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MEMORY DEPENDENT GROWTH IN SUBLINEAR VOLTERRA DIFFERENTIAL EQUATIONS

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ABSTRACT. We investigate memory dependent asymptotic growth in scalar Volterra equations with sublinear nonlinearity. In order to obtain precise results we extensively utilize the powerful theory of regular variation. By computing the growth rate in terms of a related ordinary differential equation we show that, when the memory effect is so strong that the kernel tends to infinity, the growth rate of solutions depends explicitly upon the memory of the system. Finally, we employ a fixed point argument for determining analogous results for a perturbed Volterra equation and show that, for a sufficiently large perturbation, the solution tracks the perturbation asymptotically, even when the forcing term is potentially highly non-monotone.

1. Introduction. We investigate explicit memory dependence in the asymptotic growth rates of positive solutions of the following scalar Volterra integro-differential equation

$$x'(t) = \int_{[0,t]} \mu(ds) f(x(t-s)), \quad t > 0;\ x(0) = \xi > 0, \tag{1.1}$$

where f is a positive sublinear function, i.e., $\lim_{x\to\infty} f(x)/x = 0$, and μ is a non-negative Borel measure. The relevant existence and uniqueness theory regarding equations of the form (1.1) is well known

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and guarantees a unique solution

$$x \in C(\mathbb{R}^+; (0,\infty))$$

in the framework of this article **[11,** Corollary 12.3.2], with the convention that $\mathbb{R}^+ := [0,\infty)$. By defining the function

$$M(t) = \int_{[0,t]} \mu(ds), \quad t \geq 0, \tag{1.2}$$

it follows that (1.1) is equivalent to

$$x(t) = x(0) + \int_0^t M(t-s) f(x(s))\, ds, \quad t \geq 0,\ x(0) = \xi > 0. \tag{1.3}$$

We also study the asymptotic behavior of the perturbed Volterra equation

$$x'(t) = \int_{[0,t]} \mu(ds) f(x(t-s)) + h(t), \quad t > 0;\ x(0) = \xi > 0. \tag{1.4}$$

As with the unperturbed equation, it is useful to consider an integral form of (1.4) and, by defining

$$H(t) := \int_0^t h(s)\, ds, \quad t \geq 0, \tag{1.5}$$

it follows that (1.4) can be written in integral form as

$$\begin{gathered} x(t) = x(0) + \int_0^t M(t-s) f(x(s))\, ds + H(t), \quad t \geq 0; \\ x(0) = \xi > 0. \end{gathered} \tag{1.6}$$

In **[7]**, with μ a finite measure, we demonstrate that, when f is sublinear and asymptotically increasing, the solution of (1.1) obeys

$$\lim_{t\to\infty} F(x(t))/t = \int_{[0,\infty)} \mu(ds) < \infty,$$

where

$$F(x) := \int_1^x \frac{1}{f(u)}\, du, \quad x > 0, \tag{1.7}$$

in other words, the structure of the memory does not affect the asymptotic growth rate of the solution of (1.1) when the total measure is finite; indeed, the entire mass of μ could be concentrated at 0 since the ordinary differential equation

$$y'(t) = \int_{\mathbb{R}^+} \mu(ds) \cdot f(y(t))$$

for $t \geq 0$ also obeys

$$F(y(t))/t \longrightarrow \int_{\mathbb{R}^+} \mu(ds)$$

as $t \to \infty$. This is in contrast to the linear case where the growth rate crucially depends upon the structure of the memory, cf., [**11,** Theorem 7.2.3]. In [**7**], we also show that, if $\lim_{t\to\infty} M(t) = \infty$, then

$$\lim_{t\to\infty} F(x(t))/t = \infty.$$

This result suggests that allowing the total measure to be infinite makes the long run dynamics more sensitive to the memory but that comparison with a non-autonomous ordinary differential equation may be necessary in this case.

In order to achieve precise asymptotic results for the solutions of (1.1) and (1.4), we extensively employ the theory of regular variation. We now record for the reader's convenience the definition of a regularly varying function (in the sense of Karamata) and allied notation.

Definition 1.1. Suppose that a measurable function $h : \mathbb{R} \to (0, \infty)$ obeys

$$\lim_{x\to\infty} \frac{h(\lambda\, x)}{h(x)} = \lambda^\rho, \quad \text{for all } \lambda > 0, \text{ some } \rho \in \mathbb{R}.$$

Then, h is *regularly varying at infinity* with index ρ, or $h \in \mathrm{RV}_\infty(\rho)$.

Regular variation provides a natural generalization of the class of power functions, and the application of the theory of regular variation to the study of qualitative properties of differential equations is an active area of investigation. Recent research themes in this direction are recorded in reviews such as [**13, 15**], and all properties of regularly varying functions employed may be found in the classic text [**9**]. Herein, the authors give a highly abridged list of the properties found useful in the introduction to our work [**5**], which concerns ordinary differential equations.

Many applications of regular variation in the asymptotic theory of *linear* Volterra equations deal with the situation in which it is desired to capture slow decay in the memory, as captured by a measure or kernel, or a singularity. Of course, slowly fading memory may be described in other ways, using, for instance, the theory of L^1 weighted spaces (see, e.g.,[**17**] and, for stochastic equations, [**8**]). When the kernel is integrable, it is often possible to obtain precise rates of decay in L^∞ by means of a larger class of kernels (such as the subexponential class studied in [**3**], of which regularly varying kernels are a subclass). However, for singular equations, or equations with non-integrable kernels, the full power of the theory of regular variation is often needed; in particular, for linear equations, transform methods and the Abelian and Tauberian theorems for regular variation are exploited (see, e.g.,[**4, 18**]). It should be stressed, however, that such methods are of greatest utility for linear equations; indeed, there does not seem to be especial benefit gained in this work in applying such a transform approach. Moreover, in this paper, the equation is intrinsically non-linear: $f(x)$ is not of linear order as $x \to \infty$, and regular variation arises *both* in the slow decay of μ and in the sublinear growth of f. Also, it is a general theme of the works cited above that the slow decay in the memory, combined with an appropriate type of stability, gives rise to convergence at a certain rate to equilibrium. By contrast, in this paper, solutions grow rather than decay.

With a view to applications, we believe the most interesting subclass of equations will retain the property that the asymptotic contribution to the growth rate from a moving interval of any fixed duration ($\tau > 0$, say) is negligible, in the sense that

$$\lim_{t\to\infty} \int_{[t,t+\tau)} \mu(ds) = 0 \quad \text{for each } \tau > 0. \tag{1.8}$$

It should be noted that our proofs *do not* require this stipulation; however, we mention it in order to motivate shortly a stronger hypothesis on M.

With (1.8) still in force, if μ is absolutely continuous and admits a non-negative and continuous density k such that $\mu(ds) = k(s)\,ds$, we see that $k \notin L^1(0,\infty)$ since

$$M(t) \longrightarrow \infty \quad \text{as } t \to \infty.$$

In particular, the property (1.8) is implied by $k(t) \to 0$ as $t \to \infty$. Therefore, it is perfectly possible for k to lie in another L^p space, for some $p > 1$. As an example, suppose that

$$k(t) \sim t^{-\theta} \quad \text{as } t \to \infty \text{ for } \theta \in (0,1).$$

Then, for $p > 1/\theta > 1$, $k \in L^p(0,\infty)$, while $k \notin L^1(0,\infty)$. In this sense, our work shares concerns with existing results in the literature in which the Volterra equation does not possess an integrable kernel, see e.g., **[12, 17]**.

The type of fading memory property (1.8) we suggested was of interest motivates a stronger assumption on M. First, we see that (1.8) implies

$$\frac{1}{n\tau}M(n\tau) = \frac{1}{\tau}\frac{1}{n}\sum_{j=0}^{n-1} \int_{[j\tau,j\tau+\tau)} \mu(ds) \longrightarrow 0 \quad \text{as } n \to \infty,$$

and thus, the non-negativity of μ implies that

$$M(t)/t \longrightarrow 0 \quad \text{as } t \to \infty.$$

Since $M(t) \to \infty$ as $t \to \infty$, M is non-decreasing, and $M(t)/t \to 0$ as $t \to \infty$, it is reasonable to suppose that $M \in \mathrm{RV}_\infty(\theta)$ for $\theta \in [0,1]$. We note that the inclusion of $\theta = 1$ in the parameter range does not lead to any problems in the analysis, and indeed, it transpires that our arguments are valid for all $\theta \geq 0$.

Analogously, the nonlinearity f is a positive and asymptotically increasing function such that

$$f(x) \longrightarrow \infty$$

and

$$f(x)/x \longrightarrow 0$$

as $x \to \infty$; hence, it is natural to assume that $f \in \mathrm{RV}_\infty(\beta)$ for $\beta \in [0,1)$. We can rule out some choices of the parameter β rapidly: if $\beta > 1$,

$$f(x)/x \to \infty \quad \text{as } x \to \infty,$$

and, if $\beta < 0$, f is asymptotic to a decreasing function. When $\beta = 0$, we append the hypotheses of asymptotic monotonicity and increase to infinity on f, as these are not necessarily satisfied by functions in $\text{RV}_\infty(0)$; however, otherwise, the analysis is essentially the same as when $\beta \in (0,1)$. The exclusion of the case $\beta = 1$ is largely on technical grounds: informally, when $\beta = 1$, the inverse of the increasing function F defined by (1.7) is *no longer regularly varying*; F^{-1} now belongs to the class of rapidly varying functions (which are defined below). It may also be seen from the nature of our results that the asymptotic behavior of solutions must be of a different form from those that hold when $\beta < 1$. For $\beta < 1$, no such technical problem arises, and indeed, F^{-1} is regularly varying with index $1/(1-\beta)$.

In some situations, we will consider very rapidly growing forcing terms H in the perturbed equation (1.6) which are not regularly varying. We sometimes consider forcing terms from the class of *rapidly varying functions*, and a definition of this class follows.

Definition 1.2. Suppose that a measurable function $h : \mathbb{R} \to (0,\infty)$ obeys for $\lambda > 0$:

$$\lim_{x\to\infty} \frac{h(\lambda x)}{h(x)} = \begin{cases} 0 & \lambda < 1, \\ 1 & \lambda = 1, \\ +\infty & \lambda > 1. \end{cases}$$

Then, h is *rapidly varying at infinity*, or $h \in \text{RV}_\infty(\infty)$. If, on the other hand,

$$h : \mathbb{R} \longrightarrow (0,\infty)$$

obeys for $\lambda > 0$:

$$\lim_{x\to\infty} \frac{h(\lambda x)}{h(x)} = \begin{cases} +\infty & \lambda < 1, \\ 1 & \lambda = 1, \\ 0 & \lambda > 1. \end{cases}$$

Then, we write $h \in \text{RV}_\infty(-\infty)$.

The proof of our main result for (1.1), Theorem 2.2, principally relies upon comparison methods, properties of regularly varying functions and a time change argument for delay differential equations. We first use constructive comparison methods, similar in spirit to those employed by Appleby and Buckwar **[2]** for linear equations, to establish "crude" upper and lower bounds on the solution of (1.1). The more challenging construction is that of the lower bound and is completed by comparing solutions of (1.1) with those of a related nonlinear pantograph equation using time change arguments inspired by Brunner and Maset [**10**]. Finally, we prove a convolution lemma for regularly varying functions, cf., [**1,** Theorem 3.4], which is then used, in conjunction with straightforward comparison methods, to sharpen the aforementioned "crude" upper and lower bounds and show that they coincide. Another paper which uses similar iterative methods to sharpen estimates of the growth of solutions to nonlinear convolution Volterra equations is that of Schneider [**16**].

With

$$\overline{M}(t) := \int_0^t M(s)\,ds,$$

we obtain

$$\lim_{t\to\infty} F(x(t))/\overline{M}(t) = \Lambda(\beta,\theta),$$

or that the growth rate of solutions of (1.1) depend explicitly upon both indices of regular variation, and therefore, upon the memory of the system (Theorem 2.2). The value of the parameter-dependent limit Λ can be explicitly determined in terms of the Γ function. This result is only valid for $\beta \in [0,1)$, and hence, may not hold if f is only assumed to be sublinear, i.e.,

$$\lim_{x\to\infty} f(x)/x = 0.$$

In this sense, it appears that the imposition of the hypothesis of regular variation on f and M is intrinsic to the form of the asymptotic behavior deduced, rather than being a purely technical contrivance, and the restriction to $\beta \neq 1$ also seems justified by grounds other than the complexity of the analysis needed to prove a sharp result.

The results and methods outlined above for (1.1) can also be used to yield sharp asymptotics for the perturbed equation (1.4). If H is positive, solutions to (1.4) will be positive and exhibit unbounded growth; therefore, there is no need to assume pointwise positivity of h. However, solutions of (1.4) are no longer necessarily non-decreasing, and more delicate comparison techniques are required to treat this additional difficulty.

When H is of the same order of magnitude as the solution of (2.11), we establish non-trivial upper and lower bounds on the solution and then employ a simple fixed point iteration argument to calculate the exact asymptotic growth rate of the solution in terms of a characteristic equation (Theorem 3.1). Moreover, the converse also holds; growth in the solution of (1.4) at a rate proportional to that of the solution of (2.11) is possible only when H is of the same order as that solution. In these results, the parameter θ characterizes the dependence of the growth rate upon the degree of memory in the system. When the perturbation term grows sufficiently quickly, the solution tracks H asymptotically, in the sense that

$$\lim_{t\to\infty} x(t)/H(t) = 1,$$

even when H is allowed to be highly non-monotone. Indeed, under certain restrictions, we can show that our characterization of rapid growth in the perturbation is necessary in order for $\lim_{t\to\infty} x(t)/H(t) = 1$ to prevail.

2. Main results and discussion. The following equivalence relation on the space of nonnegative continuous functions and shorthand are used throughout.

Definition 2.1. Suppose that a, $b \in C(\mathbb{R}^+;\mathbb{R}^+)$. Terms a and b are asymptotically equivalent if

$$\lim_{t\to\infty} a(t)/b(t) = 1;$$

we often write $a(t) \sim b(t)$ as $t \to \infty$ for short.

μ is a non-negative Borel measure on $\mathbb{R}^+$ with infinite total variation; more precisely,

$$\mu(E) \geq 0 \text{ for all } E \in \mathcal{B}(\mathbb{R}^+), \tag{2.1}$$
$$\int_{[0,\infty)} \mu(ds) = \lim_{t\to\infty} M(t) = \infty,$$

where M is defined as in (1.2). Our first result gives precise information on the asymptotic growth rate of the solution to (1.1). We state our result before carefully analyzing the conclusion. The proof is deferred to Section 5.

Theorem 2.2. *Suppose that the measure* μ *obeys* (2.1) *with* $M \in \mathrm{RV}_\infty(\theta)$, $\theta \geq 0$, *and that* $f \in \mathrm{RV}_\infty(\beta)$, $\beta \in [0,1)$. *When* $\beta = 0$, *let* f *be asymptotically increasing and obey* $\lim_{x\to\infty} f(x) = \infty$. *Then the solution* x *of* (1.1) *satisfies* $x \in \mathrm{RV}_\infty((1+\theta)/(1-\beta))$ *and*

$$\lim_{t\to\infty} \frac{F(x(t))}{\overline{M}(t)} = \frac{\Gamma(\theta+1)\Gamma(1+\beta\theta/1-\beta)}{\Gamma(1+\theta/1-\beta)} =: \Lambda(\beta,\theta), \tag{2.2}$$

where

$$\Gamma(x) := \int_0^\infty t^{x-1}e^{-t}\,dt \tag{2.3}$$

and

$$\overline{M}(t) := \int_0^t M(s)\,ds. \tag{2.4}$$

By Karamata's theorem, see Theorem 5.1 or [**9,** Theorem 1.5.11],

$$\lim_{t\to\infty} \overline{M}(t)/tM(t) = 1/(1+\theta).$$

Hence, the conclusion of Theorem 2.2 is equivalent to

$$\begin{aligned}\lim_{t\to\infty} \frac{F(x(t))}{t\,M(t)} &= (1+\theta)\frac{\Gamma(\theta+1)\Gamma((1+\beta\theta)/(1-\beta))}{\Gamma((1+\theta)/(1-\beta))}\\ &= \frac{1}{1-\beta}B\left(\theta+1, \frac{1+\theta\beta}{1-\beta}\right),\end{aligned}$$

where B denotes the Beta function, defined by

$$B(x,y) := \int_0^1 \lambda^{x-1}(1-\lambda)^{y-1}d\lambda,$$

cf., [**14,** page 142]. Furthermore, since $F^{-1} \in \mathrm{RV}_\infty(1/(1-\beta))$, (2.2) is also equivalent to

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t\,M(t))} = \left\{\frac{1}{1-\beta}B\left(\theta+1, \frac{1+\theta\beta}{1-\beta}\right)\right\}^{1/(1-\beta)}. \tag{2.5}$$

The next proposition records some properties of the function $\Lambda(\beta,\theta)$ which are useful when interpreting Theorem 2.2.

Proposition 2.3. *Suppose $\Lambda(\beta,\theta)$ is defined by (2.2) with $\beta \in [0,1)$ and $\theta \in [0,\infty)$. Then*

(i) $\Lambda(0,\theta) = 1$ *for fixed* $\theta \in (0,\infty)$ *and* $\Lambda(\beta,0) = 1$ *for fixed* $\beta \in (0,1)$,
(ii) $\lim_{\beta\uparrow 1}\Lambda(\beta,\theta) = 0$ *for fixed* $\theta \in (0,\infty)$ *and* $\lim_{\theta\to\infty}\Lambda(\beta,\theta) = 0$ *for fixed* $\beta \in (0,1)$,
(iii) $\beta \mapsto \Lambda(\beta,\theta)$ *is decreasing,* $\beta \in (0,1)$, θ *(fixed)* $\in (0,\infty)$,
(iv) $\theta \mapsto \Lambda(\beta,\theta)$ *is decreasing,* $\theta \in (0,\infty)$, β *(fixed)* $\in (0,1)$,
(v) $\Lambda(\beta,\theta) \in (0,1)$ *for* $\beta \in (0,1)$ *and* $\theta \in (0,\infty)$.

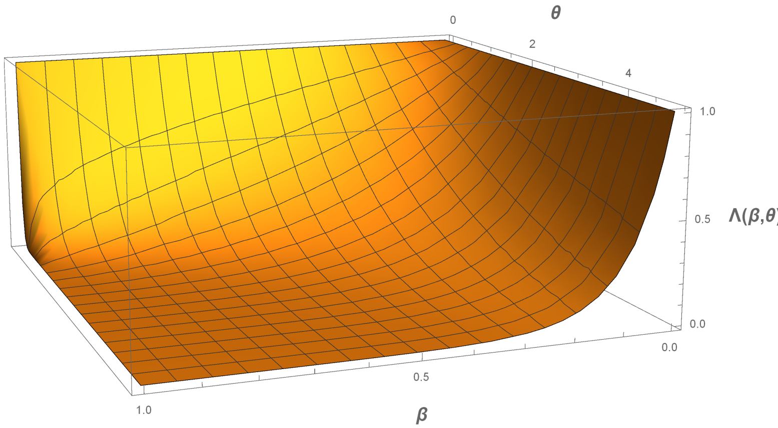

FIGURE 1. Plot of the surface $\Lambda(\beta,\theta)$ with $\beta \in [0,1)$ and $\theta \in [0,5]$.

In [**7**], the authors proved the following result which is closely related to Theorem 2.2 and whose statement is included to aid the ensuing discussion.

Theorem 2.4 ([**7,** Theorem 4])**.** *Suppose that μ is a nonnegative, finite Borel measure on $\mathbb{R}^+$, i.e.,*

$$\int_{\mathbb{R}^+} \mu(ds) = M \in (0,\infty),$$

and that there exists $\phi \in \mathcal{S}$ such that $f \in C((0,\infty);(0,\infty))$ obeys $f(x) \sim \phi(x)$ as $x \to \infty$, where

$$\mathcal{S} = \Big\{\phi \in C^1((0,\infty);(0,\infty)) : \lim_{x\to\infty} \phi'(x) = 0 \text{ and } \phi'(x) > 0 \text{ for each } x \in (0,\infty)\Big\}. \tag{2.6}$$

Then the solution x of (1.1) *obeys*

$$\lim_{t\to\infty} \frac{F(x(t))}{Mt} = 1. \tag{2.7}$$

Note that the hypotheses of Theorem 2.4 imply that f is sublinear in the sense that

$$f(x)/x \longrightarrow 0 \quad \text{as } x \to \infty.$$

If $f \in \mathrm{RV}_\infty(\beta)$ for $\beta \in (0,1)$, the hypotheses of Theorem 2.4 regarding f are satisfied and, furthermore, (2.7) is equivalent to

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(Mt)} = 1. \tag{2.8}$$

If $\theta = 0$, Theorem 2.2 yields

$$\lim_{t\to\infty} F(x(t))/\overline{M}(t) = 1,$$

or equivalently,

$$\lim_{t\to\infty} F(x(t))/tM(t) = 1$$

(Proposition 2.3 (i)). Therefore, we may reasonably think of Theorem 2.2 as a continuous extension of Theorem 2.4 to the case when μ is allowed to have infinite total variation.

If we only require that f be asymptotic to a function in $\mathcal{S}$, (2.8) implies (2.7), but the converse does not hold in general [**7,** Proposition 1]. We thus caution that "direct asymptotic information" regarding solutions (relations of the form (2.5) and (2.8)) is not always available and that "implicit asymptotic information" (relations of the form

(2.2) and (2.7)) is in many cases the best that may be achieved (cf., [**7,** Theorem 5] and [**6**]); this consideration motivates our initial formulation of Theorem 2.2.

In order to precisely highlight the technical difficulties introduced by allowing $M(t) \to \infty$ as $t \to \infty$, we now compare the proofs of Theorems 2.2 and 2.4. First, consider Theorem 2.4; since f is (asymptotically) monotone and $M < \infty$, it is straightforward to show that

$$\limsup_{t\to\infty} \frac{F(x(t))}{Mt} \le 1$$

using a comparison argument closely related to the classic Bihari inequality for nonlinear ordinary differential equations. The corresponding inferior limit is more challenging to establish and requires careful use of the hypothesis (2.6). First, using the *finiteness* of μ, the lower bound analysis of the Volterra problem may be reduced to studying a finite delay (lower) comparison equation of the form

$$\widetilde{x}'(t) = (1-\epsilon)\, M\, \phi(\widetilde{x}(t - T(\epsilon))), \qquad t > T(\epsilon) > 0; \quad \epsilon \in (0,1), \quad f \sim \phi \in \mathcal{S}, \tag{2.9}$$

with suitable initial data. This comparison will obviously be unavailable when we allow $M(t) \to \infty$ as $t \to \infty$. Now, with sublinearity naturally playing a crucial role, we establish that

$$\lim_{t\to\infty} \phi(\widetilde{x}(t-\alpha))/\phi(\widetilde{x}(t)) = 1 \quad \text{for each } \alpha > 0.$$

The auxiliary finite delay equation (2.9), and hence the Volterra equation, are thus proven to have the desired asymptotics.

For the upper bound on the growth rate, the proof of Theorem 2.2 begins in a similar manner to that of Theorem 2.4 and may be thought of as imitating the Bihari approach (with some refinement via properties of regularly varying functions). However, in contrast to the proof of Theorem 2.4, these comparison methods now only yield the correct *upper* order of magnitude of the solution; in other words, we obtain

$$\limsup_{t\to\infty} \frac{F(x(t))}{\overline{M}(t)} < C,$$

for some positive constant C. In order to obtain more precise asymptotics, we then apply a convolution lemma for regularly varying functions (Lemma 5.2) to the initial upper estimate above; this yields the improved estimate

$$\limsup_{t\to\infty} F(x(t))/\overline{M}(t) \le \Lambda(\beta,\theta),$$

cf., equation (2.2).

In proving Theorem 2.2, we cannot reduce the proof of the corresponding limit inferior to the study of a bounded delay equation such as (2.9). Instead, after employing a time change argument to simplify the nonautonomous structure of (1.3), we use a proportional delay equation of the form

$$\widetilde{x}'(t) = C\phi(\widetilde{x}(qt)), \quad t > T > 0; \tag{2.10}$$
$$q = 2^{-(2+\theta)}, \quad C > 0, \quad f \sim \phi \in \mathrm{RV}_\infty(\beta),$$

as our lower comparison equation. By contrast, when $M < \infty$, no such time change is necessary since the equation is asymptotically autonomous. The lower comparison equation (2.10) allows us to prove that

$$\liminf_{t\to\infty} x(t)/F^{-1}(tM(t)) > 0,$$

and once more we recycle our estimates (via Lemma 5.2) to show that our upper and lower bounds coincide, completing the argument.

It is also possible to glean additional insight from Theorem 2.2 by studying the second order effects of the parameters β and θ on the solution. For a fixed $\beta \in (0,1)$, a decrease in the value of θ represents an increase in the rate of decay of the measure μ. This can be made precise by supposing that the measure μ is absolutely continuous, and specifically that $\mu(ds) = m(s)\,ds$ for continuous $m \in \mathrm{RV}_\infty(\theta-1)$, $\theta \in (0,1)$. Therefore, increasing the value of θ gives more weight to values of the solution in the past (stronger memory), and we expect the growth rate of solutions of (1.1) to be slower than that of the related ordinary differential equation

$$y'(t) = M(t)f(y(t)), \quad t > 0;\ y(0) = \xi > 0. \tag{2.11}$$

The equation (2.11), in contrast, places the entire weight $M(t)$ at the present time, when the solution is largest. Hence, increasing the value of θ (putting more weight further into the past) slows the growth rate, and it is intuitive that $\Lambda(\beta,\theta)$ is decreasing in θ. Using this comparison with (2.11) once more, it is clear that Proposition 2.3 (v) must hold since solutions of (1.1) can never grow faster than those of (2.11) (if f is strictly increasing, this can be seen by inspection).

For a fixed $\theta \in (0,\infty)$, one might expect an increase in β to lead to a faster rate of growth of the solution of (1.1). Therefore, it may initially be surprising that $\Lambda(\beta,\theta)$ is decreasing in β. This counter-intuitive result is best understood by explaining the error introduced in the approximation of the right-hand side of (1.1). From (1.1),

$$\begin{aligned} x'(t) &= \int_{[0,t]} \mu(ds) f(x(t-s)) \\ &= \int_{[0,t]} \mu(ds) \frac{f(x(t-s))}{f(x(t))} f(x(t)), \quad t>0. \end{aligned}$$

The error of our upper bound on the solution is proportional to the ratio $f(x(t-s))/f(x(t))$ for $s \in (0,t)$, or $f(x(\lambda t))/f(x(t))$ for $\lambda \in (0,1)$. Since $f \circ x \in \text{RV}_\infty(\beta(1+\theta)/(1-\beta))$,

$$\lim_{t\to\infty} \frac{f(x(\lambda t))}{f(x(t))} = \lambda^{[\beta(1+\theta)]/(1-\beta)} =: \gamma(\beta).$$

When $\gamma(\beta)$ is close to one, the solution of (1.1) is close to that of (2.11), and hence, our estimate is sharp. However, $\gamma(\beta)$ is decreasing and

$$\lim_{\beta\uparrow 1} \gamma(\beta) = 0.$$

Thus, the zero limit as $\beta \uparrow 1$ in Proposition 2.3 (ii) represents the fact that the solution of (2.11) increases much faster in β than the solution to (1.1), for a fixed value of θ.

3. Results for perturbed Volterra equations. A result is presented here which illustrates how our precise understanding of the asymptotics of solutions of (1.1) can be applied to perturbed versions of the equation, such as (1.4). This result applies to perturbations of (1.1) which are of the same, or smaller, order of magnitude as solutions of the ordinary differential equation (2.11). Our assumptions on H guarantee that

$$\lim_{t\to\infty} x(t) = \infty,$$

but this limit is no longer necessarily achieved monotonically, and this is reflected in the added complexity of certain technical aspects of the proofs. The proofs of the results in this section are largely deferred to Section 5.

Theorem 3.1. *Suppose that the measure* μ *obeys* (2.1) *with* $M \in \mathrm{RV}_\infty(\theta)$*,* $\theta \geq 0$*, and that* $f \in \mathrm{RV}_\infty(\beta)$*,* $\beta \in [0,1)$*. When* $\beta = 0$*, let* f *be asymptotically increasing and obey*

$$\lim_{x\to\infty} f(x) = \infty.$$

Let x *denote the solution of* (1.4)*, and suppose that* $H \in C((0,\infty);(0,\infty))$*. Then, the following are equivalent*:

(i) $\lim_{t\to\infty} x(t)/(F^{-1}(t\,M(t))) = \zeta \in [L,\infty)$,
(ii) $\lim_{t\to\infty} H(t)/(F^{-1}(t\,M(t))) = \lambda \in [0,\infty)$,

where

$$L = \left\{ B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right)/(1-\beta)\right\}^{1/(1-\beta)},$$

and moreover,

$$\zeta = \frac{\zeta^\beta}{1-\beta}\, B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right) + \lambda. \tag{3.1}$$

We notice that, when there is a sufficiently slowly growing forcing term H, $\lambda = 0$, we exactly recover from (3.1) the asymptotic behavior of the unperturbed equation, given by (2.5). Also, in the limit as $\lambda \to 0^+$, the rate of the unperturbed equation is recovered.

Condition (ii) on H in Theorem 3.1 does not cover the case when H is of larger magnitude than the solution of the unperturbed equation (1.1), or that of (2.11). To deal with this case, we would like to know the growth rate of the solution when $\lim_{t\to\infty} H(t)/F^{-1}(t\,M(t)) = \infty$. Insight into what occurs can be gained by sending $\lambda \to \infty$ in Theorem 3.1. For $\lambda > 0$, from Theorem 3.1, we have

$$\lim_{t\to\infty} \frac{x(t)}{H(t)} = \frac{\zeta(\lambda)}{\lambda} =: \eta(\lambda),$$

where ζ depends on λ through (3.1). Since $\zeta = \zeta(\lambda)$ is the unique positive solution of (3.1), $\eta = \eta(\lambda)$ is the unique positive solution of

$\eta = 1 + K\eta^\beta \lambda^{\beta-1}$, where $K > 0$ is the λ-independent positive quantity

$$K = \frac{1}{1-\beta}\, B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right).$$

Clearly $\eta(\lambda) > 1$, and

$$\lambda \longmapsto \eta(\lambda)$$

is in C^1, by the implicit function theorem. Moreover, by implicit differentiation, $\eta'(\lambda)$ obeys

$$\eta'(\lambda)\left\{1 - \beta\frac{\eta(\lambda)-1}{\eta(\lambda)}\right\} = K(\beta-1)\eta(\lambda)^\beta \lambda^{\beta-2}.$$

Therefore, as the bracket on the left-hand side is positive,

$$\lambda \longmapsto \eta(\lambda)$$

is decreasing. Hence, for $\lambda > 1$, we have

$$\eta(\lambda) < 1 + K\eta(1)^\beta \lambda^{\beta-1};$$

thus,

$$\limsup_{\lambda\to\infty} \eta(\lambda) \le 1,$$

and furthermore,

$$\eta(\lambda) \longrightarrow 1 \quad \text{as } \lambda \to \infty.$$

In view of this discussion, we might expect that

$$\lim_{t\to\infty} H(t)/F^{-1}(t\, M(t)) = \infty$$

implies $x(t) \sim H(t)$ as $t \to \infty$, or less precisely, that sufficiently rapid growth in H forces $x(t)$ to grow at the rate $H(t)$. Therefore, it is natural to ask under what conditions we would have $x(t) \sim H(t)$ as $t \to \infty$. It is straightforward to show that a necessary condition for

$$\lim_{t\to\infty} x(t)/H(t) = 1$$

is that

$$\lim_{t\to\infty} \int_0^t \frac{M(t-s) f(H(s))\, ds}{H(t)} = 0.$$

This motivates the hypothesis

$$\lim_{t\to\infty} \frac{M(t)\int_0^t f(H(s))\,ds}{H(t)} = 0, \tag{3.2}$$

and the following result. This result requires no monotonicity in H and, as such, allows for H to undergo considerable fluctuation, a point we illustrate further in Section 4.

Theorem 3.2. *Suppose the measure* μ *obeys* (2.1) *with* $M \in \mathrm{RV}_\infty(\theta)$, $\theta \geq 0$, *and that* $f \in \mathrm{RV}_\infty(\beta)$, $\beta \in [0,1)$. *When* $\beta = 0$, *let* f *be asymptotically increasing and obey* $\lim_{x\to\infty} f(x) = \infty$. *Let* H *be a function in* $C((0,\infty);(0,\infty))$ *satisfying* (3.2). *Then, the solution,* x, *of* (1.1) *obeys* $\lim_{t\to\infty} x(t)/H(t) = 1$.

When H regularly varies at infinity the hypotheses of Theorems 3.1 and 3.2 align to give a complete classification of the asymptotics (Corollary 3.3). However, assuming regular variation of H imposes considerable regularity constraints. In particular, H is then asymptotic to an increasing function, and this restricts potential applications of Theorem 3.2 to stochastic functional differential equations.

Corollary 3.3. *Let* $M \in \mathrm{RV}_\infty(\theta)$, $\theta \geq 0$, *with* $\lim_{t\to\infty} M(t) = \infty$. *Suppose that* $f \in \mathrm{RV}_\infty(\beta)$, $\beta \in [0,1)$. *When* $\beta = 0$, *let* f *be asymptotically increasing and obey* $\lim_{x\to\infty} f(x) = \infty$. *If* $H \in \mathrm{RV}_\infty(\alpha)$, $\alpha > 0$, *then the following are equivalent*:

(i) $\lim_{t\to\infty} M(t)\int_0^t f(H(s))\,ds/H(t) = 0$,
(ii) $\lim_{t\to\infty} H(t)/F^{-1}(tM(t)) = \infty$,
(iii) $\lim_{t\to\infty} \int_0^t M(t-s)f(H(s))\,ds/H(t) = 0$.

Case $\alpha = 0$ is excluded from Corollary 3.3 since it is covered by Theorem 3.1 with $\lambda = 0$.

We now state without proof a partial converse to Theorem 3.2 with $H \in \mathrm{RV}_\infty(\alpha)$, $\alpha > 0$. The proof follows from Corollary 3.3 and estimation arguments similar to those used throughout this paper.

Theorem 3.4. *Suppose that the measure* μ *obeys* (2.1) *with* $M \in \mathrm{RV}_\infty(\theta)$, $\theta \geq 0$, *and that* $f \in \mathrm{RV}_\infty(\beta)$, $\beta \in [0,1)$. *When* $\beta = 0$, *let f be asymptotically increasing and obey* $\lim_{x\to\infty} f(x) = \infty$. *Let x denote the solution of* (1.4),

$$H \in C((0,\infty);(0,\infty)) \cap \mathrm{RV}_\infty(\alpha)$$

with $\alpha > 0$. Then, the following are equivalent:

(i)

$$\lim_{t\to\infty} \frac{M(t)\int_0^t f(H(s))\,ds}{H(t)} = 0,$$

(ii)

$$\lim_{t\to\infty} \frac{x(t)}{H(t)} = 1.$$

While discussing the hypothesis that

$$\lim_{t\to\infty} \frac{H(t)}{F^{-1}(tM(t))} = \infty$$

in the context of regular variation, it is worth remarking that this hypothesis is also satisfied for $H \in \mathrm{RV}_\infty(\infty)$, the so-called rapidly varying functions, see [**9,** page 83]. If $H \in \mathrm{RV}_\infty(\infty)$, then (3.2) holds and Theorem 3.2 can be applied; this fact is recorded in the next corollary.

Corollary 3.5. *Suppose that the measure μ obeys* (2.1) *with $M \in \mathrm{RV}_\infty(\theta)$, $\theta \geq 0$, and that $f \in \mathrm{RV}_\infty(\beta)$, $\beta \in [0,1)$. When $\beta = 0$, let f be asymptotically increasing and obey*

$$\lim_{x\to\infty} f(x) = \infty.$$

Let $x(t)$ denote the solution of (1.4), *and suppose*

$$H \in C((0,\infty);(0,\infty)) \cap \mathrm{RV}_\infty(\infty)$$

is asymptotically increasing. Then, $\lim_{t\to\infty} x(t)/H(t) = 1$.

Corollary 3.5 also holds if $H \in MR_\infty(\infty)$, a sub-class of $\mathrm{RV}_\infty(\infty)$, see [**9,** page 68] for the definition of $MR_\infty(\infty)$ since this guarantees that H is asymptotic to an increasing function (see [**9,** page 83]).

4. Examples.

4.1. Application of Theorem 2.2. The main attraction of Theorem 2.2 is that it largely reduces the asymptotic analysis of solutions of (1.1) to the computation, or asymptotic analysis, of the function F^{-1}. Under the appropriate hypotheses, Theorem 2.2 yields

$$x(t) \sim F^{-1}(t\,M(t))\left\{\frac{1}{1-\beta}B\left(\theta+1,\frac{\theta\beta+1}{1-\beta}\right)\right\}^{1/(1-\beta)}, \quad \text{as } t\to\infty.$$

In general, exact computation of F^{-1} in closed form is impossible. The following result provides the asymptotics of F^{-1} for a large class of $f \in \text{RV}_\infty(\beta)$ for $\beta \in [0,1)$ using some classical results from the theory of regular variation. Its principal appeal is that it may be applied by calculating the limit of a readily-computed function which can be found directly in terms of f, without the need for integration.

Proposition 4.1. *Suppose that $f \in \text{RV}_\infty(\beta)$, $\beta \in [0,1)$, is continuous and that*

$$\ell(x) := \big(f(x)/x^\beta\big)^{1/(1-\beta)}$$

obeys

$$\lim_{x\to\infty}\frac{\ell(x\,\ell(x))}{\ell(x)} = 1. \tag{4.1}$$

Then,

$$F(x) \sim \frac{1}{1-\beta}\frac{x}{f(x)}, \tag{4.2}$$

$$F^{-1}(x) \sim (1-\beta)^{1/(1-\beta)}\ell(x^{1/(1-\beta)})x^{1/(1-\beta)}, \quad \textit{as } x\to\infty. \tag{4.3}$$

The proof of Proposition 4.1 is a simple application of standard results using de Bruijn conjugates and is thus omitted, see [**9,** Theorem 1.5.15, Corollary 2.3.4].

The following examples illustrate the convenience of Proposition 4.1 in practice.

Example 4.2. Suppose that $f(x) \sim a\,x^\beta \log\log(x^\alpha)$ as $x \to \infty$ with $\beta \in [0,1)$, $a > 0$ and $\alpha > 0$. In this case,

$$\ell(x) \sim (a\,\log\log(x^\alpha))^{1/(1-\beta)}, \quad \text{as } x \to \infty.$$

It is straightforward to show that (4.1) holds; therefore, applying Proposition 4.1 yields

$$F^{-1}(x) \sim (1-\beta)^{1/(1-\beta)}\{a \log\log(x^{\alpha/(1-\beta)})\}^{1/(1-\beta)} x^{1/(1-\beta)}, \text{ as } x \to \infty.$$

This example is also valid with $\log\log(x)$ replaced by

$$\prod_{i=1}^{n} \log_{i-1}(x),$$

where $\log_i(x) = \log\log_{i-1}(x)$.

Example 4.3. Suppose that $f(x) \sim x^\beta(2 + \sin(\log\log(x)))$ as $x \to \infty$, with $\beta \in (0,1)$. In this case,

$$\ell(x) \sim (2 + \sin(\log\log(x)))^{1/(1-\beta)}, \quad \text{as } x \to \infty.$$

Once more, the verification of (4.1) is left to the reader. In this case, Proposition 4.1 yields

$$F^{-1}(x) \sim (1-\beta)^{1/(1-\beta)}\{2 + \sin(\log\log(x^{1/(1-\beta)}))\}^{1/(1-\beta)} x^{1/(1-\beta)}, \quad \text{as } x \to \infty.$$

4.2. Discrete measures. It may appear that our inclusion of a general measure μ in (1.1) and the hypothesis that the integral of μ is regularly varying are only compatible when μ is an absolutely continuous measure. The following proposition allows us to easily construct examples to show that our results also cover a variety of equations involving discrete measures.

Proposition 4.4. *Let $x \ge 0$ and δ_x be the Dirac measure at x on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$. Suppose that $\theta \in (0,1)$ and that $\mu_0 \in \mathrm{RV}_\infty(\theta - 1)$. Let $\tau > 0$ and*

$$\mu(ds) = \sum_{j=0}^{\lfloor t/\tau \rfloor} \mu_0(j\tau)\delta_{j\tau}(ds). \tag{4.4}$$

Hence,

$$M(t) = \int_{[0,t]} \mu(ds) = \sum_{j=0}^{\lfloor t/\tau \rfloor} \mu_0(j\tau), \tag{4.5}$$

and $M \in \mathrm{RV}_\infty(\theta)$*. Furthermore,*

$$M(t) \sim \widetilde{M}(t) := \int_0^t \widetilde{\mu}(s)\, ds \quad \text{as } t \to \infty,$$

where $\widetilde{\mu} \in \mathrm{RV}_\infty(\theta - 1)$ *is any* C^1*, a decreasing function such that* $\mu_0(s) \sim \widetilde{\mu}(s)$ *as* $s \to \infty$*.*

In the next example, we illustrate the application of our results to equations involving discrete measures.

Example 4.5. Using the notation of Proposition 4.4, suppose that

$$x'(t) = \sum_{j=0}^{\lfloor t/\tau \rfloor} \mu_0(j\tau) f(x(t-j\tau)) + \int_0^t \mu_1(s) f(x(t-s))\, ds, \quad t > 0, \tag{4.6}$$

where m is given by

$$m(E) = \int_E \mu_1(s)\, ds$$

for any Borel set $E \subset [0, \infty)$ an absolutely continuous measure. Therefore,

$$\mu(ds) = \sum_{j=0}^{\infty} \mu_0(j\tau)\delta_{j\tau}(ds) + \mu_1(s)\, ds. \tag{4.7}$$

If $\mu_0 \in \mathrm{RV}_\infty(\theta - 1)$ and $\mu_1 \in \mathrm{RV}_\infty(\alpha)$, then $M \in \mathrm{RV}_\infty(\max(\theta, \alpha + 1))$. Suppose that $\theta > \alpha + 1$ for the purposes of this example. Thus, by Proposition 4.4,

$$M(t) \sim \sum_{j=0}^{\lfloor t/\tau \rfloor} \mu_0(j\tau)$$

as $t \to \infty$, and choose

$$\mu_0(x) \sim \log \frac{x+1}{1+x}^{1-\theta} =: \widetilde{\mu}(x), \quad \theta \in (0, 1),$$

where the asymptotic relation holds as $x \to \infty$. Hence, $\widetilde{\mu} \in \text{RV}_\infty(\theta-1)$, and it follows that

$$\widetilde{M}(t) = \frac{1}{\theta}(t+1)^\theta \log(t+1) - \frac{1}{\theta^2}((t+1)^\theta - 1) \sim \frac{t^\theta}{\theta} \log(t), \quad \text{as } t \to \infty.$$

It is now straightforward to deduce the asymptotic behavior of the solution to (4.6) for $f \in \text{RV}_\infty(\beta)$ with $\beta \in [0,1)$, from Theorem 2.2.

4.3. Perturbed equations and application of Theorems 3.1 and 3.2. Using a parametrized example, we illustrate how the asymptotic behavior of solutions of (1.4) can be classified using the results of Section 3.

Example 4.6. For ease of exposition, suppose that $\beta \in (0,1)$, and let

$$\begin{aligned} f(x) &= x^\beta, \quad x \geq 0; \\ M(t) &= (1+t)^\theta - 1, \quad t \geq 0; \\ H(t) &= (1+t)^\alpha\, e^{\gamma t} - 1, \quad t \geq 0, \end{aligned}$$

with $\theta > 0$, $\alpha \in \mathbb{R}$, and $\gamma \geq 0$.

Case (i). $\gamma = 0$. In this case, $H \in \text{RV}_\infty(\alpha)$ and

$$\frac{H(t)}{F^{-1}(t\,M(t))} \sim (1-\beta)^{1/(\beta-1)} t^{\alpha-(\theta+1)/(1-\beta)}, \quad \text{as } t \to \infty.$$

If $\alpha < (\theta+1)/(1-\beta)$, then

$$\lim_{t\to\infty} \frac{H(t)}{F^{-1}(t\,M(t))} = 0,$$

and Theorem 3.1 yields the limit

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t\,M(t))} = L,$$

where

$$L = \left\{ B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right)/(1-\beta) \right\}^{1/(1-\beta)}.$$

If $\alpha = (\theta + 1)/(1 - \beta)$, then

$$\lim_{t\to\infty} \frac{H(t)}{F^{-1}(t\,M(t))} = (1-\beta)^{1/(\beta-1)} =: \lambda,$$

and Theorem 3.1 gives

$$\lim_{t\to\infty} \frac{x(t)}{F^{-1}(t\,M(t))} = \zeta,$$

where ζ satisfies (3.1).

Finally, if $\alpha > (\theta + 1)/(1 - \beta)$, then

$$\lim_{t\to\infty} H(t)/F^{-1}(t\,M(t)) = \infty.$$

Then, by Corollary 3.3, (3.2) holds and Theorem 3.2 yields $\lim_{t\to\infty} x(t)/H(t) = 1$.

Case (ii). $\gamma > 0$. In this case, $H \in \mathrm{RV}_\infty(\infty)$, and Corollary 3.5 immediately gives $\lim_{t\to\infty} x(t)/H(t) = 1$ for all $\alpha \in \mathbb{R}$, $\beta \in (0,1)$ and $\theta > 0$.

Specifically with a view to applications to stochastic functional differential equations, it is pertinent to highlight when H is required to have some form of monotonicity in the results of Section 3. When $\lambda = 0$ in Theorem 3.1, there is no monotonicity requirement on H, but $\lambda > 0$ implies that H asymptotic to the monotone increasing function F^{-1}, modulo a constant. By contrast, Theorem 3.2 allows for large "fluctuations," or irregular behavior, in H; the following examples illustrate this point.

Example 4.7. Suppose that $f \in \mathrm{RV}_\infty(\beta)$, $\beta \in (0,1)$, $M \in \mathrm{RV}_\infty(\theta)$, $\theta \geq 0$ and

$$H(t) = (1+t)^\alpha\,(2+\sin(t)) - 2, \quad \alpha > 0.$$

From Karamata's theorem,

$$\limsup_{t\to\infty} \frac{M(t)\int_0^t f(H(s))\,ds}{H(t)} = \limsup_{t\to\infty} \frac{M(t)\int_0^t f((1+s)^\alpha\,(2+\sin(s)-2))\,ds}{(1+t)^\alpha\,(2+\sin(t))-2}$$

$$\leq \limsup_{t\to\infty} \frac{(1+\epsilon)M(t)\int_0^t \phi(3\,s^\alpha)\,ds}{t^\alpha}.$$

Since

$$\frac{M(t)\int_0^t \phi(3\,s^\alpha)\,ds}{t^\alpha} \sim \frac{M(t)\,t\,f(3\,t^\alpha)}{(1+\alpha\beta)t^\alpha}, \quad \text{as } t\to\infty,$$

a sufficient condition for (3.2) to hold, and hence for Theorem 3.2 to apply, is $\alpha > (1+\theta)/(1-\beta)$. Even more rapid variation in H is permitted; for example, let $H(t) = e^t(2+\sin(t)) - 2$. In this case, asymptotic monotonicity of f and the rapid variation of e^t yield

$$\limsup_{t\to\infty} \frac{M(t)\int_0^t f(H(s))\,ds}{H(t)} \leq \limsup_{t\to\infty} \frac{M(t)\,t\,f(3e^t)}{e^t} = 0,$$

and, once more, Theorem 3.2 applies to yield $x(t) \sim H(t)$ as $t\to\infty$, where x is the solution to (1.6). By fixing $f(x) = x^\beta$, we can immediately see that it is possible to capture more general types of exponentially fast oscillation using Theorem 3.2. Choose $H(t) = e^{\sigma(t)\,t}$, where $\sigma(t)$ obeys

$$0 < \sigma_- \leq \sigma(t) \leq \sigma_+ < \infty$$

for all $t \geq 0$, for some constants σ_- and σ_+. Checking condition (3.2), we have

$$\limsup_{t\to\infty} \frac{M(t)\int_0^t f(H(s))\,ds}{H(t)} \leq \limsup_{t\to\infty} \frac{M(t)\,t\,e^{\beta\sigma_+ t}}{e^{\sigma_- t}}.$$

Thus, Theorem 3.2 applies and $x(t) \sim H(t)$ as $t\to\infty$ if $\sigma_- > \beta\sigma_+$.

5. Proofs of results. In the following proofs, we often choose to work with a monotone function approximating f; this monotone approximation will be denoted by ϕ. If f is regularly varying with a positive index, then

there exists a $\phi \in C^1((0,\infty);(0,\infty)) \cap C(\mathbb{R}^+,(0,\infty))$ such that $f(x) \sim \phi(x)$ and $\phi'(x) > 0$ for all $x > 0$,

by [**9,** Theorems 1.3.1, Theorem 1.5.13]. It is immediate that, if f is regularly varying and asymptotic to ϕ, then ϕ is also regularly varying with the same index. If $f \in \mathrm{RV}_\infty(0)$, we assume that a ϕ satisfying (5.1) exists since only a smooth, but not necessarily monotone, approximation is guaranteed in this case. The function $F(x)$ is approximated by

$$\Phi(x) := \int_1^x du/\phi(u)$$

and Φ^{-1} is the inverse function of Φ. If $f(x) \sim \phi(x)$ as $x \to \infty$, it follows trivially that $F(x) \sim \Phi(x)$ and $F^{-1}(x) \sim \Phi^{-1}(x)$, as $x \to \infty$. Since it is used frequently in our arguments, we now state a useful version of Karamata's theorem.

Theorem 5.1 (Karamata's theorem)**.** *If $\phi \in \mathrm{RV}_\infty(\beta)$ is locally bounded on $[X,\infty)$ for some $X \in \mathbb{R}^+$, then*

$$\lim_{x\to\infty} \frac{x^{\sigma+1}\phi(x)}{\int_X^x t^\sigma \phi(t)\,dt} = \sigma+\beta+1, \quad \textit{for each } \sigma \ge -(1+\beta).$$

The proof of Theorem 2.2 is decomposed into the following lemmata, the first of which provides a precise estimate on the asymptotics of the convolution of two regularly varying functions.

Lemma 5.2. *Suppose that $a \in \mathrm{RV}_\infty(\rho)$ and $b \in \mathrm{RV}_\infty(\sigma)$, where $\rho \ge 0$ and $\sigma \ge 0$, and $\lim_{t\to\infty} a(t) = \infty$. If $\sigma = 0$, let b be asymptotically increasing and obey $\lim_{t\to\infty} b(t) = \infty$. Then,*

$$\lim_{t\to\infty} \frac{\int_0^t a(s)b(t-s)\,ds}{t\,a(t)\,b(t)} = \int_0^1 \lambda^\rho (1-\lambda)^\sigma d\lambda =: B(\rho+1,\sigma+1),$$

where B denotes the Beta function.

Proof. Let ϵ, $\eta \in (0,1/2)$ be arbitrary. Define

$$\begin{aligned}
I(t) &:= \int_0^t a(s)b(t-s)\,ds \\
&= \int_0^{\epsilon t} a(s)\,b(t-s)\,ds \\
&\quad + \int_{\epsilon t}^{(1-\eta)t} a(s)b(t-s)\,ds + \int_{(1-\eta)t}^t a(s)b(t-s)\,ds \\
&=: I_1(t) + I_2(t) + I_3(t).
\end{aligned} \tag{5.1}$$

By making the substitution $s = \lambda t$,

$$\frac{I_2(t)}{t\,a(t)\,b(t)} = \frac{\int_{\epsilon t}^{(1-\eta)t} a(s)b(t-s)\,ds}{t\,a(t)\,b(t)} = \int_{\epsilon}^{1-\eta} \frac{a(\lambda t)}{a(t)}\frac{b(t(1-\lambda))}{b(t)}\,d\lambda.$$

By the uniform convergence theorem for regularly varying functions, see [**9,** Theorem 1.5.2], it follows that

$$\lim_{t\to\infty}\frac{I_2(t)}{t\,a(t)\,b(t)} = \int_{\epsilon}^{1-\eta} \lambda^{\rho}(1-\lambda)^{\sigma}\,d\lambda. \tag{5.2}$$

Since both a and b are positive functions, it is clear that $I(t) \geq I_2(t)$, and hence,

$$\liminf_{t\to\infty}\frac{I(t)}{t\,a(t)\,b(t)} \geq \int_{\epsilon}^{1-\eta} \lambda^{\rho}(1-\lambda)^{\sigma}\,d\lambda.$$

Letting η and $\epsilon \to 0^+$ then yields

$$\liminf_{t\to\infty}\frac{I(t)}{t\,a(t)\,b(t)} \geq \int_{0}^{1} \lambda^{\rho}(1-\lambda)^{\sigma}\,d\lambda. \tag{5.3}$$

By hypothesis, an increasing C^1 function β exists such that $b(t)/\beta(t) \to 1$ as $t \to \infty$. It follows that $T_1 > 0$ exists such that $t \geq T_1$ implies $b(t)/\beta(t) \leq 2$. Therefore, with $\epsilon \in (0,(1/2))$, $t \geq 2T_1$, we have that $(1-\epsilon)t \geq T_1$. Suppose that $t \geq 2T_1$, and estimate as:

$$I_1(t) = \int_0^{\epsilon t} a(s)\,b(t-s)\,ds \leq 2\beta(t)\int_0^{\epsilon t} a(s)\,ds = 2\beta(t)\,\epsilon\,t\,a(\epsilon t)\,\frac{\int_0^{\epsilon t} a(s)\,ds}{\epsilon\,t\,a(\epsilon t)}.$$

Hence, for $t \geq 2T_1$,

$$\frac{I_1(t)}{t\,a(t)\,b(t)} \leq 2\epsilon\,\frac{\beta(t)}{b(t)}\,\frac{a(\epsilon t)}{a(t)}\,\frac{\int_0^{\epsilon t} a(s)\,ds}{\epsilon\,t\,a(\epsilon t)}.$$

$a \in \mathrm{RV}_\infty(\rho)$ implies that $\lim_{t\to\infty} a(\epsilon t)/a(t) = \epsilon^{\rho}$, and similarly, by Karamata's theorem,

$$\lim_{t\to\infty}\int_0^{\epsilon t} a(s)\,ds/\epsilon\,t\,a(\epsilon t) = 1/(1+\rho).$$

Thus,

$$\limsup_{t\to\infty}\frac{I_1(t)}{t\,a(t)\,b(t)} \leq \frac{2\epsilon^{\rho+1}}{1+\rho}. \tag{5.4}$$

Finally, consider $I_3(t)$. By construction, $t \geq T_1$ implies $b(t)/\beta(t) \leq 2$, and since b, β are continuous and positive, with β bounded away from zero,

$$\sup_{0\leq t\leq T_1} \frac{b(t)}{\beta(t)} = \max_{0\leq t\leq T_1} \frac{b(t)}{\beta(t)} := B_1 < \infty.$$

Thus, there exists a $B_2 > 0$ such that $b(t) \leq B_2\,\beta(t)$ for all $t \geq 0$. Therefore,

$$\begin{aligned} I_3(t) = \int_{(1-\eta)t}^{t} a(s)\,b(t-s)\,d &\leq B_2 \int_{(1-\eta)t}^{t} a(s)\,\beta(t-s)\,ds \\ &\leq B_2\beta(\eta t) \int_{(1-\eta)t}^{t} a(s)\,ds. \end{aligned}$$

Hence,

$$\begin{aligned} \limsup_{t\to\infty} \frac{I_3(t)}{t\,a(t)\,b(t)} &\leq B_2 \limsup_{t\to\infty} \frac{\beta(\eta t)}{b(t)} \limsup_{t\to\infty} \frac{\int_{(1-\eta)t}^{t} a(s)\,ds}{t\,a(t)} \\ &= B_2\,\eta^\sigma \limsup_{t\to\infty} \frac{\int_{(1-\eta)t}^{t} a(s)\,ds}{t\,a(t)}. \end{aligned} \tag{5.5}$$

The final limit on the right-hand side of (5.5) is calculated once more by calling upon the uniform convergence theorem for regularly varying functions

$$\lim_{t\to\infty} \frac{\int_{(1-\eta)t}^{t} a(s)\,ds}{t\,a(t)} = \lim_{t\to\infty} \int_{1-\eta}^{1} \frac{a(\lambda t)}{a(t)}\,d\lambda = \int_{1-\eta}^{1} \lambda^\rho d\lambda.$$

Returning to (5.5),

$$\limsup_{t\to\infty} \frac{I_3(t)}{t\,a(t)\,b(t)} \leq B_2\,\eta^\sigma \int_{1-\eta}^{1} \lambda^\rho d\lambda = B_2\,\eta^\sigma \left(\frac{1}{\rho+1} - \frac{(1-\eta)^{\rho+1}}{\rho+1} \right). \tag{5.6}$$

Therefore, combining (5.2), (5.4) and (5.6), we obtain

$$\limsup_{t\to\infty} \frac{I(t)}{t\,a(t)\,b(t)} \leq 2\epsilon^{\rho+1} \frac{1}{1+\rho} + \int_{\epsilon}^{1-\eta} \lambda^\rho (1-\lambda)^\sigma d\lambda + B_2\,\eta^\sigma \int_{1-\eta}^{1} \lambda^\rho d\lambda.$$

Letting η and $\epsilon \to 0^+$ in the above then yields

$$\limsup_{t\to\infty} \frac{I(t)}{t\,a(t)\,b(t)} \le \int_0^1 \lambda^\rho (1-\lambda)^\sigma \, d\lambda. \tag{5.7}$$

Combining (5.7) with (5.3) gives the desired conclusion. □

The proof of Theorem 2.2 now begins in earnest by proving a "rough" lower bound on the solution which we will later refine. Lemmas 5.3, 5.4 and 5.5 are all proven under the same set of hypotheses and are presented separately purely for readability and clarity.

Lemma 5.3. *Suppose the measure μ obeys* (2.1) *with $M \in \mathrm{RV}_\infty(\theta)$, $\theta \ge 0$, and that $f \in \mathrm{RV}_\infty(\beta)$, $\beta \in [0,1)$. If $\beta = 0$, let f be asymptotically increasing and obey $\lim_{x\to\infty} f(x) = \infty$. Then the unique continuous solution x of* (1.1) *obeys*

$$\liminf_{t\to\infty} \frac{x(t)}{F^{-1}(t\,M(t))} > 0.$$

Proof. Let $\epsilon \in (0,1)$ be arbitrary. By hypothesis, a ϕ exists such that (5.1) holds, and hence, an $x_1(\epsilon) > 0$ exists such that $f(x) > (1-\epsilon)\phi(x)$ for all $x > x_1(\epsilon)$. Furthermore, $T_0(\epsilon) > 0$ exists such that $t \ge T_0$ implies $x(t) > x_1(\epsilon)$. Similarly, there exists a $T_1(\epsilon) > 0$ such that $M(t) > 0$ for all $t \ge T_1$. Since $M \in \mathrm{RV}_\infty(\theta)$, a C^1 function M_1 exists such that, for all $\epsilon \in (0,1)$, there exists a $T_2(\epsilon) > 0$ such that, for all $t \ge T_2$, $M(t) > (1-\epsilon)M_1(t)$. Let $T_3 := T_0 + T_1 + T_2$. Hence, for $t \ge 4T_3$, estimate as follows:

$$\begin{aligned}
x'(t) &= \int_{[0,t-T_3]} \mu(ds) f(x(t-s)) + \int_{(t-T_3,t]} \mu(ds) f(x(t-s)) \\
&\ge (1-\epsilon) \int_{[0,t-T_3]} \mu(ds)\phi(x(t-s)) \\
&= (1-\epsilon) \int_{[0,(t-T_3)/2]} \mu(ds)\phi(x(t-s)) \\
&\quad + (1-\epsilon) \int_{((t-T_3)/2,t-T_3]} \mu(ds)\phi(x(t-s)) \\
&\ge (1-\epsilon) \int_{[0,(t-T_3)/2]} \mu(ds)\phi(x(t-s))
\end{aligned}$$

$$\geq (1-\epsilon)M\left(\frac{1}{2}(t-T_3)\right)\phi\left(x\left(\frac{1}{2}(t+T_3)\right)\right).$$

Since $M \in \mathrm{RV}_\infty(\theta)$, $\lim_{t\to\infty} M((t-T_3)/2)/M(t-T_3) = 2^{-\theta}$. Thus, a positive constant C and a time $\widetilde{T_3} \geq 4T_3$ exist such that

$$x'(t) \geq C\,M(t-T_3)\phi\left(x\left(\frac{1}{2}(t+T_3)\right)\right) \quad \text{for all } t \geq \widetilde{T_3}. \tag{5.8}$$

Furthermore, since $t \geq \widetilde{T_3}$ implies $t - T_3 > T_2$, there exists a $C_0 > 0$ such that

$$x'(t) \geq C_0\,M_1(t-T_3)\phi\left(x((t+T_3)/2)\right), \quad \text{for all } t \geq \widetilde{T_3}. \tag{5.9}$$

Now define the C^2, positive, increasing function

$$\overline{M}_1(t) := \int_0^t M_1(s)\,ds \quad \text{for } t \geq 0.$$

Let

$$\alpha(t) := \widetilde{M}_1^{-1}(t) + T_3, \quad t \geq \widetilde{M}_1(\widetilde{T_3}). \tag{5.10}$$

For $t \geq \widetilde{M}_1(\widetilde{T_3})$,

$$\alpha(t) \geq \alpha(\widetilde{M}_1(\widetilde{T_3})) = \widetilde{T_3} + T_3 > \widetilde{T_3}$$

since α is increasing. Define $\widetilde{x}(t) := x(\alpha(t))$ for $t \geq \widetilde{M}_1(\widetilde{T_3})$. Note that $\widetilde{x} \in C^1([\widetilde{M}_1(\widetilde{T_3}),\infty);(0,\infty))$ and $\alpha'(t) = 1/M_1(\widetilde{M}_1^{-1}(t))$. For $t \geq \widetilde{M}_1(\widetilde{T_3})$, use (5.9) to compute

$$\begin{aligned}\widetilde{x}'(t) &= \alpha'(t)x'(\alpha(t))\\ &\geq \frac{C_0\,M_1(\alpha(t)-T_3)}{M_1(\widetilde{M}_1^{-1}(t))}\,\phi\left(x\left(\frac{1}{2}(\alpha(t)+T_3)\right)\right) = C_0\,\phi\left(x\left(\frac{1}{2}(\alpha(t)+T_3)\right)\right).\end{aligned} \tag{5.11}$$

Define $\tau(t) = t - \widetilde{M}_1(\widetilde{M}_1^{-1}(t)/2) > 0$, for $t \geq \widetilde{M}_1(\widetilde{T_3})$. It follows that $(\alpha(t)+T_3)/2 = \alpha(t-\tau(t))$. Hence, for $t \geq \widetilde{M}_1(\widetilde{T_3})$,

$$\widetilde{x}'(t) \geq C_0\,\phi\left(x\left(\frac{1}{2}(\alpha(t)+T_3)\right)\right) = C_0\,\phi(x(\alpha(t-\tau(t))) = C_0\,\phi(\widetilde{x}(t-\tau(t))). \tag{5.12}$$

For $t \geq \widetilde{M}_1(\widetilde{T_3})$, it is straightforward to show, using the monotonicity of $\widetilde{M}_1$, that $\tau(t) > 0$. Using that $\widetilde{M}_1 \in \mathrm{RV}_\infty(\theta+1)$, we have

$$\lim_{t\to\infty} \frac{t-\tau(t)}{t} = \lim_{t\to\infty} \frac{\widetilde{M}_1((1/2)\widetilde{M}_1^{-1}(t))}{\widetilde{M}_1(\widetilde{M}_1^{-1}(t))} = \lim_{t\to\infty} \frac{\widetilde{M}_1((1/2)\widetilde{M}_1^{-1}(t))}{\widetilde{M}_1(\widetilde{M}_1^{-1}(t))} = \left(\frac{1}{2}\right)^{\theta+1}.$$

It follows that a $T_4 > 0$ exists such that, for all $t \geq T_4$,

$$-\tau(t) > 2^{-(\theta+2)}\, t$$

for all $\epsilon > 0$ sufficiently small. Letting $T_5 := \max(T_4, \widetilde{M}_1(\widetilde{T_3}))$ we have, for $t \geq T_5$

$$\widetilde{x}'(t) \geq C_0\, \phi(\widetilde{x}(qt)), \quad q = 2^{-(\theta+2)} \in (0,1). \tag{5.13}$$

The following estimates will be needed to define a lower comparison solution. Since $\phi \circ \Phi^{-1}$ is in $\mathrm{RV}_\infty(\beta/(1-\beta))$ we have

$$\lim_{x\to\infty} \frac{(\phi\circ\Phi^{-1})(x/q)}{(\phi\circ\Phi^{-1})(x)} = \left(\frac{1}{q}\right)^{\beta/(1-\beta)}.$$

Thus, there exists $x_2 > 0$ such that for all $x \geq x_2$

$$\frac{(\phi\circ\Phi^{-1})(x/q)}{(\phi\circ\Phi^{-1})(x)} < 2\left(\frac{1}{q}\right)^{\beta/(1-\beta)}.$$

Next, let $T_5' > 0$ be so large that $\Phi(\widetilde{x}(qT_5')) - x_2 > 0$, and set $T_6 := \max(T_5, T_5') + 1$. Then,

$$\Phi(\widetilde{x}(qT_6)) > \Phi(\widetilde{x}(qT_5')) > x_2.$$

Define

$$c := \min\left(C_0\, \frac{q^{\beta/(1-\beta)}}{4}, \frac{\Phi(\widetilde{x}(qT_6)) - x_2}{2T_6(1-q)}, \frac{\Phi(\widetilde{x}(qT_6))}{2T_6} \right) \tag{5.14}$$

and

$$d := cT_6 - \Phi(\widetilde{x}(qT_6)). \tag{5.15}$$

Then, define $x_0 := cqT_6 - d = \Phi(\widetilde{x}(qT_6)) - cT_6(1-q) > x_2$. Note that $d < 0$ due to (5.14). Therefore, $1/q - 1 > 0$ and, for any $x \geq x_0$,

$x/q + (1/q - 1)d < x/q$. Hence, for $x \geq x_0$,

(5.16)
$$\frac{(\phi \circ \Phi^{-1})((x/q) + (1/q - 1)\,d)}{(\phi \circ \Phi^{-1})(x)} \leq \frac{(\phi \circ \Phi^{-1})(x/q)}{(\phi \circ \Phi^{-1})(x)} < 2\left(\frac{1}{q}\right)^{\beta/(1-\beta)}.$$

Letting $t = (x + d)/cq$ in (5.16) and noting that (5.14) implies $C_0/c \geq 4(1/q)^{\beta/(1-\beta)}$, we have

$$\text{(5.17)} \qquad \frac{(\phi \circ \Phi^{-1})(ct - d)}{(\phi \circ \Phi^{-1})(cqt - d)} < 2\left(\frac{1}{q}\right)^{\beta/(1-\beta)} < \frac{C_0}{c}, \quad \text{for all } t \geq T_6.$$

Define the lower comparison solution x_- by

$$\text{(5.18)} \qquad x_-(t) = \Phi^{-1}(ct - d), \quad t \geq qT_6.$$

Then, for $t \in [qT_6, T_6]$, by the monotonicity of Φ^{-1} and (5.15),

$$x_-(t) \leq x_-(T_6) = \Phi^{-1}(cT_6 - d) = \widetilde{x}(qT_6) \leq \widetilde{x}(t).$$

Also, $x_-(T_6) = \widetilde{x}(qT_6) < \widetilde{x}(T_6)$ since $\widetilde{x}$ is increasing. Hence,

$$\text{(5.19)} \qquad x_-(t) < \widetilde{x}(t), \quad t \in [qT_6, T_6].$$

Next, since $\Phi(x_-(t)) = ct - d$, for $t \geq T_6$,

$$\begin{aligned} x_-'(t) &= c\,(\phi \circ \Phi^{-1})(ct - d) = c\,\phi(x_-(t)) \\ &= \frac{c}{C_0}\,\frac{\phi(x_-(t))}{\phi(x_-(qt))}\,C_0\,\phi(x_-(qt)). \end{aligned}$$

Now, for $t \geq T_6$, by (5.17),

$$\frac{c}{C_0}\,\frac{\phi(x_-(t))}{\phi(x_-(qt))} = \frac{c}{C_0}\,\frac{(\phi \circ \Phi^{-1})(ct - d)}{(\phi \circ \Phi^{-1})(cqt - d)} < \frac{c}{C_0}\,\frac{C_0}{c} = 1.$$

Thus,

$$\text{(5.20)} \qquad x_-'(t) < C_0\,\phi(x_-(qt)), \quad t \geq T_6.$$

Recalling (5.13), $\widetilde{x}'(t) \geq C_0\,\phi(\widetilde{x}(qt))$ for all $t \geq T_6 > T_5$. Then, by (5.19) and (5.20), since ϕ is increasing, $\widetilde{x}(t) > x_-(t)$ for all $t \geq qT_6$. In order to see this, suppose that there is a minimal $t_0 > T_6$ such that $x_-(t_0) = \widetilde{x}(t_0)$. Thus, $x_-'(t_0) \geq \widetilde{x}'(t_0)$ and $x_-(t_0) < \widetilde{x}(t)$ for all $t \in [qT_6, t_0)$. Then, since $t_0 > T_6$ and $qt_0 > qT_6$, ϕ increasing yields

$$\widetilde{x}'(t_0) \geq C_0\,\phi(\widetilde{x}(qt_0)) > C_0\,\phi(x_-(qt_0)) > x_-'(t_0) \geq \widetilde{x}'(t_0),$$

a contradiction. Now, for $t \geq qT_6$, $\widetilde{x}(t) > x_-(t) = \Phi^{-1}(ct-d)$. Hence, for $t \geq qT_6$

$$x(\alpha(t)) = \widetilde{x}(t) > \Phi^{-1}(ct-d).$$

From the definition of α, in (5.10), $\alpha^{-1}(t) = \overline{M}_1(t-T_3)$, and therefore,

$$\begin{aligned} x(t) &= \widetilde{x}(\alpha^{-1}(t)) > \Phi^{-1}(c\,\alpha^{-1}(t)-d) \\ &= \Phi^{-1}(c\,\overline{M}_1(t-T_3)-d), \quad \overline{M}_1(t-T_3) > qT_6. \end{aligned}$$

Hence, recalling that $d<0$,

$$\Phi(x(t)) > c\overline{M}_1(t-T_3)-d > c\overline{M}_1(t-T_3), \quad \overline{M}_1(t-T_3) > qT_6. \tag{5.21}$$

Note that, for $t > 2T_3$, $t/2 < t-T_3$. Since $\overline{M}_1$ is increasing, this implies that $\overline{M}_1(t/2) \leq \overline{M}_1(t-T_3)$. Thus, (5.21) implies

$$\liminf_{t\to\infty} \frac{\Phi(x(t))}{\overline{M}_1(t)} \geq \liminf_{t\to\infty} \frac{c\,\overline{M}_1(t/2)}{\overline{M}_1(t)} = c\,2^{-(\theta+1)} > 0.$$

By Karamata's theorem, $\lim_{t\to\infty} \overline{M}_1(t)/tM_1(t) = 1/(1+\theta)$, and therefore,

$$\liminf_{t\to\infty} \frac{\Phi(x(t))}{t\,M_1(t)} \geq c\,(1+\theta)\,2^{-(\theta+1)} > 0.$$

Finally, since $\Phi^{-1} \in \mathrm{RV}_\infty(1/(1-\beta))$ and M is asymptotic to M_1, we conclude that

$$\liminf_{t\to\infty} \frac{x(t)}{\Phi^{-1}(t\,M(t))} > 0,$$

as required. □

Lemma 5.4. *Suppose that the hypotheses of Lemma* 5.3 *hold. Then, the unique continuous solution x of* (1.1) *obeys*

$$\limsup_{t\to\infty} \frac{F(x(t))}{t\,M(t)} \leq \frac{1}{1-\beta} B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right).$$

Proof. Once again, let ϕ which satisfies (5.1) obey $f(x)/\phi(x) < (1+\epsilon)$ for all $x > x_1(\epsilon)$, for any $\epsilon > 0$ and for some $x_1(\epsilon) > 0$. Due to the fact that $\lim_{t\to\infty} x(t) = \infty$, a $T_1(\epsilon)$ exists such that $t \geq T_1(\epsilon)$ implies $x(t) > x_1(\epsilon)$. Since $\lim_{t\to\infty} M(t) = \infty$, there exists a $T_2(\epsilon)$ such that $M(t) > 0$ for all $t \geq T_2$. Hence, for all $t \geq 2\max(T_1, T_2)$, (1.3) becomes

$$\frac{x(t)}{\phi(x(t))} \le \frac{x(0)}{\phi(x(t))} + \frac{\int_0^{T_1} M(t-s) f(x(s))\, ds}{\phi(x(t))} + (1+\epsilon)\, t\, M(t), \tag{5.22}$$

where the upper bound on the term

$$\int_{T_1}^{t} M(t-s)\phi(x(s))\, ds$$

was obtained by exploiting the fact that $t \mapsto x(t)$ and $t \mapsto M(t)$ are non-decreasing. By Karamata's theorem and the regular variation of ϕ,

$$\lim_{x\to\infty} (1-\beta)\phi(x)\Phi(x)/x = 1$$

holds. Thus, for all $\epsilon > 0$, an $x_2(\epsilon)$ exists such that

$$\Phi(x) < \frac{(1+\epsilon)x}{(1-\beta)\phi(x)}, \quad \text{for all } x > x_2(\epsilon).$$

Once more, the divergence of $x(t)$ yields the existence of a $T_3(\epsilon)$ such that $x(t) > x_2(\epsilon)$ for all $t \ge T_3(\epsilon)$. Letting $T_4 = 2\max(T_1, T_2, T_3)$, we obtain

$$\frac{\Phi(x(t))}{t\, M(t)} < \frac{(1+\epsilon)x(t)}{(1-\beta)\phi(x(t))\, t\, M(t)} \quad \text{for all } t \ge T_4.$$

Combining the above estimate with (5.22) yields

$$\frac{\Phi(x(t))}{t\, M(t)} < \frac{(1+\epsilon)x(0)}{(1-\beta)\phi(x(t)) t M(t)} + \frac{(1+\epsilon)\int_0^{T_1} M(t-s) f(x(s))\, ds}{(1-\beta)\phi(x(t)) t M(t)} + \frac{(1+\epsilon)^2}{1-\beta}, \quad t \ge T_4(\epsilon).$$

Hence, letting $t \to \infty$ and then sending $\epsilon \to 0^+$, we obtain

$$\limsup_{t\to\infty} \frac{\Phi(x(t))}{t\, M(t)} \le \frac{1}{1-\beta}.$$

Since $\Phi^{-1} \in \mathrm{RV}_\infty(1/(1-\beta))$, the above estimate can be restated as

$$\limsup_{t\to\infty} \frac{x(t)}{\Phi^{-1}(t\, M(t))} \le (1-\beta)^{1/(\beta-1)} < \infty.$$

We now seek to refine the "crude" upper bound on the growth of the solution obtained above. From the above construction and Lemma 5.3, we may suppose that

$$\limsup_{t\to\infty} \frac{x(t)}{\Phi^{-1}(t\,M(t))} =: \eta \in (0,\infty). \tag{5.23}$$

From (5.23), it follows that, for all $\epsilon > 0$, a $T_5(\epsilon) > 0$ exists such that, for all $t \geq T_5(\epsilon)$, $x(t) < (\eta+\epsilon)\Phi^{-1}(t\,M(t))$. From the monotonicity of ϕ, it follows that

$$\frac{\phi(x(t))}{\phi(\Phi^{-1}(t\,M(t)))} < \frac{\phi((\eta+\epsilon)\Phi^{-1}(t\,M(t)))}{\phi(\Phi^{-1}(t\,M(t)))}, \quad t \geq T_5(\epsilon).$$

Since $\phi \in \mathrm{RV}_\infty(\beta)$,

$$\limsup_{t\to\infty} \frac{\phi(x(t))}{\phi(\Phi^{-1}(t\,M(t)))} \leq (\eta+\epsilon)^\beta.$$

Thus, for all $\epsilon > 0$, a $T_6(\epsilon) > 0$ exists such that, for all $t \geq T_6$,

$$\phi(x(t)) < (1+\epsilon)(\eta+\epsilon)^\beta \phi(\Phi^{-1}(t\,M(t))).$$

Integrating this estimate yields

$$\begin{aligned} &\int_{T_6}^t M(t-s)\phi(x(s))\,ds \\ &\quad \leq (1+\epsilon)(\eta+\epsilon)^\beta \int_{T_6}^t M(t-s)\phi(\Phi^{-1}(s\,M(s)))\,ds, \quad t \geq T_6(\epsilon). \end{aligned} \tag{5.24}$$

Since $(\phi\circ\Phi^{-1})(t\,M(t)) \in \mathrm{RV}_\infty(\beta(1+\theta)/(1-\beta))$ and $M \in \mathrm{RV}_\infty(\theta)$, Lemma 5.2 can be applied to obtain

$$\lim_{t\to\infty} \frac{\int_0^t M(t-s)\phi(\Phi^{-1}(s\,M(s)))\,ds}{t\,M(t)\,\phi(\Phi^{-1}(t\,M(t)))} = B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right). \tag{5.25}$$

Hence, combining (5.24) and (5.25) yields

$$\limsup_{t\to\infty} \frac{\int_{T_6}^t M(t-s)\phi(x(s))\,ds}{t\,M(t)\,\phi(\Phi^{-1}(t\,M(t)))} \leq (1+\epsilon)(\eta+\epsilon)^\beta B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right).$$

Apply the above estimate to (1.3) as:

$$\eta = \limsup_{t\to\infty} \frac{x(t)}{\Phi^{-1}(t\,M(t))} \leq \limsup_{t\to\infty} \frac{\int_0^{T_6} M(t-s)f(x(s))\,ds}{\Phi^{-1}(t\,M(t))}$$

$$
\begin{aligned}
&+ \limsup_{t\to\infty} \frac{(1+\epsilon)\int_{T_6}^t M(t-s)\phi(x(s))\,ds}{\Phi^{-1}(t\,M(t))} \\
&\le (1+\epsilon)^2(\eta+\epsilon)^\beta B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right) \limsup_{t\to\infty} \frac{t\,M(t)\,\phi(\Phi^{-1}(t\,M(t)))}{\Phi^{-1}(t\,M(t))} \\
&= (1+\epsilon)^2(\eta+\epsilon)^\beta B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right) \limsup_{x\to\infty} \frac{x\,\phi(\Phi^{-1}(x))}{\Phi^{-1}(x)}.
\end{aligned}
$$

Letting $\epsilon \to 0^+$ and using Karamata's theorem to the remaining limit on the right-hand side,

$$
\eta^{1-\beta} = \limsup_{y\to\infty} \frac{\Phi(y)\phi(y)}{y} B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right) = \frac{1}{1-\beta} B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right),
$$

with $y = \Phi^{-1}(x)$ so that $y \to \infty$ as $x \to \infty$. Thus,

$$
\eta = \limsup_{t\to\infty} \frac{x(t)}{\Phi^{-1}(t\,M(t))} \le \left\{ \frac{1}{1-\beta} B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right) \right\}^{1/(1-\beta)}.
$$

Using $\Phi \in \mathrm{RV}_\infty(1-\beta)$ and $\Phi(x) \sim F(x)$ as $x \to \infty$, the above upper bound can be reformulated as

$$
\limsup_{t\to\infty} \frac{F(x(t))}{t\,M(t)} \le \frac{1}{1-\beta} B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right),
$$

which is the required estimate. □

Lemma 5.5. *Suppose that the hypotheses of Lemma* 5.3 *hold. Then, the unique continuous solution x of* (1.1) *obeys*

$$
\liminf_{t\to\infty} \frac{F(x(t))}{t\,M(t)} \ge \frac{1}{1-\beta} B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right).
$$

Proof. By Lemmas 5.3 and 5.4,

$$
\liminf_{t\to\infty} \frac{x(t)}{\Phi^{-1}(t\,M(t))} =: \eta \in (0,\infty).
$$

Then, for all $\epsilon \in (0,\eta) \cap (0,1)$, a $T_1(\epsilon) > 0$ exists such that, for all $t \ge T_1$,

$$
\eta - \epsilon < \frac{x(t)}{\Phi^{-1}(t\,M(t))}.
$$

Since $\lim_{t\to\infty} M(t) = \infty$, there exists a T_2 such that $M(t) > 0$ for all $t \geq T_2$. Hence,

$$x(t) > (\eta - \epsilon)\Phi^{-1}(t\,M(t)), \quad t \geq T_3 := \max(T_1, T_2). \tag{5.26}$$

Using monotonicity and regular variation of ϕ, it follows from (5.26) that

$$\liminf_{t\to\infty} \frac{\phi(x(t))}{(\phi\circ\Phi^{-1})(t\,M(t))} \geq (\eta-\epsilon)^\beta.$$

Now, since $\phi(x) \sim f(x)$ as $x \to \infty$, for all $\epsilon \in (0,\eta)\cap(0,1)$, a $T_4(\epsilon) > 0$ exists such that

$$f(x(t)) > (1-\epsilon)\phi(x(t)) > (1-\epsilon)^2(\eta-\epsilon)^\beta\,(\phi\circ\Phi^{-1})(t\,M(t)), \quad t \geq T_4(\epsilon).$$

Integration then yields

$$\int_0^t M(t-s)f(x(s))\,ds$$
$$> (1-\epsilon)^2(\eta-\epsilon)^\beta \int_{T_4}^t M(t-s)(\phi\circ\Phi^{-1})(s\,M(s))\,ds.$$

Hence, as in the proof of Lemma 5.4, applying Lemma 5.2 gives

(5.27)
$$\liminf_{t\to\infty} \frac{\int_0^t M(t-s)f(x(s))\,ds}{t\,M(t)\,(\phi\circ\Phi^{-1})(t\,M(t))} \geq (1-\epsilon)^2(\eta-\epsilon)^\beta\, B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right).$$

Now, apply the estimate from (5.27) to (1.3) as:

$$\begin{aligned}
\eta = \liminf_{t\to\infty} \frac{x(t)}{\Phi^{-1}(t\,M(t))} &\geq \liminf_{t\to\infty} \frac{\int_0^t M(t-s)f(x(s))\,ds}{\Phi^{-1}(t\,M(t))} \\
= (1-\epsilon)^2(\eta-\epsilon)^\beta\, B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right) &\liminf_{t\to\infty} \frac{t\,M(t)\,(\phi\circ\Phi^{-1})(t\,M(t))}{\Phi^{-1}(t\,M(t))} \\
= (1-\epsilon)^2(\eta-\epsilon)^\beta\, B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right) &\liminf_{x\to\infty} \frac{x\,\phi(\Phi^{-1}(x))}{\Phi^{-1}(x)}.
\end{aligned}$$

The limit of the final term on the right-hand side is $1/(1-\beta)$ by Karamata's theorem, and sending $\epsilon \to 0^+$ yields

$$\eta = \frac{\eta^\beta}{1-\beta} B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right).$$

Hence,

$$\liminf_{t\to\infty} \frac{x(t)}{F^{-1}(t\,M(t))} \geq \left\{\frac{1}{1-\beta}B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right)\right\}^{1/(1-\beta)}.$$

Since $F \in \mathrm{RV}_\infty(1-\beta)$, this can be rewritten in the form

$$\liminf_{t\to\infty} \frac{F(x(t))}{t\,M(t)} \geq \frac{1}{1-\beta}B\left(\theta+1, \frac{\theta\beta+1}{1-\beta}\right),$$

which is the desired bound. □

As with Theorem 2.2, the proof of Theorem 3.1 is split into a series of lemmata. A final consolidating argument then establishes the result, as stated in Section 2.

Lemma 5.6. *Suppose the measure* μ *obeys* (2.1) *with* $M \in \mathrm{RV}_\infty(\theta)$, $\theta \geq 0$ *and* $f \in \mathrm{RV}_\infty(\beta)$, $\beta \in [0,1)$. *If* $\beta = 0$, *let* f *be asymptotically increasing and obey* $\lim_{x\to\infty} f(x) = \infty$. *Let* $x(t)$ *denote the unique continuous solution of* (1.4), *and suppose* $H \in C((0,\infty);(0,\infty))$. *Then,*

(5.28)
$$\liminf_{t\to\infty} \frac{x(t)}{F^{-1}(t\,M(t))} \geq L := \left\{\frac{1}{1-\beta}B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right)\right\}^{1/(1-\beta)} > 0.$$

Proof. With $\epsilon \in (0,1)$ arbitrary and $T_0(\epsilon)$ and $T_1(\epsilon)$ defined as in Lemma 5.3, (1.4) admits the initial lower estimate

$$x(t) > x(0) + H(t) + (1-\epsilon)\int_T^t M(t-s)\phi(x(s))\,ds,$$
$$t \geq T(\epsilon) := T_0(\epsilon) + T_1(\epsilon).$$

Letting $y(t) = x(t+T)$ and noting that $H(t) > 0$ for $t > 0$, we obtain

$$\begin{aligned} y(t) &> x(0) + (1-\epsilon)\int_T^{t+T} M(t+T-s)\phi(x(s))\,ds \\ &= x(0) + (1-\epsilon)\int_0^t M(t-u)\phi(x(u+T))\,du \\ &= x(0) + (1-\epsilon)\int_0^t M(t-u)\phi(y(u))\,du, \quad t \geq T(\epsilon). \end{aligned}$$

Now, consider the comparison equation defined by

(5.29)
$$x_\epsilon'(t) = (1-\epsilon)\int_{[0,t]} \mu(ds)\phi(x_\epsilon(t-s)), \quad t>0,\ x_\epsilon(0)=x(0)/2.$$

In contrast to (1.4), the solution to (5.29) will be non-decreasing. Integrating (5.29) using Fubini's theorem yields

$$x_\epsilon(t) = x(0)/2 + (1-\epsilon)\int_0^t M(t-u)\phi(x_\epsilon(u))\,du, \quad t\ge 0.$$

By construction

$$x_\epsilon(t) < y(t) = x(t+T) \qquad\qquad \text{for all } t\ge 0,$$

or

$$x(t) > x_\epsilon(t-T) \qquad\qquad \text{for all } t\ge T.$$

Applying Theorem 2.2 to x_ϵ then yields

$$\lim_{t\to\infty}\frac{F(x_\epsilon(t))}{t\,M_\epsilon(t)} = \frac{1}{1-\beta}B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right),$$

where $M_\epsilon(t) = (1-\epsilon)M(t)$. Hence,

$$\lim_{t\to\infty}\frac{F(x_\epsilon(t))}{t\,M(t)} = \frac{1-\epsilon}{1-\beta}B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right).$$

Therefore,

$$\begin{aligned}\liminf_{t\to\infty}\frac{F(x(t))}{t\,M(t)} &\ge \liminf_{t\to\infty}\frac{F(x_\epsilon(t-T))}{t\,M(t)}\\ &= \liminf_{t\to\infty}\frac{F(x_\epsilon(t-T))}{(t-T)M(t-T)}\,\frac{(t-T)M(t-T)}{t\,M(t)}\\ &= \frac{1-\epsilon}{1-\beta}B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right),\end{aligned}$$

where the final equality follows from the trivial fact that $t-T\sim t$ as $t\to\infty$ and noting that M preserves asymptotic equivalence since $M\in \mathrm{RV}_\infty(\theta)$.

Finally, letting $\epsilon \to 0^+$ and using the regular variation of F^{-1} yields

$$\liminf_{t\to\infty} \frac{x(t)}{F^{-1}(t\,M(t))} \geq \left\{\frac{1}{1-\beta} B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right)\right\}^{1/(1-\beta)} = L,$$

which finishes the proof. □

Lemma 5.7. *Suppose that the hypotheses of Lemma* 5.6 *hold and*

$$\lim_{t\to\infty} H(t)/F^{-1}(t\,M(t)) = \lambda \in [0,\infty).$$

Then, with x denoting the unique continuous solution of (1.4),

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t\,M(t))} \leq U := \left(\frac{\lambda}{L^\beta} + \frac{1}{1-\beta}\right)^{1/(1-\beta)}, \tag{5.30}$$

where L is defined by (5.28).

Proof. We begin by constructing a monotone comparison solution which will majorize the solution of (1.4) and to which Lemma 5.6 can be applied. Let $\epsilon \in (0,1)$ be arbitrary, and define $T_1(\epsilon)$ and $T_2(\epsilon)$ as in the proof of Lemma 5.4.

By hypothesis,

$$\lim_{t\to\infty} H(t)/F^{-1}(t\,M(t)) = \lambda \in [0,\infty),$$

and thus, a $T(\epsilon) > 0$ exists such that $t \geq T(\epsilon)$ implies

$$H(t) < (\lambda+\epsilon)\Phi^{-1}(t\,M(t));$$

furthermore, $M \in \mathrm{RV}_\infty(\theta)$ implies that there exists an $M_1 \in C^1$ asymptotic to M and $T_0(\epsilon) > T$ such that

$$M(t) < (1+\epsilon)M_1(t) \quad \text{for all } t \geq T_0.$$

For $t \geq T_0$, since Φ^{-1} is increasing,

$$\Phi^{-1}(t\,M(t)) < \Phi^{-1}(t\,(1+\epsilon)\,M_1(t)),$$

and, since $\Phi^{-1} \in \mathrm{RV}_\infty(1/(1-\beta))$, a $T^* > T_0$ exists such that

$$\Phi^{-1}(t\,M(t)) < (1+\epsilon)^{(2-\beta)/(1-\beta)}\Phi^{-1}(t\,M_1(t))$$

for all $t \geq T^*$.

For notational convenience, define the quantity ϵ^* by letting

$$(1+\epsilon^*) := (1+\epsilon)^{(2-\beta)/(1-\beta)};$$

note that $(1+\epsilon^*) \to 1$ as $\epsilon \to 0^+$. Defining $T_2' := T^* + T_1 + T_2$, we have the estimate:

$$\begin{aligned} x(t) &< x(0) + H(t) + \int_0^{T_2'} M(t-s) f(x(s))\, ds \\ &\quad + (1+\epsilon)\int_{T_2'}^t M(t-s)\phi(x(s))\, ds \\ &\le x(0) + H(t) + M(t)\, T_2'\, F^* \\ &\quad + (1+\epsilon)\int_{T_2'}^t M(t-s)\phi(x(s))\, ds \\ &< x(0) + (\lambda+\epsilon)(1+\epsilon^*)\Phi^{-1}(t\, M_1(t)) \\ &\quad + (1+\epsilon) M_1(t)\, T_2'\, F^* + (1+\epsilon)\int_{T_2'}^t M(t-s)\phi(x(s))\, ds, \end{aligned} \tag{5.31}$$

for all $t \ge T_2'$ and where $F^* := \max_{0\le s\le T_2'} f(x(s))$. Now define the constant

$$x^* := \max_{0\le s\le T_2'} x(s)$$

and the function

$$\begin{aligned} \overline{H}(t) &:= (\lambda+\epsilon)(1+\epsilon^*)\Phi^{-1}(t\, M_1(t)) \\ &\qquad + (1+\epsilon) M_1(t)\, T_2'\, F^* - (\lambda+\epsilon)(1+\epsilon^*), \quad t \ge 0. \end{aligned}$$

Since $\Phi^{-1}(0) = 1$ and $M_1(0) = 0$, $H(0) = 0$ and, by construction, $H \in C^1((0,\infty);(0,\infty))$. The initial upper estimate (5.31) motivates the definition of the next upper comparison equation:

$$\begin{aligned} y_\epsilon'(t) &:= \overline{H}'(t) + (1+\epsilon)\int_{[0,t]} \mu(ds)\phi(y_\epsilon(t-s))\, ds, \quad t \ge 0, \\ y_\epsilon(0) &= x(0) + x^* + (\lambda+\epsilon)(1+\epsilon^*). \end{aligned}$$

Integration using Fubini's theorem quickly shows that

$$\begin{aligned} y_\epsilon(t) &= x(0) + x^* + (\lambda+\epsilon)(1+\epsilon^*) \\ &\qquad + \overline{H}(t) + (1+\epsilon)\int_0^t M(t-s)\phi(y_\epsilon(s))\, ds, \quad t \ge 0. \end{aligned}$$

Since $y_\epsilon(t)$ is non-decreasing, it is immediately clear that $x(t) \le y_\epsilon(t)$ for all $t \in [0, T_2']$. A simple time of the first breakdown argument using the estimate (5.31) then yields that $x(t) \le y_\epsilon(t)$ for all $t \ge 0$.

We now compute an explicit upper bound on $\limsup_{t\to\infty} y_\epsilon(t)/F^{-1}(t\,M(t))$. Monotonicity readily yields

$$y_\epsilon(t) \le x(0) + x^* + (\lambda+\epsilon)(1+\epsilon^*)\Phi^{-1}(t\,M_1(t)) + (1+\epsilon)M_1(t)\,T_2'\,F^* + (1+\epsilon)M(t)\,t\,\phi(y_\epsilon(t)), \quad t \ge 0.$$

Hence, with $C(t)$ suitably defined,

$$\frac{y_\epsilon(t)}{t\,M(t)\,\phi(y_\epsilon(t))} \le C(t) + \frac{(\lambda+\epsilon)(1+\epsilon^*)\Phi^{-1}(t\,M_1(t))}{t\,M(t)\,\phi(y_\epsilon(t))} + (1+\epsilon), \quad t \ge 0.$$

A short calculation reveals that $\lim_{t\to\infty} C(t) = 0$. By Karamata's theorem, a $T_3(\epsilon)$ exists such that

$$\frac{\Phi(y_\epsilon(t))}{t\,M(t)} < \frac{(1+\epsilon)C(t)}{1-\beta} + \frac{(1+\epsilon)(\lambda+\epsilon)(1+\epsilon^*)\Phi^{-1}(t\,M_1(t))}{(1-\beta)\,t\,M(t)\,\phi(y_\epsilon(t))} + \frac{(1+\epsilon)^2}{1-\beta}, \quad t \ge T_4 := T_3 + T_2'. \tag{5.32}$$

By applying Lemma 5.6 to y_ϵ, we conclude that

$$\liminf_{t\to\infty} \frac{y_\epsilon(t)}{\Phi^{-1}(t\,M(t))} =: L \in (0,\infty].$$

If $L \in (0,\infty)$, then there exists a $T_5(\epsilon)$ such that, for all $t \ge T_6 := T_5 + T_4$,

$$\begin{aligned}\frac{\Phi(y_\epsilon(t))}{t\,M(t)} &< \frac{(1+\epsilon)C(t)}{1-\beta} + \frac{(1+\epsilon)(\lambda+\epsilon)(1+\epsilon^*)\Phi^{-1}(t\,M(t))}{(1-\beta)\,t\,M(t)\,\phi((1-\epsilon)\,L\,\Phi^{-1}(t\,M(t)))} + \frac{(1+\epsilon)^2}{1-\beta} \\ &< \frac{(1+\epsilon)C(t)}{1-\beta} + \frac{(1+\epsilon)(\lambda+\epsilon)(1+\epsilon^*)\Phi^{-1}(t\,M(t))}{(1-\beta)\,t\,M(t)\,(1-\epsilon)^\beta\,L^\beta\,\phi(\Phi^{-1}(t\,M(t)))} + \frac{(1+\epsilon)^2}{1-\beta}.\end{aligned} \tag{5.33}$$

By Karamata's theorem, the following asymptotic equivalence holds

$$(1-\beta)\, t\, M(t)\, \phi\left(\Phi^{-1}(t\, M(t))\right) \sim \Phi^{-1}(t\, M(t)) \quad \text{as } t \to \infty.$$

Therefore taking the limit superior across (5.33) yields

$$\limsup_{t\to\infty} \frac{\Phi(y_\epsilon(t))}{t\, M(t)} \le \frac{(1+\epsilon)(\lambda+\epsilon)(1+\epsilon^*)}{(1-\epsilon)^\beta\, L^\beta} + \frac{(1+\epsilon)^2}{1-\beta}.$$

By letting $\epsilon \to 0^+$ and using the regular variation of Φ^{-1},

$$\limsup_{t\to\infty} \frac{x(t)}{\Phi^1(t\, M(t))} \le \left(\frac{\lambda}{L^\beta} + \frac{1}{1-\beta}\right)^{1/(1-\beta)} =: U.$$

If $L := \liminf_{t\to\infty} y_\epsilon(t)/\Phi^{-1}(t\, M(t)) = \infty$, the above construction will yield

$$\limsup_{t\to\infty} y_\epsilon(t)/\Phi^{-1}(t\, M(t)) < \infty,$$

a contradiction. Hence, $L \in (0,\infty)$, and the claim is proven. □

Lemma 5.8. *Suppose that* $\beta \in [0,1)$, $\lambda \in [0,\infty)$, *and consider the iterative scheme defined by*

(5.34)

$$x_{n+1} = g(x_n) := \frac{x_n^\beta}{1-\beta} B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right) + \lambda, \quad n \ge 1;\ x_0 \in [L, C^*],$$

with L *defined by* (5.28), U *defined by* (5.30) *and*

$$C^* := \max\left(U,\, L + \frac{\lambda}{1-\beta}\right). \tag{5.35}$$

Then, there exists a unique $x_\infty \in [L, C^*]$ *such that* $\lim_{n\to\infty} x_n = x_\infty$.

Proof. By inspection, $g \in C([L,\infty);(0,\infty))$. We calculate

$$g'(x) = \frac{\beta}{1-\beta} x^{\beta-1} B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right) > 0, \quad x > 0,$$

and similarly,

$$g''(x) = -\beta x^{\beta-2} B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right) < 0, \quad x > 0.$$

Therefore, $g'(L) = \beta > g'(x) > 0$ for all $x > L$ and $|g'(x)| \leq \beta < 1$ for all $x \in [L, \infty)$. Since g is monotone increasing, it is sufficient check that g maps $[L, C^*]$ to $[L, C^*]$ as follows. Firstly,

$$g(L) = \frac{L^\beta}{1-\beta} B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right) + \lambda = L + \lambda \in [L, C^*]. \tag{5.36}$$

By the mean value theorem, a $\xi \in [L, C^*]$ exists such that

$$\frac{g(C^*) - g(L)}{C^* - L} = g'(\xi) \leq \beta.$$

Therefore, $g(C^*) \leq \beta(C^* - L) + g(L)$, and thus, a sufficient condition for $g(C^*) \leq C^*$ is $\beta(C^* - L) + g(L) \leq C^*$ or $C^* \geq (g(L) - L\beta)/(1-\beta) = L + \lambda/(1-\beta)$, using (5.36). Thus, with C^* as defined in (5.35), $g : [L, C^*] \to [L, C^*]$. Hence, (5.34) has a unique fixed point in $[L, C^*]$ and the claim follows. □

With the preceding auxiliary results proven, we are now in a position to supply the proof of Theorem refthm.pert, as stated.

Proof of Theorem 3.1. Suppose that (ii) holds, or that

$$\lim_{t\to\infty} \frac{H(t)}{F^{-1}(t\,M(t))} = \lambda \in [0, \infty).$$

The idea here is to combine the crude bounds on the solution from Lemmas 5.6 and 5.7 with a fixed point argument based on Lemma 5.8 to complete the proof that (ii) implies (i). We compute $\limsup_{t\to\infty} x(t)/F^{-1}(t\,M(t))$ in detail only as the calculation of the corresponding inferior limit proceeds in an analogous manner. In order to begin, make the following induction hypothesis:

$$\tag{H_n} \begin{aligned} &\limsup_{t\to\infty} \frac{x(t)}{\Phi^{-1}(Mt)} \leq \zeta_n, \\ &\zeta_{n+1} := \frac{\zeta_n^\beta}{1-\beta} B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right) + \lambda, \quad n \geq 0, \end{aligned}$$

and choose $\zeta_0 := U$. (H_0) is true by Lemma 5.7. Suppose that (H_n) holds. Thus, a $T(\epsilon) > 0$ exists such that $x(t) < (\zeta_n + \epsilon)\Phi^{-1}(t\,M(t))$

for all $t \geq T$. Hence,

$$\frac{\phi(x(t))}{\phi(\Phi^{-1}(t\,M(t)))} < \frac{\phi((\zeta_n+\epsilon)\Phi^{-1}(Mt))}{\phi(\Phi^{-1}(t\,M(t)))}, \quad t \geq T.$$

The regular variation of ϕ thus yields

$$\limsup_{t\to\infty} \phi(x(t))/\phi(\Phi^{-1}(t\,M(t))) \leq (\zeta_n+\epsilon)^{\beta}.$$

Therefore, a $T_2(\epsilon) > 0$ exists such that $t \geq T_2$ implies

$$f(x(t)) < (1+\epsilon)[(\zeta_n+\epsilon)^{\beta}+\epsilon]\phi(\Phi^{-1}(t\,M(t))).$$

From (1.6),

$$\limsup_{t\to\infty} \frac{x(t)}{\Phi^{-1}(t, M(t))} = \limsup_{t\to\infty} \frac{\int_0^t M(t-s)f(x(s))ds}{\Phi^{-1}(tM(t))} + \lim_{t\to\infty} \frac{H(t)}{\Phi^{-1}(tM(t))}.$$

Using the upper bound derived from our induction hypothesis yields

$$\begin{aligned}\limsup_{t\to\infty} &\frac{x(t)}{F^{-1}(t\,M(t))} \\ &\leq (1+\epsilon)[(\zeta_n+\epsilon)^{\beta}+\epsilon] \limsup_{t\to\infty} \frac{\int_{T_2}^t M(t-s)\phi(\Phi^{-1}(s\,M(s)))}{\Phi^{-1}(t\,M(t))} + \lambda.\end{aligned}$$

Applying Karamata's theorem and Lemma 5.2,

$$\begin{aligned}\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t\,M(t))} &\leq (1+\epsilon)[(\zeta_n+\epsilon)^{\beta}+\epsilon] \\ &\quad \times \limsup_{t\to\infty} \frac{\int_{T_2}^t M(t-s)\phi(\Phi^{-1}(s\,M(s)))}{(1-\beta)t\,M(t)\,\phi(\Phi^{-1}(t\,M(t)))} + \lambda \\ &= \frac{(1+\epsilon)[(\zeta_n+\epsilon)^{\beta}+\epsilon]}{1-\beta} B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right) + \lambda.\end{aligned}$$

Letting $\epsilon \to 0^+$ yields

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t\,M(t))} \leq \frac{\zeta^{\beta}}{1-\beta} B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right) + \lambda = \zeta_{n+1},$$

proving the induction hypothesis (H_{n+1}). Hence, (H_n) holds for all n, or

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(t\,M(t))} \leq \zeta_n \quad \text{for all } n \geq 0.$$

By Lemma 5.8, $\lim_{n\to\infty}\zeta_n = \zeta$, where ζ is the unique solution in $[L, U]$ of the "characteristic" equation (3.1). Thus,

$$\limsup_{t\to\infty} \frac{x(t)}{F^{-1}(Mt)} \le \zeta.$$

In the case of the corresponding inferior limit, the only modification is to the induction hypothesis: take $\zeta_0 := L$, and the argument then proceeds as above to yield $\liminf_{t\to\infty} x(t)/F^{-1}(Mt) \ge \zeta$, completing the proof.

Now suppose that (i) holds, or that

$$\lim_{t\to\infty} x(t)/F^{-1}(t\,M(t)) = \zeta \in [L,\infty).$$

It follows that there exists a $T_3(\epsilon) > 0$ such that, for all $t \ge T_3$,

$$\phi((\zeta-\epsilon)\Phi^{-1}(t\,M(t))) < \phi(x(t)) < \phi((\zeta+\epsilon)\Phi^{-1}(t\,M(t))).$$

Hence, for $t \ge T_3$,

$$\begin{aligned}&\int_{T_3}^t M(t-s)\phi((\zeta-\epsilon)\Phi^{-1}(s\,M(s)))\,ds\\ &\quad\le \int_{T_3}^t M(t-s)\phi(x(s))\,ds\\ &\quad\le \int_{T_3}^t M(t-s)\phi((\zeta+\epsilon)\Phi^{-1}(s\,M(s)))\,ds.\end{aligned}$$

Using the regular variation of ϕ the above estimate can be reformulated as:

$$\begin{aligned}&\frac{(\zeta-\epsilon)^\beta \int_{T_3}^t M(t-s)\phi(\Phi^{-1}(s\,M(s)))\,ds}{\Phi^{-1}(t\,M(t))}\\ &\quad\le \frac{\int_{T_3}^t M(t-s)\phi(x(s))\,ds}{\Phi^{-1}(t\,M(t))}\\ &\quad\le (\zeta+\epsilon)^\beta \frac{\int_{T_3}^t M(t-s)\phi((\zeta+\epsilon)\Phi^{-1}(s\,M(s)))\,ds}{\Phi^{-1}(t\,M(t))}, \quad t \ge T_3.\end{aligned}$$

Using Lemma 5.2 and letting $\epsilon \to 0^+$ thus yields

$$\lim_{t\to\infty} \frac{\int_0^t M(t-s)\phi(x(s))ds}{\Phi^{-1}(t\,M(t))} = \frac{\zeta^\beta}{1-\beta} B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right).$$

Therefore, assuming (i) and taking the limit across (1.6), we obtain

$$\zeta = \frac{\zeta^\beta}{1-\beta} B\left(1+\theta, \frac{1+\theta\beta}{1-\beta}\right) + \lim_{t\to\infty} \frac{H(t)}{\Phi^{-1}(t\,M(t))},$$

as claimed. □

We now give the proof of Theorem 3.2 in which the perturbation is large. The reader will note that this proof makes much less use of properties of regular varying functions; in fact, we establish the asymptotic result by observing that a key functional of the solution is well approximated by a linear non-autonomous differential inequality.

Proof of Theorem 3.2. As always, $\epsilon \in (0,1)$ is arbitrary. From (5.1), a ϕ exists such that

$$\lim_{x\to\infty} f(x)/\phi(x) = 1, \qquad \lim_{x\to\infty} x\,\phi'(x)/\phi(x) = \beta,$$

see e.g., [**9,** Theorem 1.3.3]. Therefore, there exists an $x_1(\epsilon) > 0$ such that $f(x) < (1+\epsilon)\phi(x)$ for all $x \geq x_1(\epsilon)$ and $x_0(\epsilon)$ such that $\phi'(x) < (\beta+\epsilon)\phi(x)/x$ for all $x \geq x_0(\epsilon)$. Similarly, since $\lim_{t\to\infty} x(t) = \infty$, there exists a $T_1(\epsilon) > 0$ such that $x(t) > \max(x_0(\epsilon), x_1(\epsilon))$ for all $t \geq T_1(\epsilon)$. The regular variation of M means that a non-decreasing function $M_1 \in C^1$ and $T_2(\epsilon) > 0$ exist such that $(1-\epsilon)M_1(t) < M(t) < (1+\epsilon)M_1(t)$ for all $t \geq T_2(\epsilon)$. Hence,

$$(1-\epsilon)M_1(t) < \max_{T_2\leq s\leq t} M(s) < (1+\epsilon)M_1(t), \quad t \geq T_2.$$

Thus, for $t \geq T_2$

$$\begin{aligned}(1-\epsilon)M_1(t) < \max_{0\leq s\leq t} M(s) &< \max\left(\max_{0\leq s\leq T_2} M(s), \max_{T_2\leq s\leq t} M(s)\right)\\ &\leq \max\left(\max_{0\leq s\leq T_2} M(s), (1+\epsilon)M_1(t)\right).\end{aligned}$$

Therefore,

$$1-\epsilon \leq \frac{\max_{0\leq s\leq t} M(s)}{M_1(t)} \leq \max\left(\frac{\max_{0\leq s\leq T_2} M(s)}{M_1(t)}, 1+\epsilon\right),$$

and since $\lim_{t\to\infty} M_1(t) = \infty$ we conclude that $\lim_{t\to\infty} \max_{0\leq s\leq t} M(s)/M_1(t) = 1$. It follows that there exists a $T_3(\epsilon) > 0$ such that $\max_{0\leq s\leq t} M(s) < (1+\epsilon)M_1(t)$ for all $t \geq T_3(\epsilon)$.

Now, let $T = 1 + \max(T_1, T_2, T_3)$. From (1.6), with $t \geq 2T$,

$$\begin{aligned} x(t) &= x(0) + H(t) + \int_0^T M(t-s) f(x(s))\,ds \\ &\quad + \int_T^t M(t-s) f(x(s))\,ds \\ &< x(0) + H(t) + \int_0^T M(t-s) f(x(s))\,ds \\ &\quad + (1+\epsilon) \int_T^t M(t-s)\phi(x(s))\,ds \\ &= x(0) + H(t) + \int_0^T M(t-s) f(x(s))\,ds \\ &\quad + (1+\epsilon) \int_T^{t-T} M(t-s)\phi(x(s))\,ds \\ &\quad + (1+\epsilon) \int_{t-T}^t M(t-s)\phi(x(s))\,ds. \end{aligned}$$

If $s \in [T, t-T]$, then $t - s \geq T > T_1$, and, for $t \geq 2T$,

$$\begin{aligned} x(t) &< x(0) + H(t) + \int_0^T M(t-s) f(x(s))\,ds \\ &\quad + (1+\epsilon)^2 M_1(t) \int_T^{t-T} \phi(x(s))\,ds \\ &\quad + (1+\epsilon) \max_{0 \leq s \leq T} M(s) \int_{t-T}^t \phi(x(s))\,ds \end{aligned}$$

Now, as $T > T_3(\epsilon)$, $\max_{0 \leq s \leq T} M(s) < (1+\epsilon) M_1(T) < (1+\epsilon) M_1(t)$. Hence,

$$\begin{aligned} x(t) &< x(0) + H(t) + \int_0^T M(t-s) f(x(s))\,ds \\ &\quad + (1+\epsilon)^2 M_1(t) \int_T^t \phi(x(s))\,ds, \quad t \geq 2T. \end{aligned}$$

For $t \geq 2T > T$,

$$\max_{0 \leq s \leq T} M(t-s) = \max_{t-T \leq u \leq t} M(u) \leq \max_{0 \leq u \leq t} M(u) < (1+\epsilon) M_1(t).$$

Thus, for $t \geq 2T$,

$$x(t) < x(0) + H(t) + (1+\epsilon)M_1(t)\int_0^T f(x(s))\,ds + (1+\epsilon)^2 M_1(t)\int_T^t \phi(x(s))\,ds. \tag{5.37}$$

For $t \in [T, 2T]$, $x(t) \leq \max_{s\in[0,2T]} x(s) := x_1^*(\epsilon)$. Combining this with (5.37),

$$x(t) < x_1^*(\epsilon) + H(t) + (1+\epsilon)M_1(t)x_2^*(\epsilon) + (1+\epsilon)^2 M_1(t)\int_T^t \phi(x(s))\,ds, \quad t \geq 2T, \tag{5.38}$$

where $x_2^*(\epsilon) := \int_0^T f(x(s))\,ds$. Define, for $t \geq 2T$,

$$H_\epsilon(t) := x_1^*(\epsilon) + H(t) + (1+\epsilon)M_1(t)x_2^*(\epsilon). \tag{5.39}$$

Note that, by construction, $\lim_{t\to\infty} H_\epsilon(t)/H(t) = 1$. Consolidating (5.38) and (5.39), we have

$$x(t) < H_\epsilon(t) + (1+\epsilon)^2 M_1(t)\int_T^t \phi(x(s))\,ds, \quad t \geq 2T. \tag{5.40}$$

By defining

$$I_\epsilon(t) := \int_T^t \phi(x(s))\,ds, \quad t \geq 2T,$$

we can formulate an advantageous auxiliary differential inequality as follows. Since x is continuous and $\phi \in C^1(0,\infty)$, $I_\epsilon'(t) = \phi(x(t))$, $t \geq 2T$. Moreover, $\lim_{t\to\infty} I_\epsilon(t) = \infty$. By (5.40),

$$I_\epsilon'(t) = \phi(x(t)) < \phi(H_\epsilon(t) + (1+\epsilon)^2 M_1(t) I_\epsilon(t)), \quad t \geq 2T. \tag{5.41}$$

By the mean value theorem, for each $t \geq 2T$, there exists a $\xi_\epsilon(t) \in [0,1]$ such that

$$\begin{aligned}&\phi(H_\epsilon(t) + (1+\epsilon)^2 M_1(t) I_\epsilon(t)) \\ &\quad = \phi(H_\epsilon) + \phi'(H_\epsilon(t) + \xi_\epsilon(t)(1+\epsilon)^2 M_1(t) I_\epsilon(t))(1+\epsilon)^2 M_1(t) I_\epsilon(t).\end{aligned}$$

Let $a_\epsilon(t) := H_\epsilon(t) + \xi_\epsilon(t)(1+\epsilon)^2 M_1(t) I_\epsilon(t)$, $t \geq 2T$. For $t \geq 2T$,

$$a_\epsilon(t) \geq H_\epsilon(t) > x_1^*(\epsilon) := \max_{s\in[0,2T]} x(s) > x_0(\epsilon).$$

Therefore, with $\psi \in \mathrm{RV}_\infty(\beta - 1)$, a decreasing function asymptotic to $\phi(x)/x$,

$$\begin{aligned}\phi'(a_\epsilon(t)) &< (\beta+\epsilon)\frac{\phi(a_\epsilon(t))}{a_\epsilon(t} < (\beta+\epsilon)(1+\epsilon)\psi(a_\epsilon(t)) \\ &< (\beta+\epsilon)(1+\epsilon)\psi(H_\epsilon(t)), \quad t \geq 2T.\end{aligned}$$

However, since $\psi(x) \sim \phi(x)/x$, we also have $\psi(H_\epsilon(t))/(1+\epsilon) < \phi(H_\epsilon(t))/H_\epsilon(t)$, and hence,

$$\phi'(a_\epsilon(t)) < (\beta+\epsilon)(1+\epsilon)^2 \frac{\phi(H_\epsilon(t))}{H_\epsilon(t)}, \quad t \geq 2T.$$

Combining this estimate with (5.41) yields

$$I_\epsilon'(t) < \phi(H_\epsilon(t)) + (\beta+\epsilon)(1+\epsilon)^4 \frac{\phi(H_\epsilon(t))}{H_\epsilon(t)} M_1(t) I_\epsilon(t), \quad t \geq 2T.$$

Letting $\alpha_\epsilon(t) = (\beta+\epsilon)(1+\epsilon)^4 M_1(t)\, \phi(H_\epsilon(t))/H_\epsilon(t)$, this becomes

$$I_\epsilon'(t) < \phi(H_\epsilon(t)) + \alpha_\epsilon(t)\, I_\epsilon(t) \quad \text{for } t \geq 2T.$$

Thus, the variation of constants formula yields

$$I_\epsilon(t) \leq e^{\int_T^t \alpha_\epsilon(s)ds} \int_T^t e^{-\int_T^s \alpha_\epsilon(u)du} \phi(H_\epsilon(s))\, ds, \quad t \geq 2T.$$

We reformulate this as
(5.42)

$$\frac{I_\epsilon(t)}{\int_T^t \phi(H_\epsilon(s))\, ds} \leq \frac{\int_T^t e^{-\int_T^s \alpha_\epsilon(u)du} \phi(H_\epsilon(s))\, ds}{e^{-\int_T^t \alpha_\epsilon(s)ds} \int_T^t \phi(H_\epsilon(s))\, ds} =: \frac{C_\epsilon(t)}{B_\epsilon(t)}, \quad t \geq 2T.$$

Since $C_\epsilon'(t) = \phi(H_\epsilon(t)) e^{-\int_T^t \alpha_\epsilon(u)du} > 0$, we have

$$\lim_{t\to\infty} C_\epsilon(t) = C^*(\epsilon) \in (0,\infty) \quad \text{or} \quad \lim_{t\to\infty} C_\epsilon(t) = \infty.$$

Also, for $t \geq 2T$,

$$\begin{aligned}
B_\epsilon'(t) &= \phi(H_\epsilon(t))e^{-\int_T^t \alpha_\epsilon(u)du} \\
&\quad - \alpha_\epsilon(t)e^{-\int_T^t \alpha_\epsilon(u)du} \int_T^t \phi(H_\epsilon(s))\,ds \\
&= C_\epsilon'(t) - \frac{\alpha_\epsilon(t)\, C_\epsilon'(t) \int_T^t \phi(H_\epsilon(s))\,ds}{\phi(H_\epsilon(t))} \\
&= C_\epsilon'(t)\left\{1 - \frac{\alpha_\epsilon(t) \int_T^t \phi(H_\epsilon(s))ds}{\phi(H_\epsilon(t))}\right\}.
\end{aligned}$$

Therefore, recalling the definition of $\alpha_\epsilon(t)$, and rearranging,

$$\frac{B_\epsilon'(t)}{C_\epsilon'(t)} = 1 - (\beta+\epsilon)(1+\epsilon)^4\left(\frac{M_1(t)\int_T^t \phi(H_\epsilon(s))ds}{H_\epsilon(t)}\right), \quad t \geq 2T.$$

Letting $t \to \infty$ and using the hypothesis (3.2) and that $H_\epsilon(t) \sim H(t)$ and $M_1(t) \sim M(t)$ as $t \to \infty$, yields

$$\lim_{t\to\infty} B_\epsilon'(t)/C_\epsilon'(t) = 1 \quad \text{or} \quad \lim_{t\to\infty} C_\epsilon'(t)/B_\epsilon'(t) = 1.$$

Hence, a T_4 exists such that $B_\epsilon'(t) > 0$, $t \geq T_4$, and either

$$\lim_{t\to\infty} B_\epsilon(t) = B^*(\epsilon) \in (0,\infty) \quad \text{or} \quad \lim_{t\to\infty} B_\epsilon(t) = \infty.$$

Furthermore, asymptotic integration shows that $\lim_{t\to\infty} C_\epsilon(t) = \infty$ implies

$$\lim_{t\to\infty} B_\epsilon(t) = \infty$$

and $\lim_{t\to\infty} C_\epsilon(t) = C^*(\epsilon)$ implies

$$\lim_{t\to\infty} B_\epsilon(t) = B^*(\epsilon).$$

Hence,

$$\Lambda(\epsilon) := \lim_{t\to\infty} \frac{C_\epsilon(t)}{B_\epsilon(t)} = \begin{cases} 1 & \lim_{t\to\infty} C_\epsilon(t) = \infty, \\ C*(\epsilon)/B^*(\epsilon) & \lim_{t\to\infty} C_\epsilon(t) = C^*, \end{cases}$$

where the first limit is calculated using L'Hôpital's rule. Taking the limit superior across equation (5.42) then yields

(5.43)

$$\limsup_{t\to\infty} \frac{\int_T^t \phi(x(s))ds}{\int_T^t \phi(H_\epsilon(s))ds} = \limsup_{t\to\infty} \frac{I_\epsilon(t)}{\int_T^t \phi(H_\epsilon(s))ds} \le \Lambda(\epsilon) \in (0,\infty).$$

Since $H_\epsilon(t) \sim H(t)$ as $t \to \infty$ and ϕ is increasing, we can apply L'Hôpital's rule once more to compute

$$\lim_{t\to\infty} \frac{\int_T^t \phi(H_\epsilon(s))\,ds}{\int_0^t \phi(H_\epsilon(s))\,ds} = \lim_{t\to\infty} \frac{\phi(H_\epsilon(t))}{\phi(H(t))} = 1^\beta = 1,$$

using that $\phi \in \text{RV}_\infty(\beta)$. A similar argument relying on the divergence of $\phi(x(t))$ and L'Hôpital's rule yields

$$\int_T^t \phi(x(s))\,ds \sim \int_0^t \phi(x(s))\,ds \quad \text{as } t \to \infty.$$

Therefore, (5.43) is equivalent to

$$\limsup_{t\to\infty} \frac{\int_0^t \phi(x(s))ds}{\int_0^t \phi(H(s))ds} \le \Lambda(\epsilon) \in (0,\infty). \tag{5.44}$$

Hence, there exists a $\Lambda^* \in (0,\infty)$ such that

$$\limsup_{t\to\infty} \int_0^t \phi(x(s))\,ds \Big/ \int_0^t \phi(H(s))\,ds \le \Lambda^*,$$

with Λ^* independent of ϵ. Thus, a $T_6(\epsilon)$ exists such that

$$\int_0^t \phi(x(s))\,ds < (\Lambda^* + \epsilon) \int_0^t \phi(H(s))\,ds$$

for all $t \ge T_6(\epsilon)$. Letting $\overline{T} = 1 + \max(2T, T_6)$ we apply this estimate to (5.40) as:

$$\begin{aligned}\frac{x(t)}{H(t)} &< \frac{H_\epsilon(t)}{H(t)} + \frac{(1+\epsilon)^2\, M_1(t) \int_T^t \phi(x(s))\,ds}{H(t)} \\ &< \frac{H_\epsilon(t)}{H(t)} + \frac{(1+\epsilon)^2\, M_1(t)(\Lambda^* + \epsilon) \int_0^t \phi(H(s))\,ds}{H(t)}, \quad t \ge \widetilde{T}.\end{aligned}$$

Now, since $H_\epsilon(t) \sim H(t)$ as $t \to \infty$ and $M_1 \sim M$, applying (3.2) to the above estimate yields $\limsup_{t\to\infty} x(t)/H(t) \le 1$. By positivity, (1.6) admits the trivial bound $x(t) > H(t)$ for all $t \geq 0$, and hence, $\liminf_{t\to\infty} x(t)/H(t) \geq 1$, completing the proof. □

Proof of Corollary 3.3. By hypothesis, $f \circ H \in \text{RV}_\infty(\alpha\beta)$ and $M \in \text{RV}_\infty(\theta)$. Hence, $\int_0^t f(H(s))\,ds \in \text{RV}_\infty(1+\alpha\beta)$ and Karamata's theorem yields

$$\int_0^t f(H(s))\,ds \sim t\,f(H(t))/(1+\alpha\beta), \quad \text{as } t \to \infty.$$

Thus,

$$\frac{M(t)\int_0^t f(H(s))\,ds}{H(t)} \sim \frac{M(t)\,t\,f(H(t))}{(1+\alpha\beta)H(t)}, \quad \text{as } t \to \infty. \tag{5.45}$$

Lemma 5.2 yields

$$\int_0^t M(t-s)f(H(s))\,ds \sim B(1+\alpha\beta, 1+\theta)\,t\,M(t)\,f(H(t)), \quad \text{as } t \to \infty. \tag{5.46}$$

Therefore, (5.45) and (5.46) together yield

$$\frac{\int_0^t M(t-s)f(H(s))\,ds}{H(t)} \sim (1+\alpha\beta)B(1+\alpha\beta, 1+\theta)\frac{M(t)\int_0^t f(H(s))\,ds}{H(t)}, \quad \text{as } t \to \infty.$$

Hence, (i) and (iii) are equivalent. By Karamata's theorem,

$$F(H(t)) \sim \frac{H(t)}{(1-\beta)f(H(t))}$$

or

$$f(H(t))/H(t) \sim \frac{1}{(1-\beta)F(H(t))},$$

as $t \to \infty$. Hence, (5.45) may be restated as

$$\frac{M(t)\int_0^t f(H(s))\,ds}{H(t)} \sim \frac{M(t)\,t}{(1+\alpha\beta)(1-\beta)F(H(t))}, \quad \text{as } t \to \infty.$$

Thus, if (i) holds, then

$$\lim_{t\to\infty} M(t)t/F(H(t)) = 0.$$

This implies that

$$\lim_{t\to\infty} \frac{F(H(t))}{M(t)t} = \infty,$$

and hence that (iii) holds, by the regular variation of F^{-1}. The reverse implications are all also true, and (i) and (ii) are equivalent. □

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REFERENCES

1. J.A.D. Appleby, *On regularly varying and history-dependent convergence rates of solutions of a Volterra equation with infinite memory*, Adv. Diff. Eq. **2010** (2010), 1–31.

2. J.A.D. Appleby and E. Buckwar, *A constructive comparison technique for determining the asymptotic behaviour of linear functional differential equations with unbounded delay*, Differ. Eq. Dynam. Syst. **28** (2010), 271–301.

3. J.A.D. Appleby, I. Györi and D.W. Reynolds, *On exact rates of decay of solutions of linear systems of Volterra equations with delay*, J. Math. Anal. Appl. **320** (2006), 56–77.

4. J.A.D. Appleby and K. Krol, *Long memory in a linear stochastic Volterra differential equation*, J. Math. Anal. Appl. **380** (2011), 814–830.

5. J.A.D. Appleby and D.D. Patterson, *Classification of convergence rates of solutions of perturbed ordinary differential equations with regularly varying nonlinearity*, Electr. J. Qual. Theory Diff. Eq. **3** (2016), 1–38.

6. ———, *Subexponential growth rates in functional differential equations*, Discr. Contin. Dynam. Syst. (2015), 56–65.

7. ———, *Growth rates of sublinear functional and Volterra differential equations*, `arXiv:1602.06256`.

8. J.A.D. Appleby and M. Riedle, *Stochastic Volterra differential equations in weighted spaces*, J. Integral Equat. Appl. **22** (2010), 1–17.

9. N.H. Bingham, C.M. Goldie and J.L. Teugels, *Regular variation*, volume 27, Cambridge University Press, Cambridge, 1989.

10. H. Brunner and S. Maset, *Time transformations for delay differential equations*, Discr. Contin. Dynam. Syst. **25** (2009), 751–775.

11. G. Gripenberg, S.-O. Londen, and O. Staffans, *Volterra integral and functional equations*, volume 34, Cambridge University Press, Cambridge, 1990.

12. K.B. Hannsgen, *Indirect Abelian theorems and a linear Volterra equation*, Trans. Amer. Math. Soc. **142** (1969), 539–555.

13. V. Marić, *Regular variation and differential equations*, **1726**, Springer, New York, 2000.

14. F. Olver, D. Lozier, R. Boisvert and C. Clark, NIST *handbook of mathematical functions*, U.S. Department of Commerce, Washington, DC, 2010.

15. P. Řehák, *Nonlinear differential equations in the framework of regular variation*, 2014, unpublished manuscript.

16. W.R. Schneider, *The general solution of a non-linear integral equation of convolution type*, Z. Angew. Math. Phys. **33** (1982), 140–142.

17. D.F. Shea and S. Wainger, *Variants of the Wiener-Lévy theorem, with applications to stability problems for some Volterra integral equations*, American J. Math. **97** (1975), 312–343.

18. J.S. Wong and R. Wong, *Asymptotic solutions of linear Volterra integral equations with singular kernels*, Trans. Amer. Math. Soc. **189** (1974), 185–200.

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