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COUPLED VOLTERRA INTEGRAL EQUATIONS WITH BLOWING UP SOLUTIONS

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ABSTRACT. In this paper, a system of nonlinear integral equations related to combustion problems is considered. Necessary and sufficient conditions for the existence and explosion of positive solutions are given. In addition, the uniqueness of the positive solutions is shown. The main results are obtained by monotonicity methods.

1. Introduction. Nonlinear integral equations arise in models of ignition and explosive behavior in diffusive media. In applications, solutions can describe a variety of processes, including solid fuel combustion processes. There are many papers related to this topic, e.g., [**5, 13, 15**].

Some simple models of combustion may be studied with the aid of the equation

$$u(x)=\int_0^x (x-s)^{\alpha-1} g(u(s))\,ds, \quad \alpha>0, \tag{1.1}$$

where g is a nondecreasing continuous function such that $g(0)=0$. Obviously, $u\equiv 0$ is the trivial solution to (1.1). This equation describes ignition in media if a nontrivial continuous solution u exists which is positive for $x>0$. It was shown [**1, 3, 4, 6, 7, 12**] that a necessary and sufficient condition for ignition is

$$\int_0^\delta \left[\frac{u}{g(u)}\right]^{1/\alpha}\frac{du}{u}<\infty, \tag{1.2}$$

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where δ is any finite positive number. A solution u to (1.1) has explosive behavior if there exists a finite blow-up time T, that is, $u(x) \to \infty$ as $x \to T^-$. A necessary and sufficient condition for the existence of a finite blow-up time is

$$\int_0^\infty \left[\frac{u}{g(u)}\right]^{1/\alpha} \frac{du}{u} < \infty, \tag{1.3}$$

see [**1, 2, 8**].

However, in more complicated combustion models, systems of integral equations appear [**13, 14**]. Inspired by considerations in those papers we study ignition and blow-up criteria for the system of equations:

$$u(x) = \int_0^x (x-s)^{\alpha-1}[v(s)]^\gamma ds, \tag{1.4}$$

$$v(x) = \int_0^x k(x-s, u(s))\, ds, \tag{1.5}$$

where

$$k(x-s, u(s)) = \sum_{i=1}^m (x-s)^{\beta_i - 1} g_i(u(s)),$$

$\alpha, \beta_i \geq 1$, $i = 1, 2, \ldots, m$, $\gamma > 0$, and the functions g_i, $i = 1, 2, \ldots, m$, are assumed to be the same as g in (1.1).

Obviously, $u \equiv 0$, $v \equiv 0$ is the trivial solution to (1.4), (1.5). However, we shall ask about nontrivial solutions, i.e., nonnegative continuous functions u and v not identically equal to 0. Since they are nondecreasing, it follows from the convolution form of (1.4) and (1.5) that, either they are simultaneously positive for $x > 0$, or there exists a constant $c > 0$ such that

$$u(x) = v(x) = 0 \quad \text{for } 0 \leq x \leq c$$

and

$$u(x), v(x) > 0 \quad \text{for } x > c.$$

We show that, if system (1.4), (1.5) has nontrivial solutions, then it has a unique solution with u, v positive for $x > 0$, and the components of any other solution are, respectively, translations of u and v of the form $u((x-c)_+)$, $v((x-c)_+)$, where $c > 0$ is any constant and $(x-c)_+ = 0$ for $0 \leq x \leq c$ and equals $x - c$ otherwise. From now on, we deal only with nontrivial solutions with positive components for $x > 0$. We are also interested in the existence of a finite blow-up time T for such solutions.

The system (1.4), (1.5) of integral equations with $m = 1$, i.e., with a single integral term on the right hand side of equation (1.5) was studied in [**10, 11**]. Unfortunately, the methods used there allowed only consideration of the case of integer exponents α, $\beta \geq 1$. Some extensions of those results were obtained in [**9**]. In this paper, we consider a more general system and describe its nontrivial solutions.

2. Notation and statement of results. Throughout the paper, we assume that nonlinearities g_i, $i = 1, 2, \ldots, m$, in system (1.4), (1.5) are in a function class $\mathcal{G}$ consisting of nondecreasing continuous functions $g : [0, \infty) \to [0, \infty)$ which are positive for $x > 0$ with $g(0) = 0$, and which satisfy the following technical assumption

(2.1)

there exists a constant $c_g > 0$ *such that* $g(x) \leq c_g g(x/2)$ *for* $x > 0$.

We say that the solution (u, v) of system (1.4), (1.5) is nontrivial, if the functions $u(x)$ and $v(x)$ are continuous and positive for $x > 0$. Such a nontrivial solution will be called *blowing-up* at some $T > 0$, if

$$u(x) \longrightarrow \infty,$$

and consequently,

$$v(x) \to \infty, \quad \text{as } x \to T^-.$$

We emphasize that the functions $u(x)$ and $v(x)$ are nondecreasing. Integrating by parts in (1.4), (1.5) it may be seen that the derivatives $u'(x)$ and $v'(x)$ are also nondecreasing.

In order to formulate the main results of the paper we introduce an auxiliary function:

$$\Phi(x) = x \sum_{i=1}^{m} \left[\frac{g_i(x)^\gamma}{x} \right]^{1/(\alpha+\beta_i\gamma)}. \tag{2.2}$$

The existence of the solution to system (1.4), (1.5) is established in the next theorem.

Theorem 2.1. *System* (1.4), (1.5) *has a unique nontrivial solution if and only if*

$$\int_0^\delta \frac{1}{\Phi(z)}\, dz < \infty. \tag{2.3}$$

We also analyze the blow-up behavior of the solution to system (1.4), (1.5), and the appropriate result is stated in the theorem:

Theorem 2.2. *System* (1.4), (1.5) *has a blowing-up solution if and only if*

$$\int_0^\infty \frac{1}{\Phi(z)}\, dz < \infty. \tag{2.4}$$

Our first step in the study of system (1.4), (1.5) relies on its reduction to the following, single nonlinear integral equation:

$$u(x) = \int_0^x (x-s)^{\alpha-1} \left[G(u)(s)\right]^\gamma ds, \tag{2.5}$$

where

$$G(u)(s) = \int_0^s k(s-t, u(t))\, dt.$$

Remark 2.3. If $m = 1$ and $\gamma = 1$, then both the system (1.4), (1.5) and the equation (2.5) reduce to equation (1.1) with the exponent equal to $\alpha + \beta - 1$. In this case, the conditions (2.3) and (2.4) are equivalent to (1.2) and (1.3), respectively.

We introduce into our considerations the function

$$w(x) = u'(u^{-1}(x)) = 1/(u^{-1})'(x),$$

where u^{-1} is the inverse function to u. In order to find a convenient relation for w we begin with the observation that integration by parts and then substitution of $z = u(t)$ yields:

$$\int_0^s (s-t)^{\beta_i - 1} g_i(u(t))\,dt = \frac{1}{\beta_i}\int_0^s (s-t)^{\beta_i} dg_i(u(t)) = \frac{1}{\beta_i}\int_0^{u(s)} (s-u^{-1}(z))^{\beta_i} dg_i(z)$$

for $i = 1, 2, \ldots, m$.

Thus, (2.5) can be rewritten in the form

$$u(x) = \int_0^x (x-s)^{\alpha-1} G_w(u(s))^\gamma ds = \frac{1}{\alpha}\int_0^x (x-s)^\alpha d[G_w(u(s))]^\gamma, \tag{2.6}$$

where

$$G_w(z) = \sum_{i=1}^m \frac{1}{\beta_i}\int_0^z (u^{-1}(z) - u^{-1}(t))^{\beta_i} dg_i(t) = \sum_{i=1}^m \frac{1}{\beta_i}\int_0^z \left(\int_t^z \frac{1}{w(r)}\,dr\right)^{\beta_i} dg_i(t).$$

The substitution of $z = u(s)$ in (2.6) gives

$$u(x) = \frac{1}{\alpha}\int_0^{u(x)} (x - u^{-1}(s))^\alpha d[G_w(s)]^\gamma,$$

or equivalently,

$$x = \frac{1}{\alpha}\int_0^x (u^{-1}(x) - u^{-1}(s))^\alpha d[G_w(s)]^\gamma. \tag{2.7}$$

Differentiating both sides of (2.7), we obtain the sought relation

$$w(x) = \int_0^x \left(\int_s^x \frac{1}{w(z)}\,dz\right)^{\alpha-1} d[G_w(s)]^\gamma. \tag{2.8}$$

3. Auxiliary lemmas. In this section, we obtain a priori estimates for the nontrivial solution $u(x)$ of equation (2.5). They will be expressed in terms of the function Φ, given in (2.2). This function is continuous and nondecreasing with $\Phi(0) = 0$. Furthermore, it follows from (2.1) that a constant $c > 0$ exists such that

$$\Phi(x) \le c\,\Phi(x/2) \tag{3.1}$$

for $x \ge 0$.

We define the operator S related to the right hand side of (2.8) as:

$$Sw(x) = \int_0^x \left(\int_s^x \frac{1}{w(z)}\, dz \right)^{\alpha-1} d\,[G_w(s)]^\gamma\,, \quad x > 0,$$

where

$$G_w(s) = \sum_{i=1}^m \frac{1}{\beta_i} \int_0^s \left(\int_t^s \frac{1}{w(z)}\, dz \right)^{\beta_i} dg_i(t).$$

It is defined for any continuous function $w(x)$ positive for $x > 0$ such that the Stieltjes integrals $G_w(s)$ and

$$\int_0^x \left(\int_s^x \frac{1}{w(z)}\, dz \right)^{\alpha-1} d\,[G_w(s)]^\gamma$$

are convergent.

Let $g \in \mathcal{G}$, and let $\alpha, \beta \ge 1$ and $\gamma > 0$. We define

$$\Psi(x) = x \left[\frac{g(s)^\gamma}{x} \right]^{1/(\alpha+\beta\gamma)}$$

and

$$V_k(s) = \int_0^s \left(\int_t^s \frac{1}{\Psi(z)}\, dz \right)^{\beta-k} dg(t)$$

for $k = 0, 1, \ldots, n$, where $0 \le n < \beta \le n+1$.

Lemma 3.1.

(i) *The following inequalities hold*

$$\frac{1}{\Psi(x)}(x-s) \le \int_s^x \frac{dz}{\Psi(z)} \le (\alpha+\beta\gamma) s^{1-1/(\alpha+\beta\gamma)} \left(x^{1/(\alpha+\beta\gamma)} - s^{1/(\alpha+\beta\gamma)}\right) \frac{1}{\Psi(s)}. \tag{3.2}$$

(ii) *There exist constants $c_0, c_1, \ldots, c_n > 0$ such that*

$$V_k(s) \le c_k g(s)^{1-[(\beta-k)\gamma]/(\alpha+\beta\gamma)} s^{(\beta-k)/(\alpha+\beta\gamma)} \tag{3.3}$$

for $s > 0$, $k = 0, 1, 2, \ldots, n$.

Proof.

(i) The lower and upper estimates in (3.2) follow from the monotonicity of $\Psi(z)$ and $g(t)$, respectively.

(ii) We begin with $k = n$. Integrating by parts, we obtain

$$V_n(s) = J_1(t)|_0^s + (\beta - n)J_2(s),$$

where

$$J_1(t) = g(t)\biggl(\int_t^s \frac{1}{\Psi(z)}\,dz\biggr)^{\beta-n}$$

and

$$J_2(s) = \int_0^s \biggl(\int_t^s \frac{1}{\Psi(z)}\,dz\biggr)^{\beta-n-1} \frac{1}{\Psi(t)}\,g(t)\,dt.$$

It follows from the second inequality in (3.2) that

$$J_1(t) \le (\alpha+\beta\gamma)^{\beta-n} g(t)^{1-[(\beta-n)\gamma]/(\alpha+\beta\gamma)} (s^{1/(\alpha+\beta\gamma)} - t^{1/(\alpha+\beta\gamma)})^{\beta-n},$$

which shows that $J_1(t) = 0$ both at $t = 0$ and at $t = s$.

Since Ψ is nondecreasing and $-1 < \beta - n - 1 \le 0$, we observe that

$$\begin{aligned} J_2(s) &\le \frac{1}{\Psi(s)^{\beta-n-1}} \int_0^s (s-t)^{\beta-n-1} t^{-1+(1/(\alpha+\beta\gamma))} g(t)^{1-(\gamma/(\alpha+\beta\gamma))}\,dt \\ &\le \frac{1}{\Psi(s)^{\beta-n-1}} g(s)^{1-(\gamma/(\alpha+\beta\gamma))} \int_0^s (s-t)^{\beta-n-1} t^{-1+(1/(\alpha+\beta\gamma))}\,dt \\ &\le c g(s)^{1-[(\beta-n)\gamma]/(\alpha+\beta\gamma)} s^{(\beta-n)/(\alpha+\beta\gamma)}, \end{aligned}$$

where

$$c = \int_0^1 (1-z)^{\beta-n-1} z^{-1+(1/(\alpha+\beta\gamma))}\,dz.$$

Thus, we get (3.3) for $k = n$.

For $0 \le k < n$, to avoid the difficulties connected with possible divergence of the integral

$$\int_t^s \frac{dz}{\Psi(z)}, \quad \text{as } t \to 0,$$

we introduce the truncated integrals

$$V_k^\epsilon(s) = \int_\epsilon^s \left(\int_t^s \frac{1}{\Psi(z)}\, dz \right)^{\beta-k} dg(t),$$

defined for $0 \le k \le n$ and any $0 < \epsilon \le s$. We first note that

$$\begin{aligned} \frac{d}{ds} V_{k-1}^\epsilon(s) &= (\beta - k + 1) \frac{1}{\Psi(s)} \int_\epsilon^s \left(\int_t^s \frac{1}{\Psi(z)}\, dz \right)^{\beta-k} dg(t) \\ &= (\beta - k + 1) \frac{1}{\Psi(s)} V_k^\epsilon(s). \end{aligned}$$

Therefore, using this recurrence relation and the induction assumption, we obtain the estimate

$$\begin{aligned} V_{k-1}^\epsilon(s) &= (\beta - k + 1) \int_\epsilon^s \frac{1}{\Psi(z)} V_k^\epsilon(z)\, dz \\ &\le (\beta - k + 1) \int_\epsilon^s \frac{1}{\Psi(z)} V_k(z)\, dz \\ &\le (\beta - k + 1) c_k \int_\epsilon^s g(z)^{1-[(\beta-k+1)\gamma]/(\alpha+\beta\gamma)} z^{-1+(\beta-k+1)/(\alpha+\beta\gamma)}\, dz \\ &\le c_{k-1} g(s)^{1-[(\beta-(k-1))\gamma]/(\alpha+\beta\gamma)} s^{[\beta-(k-1)]/(\alpha+\beta\gamma)}, \end{aligned} \tag{3.4}$$

with $c_{k-1} = (\alpha + \beta\gamma) c_k$ valid for any $0 < \epsilon \le x$. Letting $\epsilon \to 0$ in (3.4), we get the required estimate for $V_{k-1}(s)$, which by an induction argument concludes the proof of (3.3). □

Let $a_i > 0$ for $i = 1, 2, \ldots, m$ and $p > 0$ be arbitrary numbers. We will use the following well-known inequality:

$$A_p \left(\sum_{i=1}^m a_i \right)^p \le \sum_{i=1}^m a_i^p \le B_p \left(\sum_{i=1}^m a_i \right)^p, \tag{3.5}$$

where $A_p = 1$ and $B_p = 1/m^{p-1}$ for $0 < p < 1$, $A_p = 1/m^{p-1}$ and $B_p = 1$, for $p \ge 1$.

We denote

$$\Psi_i(x) = x \left(\frac{g_i(x)^\gamma}{x} \right)^{1/(\alpha+\beta_i\gamma)},$$

$$V_{0,i}(x) = \int_0^x \left(\int_s^x \frac{1}{\Psi_i(z)}\, dz \right)^{\beta_i} dg_i(s)$$

for $i = 1, 2, \ldots, m$ and

$$V(x) = \sum_{i=1}^{m} \frac{1}{\beta_i} \int_0^x \left(\int_s^x \frac{1}{\Phi(z)}\, dz \right)^{\beta_i} dg_i(s).$$

Lemma 3.2. *There exist constants* c_V^1 *and* $c_V^2 > 0$ *such that*

$$c_V^1 s^{-\alpha+1} \Phi(s)^\alpha \le V(s)^\gamma \le c_V^2 s^{-\alpha+1} \Phi(s)^\alpha. \tag{3.6}$$

Proof. We begin with the proof of the second inequality in (3.6). Since $\Psi_i(z) \le \Phi(z)$, and consequently,

$$V(s) \le \sum_{i=1}^{m} \frac{1}{\beta_i} V_{0,i}(s),$$

it follows from the first inequality in (3.5) that there exists a constant $c > 0$ such that

$$V(s)^\gamma \le c \sum_{i=1}^{m} V_{0,i}(s)^\gamma. \tag{3.7}$$

Now, using (3.3) with $k = 0$, we see that there exist constants $c_i > 0$, $i = 1, 2, \ldots, m$, such that

$$V_{0,i}(s)^\gamma \le c_i s^{1-(\alpha/(\alpha+\beta_i\gamma))} g_i(s)^{\alpha\gamma/(\alpha+\beta_i\gamma)} = c_i s^{-\alpha+1} \Psi_i(s)^\alpha \tag{3.8}$$

for $i = 1, 2, \ldots, m$. Combining (3.8) and (3.7), then applying the second inequality in (3.5), we obtain the required upper estimate in (3.6).

Passing to the lower estimate in (3.6) we denote

$$V(s) = \sum_{i=1}^{m} \frac{1}{\beta_i} W_i(s), \tag{3.9}$$

where

$$W_i(s) = \int_0^s \left(\int_t^s \frac{1}{\Phi(z)}\, dz \right)^{\beta_i} dg_i(t)$$

for $i = 1, 2, \ldots, m$. From the monotonic properties of functions Φ and g_i, $i = 1, 2, \ldots, m$, we get the estimates

$$W_i(x) \ge \Phi(x)^{-\beta_i} \int_0^x (x - s)^{\beta_i} dg_i(t) \ge \Phi(x)^{-\beta_i} \left(\frac{x}{2}\right)^{\beta_i} g_i\left(\frac{x}{2}\right)$$

for $i = 1, 2, \ldots, m$. In view of (2.1), we see that there exist constants $c_i > 0$, $i = 1, 2, \ldots, m$, such that

$$\begin{aligned} W_i(x) &\geq c_i \Phi(x)^{-\beta_i} x^{\beta_i} g_i(x) \\ &= c_i x^{(-\alpha+1)/\gamma} \Phi(x)^{\alpha/\gamma} \left(\frac{\Psi_i(x)}{\Phi(x)} \right)^{(\alpha/\gamma)+\beta_i} \end{aligned} \tag{3.10}$$

for $i = 1, 2, \ldots, m$.

It follows from (3.9) and (3.10) that there exists a constant $c > 0$ such that

$$V(x)^\gamma \geq c x^{-\alpha+1} \Phi(x)^\alpha \left(\sum_{i=1}^{m} \left(\frac{\Psi_i(x)}{\Phi(x)} \right)^{(\alpha/\gamma)+\beta^*} \right)^\gamma, \tag{3.11}$$

where $\beta^* = \max_{1 \leq i \leq m} \beta_i$. Applying the first inequality in (3.5) to the sum on the right hand side of (3.11) and noting that

$$\Phi(x) = \sum_{i=1}^{m} \Psi_i(x),$$

we obtain the required lower estimate in (3.6), which completes the proof. □

Lemma 3.3. *There exist constants* $c_\Phi^1, c_\Phi^2 > 0$ *such that*

$$c_\Phi^1 \Phi(x) \leq S\Phi(x) \leq c_\Phi^2 \Phi(x) \tag{3.12}$$

for $x > 0$.

Proof. We begin with a discussion of the integrals

$$U_k(x) = \int_0^x \left(\int_s^x \frac{1}{\Phi(z)} \, dz \right)^{\alpha-1-k} d \, [V(s)]^\gamma$$

for $\alpha > 1$ and $k = 0, 1, 2, \ldots, n$, where n is an integer such that $n < \alpha - 1 \leq n + 1$. Our aim is to show that there exist constants $d_1, d_2, \ldots, d_n > 0$ such that

$$U_k(x) \leq d_k x^{-k} \Phi(x)^{k+1} \tag{3.13}$$

for $x > 0$ and $k = 0, 1, 2, \ldots, n$.

Let $k = n$. Integrating by parts, we get

$$U_n(x) = J(s)|_0^x + (\alpha - 1 - n)I(x), \tag{3.14}$$

where

$$J(s) = V(s)^\gamma \left(\int_s^x \frac{1}{\Phi(z)}\, dz \right)^{\alpha - n - 1}$$

and

$$I(x) = \int_0^x \left(\int_s^x \frac{1}{\Phi(z)}\, dz \right)^{\alpha - n - 2} \frac{1}{\Phi(s)} V(s)^\gamma ds.$$

It follows from estimate (3.7) and inequalities $\Psi_i(z) \le \Phi(z)$ for $i = 1, 2, \ldots, m$ that there exists a constant $c > 0$ such that

$$J(s) \le c \sum_{i=1}^m J_i(s),$$

where

$$J_i(s) = \left(\int_s^x \frac{1}{\Psi_i(z)}\, dz \right)^{\alpha - n - 1} V_{0,i}(s)^\gamma.$$

In view of (3.2) and (3.8) we conclude that there exist constants $c_i > 0$ for $i = 1, 2, \ldots, m$, such that

$$J_i(s) \le c_i (x^{1/(\alpha+\beta_i\gamma)} - s^{1/(\alpha+\beta_i\gamma)})^{\alpha-n-1} s^{1-(\alpha/(\alpha+\beta_i\gamma))} g_i(s)^{[(n+1)\gamma]/(\alpha+\beta_i\gamma)},$$

whence $J_i(s) = 0$ both at $s = 0$ and $s = x$. Finally, we see that $J(s) = 0$ both at $s = 0$ and $s = x$.

We pass to estimating $I(x)$. Since $-1 < \alpha - n - 2 \le 0$ and $\Psi_i(z) \le \Phi(z)$, an application of (3.7) and monotonicity of $\Phi(z)$ yields the inequality

$$I(x) \le c\Phi(x)^{n+2-\alpha} \sum_{i=1}^m I_i(x), \tag{3.15}$$

where

$$I_i(x) = \int_0^x (x-s)^{\alpha-n-2} \frac{1}{\Psi_i(s)} V_{0,i}(s)^\gamma ds$$

for $i = 1, 2, \ldots, m$, and $c > 0$ is some constant. We estimate the integrals $I_i(x)$, $i = 1, 2, \ldots, m$, using inequality (3.8), which results in

$$I_i(x) \le c_i x^{\alpha-1-n} x^{-(\alpha-1)/(\alpha+\beta_i\gamma)} g_i(x)^{[(\alpha-1)\gamma]/(\alpha+\beta_i\gamma)} = c_i x^{-n} \Psi_i(x)^{\alpha-1}, \tag{3.16}$$

where $c_i > 0$, $i = 1, 2, \ldots, m$, are some constants. Now, combining (3.15) and (3.16), and then applying the second inequality in (3.5), we see that there exists a constant $c > 0$ such that

$$I(x) \leq cx^{-n}\Phi(x)^{n+2-\alpha}\left(\sum_{i=1}^{m}\Psi_i(x)\right)^{\alpha-1} = cx^{-n}\Phi(x)^{n+1}. \tag{3.17}$$

Thus, we see that

$$U_n(x) = (\alpha - n - 1)\int_0^x \left(\int_s^x \frac{1}{\Phi(z)}\,dz\right)^{\alpha-n-2}\frac{1}{\Phi(s)}V(s)^\gamma ds, \tag{3.18}$$

and the required estimate for $U_n(x)$ follows from (3.17) and (3.18).

For $0 \leq k < n$, to avoid the difficulties connected with possible divergence of the integral

$$\int_t^s \frac{dz}{\Phi(z)}, \quad \text{as } t \to 0,$$

we introduce the truncated integrals

$$U_k^\epsilon(s) = \int_\epsilon^s \left(\int_t^s \frac{1}{\Phi(z)}\,dz\right)^{\alpha-1-k} dV(t)^\gamma,$$

defined for $0 \leq k \leq n$ and any $0 < \epsilon \leq s$. We first note that

$$\begin{aligned}\frac{d}{ds}U_{k-1}^\epsilon(s) &= (\alpha-k)\frac{1}{\Phi(s)}\int_\epsilon^s \left(\int_t^s \frac{1}{\Phi(z)}\,dz\right)^{\alpha-1-k} dV(t)^\gamma \\ &= (\alpha-k)\frac{1}{\Phi(s)}U_k^\epsilon(s).\end{aligned}$$

Using this recurrence relation and the induction assumption we obtain the estimate

$$\begin{aligned}U_{k-1}^\epsilon(s) &= (\alpha-k)\int_\epsilon^s \frac{1}{\Phi(z)}U_k^\epsilon(z)\,dz \\ &\leq (\alpha-k)\int_\epsilon^s \frac{1}{\Phi(z)}U_k(z)\,dz \\ &\leq c_{k-1}\int_\epsilon^s z^{-k}\Phi(z)^k dz\end{aligned} \tag{3.19}$$

with $c_{k-1} = (\alpha - k)d_k$, valid for any $0 < \epsilon \le x$. Now, using the first inequality in (3.5), we get

$$\begin{aligned} \Phi(z)^k &\le (1/A_k)\sum_{i=1}^{m}\Psi_i(z)^k \\ &= (1/A_k)z^k\sum_{i=1}^{m} z^{-k/(\alpha+\beta_i\gamma)} g_i(z)^{(k\gamma)/(\alpha+\beta_i\gamma)}. \end{aligned} \tag{3.20}$$

Letting $\epsilon \to 0$ in (3.19), it follows from (3.20) that there exists a constant $c > 0$ such that

$$U_{k-1}(s) \le c\sum_{i=1}^{m} I_i(s), \tag{3.21}$$

where

$$I_i(s) = \int_0^s z^{-k}\Psi_i(z)^k dz,$$

for $i = 1, 2, \ldots, m$. Since $g_i(z)$, $i = 1, 2, \ldots, m$, are nondecreasing, we obtain the estimates

$$I_i(s) \le c_i s^{1-k/(\alpha+\beta_i\gamma)} g_i(s)^{(k\gamma)/(\alpha+\beta_i\gamma)} = c_i z^{-k+1}\Psi_i(s)^k, \tag{3.22}$$

where $c_i > 0$, $i = 1, 2, \ldots, m$, are some constants. Now, combining (3.21) and (3.22) and then applying the second inequality in (3.5), we get the required estimate for $U_{k-1}(s)$, which, by an induction argument, ends the proof of (3.13).

Now, we are ready to justify the inequalities for our main interest.

For $\alpha = 1$, we have $S\Phi(x) = V(x)^\gamma$. Therefore, in this case, our assertion immediately follows from Lemma 3.2.

For $\alpha > 1$, we have $S\Phi(x) = U_0(x)$. Therefore, the upper estimate in (3.12) follows from the estimate of $U_0(x)$ given in (3.13).

In order to prove the lower estimate in (3.12), we first note that

$$\begin{aligned} S\Phi(x) &\ge \int_0^{x/2}\left(\int_s^x \frac{1}{\Phi(z)}\,dz\right)^{\alpha-1} d\,[V(s)]^\gamma \\ &\ge \Phi(x)^{-\alpha+1}\int_0^{x/2}(x-s)^{\alpha-1} d\,[V(s)]^\gamma \end{aligned}$$

$$\geq \Phi(x)^{-\alpha+1}\left(\frac{x}{2}\right)^{\alpha-1}[V(x/2)]^{\gamma}$$

for $x > 0$. Now, using the first inequality in (3.6) followed by inequality (3.1), we obtain our assertion. □

Remark 3.4. For $\alpha > 1$, by the same arguments as those used in obtaining (3.18), we get

$$U_0(x) = (\alpha - 1)\int_0^x \left(\int_s^x \frac{1}{\Phi(z)}\,dz\right)^{\alpha-2}\frac{1}{\Phi(s)}V(s)^{\gamma}ds$$

for $x > 0$.

Lemma 3.5. *Let $w(x)$ be a continuous and nondecreasing function, and let $c_1 > 0$ be a constant such that*

$$c_1\Phi(x) \leq w(x)$$

for $x > 0$. Then, there exists a constant c_2 such that

$$Sw(x) \leq c_2 S\Phi(x)$$

for $x > 0$.

Proof. We first note that there exists a constant $c > 0$ such that

$$G_w(s) \leq c\,G_{\Phi}(s) \tag{3.23}$$

for $s > 0$.

In the case of $\alpha = 1$, we have $S\Phi(x) = G_{\Phi}(x)^{\gamma}$ and $Sw(x) = G_w(x)^{\gamma}$. Therefore, our assertion immediately follows from (3.23).

In the case of $\alpha > 1$, we first note that

$$Sw(x) \leq c_1^{-(\alpha-1)}\int_0^x \left(\int_s^x \frac{1}{\Phi(z)}dz\right)^{\alpha-1} d[G_w(s)]^{\gamma}.$$

Now, integrating by parts in the outer integral on the right hand side and then using estimate (3.23), we see that there exists a constant $c > 0$ such that

$$Sw(x) \leq c\int_0^x \left(\int_s^x \frac{1}{\Phi(z)}\,dz\right)^{\alpha-2}\frac{1}{\Phi(s)}G_{\Phi}(s)^{\gamma}ds.$$

Hence, our assertion follows from Remark 3.4 and the estimate of $U_0(x) = S\Phi(x)$ in (3.13). □

Corollary 3.6. *Let u be a nontrivial solution to* (2.5), *and let* $w(x) = u'(u^{-1}(x))$, *where u^{-1} is inverse to u. Then, there exist constants* $c_w^1, c_w^2 > 0$ *such that*

$$c_w^1 \Phi(x) \le w(x) \le c_w^2 \Phi(x)$$

for $x > 0$.

Proof. In order to show the lower estimate we examine equation (2.8). Since the function w is nondecreasing, we have

$$\int_{x/2}^{x} \frac{1}{w(z)}\,dz \ge \frac{x}{2w(x)} \tag{3.24}$$

and

$$\begin{aligned} G_w(x/2) &\ge \int_0^{x/2} \left(\int_t^{x/2} \frac{1}{w(z)}\,dz \right)^{\beta_i} dg_i(t) \\ &\ge \left(\frac{x}{4w(x/2)} \right)^{\beta i} g_i(x/4) \\ &\ge \left(\frac{x}{4w(x)} \right)^{\beta_i} g_i(x/4) \end{aligned} \tag{3.25}$$

for $i = 1, 2, \ldots, m$ and $x > 0$.

It follows from (2.8) that

$$w(x) \ge \left(\int_{x/2}^{x} \frac{1}{w(z)}\,dz \right)^{\alpha-1} G_w(x/2)^{\gamma}$$

for $x > 0$. Now, combining (3.24), (3.25) and using (2.1), we note that there exist constants $c_i > 0$, $i = 1, 2, \ldots, m$, such that

$$w(x) \ge c_i \Psi_i(x)$$

for $i = 1, 2, \ldots, m$ and $x > 0$. Since

$$\Phi(x) = \sum_{i=1}^{m} \Psi_i(x),$$

our assertion follows from these inequalities.

Now, the upper estimate immediately follows from the proved lower estimate and Lemma 3.5. □

4. Proofs of theorems. We begin with giving the proof of Theorem 2.1.

Proof of Theorem 2.1.

Uniqueness. We begin by showing that the system (1.4), (1.5) has at most one nontrivial solution with the components u, v positive for $x > 0$. Let (u_i, v_i), $i = 1, 2$, be two nontrivial solutions of the system (1.4), (1.5) with components positive for $x > 0$. Consider a shifted solution with $u_{2,c}(x) = u_2((x-c)_+)$ and $v_{2,c}(x) = v_2((x-c)_+)$, where $c > 0$ is a constant. We observe that $u_{2,c}(x) < u_1(x)$ and $v_{2,c}(x) < v_1(x)$, at least for $0 \leq x \leq c$. Moreover, the following implication holds: if $u_{2,c}(x) < u_1(x)$ and $v_{2,c}(x) < v_1(x)$ for $0 \leq x < a$, where $a \geq c$ is a constant, then also $u_{2,c}(a) < u_1(a)$ and $v_{2,c}(a) < v_1(a)$. Hence, we conclude that $u_{2,c}(x) \leq u_1(x)$ and $v_{2,c}(x) \leq v_1(x)$ on their common interval of existence. Letting $c \to 0$, we see that $u_2(x) \leq u_1(x)$ and $v_2(x) \leq v_1(x)$ for $x > 0$. We can obtain the reverse inequality in the same way. Hence, $u_2 = u_1$ and $v_2 = v_1$.

Sufficiency. First, we shall construct a nondecreasing subsolution w of (2.5), that is, a nondecreasing function positive for $x > 0$ such that

$$w(x) \leq Tw(x) = \int_0^x (x-s)^{\alpha-1} \left[\int_0^s \sum_{i=1}^m (s-t)^{\beta_i - 1} g_i(w(t))\, dt \right]^\gamma ds. \tag{4.1}$$

Such functions w are equibounded, at least on a fixed small interval $x \in [0, \delta]$. In order to see this, we first note that the following inequality

$$\frac{w(x)}{g^*(w(x))^\gamma} \leq \phi(x) \quad \text{for } x > 0, \tag{4.2}$$

where $g^*(z) = \max_{1 \leq i \leq m} g_i(z)$ and

$$\phi(x) = \int_0^x (x-s)^{\alpha-1} \left[\sum_{i=1}^m \frac{1}{\beta_i} s^{\beta_i} \right]^\gamma ds,$$

is valid for any subsolution w. Hence, it also follows that

$$\frac{z}{g^*(z)^\gamma} \longrightarrow 0 \quad \text{as } z \to 0.$$

Now, let $M > 0$ and

$$h(z) = \inf\{s/g^*(s)^\gamma : z < s < M\} \quad \text{for } z \geq 0.$$

Since the function h is continuous and nondecreasing with $h(0) = 0$, we can choose $0 < M_0 < M$ such that $h(M_0) < h(M_0 + \epsilon)$ for $\epsilon > 0$. Due to (4.2), we have

$$w(x) \leq h^{-1}(\phi(x))$$

for $0 \leq x \leq \delta$, where h^{-1} is the inverse function to h and $\delta > 0$ is such that $\phi(x) \leq h(M_0)$ for $0 \leq x \leq \delta$. Integrating by parts in the inner integrals in (4.1), and then substituting $z = w(t)$, we obtain

$$\begin{aligned}\int_0^s (s-t)^{\beta_i - 1} g_i(w(t))\, dt &= \frac{1}{\beta_i}\int_0^s (s-t)^{\beta_i}\, dg_i(w(t)) \\ &= \frac{1}{\beta_i}\int_0^{w(s)} (s - w^{-1}(z))^{\beta_i}\, dg_i(z).\end{aligned}$$

Denote

$$H_w(z) = \sum_{i=1}^m \frac{1}{\beta_i}\int_0^z (w^{-1}(z) - w^{-1}(r))^{\beta_i}\, dg_i(r).$$

Integrating by parts in the outer integral in (4.1) and then substituting $z = w(s)$ we obtain

$$\begin{aligned}Tw(x) &= \frac{1}{\alpha}\int_0^x (x-s)^\alpha d\left[H_w(w(s))\right]^\gamma \\ &= \frac{1}{\alpha}\int_0^{w(x)} (x - w^{-1}(z))^\alpha d\left[H_w(z)\right]^\gamma.\end{aligned}$$

In order to construct a subsolution we define an auxiliary function $w_0(x)$ by its inverse

$$w_0^{-1}(x) = \int_0^x \frac{1}{\Phi(z)}\, dz$$

for $x > 0$.

We note that

$$(4.3)\quad \int_0^x (w_0^{-1}(x) - w_0^{-1}(z))^\alpha d\,[H_{w_0}(z)]^\gamma \\ \geq \int_0^{x/2} (w_0^{-1}(x) - w_0^{-1}(z))^\alpha d\,[H_{w_0}(z)]^\gamma \\ \geq (w_0^{-1}(x) - w_0^{-1}(x/2))^\alpha\,[H_{w_0}(x/2)]^\gamma$$

and

$$(4.4)\quad w_0^{-1}(x) - w_0^{-1}(x/2) \geq \frac{x}{2\Phi(x)}$$

for $x > 0$. Furthermore, due to (3.6), we have

$$(4.5)\quad H_{w_0}(x/2)^\gamma = V(x/2)^\gamma \geq cx^{-\alpha+1}\Phi(x/2)^\alpha,$$

where $c > 0$ is a constant. Now, combining (4.3), (4.5) and using (3.1), we see that there exists a constant $c > 0$ such that

$$\int_0^x (w_0^{-1}(x) - w_0^{-1}(z))^\alpha d\,[H_{w_0}(z)]^\gamma \\ \geq \int_0^{x/2} (w_0^{-1}(x) - w_0^{-1}(z))^\alpha d\,[H_{w_0}(z)]^\gamma \geq cx$$

for $x > 0$. Hence, it follows that there exists a constant $c_0 > 0$ such that

$$Tw_0(x) \geq c_0 w_0(x)$$

for $x > 0$. Now, if $0 < c_0 < 1$, we modify w_0 by taking $\widetilde{w}^{-1}(x) = c_0^{-1/(\alpha+\beta_*\gamma)} w_0^{-1}(x)$, where $\beta_* = \min_{1\leq i\leq m} \beta_i$; otherwise, we take $\widetilde{w}^{-1}(x) = w_0^{-1}(x)$. Then, we get

$$T\widetilde{w}(x) \geq \widetilde{w}(x)$$

for $x > 0$, which means that $\widetilde{w}(x)$ is a sought subsolution. Moreover, the functions $T^n\widetilde{w}$, $n = 1, 2, \ldots$, constitute a nondecreasing bounded sequence of subsolutions of (4.1). The required solution u can be obtained as the limit

$$u(x) = \lim_{n\to\infty} T^n\widetilde{w}(x) \quad \text{for } 0 \leq x \leq \delta.$$

Thus, we obtain the nontrivial solution u of (2.5), defined on a small interval $[0, \delta]$. Now, using standard arguments from the Volterra integral equation theory **[4]**, this solution can be extended to the maximal interval of existence.

Necessity. Let u be the nontrivial solution of equation (2.5). Since

$$u^{-1}(x) = \int_0^x (u^{-1})'(s)\,ds < \infty$$

for $x > 0$, our assertion follows from the upper estimate of $u'(u^{-1})$ given in Corollary 3.6. $\square$

Below, we give the proof of Theorem 2.2.

Proof of Theorem 2.2.

Necessity. Let u be the nontrivial solution of equation (2.5) blowing-up at $T < \infty$. Since

$$T = \lim_{x\to\infty} u^{-1}(x) = \int_0^\infty (u^{-1})'(s)\,ds < \infty,$$

our assertion follows from the upper estimate of $u'(u^{-1})(s)$ given in Corollary 3.6.

Sufficiency. Assume that (2.4) is true. First note that a nondecreasing, nontrivial solution $u(x)$ of equation (2.5) is blowing-up if and only if its inverse function $u^{-1}(x)$ converges to some $0 < T < \infty$, as $x \to \infty$. Since, by the left inequality in Corollary 3.6,

$$u^{-1}(x) \le 1/c_\Phi^1 \int_0^x \frac{dz}{\Phi(z)},$$

for $x > 0$, it follows from (2.4) that

$$T = \lim_{x\to\infty} u^{-1}(x) \le 1/c_\Phi^1 \lim_{x\to\infty} \int_0^x \frac{dz}{\Phi(z)} < \infty,$$

which shows that $u(x)$ is blowing-up at $T > 0$. Thus, the proof is complete. $\square$

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