4(D-l)\int_0^1 (3w_x-\psi_x)^2 \mathrm{d}x - I_\rho g_0(t)\int_0^1 (3w_t-\psi_t)^2 \mathrm{d}x \\
& + \left(D-l+\frac{(D-l)^2}{4\varepsilon_4}\right) g\circ(3w_x-\psi_x) \\
& + \varepsilon_4 I_\rho \int_0^1 (3w_t-\psi_t)^2 \mathrm{d}x - \frac{I_\rho g(0)}{4\varepsilon_4} g'\circ(3w_x-\psi_x)
\end{aligned}$$

$$
\begin{aligned}
&\le -(I_\rho g_0(t) - \varepsilon_4 I_\rho)\int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x + \varepsilon_4 G \int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x \\
&\quad + (2\varepsilon_4 D - \varepsilon_4 l)\int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x - \frac{I_\rho g(0)}{4\varepsilon_4} g' \circ (3w_x - \psi_x) \\
&\quad + \left(D - l + \frac{G(D-l)}{4\varepsilon_4} + \frac{D(D-l)}{4\varepsilon_4} + \frac{(D-l)^2}{4\varepsilon_4}\right) g \circ (3w_x - \psi_x) \\
&\le -I_\rho(g_0(t) - \varepsilon_4)\int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x + \varepsilon_4 G \int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x \\
&\quad + \varepsilon_4(2D - l)\int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x - \frac{I_\rho g(0)}{4\varepsilon_4} g' \circ (3w_x - \psi_x) \\
&\quad + (D - l)\left(1 + \frac{G + 2D - l}{4\varepsilon_4}\right) g \circ (3w_x - \psi_x).
\end{aligned}
$$

The proof is complete. □

Further, we introduce an additional functional:

$$
\begin{aligned}
I_5(t) = \frac{D\rho}{G}&\int_0^1 \varphi_t (3w_x - \psi_x)\mathrm{d}x \\
&- I_\rho \int_0^1 (3w_t - \psi_t)(\psi - \varphi_x)\,\mathrm{d}x \\
&- \frac{\rho}{G}\int_0^1 \varphi_t \int_0^t g(t-s)(3w_x - \psi_x)(s)\,\mathrm{d}s\,\mathrm{d}x.
\end{aligned}
$$

Lemma 3.6. *The functional $I_5(t)$ satisfies, for any $\varepsilon_5 > 0$,*

$$
\begin{aligned}
I_5'(t) \le -G &\int_0^1 (\psi - \varphi_x)^2 \mathrm{d}x \\
&+ \left(\frac{D\rho}{G} - I_\rho\right)\int_0^1 \varphi_t (3w_{xt} - \psi_{xt})\,\mathrm{d}x \\
&+ 18\varepsilon_5 I_\rho \int_0^1 w_t^2 \mathrm{d}x + I_\rho\left(2\varepsilon_5 + \frac{1}{4\varepsilon_5}\right)\int_0^1 (3w_t - \psi_t)^2 \mathrm{d}x \\
&+ \frac{\varepsilon_5 \rho}{G}(1 + g(0))\int_0^1 \varphi_t^2 \mathrm{d}x \\
&+ \frac{\rho g(t)}{4\varepsilon_5 G}\int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x - \frac{\rho g(0)}{4\varepsilon_5 G} g' \circ (3w_x - \psi_x).
\end{aligned}
\tag{3.6}
$$

Proof. Taking the derivative of $I_5(t)$ with respect to t and using the first two equations in (1.1), we have

$$
\begin{aligned}
I_5'(t) = &-D\int_0^1 (\psi-\varphi_x)_x(3w_x-\psi_x)\,\mathrm{d}x \\
&+\frac{D\rho}{G}\int_0^1 \varphi_t(3w_{xt}-\psi_{xt})\,\mathrm{d}x \\
&-G\int_0^1(\psi-\varphi_x)^2\mathrm{d}x - D\int_0^1(3w_{xx}-\psi_{xx})(\psi-\varphi_x)\,\mathrm{d}x \\
&+\int_0^1(\psi-\varphi_x)\int_0^t g(t-s)(3w_{xx}-\psi_{xx})(s)\,\mathrm{d}s\,\mathrm{d}x \\
&-I_\rho\int_0^1(3w_t-\psi_t)(\psi-\varphi_x)_t\,\mathrm{d}x \\
&+\int_0^1(\psi-\varphi_x)_x\int_0^t g(t-s)(3w_x-\psi_x)(s)\,\mathrm{d}s\,\mathrm{d}x \\
&-\frac{\rho}{G}\int_0^1\varphi_t\int_0^t g'(t-s)(3w_x-\psi_x)(s)\,\mathrm{d}s\,\mathrm{d}x \\
&-\frac{\rho g(0)}{G}\int_0^1\varphi_t(3w_x-\psi_x)(t)\,\mathrm{d}x.
\end{aligned}
\tag{3.7}
$$

The sixth term in (3.7) can be rewritten as follows

$$
\begin{aligned}
-I_\rho\int_0^1(3w_t-\psi_t)(\psi-\varphi_x)_t\,\mathrm{d}x = &-I_\rho\int_0^1(3w_t-\psi_t)\psi_t\,\mathrm{d}x \\
&+I_\rho\int_0^1(3w_t-\psi_t)\varphi_{xt}\,\mathrm{d}x.
\end{aligned}
\tag{3.8}
$$

Then, due to (3.7) and (3.8), and integrating by parts, we deduce

$$
\begin{aligned}
I_5'(t) = &-G\int_0^1(\psi-\varphi_x)^2 + \left(\frac{D\rho}{G}-I_\rho\right)\int_0^1\varphi_t(3w_{xt}-\psi_{xt})\,\mathrm{d}x \\
&-I_\rho\int_0^1(3w_t-\psi_t)\psi_t\,\mathrm{d}x - \frac{\rho g(t)}{G}\int_0^1\varphi_t(3w_x-\psi_x)(t)\,\mathrm{d}x \\
&+\frac{\rho}{G}\int_0^1\varphi_t\int_0^t g'(t-s)[(3w_x-\psi_x)(t)-(3w_x-\psi_x)(s)]\,\mathrm{d}s\,\mathrm{d}x.
\end{aligned}
\tag{3.9}
$$

Next, using Young's inequality and Lemma 2.1 for the last three terms of this equality, we obtain

$$\begin{aligned} I_5'(t) \le & -G\int_0^1 (\psi-\varphi_x)^2\mathrm{d}x + \left(\frac{D\rho}{G}-I_\rho\right)\int_0^1 \varphi_t(3w_{xt}-\psi_{xt})\,\mathrm{d}x \\ & + \varepsilon_5 I_\rho \int_0^1 \psi_t^2\,\mathrm{d}x + \frac{I_\rho}{4\varepsilon_5}\int_0^1 (3w_t-\psi_t)^2\mathrm{d}x + \frac{\varepsilon_5\rho g(t)}{G}\int_0^1 \varphi_t^2\,\mathrm{d}x \\ & + \frac{\rho g(t)}{4\varepsilon_5 G}\int_0^1 (3w_x-\psi_x)^2\mathrm{d}x + \frac{\varepsilon_5\rho}{G}\int_0^1 \varphi_t^2\,\mathrm{d}x \\ & - \frac{\rho g(0)}{4\varepsilon_5 G}\, g'\circ(3w_x-\psi_x). \end{aligned}$$

It is worth noting that

$$\int_0^1 \psi_t^2\,\mathrm{d}x = \int_0^1 (\psi_t - 3w_t + 3w_t)^2\mathrm{d}x.$$

Hence, by using the fact that g is non-increasing, we have

$$\begin{aligned} I_5'(t) \le & -G\int_0^1 (\psi-\varphi_x)^2\mathrm{d}x \\ & + \left(\frac{D\rho}{G}-I_\rho\right)\int_0^1 \varphi_t(3w_{xt}-\psi_{xt})\,\mathrm{d}x + 18\varepsilon_5 I_\rho\int_0^1 w_t^2\mathrm{d}x \\ & + \left(2\varepsilon_5 I_\rho + \frac{I_\rho}{4\varepsilon_5}\right)\int_0^1 (3w_t-\psi_t)^2\mathrm{d}x + \left(\frac{\varepsilon_5\rho}{G}+\frac{\varepsilon_5\rho g(t)}{G}\right)\int_0^1 \varphi_t^2\mathrm{d}x \\ & + \frac{\rho g(t)}{4\varepsilon_5 G}\int_0^1 (3w_x-\psi_x)^2\mathrm{d}x - \frac{\rho g(0)}{4\varepsilon_5 G} g'\circ(3w_x-\psi_x) \\ \le & -G\int_0^1 (\psi-\varphi_x)^2\mathrm{d}x + \left(\frac{D\rho}{G}-I_\rho\right)\int_0^1 \varphi_t(3w_{xt}-\psi_{xt})\,\mathrm{d}x \\ & + 18\varepsilon_5 I_\rho\int_0^1 w_t^2\mathrm{d}x + I_\rho\left(2\varepsilon_5+\frac{1}{4\varepsilon_5}\right)\int_0^1 (3w_t-\psi_t)^2\mathrm{d}x \\ & + \frac{\varepsilon_5\rho}{G}(1+g(0))\int_0^1 \varphi_t^2\mathrm{d}x \\ & + \frac{\rho g(t)}{4\varepsilon_5 G}\int_0^1 (3w_x-\psi_x)^2\mathrm{d}x - \frac{\rho g(0)}{4\varepsilon_5 G} g'\circ(3w_x-\psi_x). \end{aligned}$$

This completes the proof of the lemma. $\square$

Now, to estimate the second term in (3.6), we shall use the second-order energy defined by (2.4). Noting that $3w(x,0) - \psi(x,0) = 3w_0(x) - \psi_0(x)$, and using the fact that

$$
\begin{aligned}
&\frac{\partial}{\partial t}\int_0^t g(t-s)(3w_{xx} - \psi_{xx})(s)\,\mathrm{d}s \\
&= \frac{\partial}{\partial t}\int_0^t g(s)(3w_{xx} - \psi_{xx})(t-s)\,\mathrm{d}s \\
&= \int_0^t g(s)(3w_{xxt} - \psi_{xxt})(t-s)\,\mathrm{d}s + g(t)(3w_{xx} - \psi_{xx})(x,0) \\
&= \int_0^t g(t-s)(3w_{xxt} - \psi_{xxt})(s)\,\mathrm{d}s + g(t)(3w_{0xx} - \psi_{0xx}),
\end{aligned}
$$

we obtain

(3.10)
$$
\begin{aligned}
\widetilde{E}'(t) = &-4\beta\int_0^1 w_{tt}^2\mathrm{d}x - \frac{g(t)}{2}\int_0^1 (3w_{xt} - \psi_{xt})^2\mathrm{d}x \\
&+ \frac{1}{2}g' \circ (3w_{xt} - \psi_{xt}) - g(t)\int_0^1 (3w_{tt} - \psi_{tt})(3w_{0xx} - \psi_{0xx})\,\mathrm{d}x.
\end{aligned}
$$

As in **[6]**, we have the following lemma:

Lemma 3.7. *The second-order energy $\widetilde{E}(t)$ satisfies, for all $t > 0$,*

$$
\widetilde{E}(t) \le c\left(\widetilde{E}(0) + \int_0^1 (3w_{0xx} - \psi_{0xx})^2\mathrm{d}x\right). \tag{3.11}
$$

Proof. By using (2.4), (3.10) and Young's inequality, we obtain

$$
\begin{aligned}
\widetilde{E}'(t) &\le -g(t)\int_0^1 (3w_{tt} - \psi_{tt})(3w_{0xx} - \psi_{0xx})\,\mathrm{d}x \\
&\le \frac{g(t)}{2}\int_0^1 \left(I_\rho(3w_{tt} - \psi_{tt})^2 + \frac{1}{I_\rho}(3w_{0xx} - \psi_{0xx})^2\right)\mathrm{d}x \\
&\le g(t)\widetilde{E}(t) + \frac{g(t)}{2I_\rho}\int_0^1 (3w_{0xx} - \psi_{0xx})^2\mathrm{d}x,
\end{aligned}
$$

which implies

$$\frac{\partial}{\partial t}\left(\widetilde{E}(t)e^{-\int_0^t g(s)\,\mathrm{d}s}\right) \le \frac{g(t)}{2I_\rho}e^{-\int_0^t g(s)\mathrm{d}s}\int_0^1 (3w_{0xx}-\psi_{0xx})^2\mathrm{d}x$$
$$\le \frac{g(t)}{2I_\rho}\int_0^1 (3w_{0xx}-\psi_{0xx})^2\mathrm{d}x.$$

Then, a simple calculation yields

$$\begin{aligned}\widetilde{E}(t)e^{-\int_0^{+\infty} g(s)\,\mathrm{d}s} &\le \widetilde{E}(t)e^{-\int_0^t g(s)\,\mathrm{d}s}\\ &\le \widetilde{E}(0)+\frac{1}{2I_\rho}\left(\int_0^t g(s)\,\mathrm{d}s\right)\int_0^1 (3w_{0xx}-\psi_{0xx})^2\mathrm{d}x\\ &\le \widetilde{E}(0)+\frac{1}{2I_\rho}\left(\int_0^{+\infty} g(s)\,\mathrm{d}s\right)\int_0^1 (3w_{0xx}-\psi_{0xx})^2\mathrm{d}x.\end{aligned}$$

Consequently, (3.11) follows. □

Now, we are in a position to prove our main result. Let N, N_2, N_3, N_4 and $N_5>0$. We define

$$g_0(t)=\int_0^t g(s)\,\mathrm{d}s \ge \int_0^{t_0} g(s)\,\mathrm{d}s = g_1>0,\quad t\ge t_0>0$$

and

(3.12)
$$L(t)=N((E(t)+\widetilde{E}(t))+I_1(t)+N_2I_2(t)+N_3I_3(t)+N_4I_4(t)+N_5I_5(t).$$

Combining (3.1), (3.2), (3.3), (3.4), (3.5), (3.6) and (3.10), we obtain, for all $t\ge t_0$,

$$\begin{aligned}L'(t) \le& -\rho\left(1-\frac{N_5\varepsilon_5}{G}(1+g(0))\right)\int_0^1 \varphi_t^2\mathrm{d}x\\ &- I_\rho\left(N_4(g_1-\varepsilon_4)-N_2-N_5\left(2\varepsilon_5+\frac{1}{4\varepsilon_5}\right)\right)\int_0^1 (3w_t-\psi_t)^2\mathrm{d}x\\ &-\left(4N\beta-N_3\left(I_\rho+\frac{\beta}{3\varepsilon_3}\right)-18N_5\varepsilon_5I_\rho\right)\int_0^1 w_t^2\mathrm{d}x\\ &-\left(N_3D-\frac{9G}{2\varepsilon_1}\right)\int_0^1 w_x^2\mathrm{d}x - N_3\left(\frac{4\gamma}{3}-\varepsilon_3\left(G+\frac{4\beta}{3}\right)\right)\int_0^1 w^2\mathrm{d}x\end{aligned}$$

(3.13)

$$
\begin{aligned}
&- G\left(N_5 - (1+\varepsilon_1) - \frac{N_2}{4\varepsilon_2} - \frac{N_3}{4\varepsilon_3} - N_4\varepsilon_4\right)\int_0^1 (\psi-\varphi_x)^2\mathrm{d}x\\
&- \left(g(t)\left(\frac{N}{2} - \frac{N_5\rho}{4\varepsilon_5 G}\right) + N_2(l-\varepsilon_2-\varepsilon_2 G)\right.\\
&\qquad\qquad \left. - \frac{G}{2\varepsilon_1} - N_4\varepsilon_4(2D-l)\right)\int_0^1 (3w_x-\psi_x)^2\mathrm{d}x\\
&+ (D-l)\left(\frac{N_2}{4\varepsilon_2} + N_4\left(1+\frac{G+2D-l}{4\varepsilon_4}\right)\right) g\circ(3w_x-\psi_x)\\
&+ \left(\frac{N}{2} - \frac{N_4 I_\rho g(0)}{4\varepsilon_4} - \frac{N_5\rho g(0)}{4\varepsilon_5 G}\right) g'\circ(3w_x-\psi_x)\\
&- 4N\beta\int_0^1 w_{tt}^2\mathrm{d}x - \frac{Ng(t)}{2}\int_0^1 (3w_{xt}-\psi_{xt})^2\mathrm{d}x\\
&+ \frac{N}{2} g'\circ(3w_{xt}-\psi_{xt}) - Ng(t)\int_0^1 (3w_{tt}-\psi_{tt})(3w_{0xx}-\psi_{0xx})\,\mathrm{d}x\\
&+ N_5\left(\frac{D\rho}{G} - I_\rho\right)\int_0^1 \varphi_t(3w_{xt}-\psi_{xt})\,\mathrm{d}x.
\end{aligned}
$$

At this point, we must choose our constants very carefully. First, we choose $\varepsilon_1 = G$, $\varepsilon_2 = 1/N_2$, $\varepsilon_3 = 1/N_3$, $\varepsilon_4 = 1/N_4$, $\varepsilon_5 = G/(2N_5(1+g(0)))$. Then, (3.13) becomes

$$
\begin{aligned}
L'(t) \le -\frac{\rho}{2}\int_0^1 \varphi_t^2\mathrm{d}x&\\
&- I_\rho\left(N_4 g_1 - N_2 - \frac{N_5^2(1+g(0))}{2G} - \left(1+\frac{G}{1+g(0)}\right)\right)\\
&\cdot\int_0^1 (3w_t-\psi_t)^2\mathrm{d}x\\
&- \left(4N\beta - N_3\left(I_\rho + \frac{3N_3\beta}{3}\right) - \frac{9I_\rho G}{1+g(0)}\right)\int_0^1 w_t^2\mathrm{d}x\\
&- \left(N_3 D - \frac{9}{2}\right)\int_0^1 w_x^2\mathrm{d}x - \left(\frac{4N_3\gamma}{3} - \left(G+\frac{4\beta}{3}\right)\right)\int_0^1 w^2\mathrm{d}x\\
&- G\left(N_5 - \frac{N_2^2}{4} - \frac{N_3^2}{4} - 3\right)\int_0^1 (\psi-\varphi_x)^2\mathrm{d}x
\end{aligned}
$$

$$
\begin{aligned}
&- \left(g(t)\left(\frac{N}{2} - \frac{N_5^2\rho(1+g(0))}{2G^2}\right) + N_2 l \right.\\
&\left. - \left(\frac{3}{2} + G + 2D - l\right)\right) \int_0^1 (3w_x - \psi_x)^2 \mathrm{d}x\\
&+ (D-l)\left(\frac{N_2^2}{4} + N_4\left(1 + \frac{N_4(G+2D-l)}{4}\right)\right) g \circ (3w_x - \psi_x)\\
&+ \left(\frac{N}{2} - \frac{N_4^2 I_\rho g(0)}{4} - \frac{N_5^2 \rho g(0)(1+g(0))}{2G^2}\right) g' \circ (3w_x - \psi_x)\\
&- N g(t) \int_0^1 (3w_{tt} - \psi_{tt})(3w_{0xx} - \psi_{0xx})\,\mathrm{d}x\\
&+ N_5\left(\frac{D\rho}{G} - I_\rho\right) \int_0^1 \varphi_t (3w_{xt} - \psi_{xt})\,\mathrm{d}x.
\end{aligned} \tag{3.14}
$$

Then, we select N_2 large enough such that

$$N_2 l - \left(\frac{3}{2} + G + 2D - l\right) > 0.$$

Next, we choose N_3 large enough such that

$$N_3 D - \frac{9}{2} > 0 \quad \text{and} \quad \frac{4N_3\gamma}{3} - \left(G + \frac{4\beta}{3}\right) > 0.$$

Furthermore, we select N_5 large enough such that

$$N_5 - \frac{N_2^2}{4} - \frac{N_3^2}{4} - 3 > 0.$$

After that, we choose N_4 large enough such that

$$N_4 g_1 - N_2 - \frac{N_5^2(1+g(0))}{2G} - \left(1 + \frac{G}{1+g(0)}\right) > 0.$$

Finally, we select N large enough such that

$$
\begin{aligned}
4N\beta - N_3\left(I_\rho + \frac{3N_3\beta}{3}\right) - \frac{9I_\rho G}{1+g(0)} &> 0,\\
\frac{N}{2} - \frac{N_5^2\rho(1+g(0))}{2G^2} &> 0.
\end{aligned}
$$

From the above, we deduce that positive constants c_1 and c_2 exist such that (3.14) becomes

$$
\begin{aligned}
L'(t) \le {} & -c_1 \\
& \cdot \int_0^1 \Big(\varphi_t^2 + (3w_t - \psi_t)^2 + w_t^2 + (\psi - \varphi_x)^2 + (3w_x - \psi_x)^2 + w_x^2 + w^2\Big)\,\mathrm{d}x \\
& + \left(\frac{N}{2} - \frac{N_4^2 I_\rho g(0)}{4} - \frac{N_5^2 \rho g(0)(1+g(0))}{2G^2}\right) \\
& \cdot g' \circ (3w_x - \psi_x) + c_2 g \circ (3w_x - \psi_x) \\
& - N g(t) \int_0^1 (3w_{tt} - \psi_{tt})(3w_{0xx} - \psi_{0xx})\,\mathrm{d}x \\
& + N_5 \left(\frac{D\rho}{G} - I_\rho\right) \int_0^1 \varphi_t (3w_{xt} - \psi_{xt})\,\mathrm{d}x.
\end{aligned} \tag{3.15}
$$

Now, we estimate the last term on the right hand side of (3.15). This is the main difficulty in treating the case of non-equal wave speeds. In order to overcome this difficulty, as in **[6]**, we have the next lemma:

Lemma 3.8. *For any $\epsilon > 0$ and $t \ge t_0$, we have the following inequalities*:

$$
\begin{aligned}
& \left(\frac{D\rho}{G} - I_\rho\right) \int_0^1 \varphi_t (3w_{xt} - \psi_{xt})\,\mathrm{d}x \\
& \quad \le \varepsilon \int_0^1 \varphi_t^2 \mathrm{d}x + \frac{c}{\varepsilon} g(t) E(0) \\
& \quad + \frac{c}{\varepsilon}(g \circ (3w_{xt} - \psi_{xt}) - g' \circ (3w_x - \psi_x)).
\end{aligned} \tag{3.16}
$$

Proof. For all $t \ge t_0$, note that we have

$$
\left(\frac{D\rho}{G} - I_\rho\right) \int_0^1 \varphi_t (3w_{xt} - \psi_{xt})\,\mathrm{d}x = \frac{(D\rho/G) - I_\rho}{\int_0^t g(s)\,\mathrm{d}s} \int_0^1 \varphi_t
$$

$$\cdot \int_0^t g(t-s)((3w_{xt}-\psi_{xt})(t)-(3w_{xt}-\psi_{xt})(s))\,\mathrm{d}s\,\mathrm{d}x \tag{3.17}$$
$$+\frac{(D\rho/G)-I_\rho}{\int_0^t g(s)\,\mathrm{d}s}\int_0^1 \varphi_t \int_0^t g(t-s)(3w_{xt}-\psi_{xt})(s)\,\mathrm{d}s\,\mathrm{d}x.$$

Noting that

$$\frac{1}{\int_0^t g(s)\,\mathrm{d}s} \le \frac{1}{\int_0^{t_0} g(s)\,\mathrm{d}s}, \quad t \ge t_0 > 0,$$

using Young's inequality and Lemma 2.1 (for $(3w_{xt}-\psi_{xt})$ instead of $(3w_x-\psi_x)$), we obtain, for all $\varepsilon>0$ and $t\ge t_0>0$,

$$\begin{aligned} &\frac{(D\rho/G)-I_\rho}{\int_0^t g(s)\,\mathrm{d}s}\int_0^1 \varphi_t \int_0^t g(t-s)((3w_{xt}-\psi_{xt})(t)-(3w_{xt}-\psi_{xt})(s))\,\mathrm{d}s\,\mathrm{d}x \\ &\quad \le \frac{\varepsilon}{2}\int_0^1 \varphi_t^2\mathrm{d}x + \frac{c}{\varepsilon} g\circ(3w_{xt}-\psi_{xt}). \end{aligned} \tag{3.18}$$

On the other hand, noting that $1/(\int_0^t g(s)\,\mathrm{d}s) \le 1/(\int_0^{t_0} g(s)\,\mathrm{d}s)$, $t \ge t_0>0$, integrating by parts, exploiting (2.2), using Young's inequality, noting the fact that $E(t)$ is non-increasing and $(3w-\psi)(x,0) = (3w_0-\psi_0)(x)$, we obtain, for all $\varepsilon>0$ and $t\ge t_0>0$,

$$\begin{aligned} &\frac{(D\rho/G)-I_\rho}{\int_0^t g(s)\,\mathrm{d}s}\int_0^1 \varphi_t \int_0^t g(t-s)(3w_{xt}-\psi_{xt})(s)\,\mathrm{d}s\,\mathrm{d}x \\ &\quad = \frac{(D\rho/G)-I_\rho}{\int_0^t g(s)\,\mathrm{d}s}\int_0^1 \varphi_t \Big(g(0)(3w_x-\psi_x) - g(t)(3w_{0x}-\psi_{0x}) \\ &\qquad\qquad + \int_0^t g'(t-s)(3w_x-\psi_x)(s)\,\mathrm{d}s\Big)\,\mathrm{d}x \\ &\quad = \frac{(D\rho/G)-I_\rho}{\int_0^t g(s)\,\mathrm{d}s}\int_0^1 \varphi_t \Big(g(t)((3w_x-\psi_x)-(3w_{0x}-\psi_{0x})) \\ &\qquad\qquad - \int_0^t g'(t-s)((3w_x-\psi_x)(t)-(3w_x-\psi_x)(s))\,\mathrm{d}s\Big)\,\mathrm{d}x \\ &\quad \le \frac{\varepsilon}{2}\int_0^1 \varphi_t^2\mathrm{d}x + \frac{c}{\varepsilon} g(t)\int_0^1 ((3w_x-\psi_x)^2+(3w_{0x}-\psi_{0x})^2)\,\mathrm{d}x \\ &\qquad - \frac{c}{\varepsilon} g'\circ(3w_x-\psi_x) \end{aligned} \tag{3.19}$$

$$\leq \frac{\varepsilon}{2}\int_0^1 \varphi_t^2 \mathrm{d}x + \frac{c}{\varepsilon} g(t)E(0) - \frac{c}{\varepsilon} g' \circ (3w_x - \psi_x).$$

Finally, inserting (3.18) and (3.19) into (3.17), we get (3.16). □

Next, we estimate the term

$$-g(t)\int_0^1 (3w_{tt} - \psi_{tt})(3w_{0xx} - \psi_{0xx})\,\mathrm{d}x$$

on the right hand side of (3.15).

Lemma 3.9. *For any $t \geq t_0$, we have*

$$\begin{aligned} &- g(t)\int_0^1 (3w_{tt} - \psi_{tt})(3w_{0xx} - \psi_{0xx})\,\mathrm{d}x \\ &\qquad\qquad \leq cg(t)\bigg(\widetilde{E}(0) + \int_0^1 (3w_{0xx} - \psi_{0xx})^2 \mathrm{d}x\bigg). \end{aligned} \tag{3.20}$$

Proof. By using Young's inequality and (3.11), we obtain

$$\begin{aligned} &- g(t)\int_0^1 (3w_{tt} - \psi_{tt})(3w_{0xx} - \psi_{0xx})\,\mathrm{d}x \\ &\quad \leq \frac{1}{2} g(t)\int_0^1 ((3w_{tt} - \psi_{tt})^2 + (3w_{0xx} - \psi_{0xx})^2)\,\mathrm{d}x \\ &\leq cg(t)\bigg(\widetilde{E}(0) + \int_0^1 (3w_{0xx} - \psi_{0xx})^2 \mathrm{d}x\bigg). \end{aligned} \tag{3.21}$$

This completes the proof of Lemma 3.9. □

Then, inserting (3.16) and (3.20) into (3.15), choosing ε small enough such that $N_5\varepsilon < c_1$ and N large enough again such that

$$\frac{N}{2} - \frac{N_4^2 I_\rho g(0)}{4} - \frac{N_5^2 \rho g(0)(1+g(0))}{2G^2} - \frac{N_5 c}{\epsilon} > 0,$$

we deduce

$$\begin{aligned} L'(t) \leq -c\int_0^1 \bigg(&\varphi_t^2 + (3w_t - \psi_t)^2 + w_t^2 + (\psi - \varphi_x)^2 \\ &+ (3w_x - \psi_x)^2 + w_x^2 + w^2\bigg)\,\mathrm{d}x \end{aligned} \tag{3.22}$$

$$
\begin{aligned}
&+ cg \circ (3w_x - \psi_x) + cg \circ (3w_{xt} - \psi_{xt}) \\
&+ cg(t)\left(E(0) + \widetilde{E}(0) + \int_0^1 (3w_{0xx} - \psi_{0xx})^2 \mathrm{d}x \right).
\end{aligned}
$$

Now, returning to the proof of Theorem 2.3, multiplying (3.22) by $\xi(t)$, we obtain

$$
\begin{aligned}
\xi(t)E(t) \leq &-c\xi(t)L'(t) - c\left(g' \circ (3w_x - \psi_x) + g' \circ (3w_{xt} - \psi_{xt})\right) \\
&+ c\xi(t)g(t)\left(E(0) + \widetilde{E}(0) + \int_0^1 (3w_{0xx} - \psi_{0xx})\, \mathrm{d}x \right).
\end{aligned} \tag{3.23}
$$

Integrating over $(0, t)$, using (3.1), (3.11), (3.12), (3.21) and noting the fact that $\xi(t)$ and g are nonincreasing, we get, for $t > t_0 > 0$,

$$
\begin{aligned}
E(t)\int_0^t \xi(s)\mathrm{d}s &\leq \int_0^t E(s)\xi(s)\, \mathrm{d}s \\
&= \int_0^{t_0} E(s)\xi(s)\mathrm{d}s + \int_{t_0}^t E(s)\xi(s)\, \mathrm{d}s \\
&\leq t_0 E(0)\xi(0) + c\left(\xi(t_0)L(t_0) - \xi(t)L(t) + \int_{t_0}^t \xi'(s)L(s)\, \mathrm{d}s \right) \\
&\quad - c\int_{t_0}^t \left(E'(s) + \widetilde{E}'(t) + g(s)\int_0^1 (3w_{tt} - \psi_{tt})(3w_{0xx} - \psi_{0xx})\mathrm{d}x \right)\mathrm{d}s \\
&\quad + c\left(E(0) + \widetilde{E}(0) + \int_0^1 (3w_{0xx} - \psi_{0xx})^2 \mathrm{d}x \right)\int_{t_0}^t \xi(s)g(s)\, \mathrm{d}s \\
&\leq c(E(0) + E(t_0) + \widetilde{E}(t_0)) + c(E(t_0) - E(t) + \widetilde{E}(t_0) - \widetilde{E}(t)) \\
&\quad + c(\widetilde{E}(0) + \int_0^1 (3w_{0xx} - \psi_{0xx})\, \mathrm{d}x)\int_{t_0}^t g(s)\, \mathrm{d}s \\
&\quad + c\left(E(0) + \widetilde{E}(0) + \int_0^1 (3w_{0xx} - \psi_{0xx})^2 \mathrm{d}x \right)\int_0^t g(s)\, \mathrm{d}s \\
&\leq c\left(E(0) + \widetilde{E}(0) + \int_0^1 (3w_{0xx} - \psi_{0xx})^2 \mathrm{d}x \right)\left(1 + \int_0^t g(s)\, \mathrm{d}s \right),
\end{aligned} \tag{3.24}
$$

which yields (2.5). This completes the proof of Theorem 2.3.

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