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ADAPTIVE TIME DOMAIN BOUNDARY ELEMENT METHODS WITH ENGINEERING APPLICATIONS

HEIKO GIMPERLEIN, MATTHIAS MAISCHAK AND ERNST P. STEPHAN

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ABSTRACT. Time domain Galerkin boundary elements provide an efficient tool for numerical solution of boundary value problems for the homogeneous wave equation. We review recent advances in their a posteriori error analysis and the resulting adaptive mesh refinement procedures, as well as basic algorithmic aspects of these methods. Numerical results for adaptive mesh refinements are discussed in two and three dimensions, as are benchmark problems in a half-space related to the transient emission of traffic noise.

1. Introduction. Efficient and accurate computational methods for simulating sound emission in space and time are of interest from the modeling of environmental noise to the acoustics of concert halls. This survey reviews time domain Galerkin boundary element methods for acoustic wave problems as studied in [**16, 25, 26, 29, 42**], with references for related works. We specifically emphasize algorithmic aspects and recent progress towards space-time adaptive mesh refinements as well as applications to tire noise. Time domain boundary element methods prove to be stable and accurate in long-time computations and are competitive with frequency domain methods for realistic problems from the sound emission of tires.

Computations in time domain are of particular interest for problems beyond the reach of frequency domain methods, such as the simulation of transient dynamics and moving sound sources or nonlinear and

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dynamical contact problems. They can also be applied to obtain results in frequency domain, for all frequencies in one computation, with the help of the fast Fourier transform (FFT) to translate between time and frequency. This approach proves competitive if a broad band of frequencies is of interest.

Let $d = 2, 3$ and $\Omega^i \subset \mathbb{R}^d$ be a bounded polygonal domain. For simplicity, we assume that the exterior domain $\Omega^e = \mathbb{R}^d \setminus \overline{\Omega^i}$ is connected and that the boundary $\Gamma = \partial\Omega$ is a Lipschitz manifold. Our emphasis will be on case $d = 3$.

We aim to find a weak solution to an acoustic initial-boundary problem for the wave equation in Ω^e:

$$\begin{aligned} \frac{\partial^2 u}{\partial t^2} - \Delta u &= 0 \quad \text{in } \mathbb{R}^+ \times \Omega^e, \\ \frac{\partial u}{\partial n} - \alpha \frac{\partial u}{\partial t} &= g \quad \text{on } \mathbb{R}^+ \times \Gamma, \\ u(0,x) = \frac{\partial u}{\partial t}(0,x) &= 0 \quad \text{in } \Omega^e. \end{aligned} \tag{1.1}$$

Here, n denotes the inward unit normal vector to $\partial\Omega^e$, g lies in a suitable Sobolev space, $\alpha \in L^\infty(\Gamma)$. Note that, in the case of an incoming wave u^{inc} scattered by Ω^i, the right hand side is

$$g = -\frac{\partial u^{\text{inc}}}{\partial n} + \alpha \frac{\partial u^{\text{inc}}}{\partial t}.$$

In order for equation (1.1) to be well-posed, α should have a nonnegative real part so that waves are not amplified at reflection. We also consider the simpler Dirichlet problem on Γ, for which, instead of the absorbing boundary condition, $u|_{\mathbb{R}^+ \times \Gamma}$ is given.

For the boundary element methods discussed here, the acoustic and Dirichlet boundary problems are reformulated as time-dependent integral equations on $\mathbb{R}^+ \times \Gamma$. The integral equations are then numerically approximated by a Galerkin method in space-time. We present from **[25, 26]** an a priori and an a posteriori error analysis for methods based on integral formulations of the first kind. Computational experiments explore adaptive mesh refinements given in **[26]** and illustrate the methods for real-world problems from the sound emission of tires.

For the sound emission of tires, the wave equation also needs to be considered in a half space, $\Omega^i \subset \mathbb{R}^d_+$. One may consider Ω^i as a solid tire, either in contact with the street (on $\partial\Omega^i \cap \partial\mathbb{R}^d_+$) or elevated above it ($\partial\Omega^i \cap \partial\mathbb{R}^d_+ = \emptyset$). We will concentrate on the latter case, as it simplifies notation. The boundary of $\Omega^e = \mathbb{R}^d_+ \backslash \overline{\Omega^i}$ decomposes into the boundary $\Gamma = \partial\Omega^e \cap \partial\Omega^i$ of the obstacle and the boundary $\Gamma_\infty = \partial\Omega^e \cap \partial\mathbb{R}^d_+$ of the half-space.

In this case, the wave equation (1.1) is supplemented by acoustic boundary conditions on Γ_∞:

$$\frac{\partial u}{\partial n} - \alpha_\infty \frac{\partial u}{\partial t} = 0 \quad \text{on } \mathbb{R}^+ \times \Gamma_\infty, \tag{1.2}$$

where $\operatorname{Re} \alpha_\infty \geq 0$ [**25**].

1.1. Related work. Hyperbolic time domain boundary integral equations and their numerical approximation are attributed to Friedman and Shaw [**24**], respectively, Cruse and Rizzo [**18**]. The first modern boundary element methods and the basic algorithmic approaches were developed by Mansur [**38**], while mathematical analysis of time-dependent Galerkin boundary element methods was initiated by Bamberger and Ha-Duong [**11**]. Relevant works on the numerical implementation of the resulting marching-in-on-time scheme include Terrasse [**21, 33, 49**], which made the methods competitive for commercial applications.

As a main challenge in the stable implementation of time domain integral methods, the fundamental solution to the wave equation is singular, a Dirac distribution in odd dimensions supported on the light cone. The discretization and accurate computation of entries in the Galerkin matrix has been considered in detail by Maischak, Ostermann and Stephan [**37, 47**] as well as Ostermann [**42**] for further algorithmic details; see also [**6**] for an alternative approach.

The analysis initiated by Bamberger and Ha-Duong is based on frequency domain. Using the Laplace transform to translate between frequency and time domains, well-posedness and convergence of numerical approximations can be analyzed for the infinite time interval $[0, \infty)$. Recent work by Aimi et al. [**2, 3, 5, 7**] emphasizes formulations directly related to the conserved energy of the wave equation on a finite time interval $[0, T)$. At the expense of a slightly more involved weak formulation, intrinsic coercivity directly implies the stability and convergence of these methods.

A detailed exposition of the mathematical background of time domain integral equations and their discretizations is available in [**45**]; also, see [**17, 32**] for more concise introductions and [**22**] for recent progress.

Recent interest has especially centered around fast methods and adaptivity and interface problems, including coupling to finite elements with possibly different time discretizations. In particular, we refer to the work of Sylvand on fast multipole methods [**48**]. First steps towards adaptive mesh refinements will be discussed in this article. They concern space [**26**], time [**44**] and space-time in dimension 2 [**29**], but the optimal algorithmic implementation of these methods is only beginning to be understood.

For interface problems, Abboud, et al. [**1**] initiated the mathematical analysis of FEM-BEM coupling in the time domain, coupling discontinuous finite elements to time domain integral equations. A subsequent work by Banjai, Lubich and Sayas [**13**] provides a fundamental general analysis of the coupling between different discretizations, including convolution quadrature. Energy-based formulations of FEM-BEM coupling have been investigated by Aimi and collaborators [**4, 8**], while the authors study adaptivity in the context of fluid-structure interaction [**28**]. Certain truly transient phenomena studied by engineers cannot be simulated in the frequency domain because they involve nonlinear contact and damage, see [**34, 46**] for time domain BEM approaches to such problems. Their mathematical analysis remains a difficult challenge for future work.

In the engineering literature, fast methods are being developed and studied, especially in the group of Michielssen, see e.g., [**52**]. We finally mention the alternative Ansatz functions in time that have been explored in [**19, 20**].

As an alternative to time domain boundary elements, the past few years have seen rapid progress for convolution quadrature methods [**14, 15, 45**]. Convolution quadrature exploits the convolution structure in time for integral equations to approximate them through the frequency domain by an inverse Laplace transform. Given a frequency domain solver, their implementation does not struggle with the careful, accurate computation of distributional integrals like time domain boundary elements do. However, for long-time simulations and certain nonlinear problems with constraints, such as dynamic contact and friction problems, the variational nature of Galerkin time domain methods may be advantageous.

Apart from wave propagation problems in $\mathbb{R}^d$, applications like the sound emission and propagation above a street may naturally lead to problems posed in a half-space [**16**]. Here, current work is motivated by the exact fundamental solutions obtained by Ochmann [**41**], as they allow acoustic Robin boundary conditions on the infinite boundary of the half-space. Further engineering applications involve wave propagation in moving coordinate systems or with moving sources [**9, 43**].

2. Boundary integral formulations. Similar to elliptic problems, the initial-boundary value problem (1.1) for the wave equation can be formulated as an integral equation of either the first or second kind on Γ.

We introduce the single layer *potential* in time domain as

$$S\varphi(t,x) = \int_{\mathbb{R}^+\times\Gamma} G(t-\tau,x,y)\,\varphi(\tau,y)\,d\tau\,ds_y,$$

where G is a fundamental solution to the wave equation. Specifically in three dimensions, it is given by

$$S\varphi(t,x) = \frac{1}{4\pi}\int_\Gamma \frac{\varphi(t-|x-y|,y)}{|x-y|}\,ds_y.$$

We similarly define the double-layer potential as

$$D\varphi(t,x) = \int_{\mathbb{R}^+\times\Gamma} \frac{\partial G}{\partial n_y}(t-\tau,x,y)\,\varphi(\tau,y)\,d\tau\,ds_y.$$

For acoustic boundary conditions, we require the *single-layer operator* V, its normal derivative K', the double-layer operator K and hypersingular operator W for $x\in\Gamma$, $t>0$:

$$V\varphi(t,x) = 2\int_{\mathbb{R}^+\times\Gamma} G(t-\tau,x,y)\,\varphi(\tau,y)\,d\tau\,ds_y,$$

$$K\varphi(t,x) = 2\int_{\mathbb{R}^+\times\Gamma} \frac{\partial G}{\partial n_y}(t-\tau,x,y)\,\varphi(\tau,y)\,d\tau\,ds_y,$$

$$K'\varphi(t,x) = 2\int_{\mathbb{R}^+\times\Gamma} \frac{\partial G}{\partial n_x}(t-\tau,x,y)\,\varphi(\tau,y)\,d\tau\,ds_y,$$

$$W\varphi(t,x) = -2\int_{\mathbb{R}^+\times\Gamma} \frac{\partial^2 G}{\partial n_x \partial n_y}(t-\tau,x,y)\,\varphi(\tau,y)\,d\tau\,ds_y.$$

For the absorbing half-space, the *single-layer potential* S *and boundary integral operators* V, K, K', W are analogously defined in terms of an appropriate Green's function which satisfies the acoustic boundary condition (1.2) on Γ_∞. Explicit formulas have been obtained by Ochmann **[40, 41]**, in particular, for $d = 3$:

$$\begin{aligned} V\varphi(t,x) &= \frac{1}{2\pi}\int_\Gamma \frac{\varphi(t-|x-y|,y)}{|x-y|}\,ds_y \\ &\quad + \frac{1}{2\pi}\int_\Gamma \frac{\varphi(t-|x-y'|,y)}{|x-y'|}\,ds_y - \frac{2\alpha_\infty}{\pi} \\ &\quad \times \int_0^\infty\!\!\int_\Gamma \frac{\partial}{\partial s}\left[\frac{H(t-s-|x-y'|)}{\sqrt{(t-s+\alpha_\infty(x_3+y_3))^2+(\alpha_\infty^2-1)R^2}}\right]\varphi(s,y)\,ds_y ds. \end{aligned}$$

Here, y' denotes the reflection of $y=(y_1,y_2,y_3)\in\Gamma$ on the street $\partial\mathbb{R}^3_+$: $y'=(y_1,y_2,-y_3)$. Furthermore, $R^2=(x_1-y_1)^2+(x_2-y_2)^2$, and H is the Heaviside function.

The *boundary integral operators* are considered between space-time anisotropic Sobolev spaces $H^s_\sigma(\mathbb{R}^+,\widetilde{H}^r(\Gamma))$. In order to define them, if $\partial\Gamma\neq\emptyset$, first extend Γ to a closed, orientable Lipschitz manifold $\widetilde{\Gamma}$.

The definition of the usual Sobolev spaces of supported distributions on Γ are:

$$\widetilde{H}^r(\Gamma) = \{u \in H^r(\widetilde{\Gamma}) : \operatorname{supp} u \subset \overline{\Gamma}\}, \qquad r\in\mathbb{R}.$$

Furthermore, $H^r(\Gamma)$ is the quotient space $H^r(\widetilde{\Gamma})/\widetilde{H}^r(\widetilde{\Gamma}\setminus\overline{\Gamma})$.

In order to define an explicit family of Sobolev norms, we introduce a partition of unity α_i subordinate to a covering of $\widetilde{\Gamma}$ by open sets B_i. For diffeomorphisms φ_i mapping each B_i into the unit cube $\subset\mathbb{R}^d$, a family of Sobolev norms is induced from $\mathbb{R}^d$:

$$||u||_{r,\omega,\tilde{\Gamma}} = \left(\sum_{i=1}^p \int_{\mathbb{R}^d} (|\omega|^2+|\xi|^2)^r |\mathcal{F}\left\{(\alpha_i u)\circ\varphi_i^{-1}\right\}(\xi)|^2 d\xi\right)^{1/2}.$$

The norms for different $\omega\in\mathbb{C}\setminus\{0\}$ are equivalent, and $\mathcal{F}$ denotes the

Fourier transform. They induce norms on $H^r(\Gamma)$:

$$||u||_{r,\omega,\Gamma} = \inf_{v \in \tilde{H}^r(\tilde{\Gamma} \setminus \overline{\Gamma})} ||u+v||_{r,\omega,\tilde{\Gamma}},$$

and on $\widetilde{H}^r(\Gamma)$:

$$||u||_{r,\omega,\Gamma,*} = ||e_+ u||_{r,\omega,\tilde{\Gamma}}.$$

e_+ extends distribution u by 0 from Γ to $\widetilde{\Gamma}$. It is stronger than $||u||_{r,\omega,\Gamma}$ whenever $r \in 1/2 + \mathbb{Z}$.

We now define a class of space-time anisotropic Sobolev spaces:

Definition 2.1. For $s, r \in \mathbb{R}$, define

$$\begin{aligned} &H^s_\sigma(\mathbb{R}^+, H^r(\Gamma)) \\ &\quad = \{u \in \mathcal{D}'_+(H^r(\Gamma)) : e^{-\sigma t} u \in \mathcal{S}'_+(H^r(\Gamma)) \text{ and } ||u||_{s,r,\Gamma} < \infty\}, \end{aligned}$$

$$\begin{aligned} &H^s_\sigma(\mathbb{R}^+, \widetilde{H}^r(\Gamma)) \\ &\quad = \{u \in \mathcal{D}'_+(\widetilde{H}^r(\Gamma)) : e^{-\sigma t} u \in \mathcal{S}'_+(\widetilde{H}^r(\Gamma)) \text{ and } ||u||_{s,r,\Gamma,*} < \infty\}. \end{aligned}$$

$\mathcal{D}'_+(E)$, respectively, $\mathcal{S}'_+(E)$, denotes the spaces of distributions, respectively, tempered distributions, on $\mathbb{R}$ with support in $[0, \infty)$, taking values in $E = H^r(\Gamma), \widetilde{H}^r(\Gamma)$. The relevant norms are given by:

$$\begin{aligned} \|u\|_{s,r,\Gamma} &= \left(\int_{-\infty+i\sigma}^{+\infty+i\sigma} |\omega|^{2s} \|\widehat{u}(\omega)\|^2_{r,\omega,\Gamma} d\omega \right)^{1/2}, \\ \|u\|_{s,r,\Gamma,*} &= \left(\int_{-\infty+i\sigma}^{+\infty+i\sigma} |\omega|^{2s} \|\widehat{u}(\omega)\|^2_{r,\omega,\Gamma,*} d\omega \right)^{1/2}. \end{aligned}$$

For $|r| \leq 1$, the spaces are independent of the choice of α_i and φ_i, see **[25, 32]** for a more detailed discussion.

The representation formula uses S and D to express a solution to the wave equation in terms of its Dirichlet and Neumann data on Γ:

Theorem 2.2. *Let* $u \in L^2(\mathbb{R}^+, H^1(\Omega)) \cap H^1_0(\mathbb{R}^+, L^2(\Omega))$ *be the solution of equation* (1.1) *for a Lipschitz boundary* Γ. *Then*:

$$u(t,x) = S\varphi(t,x) - Dp(t,x),$$

where $\varphi = [u]$ *is the jump of* u *across* Γ *and* $p = [\partial u/\partial n]$ *is the jump of the normal flux.*

As shown in **[33]** by reformulation to an interior problem, the initial boundary value problem (1.1) is equivalent to a system of integral equations of the first kind on Γ,

$$\begin{cases} K'p - W\varphi + \alpha(\partial\varphi/\partial t) = F \\ p + \alpha(V\partial_t p + K\partial_t\varphi) = G. \end{cases} \tag{2.1}$$

Here, $\varphi = [u]$ and $p = [\partial u/\partial n]$ as above, and, for an incoming wave u^{inc} scattered by Ω^i, we have

$$F = -2\frac{\partial u^{\mathrm{inc}}}{\partial n} \quad \text{and} \quad G = -2\alpha\frac{\partial u^{\mathrm{inc}}}{\partial t}.$$

If $\alpha^{-1} \in L^\infty(\Gamma)$, we may pair these equations with test functions $\partial_t\psi$, respectively, q/α, to obtain the space-time variational formulation:

Find $\Phi = (\varphi, p) \in H^1_\sigma(\mathbb{R}^+, \widetilde{H}^{1/2}(\Gamma)) \times H^1_\sigma(\mathbb{R}^+, L^2(\Gamma))$ such that, for all $\Psi = (\psi, q) \in H^1_\sigma(\mathbb{R}^+, \widetilde{H}^{1/2}(\Gamma)) \times H^1_\sigma(\mathbb{R}^+, L^2(\Gamma))$:

$$a(\Phi, \Psi) = l(\Psi). \tag{2.2}$$

Here,

$$l(\Psi) = \int_0^\infty \int_\Gamma F\partial_t\psi \, ds_x \, d_\sigma t + \int_0^\infty \int_\Gamma \frac{Gq}{\alpha} \, ds_x \, d_\sigma t, \tag{2.3}$$

and $a(\Phi, \Psi)$ is given by

$$\int_0^\infty \int_\Gamma \Big(\alpha(\partial_t\varphi)(\partial_t\psi) + \frac{1}{\alpha}pq + K'p(\partial_t\psi) - W\varphi(\partial_t\psi) + V(\partial_t p)q + K(\partial_t\varphi)q\Big) \, ds_x \, d_\sigma t, \tag{2.4}$$

for $d_\sigma t = e^{-2\sigma t}dt$, $\sigma > 0$. The complementary Neumann problem, $\alpha = 0$, is discussed in **[16, 27]**.

A single-layer ansatz $u = 2S\phi$ leads to the integral formulation $V\partial_t\phi = \partial_t f$ for the Dirichlet problem. Its variational formulation reads:

Find $\phi \in H^1_\sigma(\mathbb{R}^+, \widetilde{H}^{-1/2}(\Gamma))$ such that

$$b(\phi, \psi) = \langle \partial_t f, \psi \rangle, \quad \text{for all } \psi \in H^1_\sigma(\mathbb{R}^+, \widetilde{H}^{-1/2}(\Gamma)), \tag{2.5}$$

where

$$b(\phi,\psi) = \int_0^\infty \int_\Gamma (V\partial_t\phi)\,\psi\,ds_x\,d_\sigma t,$$
$$\langle\partial_t f,\psi\rangle = \int_0^\infty \int_\Gamma (\partial_t f)\,\psi\,ds_x\,d_\sigma t.$$

Adapting fundamental observations in **[11, 32]** to our situation, the bilinear forms $a(\Phi,\Psi)$ and $b(\phi,\psi)$ are continuous and, in a weak sense, coercive. They are related to the physical energy of the system. As a consequence, both the acoustic and Dirichlet problems admit unique solutions for sufficiently smooth data, see **[25]** for details.

In addition to the variational formulations as integral equations of the first kind, for computations, an integral equation of the second kind will prove useful. We will only state the Neumann case, $\alpha = 0$. Here a single-layer ansatz $u = S\varphi$ leads to the integral equation $(-I + K')\varphi = 2g$ and the weak formulation:

Find $\varphi \in H^1/2_\sigma([0,\infty), \widetilde{H}^{-1/2}(\Gamma))$ such that, for all test functions $\psi \in H^{1/2}_\sigma([0,\infty), H^{-1/2}(\Gamma))$, the following holds:

$$\int_0^\infty \int_\Gamma (-I + K')\,\varphi\,\psi\,ds_x\,d_\sigma t = 2\int_0^\infty \int_\Gamma g\,\psi\,ds_x\,d_\sigma t. \tag{2.6}$$

As it is equivalent to the original initial boundary value problem, this formulation also admits a unique solution for smooth right hand sides. However, while the integral equations of the first kind were related to the energy and coercive, this may not be the case for equation (2.6).

As written, the above integral equations (2.2), (2.5) and (2.6) formally hold both in the whole space $\mathbb{R}^d$ and the half space $\mathbb{R}^d_+$, with layer potentials defined in terms of the Green's function for the appropriate domain as above. The choice of the Green's function assures that, even for the absorbing half-space, we obtain an equation on Γ, not on the unbounded $\partial\Omega^e$.

3. Discretization. If Γ is not polygonal, we approximate it by a piecewise polygonal curve, respectively, surface, and again write Γ for the approximation. For simplicity, when $d = 3$, we will use here a surface composed of N triangular facets Γ_i such that $\Gamma = \cup_{i=1}^N \Gamma_i$. When $d = 2$, we assume $\Gamma = \cup_{i=1}^N \Gamma_i$ is comprised of line segments

Γ_i. In each case, the elements Γ_i are closed with $\operatorname{int}(\Gamma_i) \neq \varnothing$, and for distinct $\Gamma_i, \Gamma_j \subset \Gamma$, the intersection $\operatorname{int}(\Gamma_i) \cap \operatorname{int}(\Gamma_j) = \varnothing$.

For the time discretization, we consider a uniform decomposition of the time interval $[0, \infty)$ into subintervals $I_n = [t_{n-1}, t_n)$ with time step $|I_n| = \Delta t$, such that $t_n = n\Delta t$, $n = 0, 1, \ldots$.

We choose a basis $\varphi_1^p, \ldots, \varphi_{N_s}^p$ of the space V_h^p of piecewise polynomial functions of degree p in space (continuous and vanishing at $\partial\Gamma$ if $p \geq 1$) and a basis $\beta^{1,q}, \ldots, \beta^{N_t,q}$ of the space $V_{\Delta t}^q$ of piecewise polynomial functions of degree q in time (continuous and vanishing at $t = 0$ if $q \geq 1$).

Let $\mathcal{T}_S = \{T_1, \ldots, T_{N_s}\}$ be the spatial mesh for Γ and

$$\mathcal{T}_T = \{[0, t_1), [t_1, t_2), \ldots, [t_{N_t-1}, T)\}$$

the time mesh for a finite subinterval $[0, T)$.

We consider the tensor product of the approximation spaces in space and time, V_h^p and $V_{\Delta t}^q$, associated to the space-time mesh $\mathcal{T}_{S,T} = \mathcal{T}_S \times \mathcal{T}_T$, and we write

$$V_{h,\Delta t}^{p,q} = V_h^p \otimes V_{\Delta t}^q.$$

These approximation spaces lead to Galerkin formulations for the acoustic and Dirichlet problems (2.2), (2.5) and (2.6), e.g., the Galerkin formulation of (2.5) reads:

Find $\phi_{h,\Delta t} \in V_{h,\Delta t}^{p,q}$ such that

$$b(\phi_{h,\Delta t}, \psi_{h,\Delta t}) = \langle (\partial_t f)_{h,\Delta t}, \psi_{h,\Delta t} \rangle \quad \text{for all } \psi_{h,\Delta t} \in V_{h,\Delta t}^{p,q}. \tag{3.1}$$

In **[25, 39]**, we discuss a priori error estimates and the convergence of Galerkin approximations for (2.2) and (2.5) in a half space. Analogous results for the whole space and non-polygonal Γ date back to **[11]**, in slightly different Sobolev norms.

The basic estimate for the Dirichlet problem is:

Theorem 3.1 (**[25]**). *For solutions $\phi \in H_\sigma^1(\mathbb{R}^+, \widetilde{H}^{-1/2}(\Gamma))$ of* (2.5) *and $\phi_{h,\Delta t} \in V_{h,\Delta t}^{p,q}$ of equation* (3.1) *the next a priori estimate holds*:

$$\|\phi - \phi_{h,\Delta t}\|_{0,-1/2,\Gamma,*} \lesssim ||(\partial_t f)_{h,\Delta t} - \partial_t f||_{0,1/2,\Gamma}$$

$$+ \inf_{\psi_{h,\Delta t} \in V_{h,\Delta t}^{p,q}} \left\{ \left(1 + \frac{1}{\Delta t}\right) \|\phi - \psi_{h,\Delta t}\|_{0,-1/2,\Gamma} + \frac{1}{\Delta t} \|\partial_t \phi - \partial_t \psi_{h,\Delta t}\|_{0,-1/2,\Gamma} \right\}.$$

If, in addition, $\phi \in H_\sigma^s(\mathbb{R}^+, H^m(\Gamma))$*, then*

$$\|\phi - \phi_{h,\Delta t}\|_{0,-1/2,\Gamma,*} \lesssim ||(\partial_t f)_{h,\Delta t} - \partial_t f||_{0,1/2,\Gamma} + \left((h^{\alpha_1} + \Delta t^{\beta_1}) \left(1 + \frac{1}{\Delta t}\right) + (h^{\alpha_2} + \Delta t^{\beta_2}) \frac{1}{\Delta t} \right) ||\phi||_{s,m,\Gamma},$$

where

$$\alpha_1 = m + \min\left\{\frac{1}{2}, -\frac{m}{2(m+s)}\right\},$$
$$\beta_1 = m + s + \min\left\{\frac{1}{2}, \frac{m+s}{2m}\right\},$$
$$\alpha_2 = m + \min\left\{\frac{1}{2}, -\frac{m}{2(m+s-1)}\right\},$$
$$\beta_2 = m + s + \min\left\{-\frac{1}{2}, -1 + \frac{m+s-1}{2m}\right\},$$

and $m \geq -1/2$*,* $s \geq 0$*.*

For the acoustic problem, we introduce the norm

$$|||p, \varphi||| = \left(||p||^2_{0,0,\Gamma} + ||\varphi||^2_{0,1/2,\Gamma,*} + ||\partial_t \varphi||^2_{0,0,\Gamma} \right)^{1/2}.$$

Theorem 3.2 ([**25**])**.** *Assume, for simplicity, that* $\alpha^{-1} \in L^\infty(\Gamma)$*. For the solutions*

$$\Phi = (p, \varphi) \in H_\sigma^1(\mathbb{R}^+, \widetilde{H}^{1/2}(\Gamma)) \times H_\sigma^1(\mathbb{R}^+, L^2(\Gamma))$$

of equation (2.2) *and*

$$\Phi_{h,\Delta t} = (p_{h,\Delta t}, \varphi_{h,\Delta t}) \in V_{h,\Delta t}^{\tilde{p},\tilde{q}} \times V_{h,\Delta t}^{p,q}$$

of its discretization, the a priori estimate holds:

$$
\begin{aligned}
&|||p - p_{h,\Delta t}, \varphi - \varphi_{h,\Delta t}||| \\
&\quad \lesssim ||F_{h,\Delta t} - F||_{0,0,\Gamma} + ||G_{h,\Delta t} - G||_{0,0,\Gamma} + \max\left(\frac{1}{\Delta t}, \frac{1}{\sqrt{h}}\right) \\
&\quad \times \inf_{(q_{h,\Delta t},\psi_{h,\Delta t}) \in V^{\tilde{p},\tilde{q}}_{h,\Delta t} \times V^{p,q}_{h,\Delta t}} \left(||p - q_{h,\Delta t}||_{1,0,\Gamma} + ||\varphi - \psi_{h,\Delta t}||_{1,1/2,\Gamma}\right).
\end{aligned}
$$

As for the Dirichlet problem, better estimates are obtained under smoothness assumptions, $\varphi \in H^{s_1}_\sigma(\mathbb{R}^+, H^{m_1}(\Gamma))$, $p \in H^{s_2}_\sigma(\mathbb{R}^+, H^{m_2}(\Gamma))$, **[25]**.

We refer to **[27]** for an analysis of the Neumann problem. While computationally convenient, the analysis of numerical methods based on equation (2.6) remains open. In particular, schemes based on equation (2.6) are not known to be stable, or to admit unique discrete solutions.

4. A posteriori error estimates. Computable error indicators are a key ingredient to design adaptive mesh refinements. For time-dependent boundary element methods such efficient and reliable estimates of residual type have been obtained in **[26]**, see also **[29, 30, 31, 44]** for alternative error indicators and relevant estimates for the boundary integral operators. In the case of the Dirichlet problem, we obtain **[26]**:

Theorem 4.1 (**[26]**). *Let* $\phi, \phi_{h,\Delta t} \in H^1_0([0,T], H^{-1/2}(\Gamma))$ *be the solutions to equation* (2.5)*, respectively, equation* (3.1)*. Assume that*

$$R = \dot{f} - V\dot{\phi}_{h,\Delta t} \in H^0([0,T], H^1(\Gamma)).$$

Then

$$
\begin{aligned}
\|\phi - \phi_{h,\Delta t}\|^2_{0,-1/2,\Gamma} &\lesssim \|R\|_{0,1,\Gamma}\left(\Delta t\|\partial_t R\|_{0,0,\Gamma} + \|h \cdot \nabla R\|_{0,0,\Gamma}\right) \\
&\lesssim \max\{\Delta t, h\}(\|\partial_t R\|_{L^2([0,T],L^2(\Gamma))} \\
&\qquad + \|\nabla R\|_{L^2([0,T],L^2(\Gamma))})^2.
\end{aligned}
$$

Remark 4.2. The estimate generalizes to arbitrary subspaces V in place of $V^{p,q}_{h,\Delta t}$, in particular, discretizations with smooth ansatz functions in time are of interest **[44]**.

a) As the single-layer potential maps $H^1([0,T],L^2(\Gamma))$ continuously to $H^0([0,T],H^1(\Gamma))$, $V\dot{\phi}_{h,\Delta t}$ belongs to $H^0([0,T],H^1(\Gamma))$, if, for example, $\phi_{h,\Delta t} \in H^2([0,T],L^2(\Gamma))$. The a posteriori estimate is therefore only valid for sufficiently smooth discretizations, e.g., constructed from C^1-continuous splines.

b) In practice, we will use here

$$\Delta t\|\partial_t R\|_{0,0,\Gamma} + \|h \cdot \nabla R\|_{0,0,\Gamma}$$

as an error indicator.

For the acoustic problem, a simple error estimate reads as follows:

Theorem 4.3 ([**26**]). *Let*

$$(\varphi,p),(\varphi_{h,\Delta t},p_{h,\Delta t}) \in H_0^1([0,T],H^{1/2}(\Gamma)) \times H^1([0,T],L^2(\Gamma))$$

be the solutions to equation (2.2) *and its discretized variant, and assume that*

$$R_1 = F - \alpha\dot{\varphi}_{h,\Delta t} + 2K' p_{h,\Delta t} - 2W\varphi_{h,\Delta t} \in L^2([0,T],L^2(\Gamma)),$$
$$R_2 = G + \alpha^{-1}p_{h,\Delta t} + 2V\dot{p}_{h,\Delta t} + 2K\dot{\varphi}_{h,\Delta t} \in L^2([0,T],L^2(\Gamma)).$$

Then

$$|||p - p_{h,\Delta t},\varphi - \varphi_{h,\Delta t}||| \lesssim \|R_1\|_{0,0,\Gamma} + \|R_2\|_{0,0,\Gamma}.$$

5. Algorithmic considerations. For piecewise constant test functions the Galerkin discretization in space and time leads to a block-lower-triangular system of equations, which can be solved by blockwise forward substitution.

For example, the Dirichlet problem yields an algebraic system of the form

$$\sum_{m=1}^{n} V^{n-m}b^m = 2(f^{n-1} - f^n)$$

in time step $n = 1, 2, 3, \ldots$. It can be solved by forward substitution, giving rise to the *marching in on time* (MOT) scheme

$$V^0 b^n = 2(f^{n-1} - f^n) - \sum_{m=1}^{n-1} V^{n-m}b^m.$$

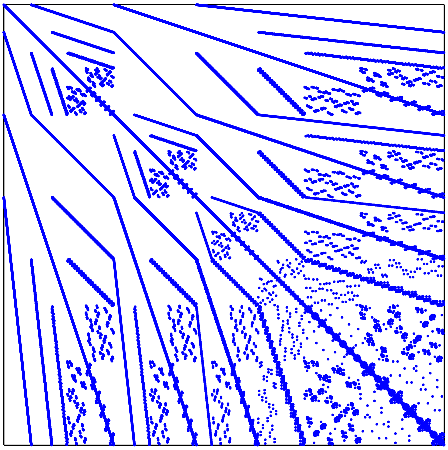


FIGURE 1. Sparsity pattern of the Galerkin matrix V^0 for a uniform discretization of the sphere [**42**].

The Galerkin solution of equation (3.1) is then given by:

$$\dot{\varphi}_{h,\Delta t}(x,t) = \sum_{m=1}^{N_t} \sum_{i=1}^{N_s} b_i^m \beta^{m,0}(t) \varphi_i^p(x).$$

The above fully discrete systems involve the computation of a series of matrices, that (if $\alpha_\infty = 0$) are sparsely populated, because the Dirac-delta fundamental solution restricts the number of interacting elements per time step. Figure 1 shows the distribution of nonzero matrix entries for a typical matrix V^0, when Γ is an approximation of S^2 by 5120 triangles. Note that the computation of each matrix depends only upon the time difference. Furthermore, for bounded surfaces Γ, the matrices V^{n-m} vanish whenever the time difference $l := n - m$ satisfies

$$l > \left[\frac{\operatorname{diam} \Gamma}{\Delta t} \right],$$

i.e., the light cone has traveled through the entire surface Γ.

5.1. An *hp*-composite quadrature of matrix elements. The most time consuming part in the MOT algorithm is the matrix computation although the resulting matrices are sparse in each time step. An efficient *hp* composite Gauss-quadrature allows for computation of the entries in V^l, and similarly for the other layer operators [**37, 42, 47**].

Recall the form of the matrix entries of V^l in $\mathbb{R}^3$ as an example:

$$\frac{1}{2\pi}\iiint_{\mathbb{R}^+\times\Gamma\times\Gamma}\frac{\varphi_i^p(y)\partial_t\beta^{n,q}(t-|x-y|)}{|x-y|}\,\varphi_j^p(x)\beta^{m,q}(t)\,ds_y\,ds_x\,d_\sigma t.$$

The time integrals are first evaluated analytically and result in an integration domain

$$E=\{(x,y)\in\Gamma\times\Gamma : r_{\min}\le|x-y|\le r_{\max}\}$$

of the form of a light cone, $r_{\min}$ and $r_{\max}$, depending upon t_m and t_n. It remains to evaluate terms such as

$$G_{ij}^\nu=\iint_E k_\nu(x-y)\varphi_i^p(y)\varphi_j^p(x)\,ds_y\,ds_x, \tag{5.1}$$

where $k_\nu(x-y)=|x-y|^\nu$ denotes a weakly singular kernel function. Our numerical quadrature separates the outer spatial integration from the one which is singular inner. Define the domain of influence of $x\in\mathbb{R}^3$ by

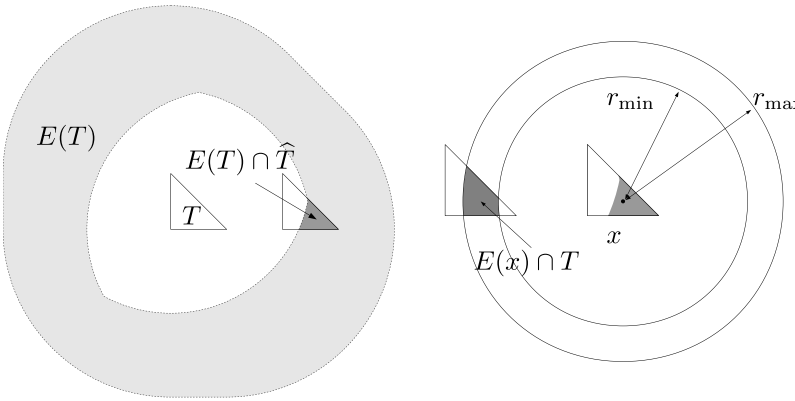
$$E(x):=B_{r_{\max}}(x)\setminus B_{r_{\min}}(x)=\left\{y\in\mathbb{R}^3 : r_{\min}\le|x-y|\le r_{\max}\right\},$$

as in Figure 2 (b). Figure 2 (a) similarly sketches the domain of influence of a triangle T,

$$E(T):=\bigcup_{x\in T}E(x)=\{y\in\mathbb{R}^3 : r_{\min}\le|x-y|\le r_{\max},\ x\in T\}.$$

Defining $E(T_j,T_i):=E\cap(T_j\times T_i)$, we rewrite equation (5.1) as

$$\begin{aligned}G_{ij}^\nu&=\sum_{\substack{T_{i'}\subset\operatorname{supp}\varphi_i\\T_{j'}\subset\operatorname{supp}\varphi_j}}\ \iint_{E(T_{j'},T_{i'})}k_\nu(x-y)\varphi_i^p(y)\varphi_j^p(x)\,ds_y\,ds_x\\&=\sum_{\substack{T_{i'}\subset\operatorname{supp}\varphi_i\\T_{j'}\subset\operatorname{supp}\varphi_j}}\ \int_{T_{j'}\cap E(T_{i'})}\varphi_j^p(x)P_{i,i'}^p(x)\,ds_x,\end{aligned}$$



(a) Outer integral: Domain of influence of triangle $\widehat{T}$ intersected with triangle T.

(b) Inner integral: Domain of influence $E(x)$ of point $x \in E(T) \cap \widehat{T}$.

FIGURE 2. Domains of influence and the illumination of test and trial elements $\widehat{T}$ and T during the evaluation of the inner and outer integral.

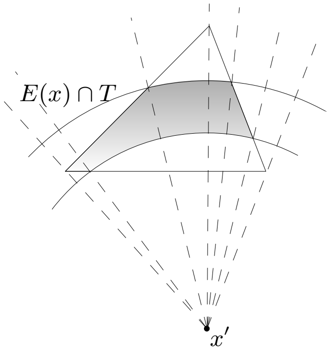
with a retarded potential $P_{i,i'}$ given by

$$P_{i,i'}(x) := \int_{E(x)\cap T_{i'}} k_\nu(x-y)\varphi_i^p(y)\, ds_y.$$

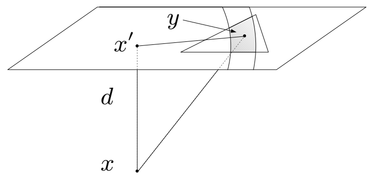
To simplify notation, we explain the quadrature for a simplified integral. Given triangles T and $\widehat{T}$ and basis functions φ and $\widehat{\varphi}$ defined on T and $\widehat{T}$, respectively, a typical entry in the Galerkin matrix reads

$$\int_{E(T)\cap\widehat{T}} P\varphi(x)\widehat{\varphi}(x)\, ds_x, \quad P\varphi(x) := \int_{E(x)\cap T} k_\nu(x-y)\varphi(y)\, ds_y. \tag{5.2}$$

We evaluate the outer and inner integrals step-by-step, decomposing the integration domain and using a grading strategy for the different singularities. It is crucial to take into account the cut-off behavior due to the different domains of influence, and below, we recall the rigorous error analysis.



(a) Decomposition of $E(x) \cap T$ w.r.t. x' into $n_d = 5$ subelements.



(b) Projection of x onto the triangle plane.

FIGURE 3.

5.1.1. Composite inner quadrature. In order to calculate $P\varphi$ as defined in equation (5.2), we first seek a parametric representation of $E(x) \cap T$. Let x' denote the orthogonal projection of x onto the triangular plane $\mathcal{E}_T$, and set $d := |x - x'|$, cf., Figure 3 (b). With $r'_{\min\,/\,\max} := (r^2_{\min\,/\,\max} - d^2)^{1/2}$, we obtain

$$\begin{aligned} E(x) \cap \mathcal{E}_T &= (B_{r'_{\min}}(x') \setminus B_{r'_{\max}}(x')) \cap \mathcal{E}_T \\ &= \{ y \in \mathcal{E}_T \ : \ r'_{\min} \le |x' - y| \le r'_{\max} \}, \\ E(x) \cap T &= (B_{r'_{\min}}(x') \setminus B_{r'_{\max}}(x')) \cap T. \end{aligned}$$

We introduce polar coordinates (r, θ) around x' and decompose

$$E(x) \cap T = \bigcup_{l=1}^{n_d} D_l,$$

$$D_l := \{ (r, \theta) : \theta \in (\theta_l, \theta_{l+1}) \text{ and } r \in (r_{1,l}(\theta), r_{2,l}(\theta)) \},$$

where it can be shown that $n_d \le 12$ and

$$r_{1,l} := \begin{cases} r'_{\min} & e \in B_{r'_{\min}(x)} \\ r_e(\theta) & \text{else}, \end{cases}$$

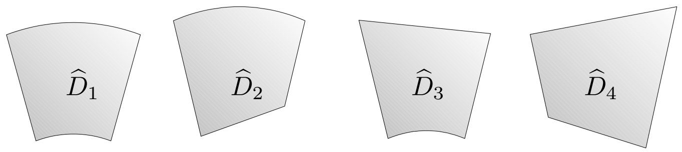


FIGURE 4. Generic integration domains.

$$r_{2,l} := \begin{cases} r'_{\max} & e \notin B_{r'_{\max}(x)} \\ r_e(\theta) & \text{else.} \end{cases}$$

Here, $r_e(\theta)$ is the parametrization of the intersected triangle edge e in polar coordinates with respect to x'. In terms of the normal vector n of e and the end point v of e,

$$r_e(\theta) = \frac{v \cdot n}{n_1 \cos\theta + n_2 \sin\theta}.$$

Four generic cases of decomposition types are sketched in Figure 4:

$$\begin{aligned} \widehat{D}_1 &:= \{(r,\theta) : \theta \in (\theta_1,\theta_2) \quad \text{and} \quad r \in (r_{\min}, r_{\max})\}, \\ \widehat{D}_2 &:= \{(r,\theta) : \theta \in (\theta_1,\theta_2) \quad \text{and} \quad r \in (r_{e_1}(\theta), r_{\max})\}, \\ \widehat{D}_3 &:= \{(r,\theta) : \theta \in (\theta_1,\theta_2) \quad \text{and} \quad r \in (r_{\min}, r_{e_2}(\theta))\}, \\ \widehat{D}_4 &:= \{(r,\theta) : \theta \in (\theta_1,\theta_2) \quad \text{and} \quad r \in (r_{e_1}(\theta), r_{e_2}(\theta))\}. \end{aligned}$$

From equation (5.2), we obtain

$$P\varphi(x) = \sum_{l=1}^{n_d} \int_{\hat{D}_l} (d^2 + r^2)^{\nu/2} \varphi(r,\theta) r \, dr \, d\theta,$$

where $d > 0$ and φ is sufficiently regular. For each of the domains $\widehat{D}_l$, we can write the integral as

$$\begin{aligned} I^{(\hat{D}_l)} f &:= \int_{\theta_1}^{\theta_2} \int_{r_1(\theta)}^{r_2(\theta)} f(r,\theta) \, dr \, d\theta, \\ f(r,\theta) &:= (d^2 + r^2)^{\nu/2} \varphi(r,\theta) r. \end{aligned} \tag{5.3}$$

We introduce our quadrature method by

$$Q_n^{[a,b]} f := \sum_{i=1}^{n} w_i f(x_i)$$

to denote the Gauß-Legendre quadrature rule with n quadrature points for evaluating

$$\int_a^b f\, dx.$$

Given a subdivision of $[a, b]$ into m subintervals I_j, a variable order composite Gauß rule with degree vector $\mathbf{n} = (n_1, \ldots, n_m)$ is defined by

$$Q_{n,m,\sigma} f := \sum_{j=1}^{m} Q_{n_j}^{I_j} f.$$

We use a geometric subdivision of $[a, b]$ with m levels and grading parameter $\sigma \in (0, 1)$:

$$[a, b] = \bigcup_{j=1}^{m} I_j,$$

where, for $j = 1, \ldots, m$, we let

$$\begin{aligned} I_j &:= [x_{j-1}, x_j], \\ x_0 &:= a, \\ x_j &:= a + (b - a)\sigma^{m-j}. \end{aligned}$$

For $n_r = (n_1^{(r)}, \ldots, n_m^{(r)})$, $m_r \geq 1$ and $\sigma_r \in (0, 1]$, the integral (5.3) is then computed as

$$Q^{\hat{D}_l} f := Q_{n_\theta}^{[\theta_1,\theta_2]}(Q_{n_r,m_r,\sigma_r}^{[r_1(\theta),r_2(\theta)]} f).$$

5.1.2. Error analysis for the evaluation of equation (5.3). A detailed analysis **[42]** shows that the integrand belongs to the countably normed, weighted space $B_\beta^0(T)$ of Babuska **[10]**.

Definition 5.1 (Countably normed space $B_\beta^l(T)$)**.** We say $u \in B_\beta^l(T)$ with respect to a weight function $\Phi_{\beta,\alpha,l}$, if $u \in H^{l-1}(T)$, and if

$$\|\Phi_{\beta,\alpha,l} D^\alpha u\|_{L^2(\Omega)} \leq C d^{|\alpha|-l} (|\alpha| - l)!$$

for $|\alpha| = l, l+1, \dots$. Here, the constants $C > 0$ and $d \geq 1$ are independent of $|\alpha|$.

If the number of angular quadrature points, n_θ, is chosen proportional to m_r, we obtain the next theorem on the accuracy of the quadrature in our TDBEM:

Theorem 5.2 ([**42**]). *Set a function $f \in B^0_\beta(T)$ with a weight function $\Phi_{\beta,\alpha,0}(r) = r^{|\alpha|+\beta}$, and let $\max(1, \sqrt{r_{\max}})(\theta_2 - \theta_1) < eC_\theta$. Then the following holds for $\widehat{D}_l$:*

$$|I^{(\hat{D}_l)} f - Q^{(\hat{D}_l)} f| \leq Ce^{-b\sqrt[3]{N}}$$

for $l = 1, \dots, 4$. Here, N denotes the total number of quadrature points, and b and C are positive constants independent of N, dependent on the grading factor σ_r, the number of levels m_r, and on f. Also,

$$C_\theta := \begin{cases} 1 & \text{for } \widehat{D}_1, \\ \min\limits_{\theta \in (\theta_1, \theta_2)} |\cos(\theta - \theta_1^*)| & \text{for } \widehat{D}_2, \\ \min\limits_{\theta \in (\theta_1, \theta_2)} |\cos(\theta - \theta_2^*)| & \text{for } \widehat{D}_3, \\ \min\limits_{\theta \in (\theta_1, \theta_2)} (|\cos(\theta - \theta_1^*)|, |\cos(\theta - \theta_2^*)|) & \text{for } \widehat{D}_4, \end{cases}$$

and θ_i^* denotes the angle corresponding to the edge normal n_i, $i = 1, 2$.

6. Numerical experiments for tires. The numerical experiments in this section will use the discretization of the Neumann problem, equation (2.6) in $\mathbb{R}^3_+$, with $\alpha_\infty = 0$. It illustrates selected results from [**16**] for ansatz and test functions which are piecewise constant in space and time. We use $\sigma = 0$ for the computations.

6.1. Validation on a problem with a known solution. Considering a wave problem with a known solution p in the exterior of a unit ball in $\mathbb{R}^3_+$ allows us to analyze the convergence properties of our method. For some fixed $0 < R < 1$, one obtains a radial pulse which solves equation (1.1) outside a unit sphere at a distance h above the street:

$$u(t,x) = \frac{r(h) - t}{2r(h)} \left[1 + \cos\left(\frac{\pi(r(h) - t)}{R}\right)\right] H(R - |r(h) - t|)$$

$$+\frac{r(-h)-t}{2r(-h)}\left[1+\cos\left(\frac{\pi(r(-h)-t)}{R}\right)\right]H(R-|r(-h)-t|).$$

Here, $H(t)$ denotes the Heaviside function and $r(h)=\|x_1,x_2,x_3-h-1\|$ and $r(-h)=\|x_1,x_2,x_3+h+1\|$. By a modification of **[44]** to the half-space, the density for the single layer potential ansatz is

$$\varphi(t,x)=-2\sum_{k=0}^{\lfloor t/2\rfloor} f_1(t-2k)+2\sum_{k=0}^{\lfloor t/2\rfloor}\int_{2k}^{t} e^{-(s-2k)} f_1(t-s)\,ds, \tag{6.1}$$

where

$$f_1(t)=\left[\frac{t}{2r(h)^2}\left(1+\cos\left(\frac{\pi(r(h)-t)}{R}\right)\right)\right.$$
$$\left.-\frac{\pi}{R}\frac{r(h)-t}{2r(h)}\sin\left(\frac{\pi(r(h)-t)}{R}\right)\right]H(R-|r(h)-t|).$$

Figure 4 shows the relative discretization errors

$$\frac{\|\varphi_{\Delta t,h}-\varphi\|_{L^2([0,10];L^2(\Gamma))}}{\|\varphi\|_{L^2([0,10];L^2(\Gamma))}}$$

and

$$\frac{\|u_{\Delta t,h}(t,x_0)-u(t,x_0)\|_{L^2([0,10])}}{\|u(t,x_0)\|_{L^2([0,10])}},$$

with $\varphi_{\Delta t,h}$ the TD-BEM Galerkin approximation of φ and $p_{\Delta t,h}=S\varphi_{\Delta t,h}$ on uniform meshes. Here, $x_0=(0,0,2.8)^\top$ for $h=0.63$, $R=0.9$. The figure shows a convergence rate of 0.4 for the density, respectively, 0.65 for the sound pressure, with respect to the degrees of freedom (dof), i.e., the product of the number of time steps and the number of triangles. The ratio of mesh size h and time step size Δt is $\Delta t/h\approx 0.38$.

6.2. Vibrating tire. Cyclic deformations of a moving tire enter the computations through the right hand side f in equation (2.6). Physically, the right hand side f is the result of the tire vibrations

$$f=-2\rho\frac{\partial^2 v_n}{\partial t^2}.$$

Here, v_n describes the displacement of the tire in the outer normal direction and ρ the density of air. In **[16]**, f is determined from

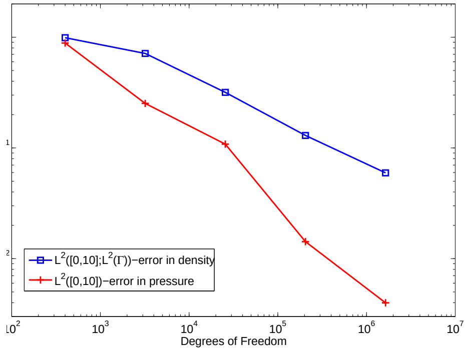


FIGURE 5. Relative L^2-errors of density $\varphi_{\Delta t,h}$ and pressure $p_{\Delta t,h}$ [**16**].

the particle velocity $\partial u/\partial\tau$ on Γ, as supplied by the work group of W. Kropp at the Chalmers University in Gothenburg at the LeiStra3 cooperation. These particle velocities are given for 513 equidistant frequency points between 0 Hz and 1809.4 Hz in each of the 6,027 nodes of the triangulation in Figure 6.

Figure 7 displays the A-weighted sound pressure level of the radiated acoustic wave. The simulation parameters are $\Delta t = 0.01$ averaged over 321 points in the hemisphere

$$\{x \in \mathbb{R}^3_+ : \|x\|_2 = 1\}$$

[**16**]. These curves are obtained by a fast Fourier transform (FFT) of the calculated sound pressure level for times $t \geq t_0$, with $t_0 = 0, 0.005, 0.02$. The blue reference curve [**51**] is calculated by a Burton-Miller stabilized BEM collocation method for the Helmholtz equation with piecewise constant trial functions.

Further computational results in [**16**] for truck tires and the sound amplification in the horn geometry underline that the methods presented in this paper are competitive for industrial scale transient and broad-band frequency domain computations.

7. Adaptive mesh refinements. Fully space-time adaptive methods were explored by Gläfke [**29**] for two-dimensional problems. He

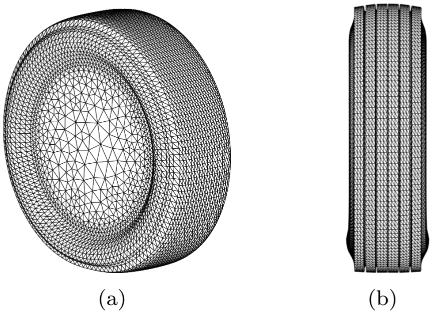


FIGURE 6. Discretization of car, respectively, truck, tires used for computations [**16**].

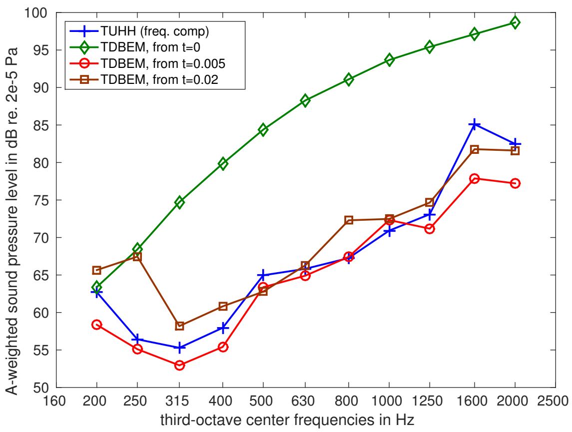


FIGURE 7. Comparison of the A-weighted sound pressure level averaged over 321 points and frequency bands for TDBEM and frequency domain BEM [**16**].

does not treat the temporal domain separately from the spatial domain but refines the mesh of the space-time cylinder. More precisely, the rectangular space-time elements are refined into four equally-sized rectangles. The flexibility of this approach comes with an additional computational cost: the MOT scheme no longer applies and one must store and solve the full space-time system in one step.

Error indicators η, such as those from a posteriori error estimates derived in Theorem 4 of Section 4, lead to an adaptive algorithm, based on the four steps:

$$\textbf{SOLVE} \longrightarrow \textbf{ESTIMATE} \longrightarrow \textbf{MARK} \longrightarrow \textbf{REFINE}.$$

Space-time adaptive algorithm in two dimensions [29]:
Input: Mesh $\mathcal{T} = (\mathcal{T}_S \times \mathcal{T}_T)_0$, refinement parameter $\theta \in (0,1)$, tolerance $\epsilon > 0$, data f.

(1) Solve $V\dot{\varphi}_{h,\Delta t} = \dot{f}$ on $\mathcal{T}$.
(2) Compute the error indicators $\eta(\Box)$ in each space-time rectangle.
(3) Stop if $\sum_i \eta^2(\Box_i) < \epsilon^2$.
(4) Mark all $\Box \in \mathcal{T}$ which satisfy refinement criterion based on θ.
(5) Refine each marked $\Box$ into four new rectangles to obtain a new mesh $\mathcal{T}$.
(6) Go to 1.

Output: Approximation of $\dot{\varphi}$.

In the following experiment, Gläfke uses a box pulse of the form

$$H(x_1 + x_2 + 2\alpha t + \lambda) - H(x_1 + x_2 + 2\alpha t)$$

as the incident signal of the scattering problem with scatterer $[1,1]^2$. Here, $\lambda = 0.05$ and $\alpha = 1/\sqrt{2}$. The box pulse is non-smooth, which appears to have an affect on the regularity of the solution of the problem. Convergence order for the adaptive version turns out to be genuinely higher than that of the uniform version, as well as for large degrees of freedom. The meshes that result from the adaptive algorithm, as shown in Figure 8, are heavily refined along the part of the surface of the space-time cylinder where the box pulse moves along the scatterer.

Space-time adaptive methods in three dimension, on the other hand, are still in their infancy. As a test case, in **[26]** we concentrate on time-independent geometric singularities of the solution, e.g., in the horn geometry between the tire and the street. In this case we expect to have time-independent meshes, refined near the singularities, which do not require an update of the Galerkin matrices in every time step.

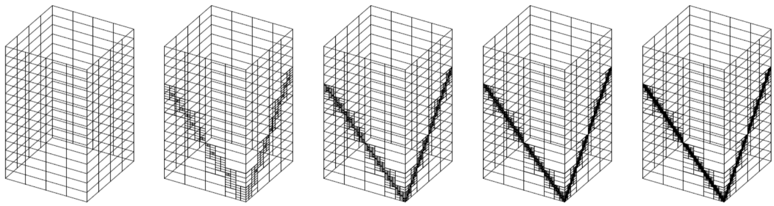


FIGURE 8. Adaptive mesh refinements for a box pulse in 2d **[29]**.

From the discrete solution $\dot{\varphi}_{h,\Delta t}$ of the Dirichlet problem equation (3.1) and $\dot{f}$ we determine in every triangle $\triangle$ the time integrated local error indicator

$$\eta(\triangle)^2 = \int_0^T \int_\triangle [h\nabla_\Gamma(\dot{f} - V\dot{\varphi}_{h,\Delta t})]^2,$$

where the time integral is approximated by a Riemann sum.

The error indicators $\eta(\triangle)$ lead to the next algorithm.

Adaptive algorithm [26]:
Input: Mesh $\mathcal{T} = \mathcal{T}_0$, refinement parameter $\theta \in (0, 1)$, tolerance $\epsilon > 0$, data f.

(1) Solve $V\dot{\varphi}_{h,\Delta t} = \dot{f}$ on $\mathcal{T}$.
(2) Compute the error indicators $\eta(\triangle)$ in each triangle $\triangle \in \mathcal{T}$.
(3) Find $\eta_{\max} = \max_\triangle \eta(\triangle)$.
(4) Stop if $\sum_i \eta^2(\triangle_i) < \epsilon^2$.
(5) Mark all $\triangle \in \mathcal{T}$ with $\eta(\triangle_i) > \theta\eta_{\max}$.
(6) Refine each marked triangle into four new triangles to obtain a new mesh $\mathcal{T}$
(and project the new nodes onto the sphere). Choose Δt such that $\Delta t/\Delta x \leq 1$ for all triangles.
(7) Go to 1.

Output: Approximation of $\dot{\varphi}$.

According to the a posteriori estimates derived in **[26]**, the error between the approximate and the actual solution to the problem is bounded by a multiple of ϵ, up to quantities involving time-derivatives of the residual $\dot{f} - V\dot{\varphi}_{h,\Delta t}$.

We consider the Dirichlet problem for the wave equation in the exterior of the three-dimensional (discretised) unit ball with a singular right hand side. We choose the right-hand side as $\dot{f}(t,x) = 2$ if $x_1 > 0$, and 0 otherwise. The function $\dot{f}$ is a toy example for a time-independent singularity, similar to the singular horn-like geometry where a tire meets a street, see **[16]**. We expect adaptive mesh refinements to be concentrated around the line of discontinuity of $\dot{f}$, given by $x_1 = 0$. For simplicity, we neglect the error of the surface approximation.

The numerical experiment depicted in Figure 9 shows the mesh generated by the above adaptive algorithm after three mesh refinements, beginning with an initial icosahedral triangulation of the sphere with 80 nodes. Most refinements are near the discontinuity of f, as expected.

The above experiment presents only a first step towards space-time adaptive TDBEM, for the case of the geometric singularities relevant to sound radiation of tires. The optimal use of space-time adaptivity and its application to the acoustic boundary conditions remain to be explored.

8. Outlook. Engineering problems, such as sound emission of car and truck tires or scattering problems, motivate the analysis of coupled finite and boundary elements. The first work in this direction investigates the coupling of different time and spatial discretizations for scalar wave equations **[1, 4, 8, 13]**. Waves scattered by an immersed elastic object in a fluid provide a key example of practical interest. A basic well-posedness theory for the time-dependent problem can be found in **[23, 36]**. The a priori and a posteriori analysis of numerical discretizations based on Galerkin TDBEM **[28]**, respectively, convolution quadrature **[35]**, has recently been considered.

For large-scale engineering computations, the efficient assembly of the space-time Galerkin matrix proves crucial. Fast multipole methods based on perturbative expansions of the Green's function in the far field are being developed, in particular, by Sylvand **[48]**; also see **[12]** for related work on the case of the Helmholtz-based convolution quadrature.

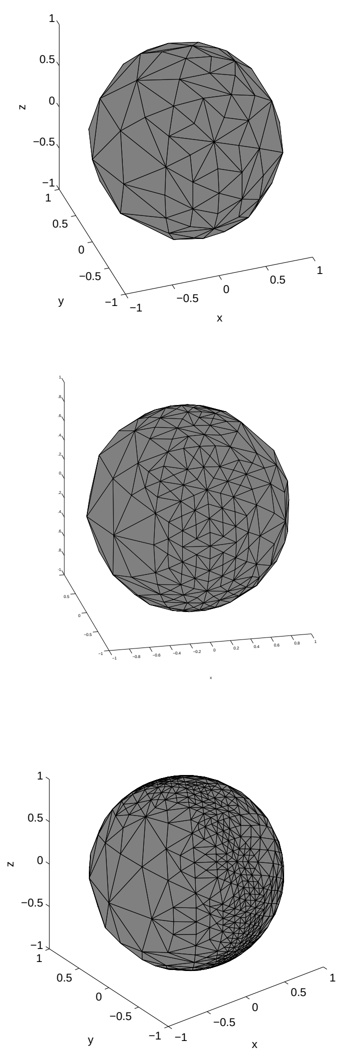


FIGURE 9. The first three adaptively generated meshes for $V\dot{\varphi} = \dot{f}$ starting from an icosahedral triangulation with 80 nodes, $\theta = 0.9$ **[26]**.

In this article, we have provided a survey of recent advances in time domain boundary element methods for the wave equation and applications to engineering problems. This approach proves efficient and highly accurate for scattering and emission problems, and we demonstrate its relevance to applications in traffic noise. The a posteriori estimates presented in this survey lead to fast space-time adaptive mesh refinements. They encompass a first step towards high order *hp*-adaptive methods in space and time.

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Heriot-Watt University, Maxwell Institute for Mathematical Sciences and Department of Mathematics, Edinburgh EH14 4AS, UK AND University of Paderborn, Institute for Mathematics, Warburger Str. 100, Paderborn, 33098 Germany
Email address: **h.gimperlein@hw.ac.uk**

Brunel University, Department of Mathematics, Uxbridge, UB8 3PH, United Kingdom
Email address: **matthias.maischak@brunel.ac.uk**

Leibniz University Hannover, Institute of Applied Mathematics, Hannover, 30167 Germany
Email address: **stephan@ifam.uni-hannover.de**