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SPLIT-STEP COLLOCATION METHODS FOR STOCHASTIC VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT. In this paper, a split-step collocation method is proposed for solving linear stochastic Volterra integral equations (SVIEs) with smooth kernels. The Hölder condition and the conditional expectations of the exact solutions are investigated. The solvability and mean-square boundedness of numerical solutions are proved and the strong convergence orders of collocation solutions and iterated collocation solutions are also shown. In addition, numerical experiments are provided to verify the conclusions.

1. Introduction. Stochastic Volterra integral equations (SVIEs) are widely used in many fields such as mechanics, biology, finance and medicine. Hence, the research of SVIEs is very applicable, and over the past decades, many mathematicians have obtained significant results. The existence and uniqueness results of solutions of SVIEs were investigated in [**8, 18**].

Since, in many applications, there are no closed solutions, it is important to provide approximate solutions by some numerical approaches [**9, 20, 21**]. Recently, many numerical methods for Volterra integral equations (VIEs) have been proposed, such as quadrature, Runge-Kutta and block methods, Chebyshev polynomial methods, collocation methods, etc., in [**1, 2, 7, 14**]. On the other hand, there are also some research results on numerical methods for SVIEs, such as stochastic operational matrices of block pulse functions in [**10, 12, 13**], triangu-

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lar and operational matrices of integrations in **[11]**, the generalized hat basis functions in **[5]** and delta function approximations in **[17]**.

Split-step methods represent a class of fully implicit methods which allow the incorporation of implicitness in the stochastic part of the system with relatively little additional cost. Such an advantage makes them very attractive for solving equations with noise. Originally proposed by Higham **[6]**, he applied the split-step backward Euler (SSBE) method to nonlinear autonomous stochastic differential equations (SDEs). Under the one-sided Lipschitz condition, the 1/2 strong convergence order is proved. The SSBE method is a special case of stochastic split-step θ (SSθ) methods introduced in **[3]**, while the strong convergence order 1/2 is the same for SSθ methods with $0 \leqslant \theta \leqslant 1$.

In this paper, we consider a one-stage collocation method for linear SVIEs of the second kind:

$$\begin{aligned} X(t) = g(t) &+ \int_0^t K(t,s)X(s)\,\mathrm{d}s \\ &+ \int_0^t \sigma(t,s)X(s)\,\mathrm{d}W(s), \quad t \in I := [0,T], \end{aligned} \tag{1.1}$$

where $\Delta := \{(t,s) : 0 \leqslant s \leqslant t \leqslant T\}$, $g \in C^1(I)$ is a deterministic continuous function, $K \in C^1(\Delta)$ and $\sigma \in C^1(\Delta)$ are two deterministic kernels and $\{W(t), t \geqslant 0\}$ is a standard Brownian motion defined on a complete probability space $(\Omega, \mathscr{F}, P)$ with a filtration $\{\mathscr{F}_t\}_{t\geq 0}$ under the usual conditions.

This paper is organized as follows. In Section 2, we extend collocation methods and iterated collocation methods for deterministic VIEs to SVIEs under the conditional expectation sense, which, based on the definition of the Itô integral, is implemented in the split-step approach. In Section 3, some fundamental properties of exact solutions to linear SVIEs are discussed, such as mean-square boundedness, the Hölder condition and properties of conditional expectations of the exact solutions. In Section 4, the solvability, the mean-square boundedness and the strong convergence order of numerical solutions is investigated. In Section 5, some numerical examples for strong convergence order are presented.

2. Collocation methods. In this section, we will establish a collocation method for linear SVIEs by combining the approaches of one-stage collocation methods for deterministic VIEs in [**1**] with the conditional expectation for continuous numerical solutions of semi-implicit Euler methods to stochastic pantograph differential equations in [**19**].

2.1. Collocation methods for deterministic VIEs. For the deterministic VIEs,

$$u(t) = g(t) + \int_0^t K(t,s)u(s)\,\mathrm{d}s, \quad t \in I := [0,T], \tag{2.1}$$

the one-stage collocation methods approximate the exact solution by a piecewise constant function

$$u_h \in S_0^{(-1)}(I_h) := \{v\big|_{(t_n,t_{n+1})} \text{ is a constant for } 0 \leqslant n \leqslant N-1\},$$

which satisfies the collocation equation

$$u_h(t_{n,1}) = g(t_{n,1}) + \int_0^{t_{n,1}} K(t_{n,1},s)u_h(s)\,\mathrm{d}s, \tag{2.2}$$

where $h = T/N$ is the stepsize, $I_h = \{t_n = nh : 0 \leqslant n \leqslant N\}$, $t_{n,1} = t_n + c_1 h$, $n = 0,1,\dots,N-1$, $0 \leqslant c_1 \leqslant 1$. Indeed, the collocation equation is exactly the same as the original equation at the collocation points $t_{n,1}$. The iterated collocation solution u_h^{it} corresponding to the collocation solution u_h is defined by

$$u_h^{it}(t) = g(t) + \int_0^t K(t,s)u_h(s)\,\mathrm{d}s, \quad t \in I.$$

It trivially satisfies

$$u_h^{it}(t_{n,1}) = u_h(t_{n,1}) \quad \text{for all } n = 0,1,\dots,N-1.$$

In the special case $c_1 = 1$, $u_h(t) \equiv u_n$ for $t \in (t_n, t_{n+1})$ satisfies

$$u_n = g(t_{n+1}) + F_n(t_{n+1}) + h\bigg(\int_0^1 K(t_{n+1}, t_n + \theta h)\,\mathrm{d}\theta\bigg)u_n, \tag{2.3}$$

and the iterated collocation solution is given by

$$u_h^{it}(t) = g(t) + F_n(t) + \bigg(\int_{t_n}^t K(t,s)\,\mathrm{ds}\bigg)\mathrm{u_n}, \tag{2.4}$$

where

$$F_n(t) := \sum_{\ell=0}^{n-1} h\Big(\int_0^1 K(t, t_\ell + \theta h)\,\mathrm{d}\theta\Big) u_\ell.$$

2.2. Collocation methods for SVIEs. Different from the deterministic case, the numerical solutions to (1.1) must be $\mathscr{F}_t$-adapted stochastic processes based on the definition of the Itô integral. From the right continuity of the filtration $\mathscr{F}_t$, i.e., $\cap_{t>t_n}\mathscr{F}_t = \mathscr{F}_{t_n}$, a suitable collocation space for one-stage collocation methods is

$$\mathcal{S}_0^{(-1)}(I_h) := \Big\{ v\big|_{(t_n,t_{n+1})} \text{ is an } \mathscr{F}_{t_n}\text{-measurable constant random variable, } 0 \leqslant n \leqslant N-1 \Big\}.$$

The one-stage collocation methods are established as follows.

(i) When the kernel in the diffusion term is dependent upon s, the Itô integral is difficult to implement during the numerical process. Hence, according to Maruyama's approach, it is approximated by

$$\int_0^t \sigma(t,s)X_h(s)\,\mathrm{d}W(s) \approx \int_0^t \sigma(t,s_h)X_h(s)\,\mathrm{dW(s)},$$

where $s_h = t_n$ for $s \in (t_n, t_{n+1})$.

(ii) The collocation equation for SVIEs cannot be defined in the same form as the deterministic case since, in general, the left-handed is $\mathscr{F}_{t_n}$-measurable but the right-handed is $\mathscr{F}_{t_{n+1}}$-measurable. In order to overcome this the authors introduce the conditional expectation to the numerical scheme in [**19**]. The version for SVIEs is presented in the following form

$$X_h(t_{n,1}) = \mathbb{E}\Big(g(t_{n,1}) + \int_0^{t_{n,1}} K(t_{n,1},s)X_h(s)\,\mathrm{d}s + \int_0^{t_{n,1}} \sigma(t_{n,1},s_h)X_h(s)\,\mathrm{d}W(s)|\mathscr{F}_{t_n}\Big), \tag{2.5}$$

and the iterated collocation solution is defined by

$$X_h^{it}(t) = g(t) + \int_0^t K(t,s)X_h(s)\,\mathrm{d}s + \int_0^t \sigma(t,s_h)X_h(s)\,\mathrm{d}W(s) \quad \text{for all } t \in I. \tag{2.6}$$

2.3. Implementation. For the deterministic functions g and K, it follows from Fubini's theorem that

$$\mathbb{E}\left(g(t) + \int_0^t K(t,s)X_h(s)\,\mathrm{d}s \mid \mathscr{F}_{t_n}\right) = g(t) + \int_0^t K(t,s)X_h(s)\,\mathrm{d}s \text{ for } t \in (t_n, t_{n+1}).$$

Since $X_h(t)$ is an $\mathscr{F}_{t_n}$-measurable for $t \in (t_n, t_{n+1})$,

$$\mathbb{E}\left(\int_0^{t_n} \sigma(t,s_h)X_h(s)\,\mathrm{d}W(s) \mid \mathscr{F}_{t_n}\right) = \int_0^{t_n} \sigma(t,s_h)X_h(s)\,\mathrm{d}W(s)$$

and, by the properties of conditional expectations,

$$\begin{aligned}\mathbb{E}&\left(\int_{t_n}^t \sigma(t,s_h)X_h(s)\,\mathrm{d}W(s) \mid \mathscr{F}_{t_n}\right)\\ &= \mathbb{E}\left(\sigma(t,t_n)X_h(t_n)(W(t)-W(t_n)) \mid \mathscr{F}_{t_n}\right)\\ &= \sigma(t,t_n)X_h(t_n)\mathbb{E}\left(W(t)-W(t_n) \mid \mathscr{F}_{t_n}\right) = 0.\end{aligned}$$

The last equality comes from the independence increment of the Brownian motion and $\mathbb{E}(W(t) - W(t_n)) = 0$. Hence, the special case $c_1 = 1$ of collocation methods is implemented as follows:

(i) $X_h(t) = X_n^*$ is an $\mathscr{F}_{t_n}$-measurable constant random variable for $t \in (t_n, t_{n+1})$;

(ii) the collocation equation (2.5) has the form

$$X_n^* = g(t_{n+1}) + F_n(t_{n+1}) + h\left(\int_0^1 K(t_{n+1}, t_n + \theta h)\,\mathrm{d}\theta\right)X_n^*, \tag{2.7}$$

where

$$F_n(t) = h\sum_{\ell=0}^{n-1}\left(\int_0^1 K(t, t_\ell + \theta h)\,\mathrm{d}\theta\right)X_\ell^* + \sum_{\ell=0}^{n-1}\sigma(t,t_\ell)X_\ell^*\Delta W_\ell;$$

(iii) the iterated collocation equation X_h^{it}, an $\mathscr{F}_t$-adapted stochastic process is defined by

(2.8)
$$X_h^{it}(t) = g(t)+F_n(t)+\left(\int_{t_n}^{t} K(t,s)\,\mathrm{d}s\right)X_n^* + \sigma(t,t_n)X_n^*(W(t)-W(t_n)).$$

Remark 2.1. Let $K(t,s) \equiv \lambda$, $\sigma(t,s) \equiv \mu$ and $g(t) = X_0$ be constants. Then, the collocation equation (2.7) and the iterated collocation equation (2.8) read

$$X_n^* = X_0 + \lambda h X_n^* + \sum_{\ell=0}^{n-1} \lambda h X_\ell^* + \sum_{\ell=0}^{n-1} \mu X_\ell^* \Delta W_\ell,$$
$$X_h^{it}(t_{n+1}) = X_0 + \sum_{\ell=0}^{n} \lambda h X_\ell^* + \sum_{\ell=0}^{n} \mu X_\ell^* \Delta W_\ell.$$

Denote $X_h^{it}(t_n) = X_n$. Then, the above scheme yields

$$\begin{aligned} X_n^* &= X_n + \lambda h X_n^*, \\ X_{n+1} &= X_n^* + \mu X_n^* \Delta W_n, \end{aligned} \tag{2.9}$$

which is the same as the scheme of the SSBE method for the SDE:

$$\begin{aligned} \mathrm{d}X(t) &= \lambda X(t)\,\mathrm{d}s + \mu X(t)\,\mathrm{d}W(t), \quad t \in I, \\ X(0) &= X_0. \end{aligned} \tag{2.10}$$

Hence, the method (2.7)–(2.8) is also named by a split-step collocation method.

Remark 2.2. Let the diffusion kernel $\sigma(t,s) \equiv 0$. Then, the collocation equation (2.7) and the iterated collocation equation (2.8) are the same as the corresponding equations (2.3) and (2.4) for the deterministic case.

Remark 2.3. For deterministic equations, the iterated collocation solution coincides with the collocation solution at the collocation points. This is, in general, not true for SVIEs, since $X_h(t_{n+1})$ is $\mathscr{F}_{t_n}$-measurable and $X_h^{it}(t_{n+1})$ is $\mathscr{F}_{t_{n+1}}$-measurable, while it always holds that

$$X_h(t_{n+1}) = \mathbb{E}\left(X_h^{it}(t_{n+1})|\mathscr{F}_{t_n}\right),$$

since

$$\begin{aligned}\mathbb{E}\left(X_h^{it}(t_{n+1}) \mid \mathscr{F}_{t_n}\right) &= \mathbb{E}\bigg(g(t_{n+1}) + \int_0^{t_{n+1}} K(t_{n+1},s)X_h(s)\,\mathrm{d}s \\ &\quad + \int_0^{t_{n+1}} \sigma(t_{n+1},s_h)X_h(s)\,\mathrm{d}W(s) \mid \mathscr{F}_{t_n}\bigg) \\ &= g(t_{n+1}) + \int_0^{t_{n+1}} K(t_{n+1},s)X_h(s)\,\mathrm{d}s \\ &\quad + \int_0^{t_n} \sigma(t_{n+1},s_h)X_h(s)\,\mathrm{d}W(s) \\ &= X_h(t_{n+1}).\end{aligned}$$

Remark 2.4. Similarly to the above discussion, the numerical solutions of (2.9) to the linear SDE (2.10) satisfy $X_n^* = \mathbb{E}(X_{n+1}|\mathscr{F}_{t_n})$.

3. Properties of the exact solutions. In this section, we discuss the mean-square boundedness and the Hölder condition of the exact solutions to (1.1). In order to illustrate the approximation of collocation solutions, we also introduce the conditional expectations of the exact solutions and investigate some fundamental properties between $X(t)$ and the conditional expectations.

3.1. Hölder conditions. Although the exact solutions to SVIEs (1.1) are in general not a martingale, the mean-square boundedness for continuous g, K and σ is also established in a similar manner to SDEs.

Theorem 3.1. *Assume that $g \in C(I)$, $K, \sigma \in C(\Delta)$. Then, the exact solutions to* (1.1) *are bounded in the mean square sense, i.e., there exists a constant $C_1 > 0$ independent of t, such that*

$$\mathbb{E}|X(t)|^2 \leqslant C_1 \quad \textit{for all } 0 \leqslant t \leqslant T.$$

Proof. Let $\tau_n = \inf\{t : |X(t)| > n\}$ be the stopping time and $X_n(t) = X(t \wedge \tau_n)$. Then, we obtain from (1.1), the Hölder inequality and Itô isometric formula that

$$
\begin{aligned}
\mathbb{E}\left(|X_n(t)|^2\right) &= \mathbb{E}\Bigg(\bigg|g(t\wedge\tau_n) + \int_0^{t\wedge\tau_n} K(t\wedge\tau_n, s)X(s)\,\mathrm{d}s \\
&\qquad + \int_0^{t\wedge\tau_n} \sigma(t\wedge\tau_n, s)X(s)\,\mathrm{d}W(s)\bigg|^2\Bigg) \\
&\le 3\mathbb{E}\Bigg(M^2 + \int_0^t |K(t,s)|^2\mathrm{d}s\cdot\int_0^t |X(s\wedge\tau_n)|^2\mathrm{d}s \\
&\qquad + \int_0^t |\sigma(t\wedge\tau_n, s\wedge\tau_n)X(s\wedge\tau_n)|^2\mathrm{d}s\Bigg) \\
&\le 3M^2 + (3M^2T + 3M^2)\mathbb{E}\left(\int_0^t |X_n(s)|^2\,\mathrm{d}s\right),
\end{aligned}
$$

where $M = \max_{0\leqslant s\leqslant t\leqslant T}\{|g(t)|, |K(t,s)|, |\sigma(t,s)|\}$. From Gronwall's inequality, we have

$$\mathbb{E}(|X_n(t)|^2) \le 3M^2\exp((3M^2T + M^2)T).$$

Hence, by Chebyshev's inequality,

$$P(\tau_n < t) = P(X_n(t) \ge n) \leqslant \frac{1}{n^2}(3M^2\exp(3M^2T + M^2)T),$$

which implies that

$$\sum_{n=0}^{\infty} P(\tau_n < t) < \infty.$$

Hence, $\lim_{n\to\infty}\tau_n \ge t$ almost surely. Therefore, the proof is completed by the control convergence theorem. □

For smooth kernels, a 1/2 order Hölder condition of the exact solutions is provided in the next theorem.

Theorem 3.2. *Assume that $g \in C^1(I)$ and $K, \sigma \in C^1(\Delta)$. Then, the exact solutions to* (1.1) *satisfy a Hölder condition, i.e., there exists a constant $C_2 > 0$ independent of t and $\bar{t}$, such that*

$$\mathbb{E}(|X(t) - X(\bar{t})|^2) \leqslant C_2|t - \bar{t}| \quad \textit{for all } 0 \leqslant \bar{t} \leqslant t \leqslant T.$$

Proof. From $g \in C^1(I)$ and $K, \sigma \in C^1(\Delta)$, we have

$$
\begin{aligned}
|g(t) - g(\bar{t})|^2 &= O((t-\bar{t})^2), \\
|K(t,s) - K(\bar{t},s)|^2 &= O((t-\bar{t})^2), \qquad (3.1)
\end{aligned}
$$

$$|\sigma(t,s)-\sigma(\bar{t},s)|^2 = O((t-\bar{t})^2).$$

From (1.1) and the elementary inequality $(a+b+c+d+e)^2 \leqslant 5(a^2+b^2+c^2+d^2+e^2)$, we obtain

$$\begin{aligned}
&\mathbb{E}(|X(t)-X(\bar{t})|^2)\\
&\quad= \mathbb{E}\bigg(|g(t)-g(\bar{t}) + \int_0^t K(t,s)X(s)\,\mathrm{d}s - \int_0^{\bar{t}} K(\tau,s)X(s)\mathrm{d}s\\
&\qquad + \int_0^t \sigma(t,s)X(s)\,\mathrm{d}W(s) - \int_0^{\bar{t}} \sigma(\bar{t},s)\,\mathrm{d}W(s)|^2\bigg)\\
&\leq 5\mathbb{E}(|g(t)-g(\bar{t})|^2) + 5\mathbb{E}\bigg(\bigg|\int_0^{\bar{t}} (K(t,s)-K(\bar{t},s))X(s)\,\mathrm{d}s\bigg|^2\bigg)\\
&\quad + 5\mathbb{E}\bigg(\bigg|\int_{\bar{t}}^t K(t,s)X(s)\,\mathrm{d}s\bigg|^2\bigg) + 5\mathbb{E}\bigg(\bigg|\int_{\bar{t}}^t \sigma(t,s)X(s)\,\mathrm{d}W(s)\bigg|^2\bigg)\\
&\quad + 5\mathbb{E}\bigg(\bigg|\int_0^{\bar{t}} (\sigma(t,s)-\sigma(\bar{t},s))X(s)\,\mathrm{d}W(s)\bigg|^2\bigg).
\end{aligned}$$

From the Hölder inequality and the property of the Itô integral, the following is obtained.

$$\begin{aligned}
\mathbb{E}(|X(t)-X(\bar{t})|^2) \leq{}& 5\mathbb{E}(|g(t)-g(\bar{t})|^2)\\
&+ 5\mathbb{E}\bigg(\int_0^{\bar{t}} |K(t,s)-K(\bar{t},s)|^2\mathrm{d}s \cdot \int_0^{\bar{t}} |X(s)|^2\,\mathrm{d}s\bigg)\\
&+ 5\mathbb{E}\bigg(\int_{\bar{t}}^t |K(t,s)|^2\mathrm{d}s \cdot \int_{\bar{t}}^t |X(s)|^2\mathrm{d}s\bigg)\\
&+ 5\mathbb{E}\bigg(\int_{\bar{t}}^t |\sigma(t,s)X(s)|^2\mathrm{d}s\bigg)\\
&+ 5\mathbb{E}\bigg(\int_0^{\bar{t}} |\sigma(t,s)-\sigma(\bar{t},s)|^2|X(s)|^2\mathrm{d}s\bigg).
\end{aligned}$$

According to (3.1), Theorem 3.1 and the property of the Itô integral, we get

$$\mathbb{E}(|X(t)-X(\bar{t})|^2) \leqslant C_2|t-\bar{t}|.$$

The proof is complete. □

3.2. Conditional expectation of the exact solution. Since the collocation equations for SVIEs are defined in the sense of conditional expectations, in this subsection, we discuss the conditional expectation of $X(t)$ under $\mathscr{F}_\tau$, i.e., $Y(t,\tau) = \mathbb{E}(X(t) \mid \mathscr{F}_\tau)$ for any given $t, \tau \in [0,T]$. It follows from Theorem 3.1 that

$$\mathbb{E}\left(\int_\tau^t \sigma(t,s)X(s)\,\mathrm{d}W(s) \mid \mathscr{F}_\tau \right) = 0.$$

Hence, $Y(t,\tau) = X(t)$ for $0 \leqslant t \leqslant \tau$, and $Y(t,\tau)$, $t \geqslant \tau$, is a solution of

$$Y(t,\tau) = g(t) + F_\tau(t) + \int_\tau^t K(t,s)Y(s,\tau)\,\mathrm{d}s, \tag{3.2}$$

where

$$F_\tau(t) = \int_0^\tau K(t,s)X(s)\,\mathrm{d}s + \int_0^\tau \sigma(t,s)X(s)\,\mathrm{d}W(s).$$

From the orthogonality of $X(t) - Y(t,\tau)$ and $Y(t,\tau)$, Theorem 3.1 implies that $\mathbb{E}(|Y(t,\tau)|^2) \leqslant \mathbb{E}(|X(t)|^2) \leqslant C_1$ for all $0 \leqslant \tau,\ t \leqslant T$.

Remark 3.3. Let $K(t,s) \equiv \lambda$, $\sigma(t,s) \equiv \mu$ and $g(t) = X_0$ be constants. Then, for $0 \leqslant \tau < t \leqslant T$, (3.2) reduces to

$$Y(t,\tau) = X(\tau) + \int_\tau^t \lambda Y(s,\tau)\,\mathrm{d}s,$$

whose solution is $Y(t,\tau) = X(\tau)e^{\lambda(t-\tau)}$. Hence, $\mathbb{E}|Y(t,\tau) - Y(\widetilde{t},\tau)|^2 \leqslant C|t-\widetilde{t}|^2$.

Now, we discuss the estimation between $X(t)$ and $Y(t,\tau)$.

Theorem 3.4. *Assume that $g \in C^1(I)$, $K, \sigma \in C^1(\Delta)$. Then there exists a constant $C_3 > 0$ independent of t and τ, such that*

$$\mathbb{E}|X(t) - Y(t,\tau)|^2 \leqslant C_3|t-\tau| \quad \textit{for all } 0 \leqslant \tau, t \leqslant T.$$

Proof. From the definition of $Y(t,\tau)$, we only need consider the case $0 \leqslant \tau < t \leqslant T$. Let $Z(t,\tau) = X(t) - Y(t,\tau)$. Then,

$$Z(t,\tau) = \int_\tau^t K(t,s)Z(s,\tau)\,\mathrm{d}s + \int_\tau^t \sigma(t,s)X(s)\,\mathrm{d}W(s).$$

Taking the expectation on both sides and using the Hölder inequality and the property of the Itô integral, the following is easily obtained:

$$\begin{aligned}\mathbb{E}(|Z(t,\tau)|^2) \leq 2\mathbb{E}\bigg|\int_\tau^t K(t,s)Z(s,\tau)\,\mathrm{d}s\bigg|^2 \\ + 2\mathbb{E}\bigg(\bigg|\int_\tau^t \sigma(t,s)X(s)\,\mathrm{d}W(s)\bigg|^2\bigg) \\ \leq 2\mathbb{E}\bigg(\int_\tau^t |K(t,s)|^2\mathrm{d}s \cdot \int_\tau^t |Z(t,\tau)|^2\mathrm{d}s\bigg) \\ + 2\mathbb{E}\bigg(\int_\tau^t |\sigma(t,s)X(s)|^2\mathrm{d}s\bigg) \\ \leq 2M^2T\bigg(\int_\tau^t |Z(t,\tau)|^2\mathrm{d}s\bigg) + 2M^2C_1|t-\tau|.\end{aligned}$$

According to Gronwall's inequality, we get

$$\mathbb{E}(|Z(t,\tau)|^2) \leqslant 2M^2C_1e^{2M^2T^2}|t-\tau|.$$

Therefore, the proof is complete with $C_3 = 2M^2C_1e^{2M^2T^2}$. □

From the definition, $Y(t,\tau)$ satisfies a Hölder condition with order at least 1/2. While, for $t, \widetilde{t} \geq \tau$, the general result of the one order Hölder condition of $Y(t,\tau)$ in Remark 3.3 is proved in the following theorem.

Theorem 3.5. *Assume that $g \in C^1(I)$, $K, \sigma \in C^1(\Delta)$. Then, there exists a constant $C_4 > 0$, independent of $t, \widetilde{t}$ and τ, such that*

$$\mathbb{E}(|Y(t,\tau) - Y(\widetilde{t},\tau)|^2) \leqslant C_4|t-\widetilde{t}|^2 \quad \textit{for all } \tau \leqslant t \leqslant \widetilde{t} \leqslant T. \tag{3.3}$$

Proof. Let $\tau \leqslant t \leqslant \widetilde{t} \leqslant T$. Then, by (3.2), we get

$$\begin{aligned}&\mathbb{E}(|Y(t,\tau) - Y(\widetilde{t},\tau)|^2) \\ &\quad \leqslant 4\mathbb{E}(|g(t) - g(\widetilde{t})|^2) + 4\mathbb{E}\bigg|\int_{\widetilde{t}}^t K(t,s)Y(s,\tau)\,\mathrm{d}s\bigg|^2\end{aligned}$$

$$+ 4\mathbb{E}\left|\int_0^{\tilde{t}} (K(\widetilde{t},s) - K(t,s))Y(s,\tau)\,\mathrm{d}s\right|^2$$
$$+ 4\mathbb{E}\left|\int_0^{\tau} (\sigma(t,s) - \sigma(\widetilde{t},s))Y(s,\tau)\,\mathrm{d}W(s)\right|^2.$$

Hence, (3.1) and the Hölder inequality imply that

$$\begin{aligned}
&\mathbb{E}(|Y(t,\tau) - Y(\widetilde{t},\tau)|^2) \\
&\quad \leqslant O(|t-\widetilde{t}|^2) + 4\mathbb{E}\left(\left|\int_t^{\tilde{t}} K(\widetilde{t},s)Y(s,\tau)\,\mathrm{d}s\right|^2\right) \\
&\quad \leqslant O(|t-\widetilde{t}|^2) + 4M^2|t-\widetilde{t}|\mathbb{E}\left(\int_t^{\tilde{t}} |Y(s,\tau)|^2\,\mathrm{d}s\right) \\
&\quad \leqslant O(|t-\widetilde{t}|^2) + 4\overline{C}|t-\widetilde{t}|^2 M^2 = C_4|t-\widetilde{t}|^2.
\end{aligned}$$

Therefore, the proof is complete. □

Corollary 3.6. *Assume that $g \in C^1(I)$, $K, \sigma \in C^1(\Delta)$. Then,*

$\mathbb{E}|X(t) - Y(s,\tau)|^2 \leqslant 2C_3|t-\tau| + 2C_4|t-s|^2$ *for all* $0 \leqslant \tau \leqslant s,\ t \leqslant T$.

Proof. From Theorems 3.4 and 3.5, the desired result is reached by (3.4)

$$\begin{aligned}
\mathbb{E}|X(t) - Y(s,\tau)|^2 &= \mathbb{E}|X(t) - Y(t,\tau) + Y(t,\tau) - Y(s,\tau)|^2 \\
&\leqslant 2\mathbb{E}|X(t) - Y(t,\tau)|^2 + 2\mathbb{E}|Y(t,\tau) - Y(s,\tau)|^2.
\end{aligned}$$

The proof is complete. □

4. Numerical analysis of collocation methods.

In this section, we investigate the solvability, mean-square boundedness and strong convergence order of the collocation and iterated collocation solutions.

4.1. Solvability and mean-square boundedness.

We first provide the solvability of collocation methods for sufficiently small stepsize.

Theorem 4.1. *Assume that $g \in C^1(I)$, $K, \sigma \in C^1(\Delta)$. Then there exists an $\overline{h} > 0$ such that each equation of* (2.7) *has a unique solution for $h \in (0, \overline{h})$.*

Proof. Suppose that the collocation solutions exist up to $[0, t_n]$. Then, from (2.7), we have

$$\left(1 - h\int_0^1 K(t_{n+1}, t_n + \theta h)\,\mathrm{d}\theta\right) X_n^* = g(t_{n+1}) + F_n(t_{n+1}).$$

Since

$$\int_0^1 K(t_{n+1}, t_n + \theta h)\,\mathrm{d}\theta$$

is bounded by

$$\|K\|_\infty = \sup_{(t,s)\in\Delta} |K(t,s)|,$$

as long as $0 < h < \overline{h} = 1/\|K\|_\infty$, there exists a unique solution to (2.7). Therefore, the proof is complete. □

Remark 4.2. The condition in Theorem 4.1 is a sufficient condition but not necessary. In order to see this consider a constant kernel $K(t,s) \equiv \lambda < 0$. Then, the collocation solutions uniquely exist for $h > 0$.

Next, we discuss the mean-square boundedness of the collocation solutions $X_h(t)$.

Theorem 4.3. *Let $g \in C(I)$, $K, \sigma \in C(\Delta)$ and $h < 1/(6M^2(T+1))$. Then, there exists a constant $C_5 > 0$, independent of t, such that*

$$\mathbb{E}(|X_h(t)|^2) \leqslant C_5 \quad \text{for all } t \in I. \tag{4.1}$$

Proof. From the elementary inequality $(a+b+c)^2 \leqslant 3(a^2+b^2+c^2)$, we have

$$\begin{aligned}
&\mathbb{E}(|X_h(t_{n+1})|^2) \\
&\quad = \mathbb{E}\bigg(\bigg|g(t_{n+1}) + \int_0^{t_{n+1}} K(t_{n+1}, s) X_h(s)\,\mathrm{d}s \\
&\qquad\qquad + \int_0^{t_n} \sigma(t_{n+1}, s_h) X_h(s)\,\mathrm{d}W(s)\bigg|^2\bigg)
\end{aligned}$$

$$\leqslant 3\mathbb{E}(|g(t_{n+1})|^2) + 3\mathbb{E}\bigg(\bigg|\int_0^{t_{n+1}} K(t_{n+1},s)X_h(t)\,\mathrm{d}s\bigg|^2\bigg) + 3\mathbb{E}\bigg(\bigg|\int_0^{t_n} \sigma(t_{n+1},s_h)X_h(t)\,\mathrm{d}W(s)\bigg|^2\bigg).$$

Using mean-square boundedness, Hölder inequality and the property of the Itô integral in **[15]**, we obtain that

$$\begin{aligned}
&\mathbb{E}(|X_h(t_{n+1})|^2)\\
&\quad\leqslant 3M^2 + 3T\mathbb{E}\bigg(\int_0^{t_{n+1}} |K(t_{n+1},s)X_h(s)|^2\mathrm{d}s\bigg)\\
&\qquad + 3\mathbb{E}\bigg(\int_0^{t_n} |\sigma(t_{n+1},s_h)X_h(s)|^2\mathrm{d}s\bigg)\\
&\quad\leqslant 3M^2 + 3TM^2\mathbb{E}\bigg(\int_0^{t_{n+1}} |X_h(s)|^2\mathrm{d}s\bigg)\\
&\qquad + 3M^2\mathbb{E}\bigg(\int_0^{t_n} |X_h(s)|^2\mathrm{d}s\bigg)\\
&\quad= 3M^2 + 3M^2(T+1)\mathbb{E}\bigg(\int_0^{t_{n+1}} |X_h(s)|^2\mathrm{d}s\bigg),
\end{aligned}$$

which is equivalent to

$$\mathbb{E}(|X_n^*|^2) \leqslant 3M^2 + 3M^2(T+1)h\mathbb{E}\bigg(\sum_{j=0}^{n} |X_j^*|^2\bigg).$$

It follows from the condition on the stepsize that

$$\mathbb{E}(|X_n^*|^2) \leqslant 6M^2 + 6M^2(T+1)h\mathbb{E}\bigg(\sum_{j=0}^{n-1} |X_j^*|^2\bigg),$$

which implies by the discrete Gronwall inequality in **[1]** that

$$\mathbb{E}(|X_h(t)|^2) \leqslant C_5 \quad \text{for all } t\in I, \tag{4.2}$$

where $C_5 = 6M^2\exp(6M^2(T+1)T)$. The proof is complete. □

Corollary 4.4. *Let $g\in C(I)$ and $K,\sigma\in C(\Delta)$. Then, there exists a constant $C_6>0$ such that*

$$\mathbb{E}(|X_h^{it}(t)|^2) \leqslant C_6 \quad \textit{for all } t\in I.$$

Proof. This result is trivial from (2.7), (2.8) and Theorem 4.3. □

4.2. Strong convergence order. We first present the strong convergence order of the collocation solutions $X_h(t)$ to the conditional expectations $Y(t, t_n)$.

Theorem 4.5. *Assume that $g \in C^1(I)$, $K, \sigma \in C^1(\Delta)$ and $h < 1/6M^2(T+1)$. Then, there exists a constant $C_7 > 0$, independent of t_n, such that*

$$\mathbb{E}(|Y(t_{n+1}, t_n) - X_h(t_{n+1})|^2) \leqslant C_7 h \quad \textit{for all } n = 0, 1, \ldots, N-1, \tag{4.3}$$

where $Y(t_{n+1}, t_n)$ is defined by (3.2).

Proof. According to (2.5) and (3.2), we know

$$\begin{aligned}
&\mathbb{E}(|Y(t_{n+1}, t_n) - X_h(t_{n+1})|^2) \\
&= \mathbb{E}\bigg(\bigg|\int_0^{t_{n+1}} K(t_{n+1}, s)(Y(s, t_n) - X_h(s))\,\mathrm{d}s \\
&\qquad + \int_0^{t_n} \sigma(t_{n+1}, s)(Y(s, t_n) - X_h(s))\,\mathrm{d}W(s) \\
&\qquad + \int_0^{t_n} (\sigma(t_{n+1}, s) - \sigma(t_{n+1}, s_h))X_h(s)\,\mathrm{d}W(s)\bigg|^2\bigg).
\end{aligned}$$

By the mean-square boundedness, the Hölder inequality, (3.1) and the property of the Itô integral, we have

$$\begin{aligned}
&\mathbb{E}(|Y(t_{n+1}, t_n) - X_h(t_{n+1})|^2) \\
&\leqslant 3TM^2\mathbb{E}\bigg(\int_0^{t_{n+1}} |Y(s, t_n) - X_h(s)|^2 \mathrm{d}s\bigg) \\
&\quad + 3M^2\mathbb{E}\bigg(\int_0^{t_n} |Y(s, t_n) - X_h(s)|^2 \mathrm{d}s\bigg) \\
&\quad + 3\mathbb{E}\bigg(\int_0^{t_n} |(\sigma(t_{n+1}, s) - \sigma(t_{n+1}, s_h))X_h(s)|^2 \mathrm{d}s\bigg) \\
&\leqslant 3(T+1)M^2\mathbb{E}\bigg(\sum_{\ell=0}^{n} \int_{t_\ell}^{t_{\ell+1}} |Y(s, t_n) - Y(t_{\ell+1}, t_\ell) \\
&\qquad + Y(t_{\ell+1}, t_\ell) - X_h(t_{\ell+1})|^2 \mathrm{d}s\bigg) + O(h^2)
\end{aligned}$$

$$\le 6(T+1)M^2\mathbb{E}\bigg(\sum_{\ell=0}^{n}\int_{t_\ell}^{t_{\ell+1}}|Y(s,t_n)-Y(t_{\ell+1},t_\ell)|^2\mathrm{d}s\bigg)$$
$$+6(T+1)M^2\mathbb{E}\bigg(\sum_{\ell=0}^{n}\int_{t_\ell}^{t_{\ell+1}}|Y(t_{\ell+1},t_\ell)-X_h(t_{\ell+1})|^2\mathrm{d}s\bigg)+O(h^2),$$

which implies that

$$\begin{aligned}
&\mathbb{E}(|Y(t_{n+1},t_n)-X_h(t_{n+1})|^2)\\
&\quad\le 6(T+1)M^2\mathbb{E}\bigg(\sum_{\ell=0}^{n-1}\int_{t_\ell}^{t_{\ell+1}}|X(s)-Y(s,t_\ell)\\
&\qquad\qquad +Y(s,t_\ell)-Y(t_{\ell+1},t_\ell)|^2\mathrm{d}s\\
&\qquad\qquad +\int_{t_n}^{t_{n+1}}|Y(s,t_n)-Y(t_{n+1},t_n)|^2\mathrm{d}s\bigg)\\
&\qquad +6(T+1)M^2\mathbb{E}\bigg(\sum_{\ell=0}^{n}\int_{t_\ell}^{t_{\ell+1}}|Y(t_{\ell+1},t_\ell)-X_h(t_{\ell+1})|^2\mathrm{d}s\bigg)\\
&\quad +O(h^2).
\end{aligned}$$

According to Theorems 3.4 and 3.5, we obtain that

$$\begin{aligned}
&\mathbb{E}(|Y(t_{n+1},t_n)-X_h(t_{n+1})|^2)\\
&\quad\le 12(T+1)M^2C_3Th+12(T+1)M^2C_4Th^2+6(T+1)M^2C_4h^3\\
&\qquad +6(T+1)M^2\mathbb{E}\bigg(\sum_{\ell=0}^{n}\int_{t_\ell}^{t_{\ell+1}}|Y(t_{\ell+1},t_\ell)-X_h(t_{\ell+1})|^2\mathrm{d}s\bigg)+O(h^2).
\end{aligned}$$

Therefore, the proof is completed in a similar way to the proof of Theorem 4.3. □

Corollary 4.6. *The solutions X_n^* of the* SSBE *method for* SDEs *converge to the conditional expectations $\mathbb{E}(X(t_{n+1})|\mathscr{F}_{t_n})$ with the strong order $1/2$.*

In the next theorem, the strong convergence orders of collocation solutions and iterated collocation solutions for SVIEs are established, not only at the mesh points, but also in the entire interval.

Theorem 4.7. *Assume that $g\in C^1(I)$ and $K,\sigma\in C^1(\Delta)$. Then, both the collocation solutions and the iterated collocation solutions uniformly* *converge to the exact solution to* (1.1) *with the strong order* 1/2, *i.e., there exists a constant* C *such that, for sufficiently small stepsize* $h > 0$,

$$\begin{aligned} \mathbb{E}(|X(t) - X_h(t)|^2) &\leqslant Ch \quad \textit{for all } t \in I, \\ \mathbb{E}(|X(t) - X_h^{it}(t)|^2) &\leqslant Ch \quad \textit{for all } t \in I. \end{aligned} \tag{4.4}$$

Proof. It follows from Corollary 3.6 and Theorem 4.5 that, for any given $t \in [t_n, t_{n+1}]$,

$$\begin{aligned} &\mathbb{E}|X(t) - X_h(t)|^2 \\ &\quad \leqslant 2\mathbb{E}|X(t) - Y(t_{n+1}, t_n)|^2 + 2\mathbb{E}|Y(t_{n+1}, t_n) - X_h(t_{n+1})|^2 \\ &\quad \leqslant 4(C_3 + C_4)h + 2C_7 h \end{aligned}$$

for sufficiently small stepsize. For the iterated collocation solutions, it follows from (1.1) and (2.8) that

$$\begin{aligned} &\mathbb{E}(|X(t) - X_h^{it}(t)|^2) \\ &\quad = \mathbb{E}\bigg(\int_0^t K(t,s)(X(s) - X_h(s))\,\mathrm{d}s \\ &\qquad\quad + \int_0^t \sigma(t,s)(X(s) - X_h(s))\,\mathrm{d}W(s) \\ &\qquad\qquad + \int_0^t (\sigma(t,s) - \sigma(t,s_h)X_h(s)\,\mathrm{d}W(s) \bigg)^2 \\ &\quad \leqslant 3M^2(T+1) \int_0^t \mathbb{E}(|X(s) - X_h(s)|^2)\,\mathrm{d}s + O(h^2) = O(h). \end{aligned}$$

The proof is complete. □

Remark 4.8. For a special case of SVIEs

$$X(t) = g(t) + \int_0^t K(t,s)X(s)\,\mathrm{d}s + \int_0^t \sigma(s)X(s)\,\mathrm{d}W(s),$$

the method (2.7)–(2.8) reads

$$X_n^* = g(t_{n+1}) + \sum_{\ell=0}^{n} \bigg(\int_{t_\ell}^{t_{\ell+1}} K(t_{n+1}, s)\,\mathrm{d}s \bigg) X_\ell^* + \sum_{\ell=0}^{n-1} \sigma(t_\ell) X_\ell^* \Delta W_\ell,$$

$$X_h^{it}(t) = g(t) + \sum_{\ell=0}^{n-1}\left(\int_{t_\ell}^{t_{\ell+1}} K(t,s)\,\mathrm{d}s\right)X_\ell^* + \sum_{\ell=0}^{n-1}\sigma(t_\ell)X_\ell^*\Delta W_\ell$$
$$+ \int_{t_n}^{t} K(t,s)\,\mathrm{d}s X_n^* + \sigma(t_n)X_n^*(W(t)-W(t_n)), \quad t \in I.$$

Theorem 4.7 shows that the uniform strong convergence order of the iterated collocation solutions is 1/2.

5. Numerical examples. The computations associated with the examples were performed using Matlab. The uniform convergence order of the iterated collocation solutions is never higher than the local convergence order. Hence, in this section, we only compute the strong order at the end-point T. Since the closed solution is not available, the approximation solution of the EM method with tiny stepsize is regarded as the reference solution. Let $X_h^{it}(t)$ be the iterated collocation solutions, and let the errors and strong orders be defined as

$$E_h = \frac{1}{M}\sum_{i=1}^{M} |(X_h^i)^{it}(T) - (X_{2^{-12}}^i)^{it}(T)|$$

and

$$p = \frac{\log(E_h) - \log(E_{h/2})}{\log 2},$$

where $(X_h^i)^{it}$ is the ith-sample path of the iterated collocation solutions to the exact solution $X^i(T)$ with the stepsize h, $i = 1, \ldots, M$ and $M = 4000$ is the number of samples.

Example 5.1. Consider the following linear SDEs

$$\mathrm{d}X(t) = X(t)\,\mathrm{d}t + X(t)\,\mathrm{d}W(t), \qquad X(0) = 1,$$
$$\mathrm{d}X(t) = (1 + X(t))\,\mathrm{d}t + tX(t)\,\mathrm{d}W(t), \qquad X(0) = 0,$$

of which the equivalent form is comprised of the linear SVIEs

$$X(t) = 1 + \int_0^t X(s)\,\mathrm{d}s + \int_0^t X(s)\,\mathrm{d}W(s), \tag{5.1}$$

$$X(t) = t + \int_0^t X(s)\,\mathrm{d}s + \int_0^t sX(s)\,\mathrm{d}W(s). \tag{5.2}$$

Applying the collocation method to (5.1) and (5.2), we list the errors and orders in Table 1, which illustrates that the strong convergence is 1/2. It is consistent with the results for SDEs in **[6]** as well as with the conclusions of Theorem 4.7.

TABLE 1. The errors and orders for Example 5.1.

	(5.1)		(5.2)	
N	E_h	p	E_h	p
2^5	0.3172	-	0.0120	-
2^6	0.1689	0.45	0.0056	0.55
2^7	0.0639	0.70	0.0023	0.63
2^8	0.0317	0.51	0.0012	0.49
2^9	0.0151	0.53	0.0005	0.56

Example 5.2. Applying the collocation method to the following linear SVIEs:

$$X(t) = 1 + \int_0^t (t-s)X(s)\,\mathrm{d}s + \int_0^t X(s)\,\mathrm{d}W(s), \tag{5.3}$$

$$X(t) = 1 + \int_0^t (t-s)X(s)\,\mathrm{d}s + \int_0^t sX(s)\,\mathrm{d}W(s), \tag{5.4}$$

we list the errors and orders in Table 2. The numerical results show that the strong convergence is 1/2 when the kernel in the diffusion term is independent of t.

TABLE 2. The errors and orders for Example 5.2.

	(5.3)		(5.4)	
N	E_h	p	E_h	p
2^5	0.0703	-	0.0088	-
2^6	0.038	0.44	0.0046	0.47
2^7	0.0145	0.69	0.002	0.60
2^8	0.0072	0.50	0.001	0.50
2^9	0.0034	0.54	0.0005	0.50

Example 5.3. Now consider SVIEs with a diffusion kernel depending on t, i.e.,

$$X(t) = 1 + \int_0^t (t-s)X(s)\,\mathrm{d}s + \int_0^t tX(s)\,\mathrm{d}W(s), \tag{5.5}$$

$$X(t) = 1 + \int_0^t (t-s)X(s)\,\mathrm{d}s + \int_0^t (1+t-s)X(s)\,\mathrm{d}W(s). \tag{5.6}$$

In this case, the Itô integration is not a Martingale, while the numerical results in Table 3 illustrate that the convergence order is still 1/2.

TABLE 3. The errors and orders of convergence for Example 5.3.

	(5.5)		(5.6)	
N	E_h	p	E_h	p
2^5	0.0172	-	0.2025	-
2^6	0.0088	0.48	0.1022	0.49
2^7	0.0037	0.62	0.0377	0.72
2^8	0.0019	0.48	0.0188	0.50
2^9	0.0009	0.54	0.0089	0.54

Example 5.4. Finally, consider the SVIEs with a nonsmooth kernel

$$X(t) = 1 + \int_0^t \sqrt{t-s}X(s)\,\mathrm{d}s + \int_0^t \frac{3}{10}X(s)\,\mathrm{d}W(s).$$

In this example, the errors and the orders are computed with stepsize $h = 1/25,\ 1/50,\ 1/100,\ 1/200$ and the tiny stepsize $h = 1/1000$ for the EM method. The numerical digits in Table 4 suggest the convergence order of collocation methods for SVIEs with a nonsmooth kernel is still 1/2.

TABLE 4. The errors and orders of convergence for Example 5.4.

N	E_h	p
25	0.5982×10^{-3}	–
50	0.2677×10^{-3}	0.5801
100	0.1190×10^{-3}	0.5846
200	0.0519×10^{-3}	0.5988

6. Conclusions and future work. In this paper, we have established stochastic collocation methods for SVIEs in a piecewise constant random space by means of the conditional expectation, which is also called the split-step collocation methods. In addition to the solvability and the mean-square boundedness of collocation solutions and iterated collocation solutions, we have proved that the uniform strong convergence orders of collocation and iterated collocation solutions are $1/2$. In future research, we plan to apply the method to SVIEs with non-smooth or weakly singular kernels $K(t,s) = (t-s)^{\alpha}$, $\alpha \in (-1,1)$, such as Example 5.4, where there are many real world applications.

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