EXISTENCE OF SOLUTIONS AND CONTROLLABILITY FOR IMPULSIVE FRACTIONAL ORDER DAMPED SYSTEMS

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ABSTRACT. In this paper, we are concerned with the controllability of linear and nonlinear Caputo impulsive fractional order damped systems in Banach spaces. Our main purpose is to establish some necessary and sufficient conditions for controllability for this kind of impulsive control system by using Mittag-Leffler matrix functions and the Schauder fixed point theorem.

1. Introduction. Control theory is an area of application-oriented mathematics which deals with basic principles underlying the analysis and design of control systems. It is well known that the study of controllability plays an important role in control theory and engineering. Although most dynamical systems are analyzed in either the continuous- or discrete-time domain, many real systems in physics, chemistry, biology, engineering, and information science, may experience abrupt changes at certain instances during continuous dynamical processes. These kinds of impulsive behaviors can be modeled by impulsive systems. Instantaneous impulsive fractional differential equations are used to describe some practical dynamical systems including evolutionary processes which are characterized by abrupt changes of the state in certain instances. The theory of instantaneous impulsive fractional differential equations has found its extensive applications in realistic mathematical models and has emerged as an important area of investigation in recent years. The study of impulsive control

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systems has received increasing interest and has had rapid development in the past few years, since dynamical systems with impulsive effects have great importance in applied sciences (see, for example, [3, 4, 6, 8, 9, 10, 11, 12, 16, 17, 18, 20, 21, 22, 23, 24, 25, 26] and the references therein).

In this paper, we study controllability results for the following fractional order damped impulsive linear control systems:

$$(1.1) \quad \begin{cases} {}^cD_t^\alpha x(t) = A^cD_t^\beta x(t) + Bu(t) & t \in J', \\ \Delta x(t_i) := I_i(x(t_i^-)), \ \Delta x'(t_i) := J_i(x'(t_i^-)) & i = 1, 2, \ldots, k, \\ x(0) = x_0, \ x'(0) = x_0', \end{cases}$$

and the following fractional order damped impulsive nonlinear control systems:

$$(1.2) \quad \begin{cases} {}^cD_t^\alpha x(t) = A^cD_t^\beta x(t) + Bu(t) + f(t, x(t)) & t \in J', \\ \Delta x(t_i) := I_i(x(t_i^-)), \ \Delta x'(t_i) := J_i(x'(t_i^-)) & i = 1, 2, \ldots, k, \\ x(0) = x_0, \ x'(0) = x_0', \end{cases}$$

where $0 < \beta \le 1 < \alpha \le 2$, ${}^cD_t^\alpha$ and ${}^cD_t^\beta$ denote the Caputo fractional derivatives of orders α and β with lower limit zero, respectively. A and B are $n \times n$ and $n \times m$ matrices, respectively.

$$J = [0, T], \qquad J' = J - \{t_1, t_2, \cdots, t_k\}$$

and

$$0 = t_0 < t_1 < t_2 < \cdots < t_i < \cdots < t_k < t_{k+1} = T < \infty,$$

$f : J \times \mathbb{R}^n \to \mathbb{R}^n$, $I_i, J_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2, \ldots, k$. $x(t) \in \mathbb{R}^n$ is the state variable. $u(t) \in \mathbb{R}^m$ is the control variable. $\Delta x(t_i) := x(t_i^+) - x(t_i^-)$, where $x(t_i^+) = \lim\limits_{\epsilon \to 0^+} x(t_i + \epsilon)$ and $x(t_i^-) = \lim\limits_{\epsilon \to 0^-} x(t_i + \epsilon)$ represent the right and left limits of $x(t)$ at $t = t_i$, respectively. $\Delta x'(t_i)$ is similarly defined.

The fractional order damped control systems without impulse effect in finite-dimensional spaces above were considered by Balachandran, Govindaraj, Reiver and Trujillo [2]. They studied the controllability of linear and nonlinear fractional order damped systems. In this paper, we deal with the controllability of linear and nonlinear Caputo impulsive fractional order damped systems. Our main purpose is to establish

some necessary and sufficient conditions of controllability for this type of impulsive control system by using Mittag-Leffler matrix functions and the Schauder fixed point theorem.

This paper is organized as follows. In Section 2, we give some notation and recall some preliminary material. In Section 3, we give the solution expression of impulsive fractional systems. In Section 4, we study the controllability of systems (1.1) and (1.2).

2. Preliminaries. In this section, we introduce some basic definitions and preliminaries which are used throughout this paper. Let $\mathbb{R}^n$ be the n-dimensional Euclidean space, $\mathbb{R}_+ = [0, \infty)$. Let $C(J, \mathbb{R}^n)$ be the Banach space of all continuous functions from J into $\mathbb{R}^n$ with the norm

$$\|u\|_c := \sup\{\|u(t)\|_{\mathbb{R}^n} : t \in J\} \quad \text{for } u \in C(J, \mathbb{R}^n).$$

For brevity, $\|u(t)\|_{\mathbb{R}^n}$ is taken as $\|u(t)\|$. We also introduce the spaces:

$$PC(J, \mathbb{R}^n) = \{u : J \to \mathbb{R}^n \mid u \in C((t_i, t_{i+1}], \mathbb{R}^n), u(t_i^-), u(t_i^+) \text{ exist}$$
$$\text{and } u(t_i^-) = u(t_i), \ i = 0, 1, 2, \ldots, k\},$$

with the norms

$$\|u\|_{PC} := \sup\{\|u(t)\| : t \in J\},$$
$$PC^1(J, \mathbb{R}^n) = \{u \in PC(J, \mathbb{R}^n) : u' \in PC(J, \mathbb{R}^n)\},$$

and

$$\|u\|_{PC^1} := \max\{\|u\|_{PC}, \|u'\|_{PC}\}.$$

Obviously, $PC(J, \mathbb{R}^n)$ and $PC^1(J, \mathbb{R}^n)$ are Banach spaces. The Banach space of all Lebesgue p-integrable functions from J into $\mathbb{R}^n$ with

$$\|f\|_{L^p} := \left(\int_J \|f(t)\|^p dt \right)^{1/p}$$

are denoted by $L^p(J, \mathbb{R}^n)$.

Firstly, let us recall the following basic definitions from fractional calculus. For more details, see [7, 15].

Definition 2.1. The fractional integral of order α with lower limit 0 for a function $f : [0, \infty) \rightarrow \mathbb{R}$ is defined as

$$(2.1) \qquad I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) \, ds, \quad \alpha > 0, \ t > 0,$$

provided the right-hand side is pointwise defined on $[0, \infty)$, where Γ is the Gamma function.

Definition 2.2. The Caputo derivative of order α with lower limit 0 for a function $f : [0, \infty) \rightarrow \mathbb{R}$ can be written as:

$$^c D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) \, dt,$$
$$0 < n - 1 < \alpha \leq n.$$

Its Laplace transform is defined by

$$L\{^c D_t^\alpha f(t)\}(s) = s^\alpha F(s) - \sum_{k=0}^{n-1} f^{(k)}(0^+) s^{\alpha-1-k}.$$

In particular, if $0 < \alpha \leq 1$, then

$$L\{^c D_t^\alpha f(t)\}(s) = s^\alpha F(s) - f(0^+) s^{\alpha-1},$$

and, if $1 < \alpha \leq 2$, then

$$L\{^c D_t^\alpha f(t)\}(s) = s^\alpha F(s) - f(0^+) s^{\alpha-1} - f'(0^+) s^{\alpha-2}.$$

Next, let us proceed to the definition and properties of the Mittag-Leffler function which can be found in [**7, 13, 14, 15**].

Definition 2.3. The two-parameter Mittag-Leffler function is defined by:

$$E_{\alpha,\beta}(Z) = \sum_{k=0}^\infty \frac{Z^k}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \ \beta > 0,$$

where $Z \in \mathbb{C}$. The Laplace transform of the Mittag-Leffler function is:

$$L\{t^{\beta-1} E_{\alpha,\beta}(\pm a t^\alpha)\}(s) = \frac{s^{\alpha-\beta}}{s^\alpha \mp a}, \quad \text{Re}\,(s) > |a|^{1/\alpha}.$$

In particular, for $\beta = 1$,

$$E_{\alpha,1}(Z) = E_\alpha(Z) = \sum_{k=0}^{\infty} \frac{Z^k}{\Gamma(\alpha k + 1)}, \quad Z \in \mathbb{C}.$$

Remark 2.4. Let $A \in \mathbb{C}^{n \times n}$. The matrix extension of the aforementioned Mittag-Leffler function has the following representation:

$$E_{\alpha,\beta}(At^\alpha) = \sum_{k=0}^{\infty} \frac{A^k t^{k\alpha}}{\Gamma(\alpha k + \beta)}, \quad \alpha > 0, \ \beta > 0$$

and

$$L\{t^{\beta-1} E_{\alpha,\beta}(At^\alpha)\}(s) = (s^\alpha I - A)^{-1} s^{\alpha-\beta}, \quad \text{Re}\,(s) > \|A\|^{1/\alpha}.$$

In order to define the solution of systems (1.1) and (1.2), we need:

Lemma 2.5 (cf. [9]). *If $f \in PC^n(J, \mathbb{R})$, $n - 1 < \alpha < n$, then we have the following general expression:*

$$I_t^\alpha \,{}^c D_t^\alpha f(t)$$
$$= \begin{cases} f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!} t^j & t \in [0, t_1], \\ f(t) - \sum_{j=0}^{n-1} \frac{f^{(j)}(0)}{j!} t^j & \\ \quad - \sum_{m=1}^{i} \sum_{j=0}^{n-1} \frac{\Delta f^{(j)}(t_m)}{j!} (t - t_m)^j & t \in (t_i, t_{i+1}], \ i = 1, 2, \ldots, k. \end{cases}$$

3. The solution of impulsive fractional systems. In the present section, we turn our attention to the solution formula of the impulsive fractional control systems. Now, according to Lemma 2.5, we have the following.

Theorem 3.1. *Let $\alpha \in (1, 2]$, $\beta \in (0, 1]$, $B \in \mathbb{R}^{n \times m}$ and $u \in L^{1/p}(J, \mathbb{R}^m)$, $p \in (0, \alpha - 1)$, if $x \in PC^1(J, \mathbb{R}^n)$, and x is a solution*

of the following problem:

(3.1)
$$\begin{cases} {}^cD_t^\alpha x(t) = A{}^cD_t^\beta x(t) + Bu(t) & t \in J', \\ \Delta x(t_i) = I_i(x(t_i^-)), \Delta x'(t_i) = J_i(x'(t_i^-)) & i = 1, 2, \ldots, k, \\ x(0) = x_0, x'(0) = x_0'. \end{cases}$$

Then, x satisfies the following equation:
(3.2)
$$x(t) = \begin{cases} E_{\alpha-\beta}(At^{\alpha-\beta})x_0 + tE_{\alpha-\beta,2}(At^{\alpha-\beta})x_0' \\ \quad -At^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(At^{\alpha-\beta})x_0 \\ \quad +\int_0^t(t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})Bu(s)\,ds, \quad t \in [0,t_1], \\ E_{\alpha-\beta}(At^{\alpha-\beta})x_0 - At^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(At^{\alpha-\beta})x_0 \\ \quad +tE_{\alpha-\beta,2}(At^{\alpha-\beta})x_0' + \sum_{j=1}^i E_{\alpha-\beta}(A(t-t_j)^{\alpha-\beta})I_j(x(t_j^-)) \\ \quad -A\sum_{j=1}^i (t-t_j)^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(A(t-t_j)^{\alpha-\beta})I_j(x(t_j^-)) \\ \quad +\sum_{j=1}^i (t-t_j)E_{\alpha-\beta,2}(A(t-t_j)^{\alpha-\beta})J_j(x'(t_j^-)) \\ \quad +\int_0^t(t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})Bu(s)\,ds, \\ \qquad\qquad\qquad\qquad t \in (t_i, t_{i+1}], \quad i = 1, 2, \ldots, k. \end{cases}$$

Proof. We observe that $x(\cdot)$ can be decomposed to $v(\cdot) + w(\cdot)$, where v is the continuous solution for

(3.3)
$$\begin{cases} {}^cD_t^\alpha v(t) = A{}^cD_t^\beta v(t) + Bu(t) & t \in J, \\ v(0) = x_0, v'(0) = x_0', \end{cases}$$

and w is the solution for

(3.4)
$$\begin{cases} {}^cD_t^\alpha w(t) = A{}^cD_t^\beta w(t) & t \in J', \\ \Delta w(t_i) = I_i(x(t_i^-)), \Delta w'(t_i) = J_i(x'(t_i^-)) & i = 1, 2, \ldots, k, \\ w(0) = 0, w'(0) = 0. \end{cases}$$

Firstly, by [2], we get that the solution of equation (3.3) is given by:

$$v(t) = E_{\alpha-\beta}(At^{\alpha-\beta})x_0 - At^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(At^{\alpha-\beta})x_0$$
$$+ tE_{\alpha-\beta,2}(At^{\alpha-\beta})x_0'$$

$$+ \int_0^t (t-s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta}) Bu(s)\, ds, \quad t \in J.$$

Next, we calculate the solution of equation (3.4).

In order to do this, we apply the Riemann-Liouville fractional integral operator on both sides of the equation (3.4), to obtain

$$I_t^\alpha\, {}^c D_t^\alpha w(t) = A I_t^\alpha\, {}^c D_t^\beta w(t) = A I_t^{\alpha-\beta}(I_t^\beta\, {}^c D_t^\beta w(t)).$$

If $t \in [0, t_1]$, by Lemma 2.5, we have

$$w(t) - w(0) - w'(0)t = A I_t^{\alpha-\beta} w(t) - A I_t^{\alpha-\beta} w(0).$$

Since $w(0) = 0, w'(0) = 0$, then

$$(3.5) \qquad w(t) = \frac{A}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} w(s)\, ds.$$

If $t \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots, k$, again by Lemma 2.5, we have

$$w(t) - w(0) - w'(0)t - \sum_{j=1}^i \Delta w(t_j) - \sum_{j=1}^i \Delta w'(t_j)(t - t_j)$$

$$= A I_t^{\alpha-\beta}[w(t) - w(0) - \sum_{j=1}^i \Delta w(t_j)].$$

It follows from equation (3.4) that

$$(3.6) \qquad w(t) = \sum_{j=1}^i \Delta w(t_j) + \sum_{j=1}^i \Delta w'(t_j)(t - t_j) + A I_t^{\alpha-\beta} w(t)$$

$$- \frac{A(t-t_j)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \sum_{j=1}^i \Delta w(t_j)$$

$$= \sum_{j=1}^i I_j(x(t_j^-)) - \frac{A(t-t_j)^{\alpha-\beta}}{\Gamma(\alpha-\beta+1)} \sum_{j=1}^i I_i(x(t_j^-))$$

$$+ \sum_{j=1}^i J_j(x'(t_j^-))(t - t_j)$$

$$+ \frac{A}{\Gamma(\alpha - \beta)} \int_0^t (t - s)^{\alpha - \beta - 1} w(s) \, ds.$$

Equations (3.5) and (3.6) can be rewritten as

$$(3.7) \qquad w(t) = \sum_{j=1}^{i} I_j(x(t_j^-))\chi_j(t) + \sum_{j=1}^{i} J_j(x'(t_j^-))\chi_j(t)(t - t_j)$$

$$- \frac{A(t - t_j)^{\alpha - \beta}}{\Gamma(\alpha - \beta + 1)} \sum_{j=1}^{i} I_j(x(t_j^-))\chi_j(t)$$

$$+ \frac{A}{\Gamma(\alpha - \beta)} \int_0^t (t - s)^{\alpha - \beta - 1} w(s) \, ds, \quad t \in (t_i, t_{i+1}],$$

where

$$\chi_j(t) = \begin{cases} 0 & t \le t_j, \\ 1 & t > t_j. \end{cases}$$

Let $\lambda > 0$. We adopt the idea in [25] and apply the Laplace transformation to (3.7). Then

$$W(\lambda) = \sum_{j=1}^{i} \frac{e^{-\lambda t_j}}{\lambda} I_j(x(t_j^-)) + \sum_{j=1}^{i} \frac{e^{-\lambda t_j}}{\lambda^2} J_j(x'(t_j^-))$$

$$- \sum_{j=1}^{i} \frac{A e^{-\lambda t_j}}{\lambda^{\alpha - \beta + 1}} I_j(x(t_j^-)) + \frac{A}{\lambda^{\alpha - \beta}} W(\lambda),$$

which implies

$$W(\lambda) = \sum_{j=1}^{i} \lambda^{\alpha - \beta - 1} (\lambda^{\alpha - \beta} I - A)^{-1} e^{-\lambda t_j} I_j(x(t_j^-))$$

$$+ \sum_{j=1}^{i} \lambda^{\alpha - \beta - 2} (\lambda^{\alpha - \beta} I - A)^{-1} e^{-\lambda t_j} J_j(x'(t_j^-))$$

$$- \sum_{j=1}^{i} \lambda^{-1} A (\lambda^{\alpha - \beta} I - A)^{-1} e^{-\lambda t_j} I_j(x(t_j^-)).$$

Note that the Laplace transform of $t^{\beta - 1} E_{\alpha, \beta}(A t^\alpha)$ is $\lambda^{\alpha - \beta}(\lambda^{\alpha - \beta} I - A)^{-1}$. Thus, we can calculate the solution of (3.4) as

$$w(t) = \sum_{j=1}^{i} \chi_j(t) E_{\alpha-\beta}(A(t-t_j)^{\alpha-\beta}) I_j(x(t_j^-))$$

$$+ \sum_{j=1}^{i} \chi_j(t)(t-t_j) E_{\alpha-\beta,2}(A(t-t_j^{\alpha-\beta}) J_j(x'(t_j^-))$$

$$- A \sum_{j=1}^{i} \chi_j(t)(t-t_j)^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(A(t-t_j)^{\alpha-\beta}) I_j(x(t_j^-)).$$

By the above, the solution of equation (3.1) can be given by:

$$x(t) = E_{\alpha-\beta}(At^{\alpha-\beta})x_0 - At^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(At^{\alpha-\beta})x_0$$

$$+ tE_{\alpha-\beta,2}(At^{\alpha-\beta})x_0' + \sum_{j=1}^{i} \chi_j(t) E_{\alpha-\beta}(A(t-t_j)^{\alpha-\beta}) I_j(x(t_j^-))$$

$$+ \sum_{j=1}^{i} \chi_j(t)(t-t_j) E_{\alpha-\beta,2}(A(t-t_j^{\alpha-\beta}) J_j(x'(t_j^-))$$

$$- A \sum_{j=1}^{i} \chi_j(t)(t-t_j)^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(A(t-t_j)^{\alpha-\beta}) I_j(x(t_j^-))$$

$$+ \int_0^t (t-s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta}) Bu(s)\, ds, \quad t \in (t_i, t_{i+1}],$$

that is,

$$x(t) = \begin{cases} E_{\alpha-\beta}(At^{\alpha-\beta})x_0 - At^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(At^{\alpha-\beta})x_0 + tE_{\alpha-\beta,2}(At^{\alpha-\beta})x_0' \\ \quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta}) Bu(s)\, ds, \ t \in [0, t_1], \\ E_{\alpha-\beta}(At^{\alpha-\beta})x_0 - At^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(At^{\alpha-\beta})x_0 + tE_{\alpha-\beta,2}(At^{\alpha-\beta})x_0' \\ \quad + \sum_{j=1}^{i} E_{\alpha-\beta}(A(t-t_j)^{\alpha-\beta}) I_j(x(t_j^-)) \\ \quad + \sum_{j=1}^{i} (t-t_j) E_{\alpha-\beta,2}(A(t-t_j)^{\alpha-\beta}) J_j(x'(t_j^-)) \\ \quad - A \sum_{j=1}^{i} (t-t_j)^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(A(t-t_j)^{\alpha-\beta}) I_j(x(t_j^-)) \\ \quad + \int_0^t (t-s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta}) Bu(s)\, ds, \\ \quad\quad t \in (t_i, t_{i+1}], \quad i = 1, 2, \ldots, k. \end{cases}$$

The proof is completed. $\qquad\qquad\square$

Remark 3.2. The concept of the solution to the homogeneous impulsive problem (3.4) is crucial. Recently, some different views on it, cf. [**19**], have arisen. However, the authors in [**5**] have given a clear and correct definition for the concept of the solution of the homogeneous impulsive problem (3.4).

4. Main results. In this section, we present our main results on controllability of linear system (1.1) and nonlinear system (1.2).

First, following [**1**], we give the concept of controllability as follows.

Definition 4.1. System (1.1) is said to be controllable on $[0, t_f]$, $t_f \in (0, T]$, if, for each vector x_0, x_0', $x_{t_f} \in \mathbb{R}^n$, there exists a control $u \in L^{1/p}([0, t_f], \mathbb{R}^m)$, $p \in (0, \alpha - 1)$, such that the corresponding solution of (1.1) satisfies $x(t_f) = x_{t_f}$.

4.1. Linear system case. We first give a necessary and sufficient condition of controllability for linear system (1.1).

Theorem 4.2. *Linear system* (1.1) *is controllable on* $[0, t_f]$ *if and only if the* $n \times n$ *Gramian matrix*

$$W[0, t_f] = \int_0^{t_f} (t_f - s)^{2\alpha - 2}[E_{\alpha-\beta,\alpha}(A(t_f - s)^{\alpha-\beta})B]$$
$$\times [E_{\alpha-\beta,\alpha}(A(t_f - s)^{\alpha-\beta})B]^* ds$$

is non-singular. Here, $*$ *denotes the matrix transpose.*

Proof. Sufficiency. Suppose that $W[0, t_f]$ is non-singular. Then its inverse is well defined. For $t_f \in [0, t_1]$, define the control function as

(4.1) $u(t) = (t_f - t)^{\alpha-1} B^* E_{\alpha-\beta,\alpha}(A^*(t_f - t)^{\alpha-\beta}) W^{-1}[0, t_f] y_{t_f},$

where

(4.2) $y_{t_f} = x_{t_f} - E_{\alpha-\beta}(At_f^{\alpha-\beta}) x_0 - At_f^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(At_f^{\alpha-\beta}) x_0$
$$+ t_f E_{\alpha-\beta,2}(At_f^{\alpha-\beta}) x_0'.$$

Substituting $t = t_f$ in (3.2) and inserting (4.1) and (4.2), we have

$$
\begin{aligned}
x(t_f) &= E_{\alpha-\beta}(At_f^{\alpha-\beta})x_0 - At_f^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(At_f^{\alpha-\beta})x_0 \\
&\quad + t_f E_{\alpha-\beta,2}(At_f^{\alpha-\beta})x_0' + \int_0^{t_f}(t_f-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t_f-s)^{\alpha-\beta}) \\
&\quad \times B(t_f - s)^{\alpha-1}B^*E_{\alpha-\beta,\alpha}(A^*(t_f - s)^{\alpha-\beta})W^{-1}[0,t_f]y_{t_f}\,ds \\
&= x_{t_f} - y_{t_f} + W[0,t_f]W^{-1}[0,t_f]y_{t_f} = x_{t_f}.
\end{aligned}
$$

Thus, system (1.1) is controllable on $[0, t_f]$ if $t_f \in [0, t_1]$.

Next, for $t_f \in (t_1, t_2]$, define the control function as

$$
\begin{aligned}
(4.3)\quad u(t) &= (t_f - t)^{\alpha-1}B^*E_{\alpha-\beta,\alpha}(A^*(t_f - t)^{\alpha-\beta})W^{-1}[0,t_f] \\
&\quad \Big[y_{t_f} - E_{\alpha-\beta}(A(t_f - t_1)^{\alpha-\beta})I_1(x(t_1^-)) \\
&\qquad + A(t_f - t_1)^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(A(t_f - t_1)^{\alpha-\beta})I_1(x(t_1^-)) \\
&\qquad - (t_f - t_1)E_{\alpha-\beta,2}(A(t_f - t_1)^{\alpha-\beta})J_1(x'(t_1^-))\Big].
\end{aligned}
$$

Substituting $t = t_f$ in equation (3.2) and inserting (4.2) and (4.3), we obtain

$$
\begin{aligned}
x(t_f) &= E_{\alpha-\beta}(At_f^{\alpha-\beta})x_0 - At_f^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(At_f^{\alpha-\beta})x_0 \\
&\quad + t_f E_{\alpha-\beta,2}(At_f^{\alpha-\beta})x_0' + E_{\alpha-\beta}(A(t_f - t_1)^{\alpha-\beta})I_1(x(t_1^-)) \\
&\quad + (t_f - t_1)E_{\alpha-\beta,2}(A(t_f - t_1)^{\alpha-\beta})J_1(x'(t_1^-)) \\
&\quad - A(t_f - t_1)^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(A(t_f - t_1)^{\alpha-\beta})I_1(x(t_1^-)) \\
&\quad + \int_0^{t_f}(t_f - s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t_f - s)^{\alpha-\beta})B(t_f - s)^{\alpha-1} \\
&\quad \times B^*E_{\alpha-\beta,\alpha}(A^*(t_f - s)^{\alpha-\beta})W^{-1}[0,t_f] \\
&\quad \times \Big[y_{t_f} + A(t_f - t_1)^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(A(t_f - t_1)^{\alpha-\beta})I_1(x(t_1^-)) \\
&\qquad - E_{\alpha-\beta}(A(t_f - t_1)^{\alpha-\beta})I_1(x(t_1^-)) \\
&\qquad - (t_f - t_1)E_{\alpha-\beta,2}(A(t_f - t_1)^{\alpha-\beta})J_1(x'(t_1^-))\Big]\,ds \\
&= x_{t_f} - y_{t_f} + E_{\alpha-\beta}(A(t_f - t_1)^{\alpha-\beta})I_1(x(t_1^-)) \\
&\quad - A(t_f - t_1)^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(A(t_f - t_1)^{\alpha-\beta})I_1(x(t_1^-)) \\
&\quad + (t_f - t_1)E_{\alpha-\beta,2}(A(t_f - t_1)^{\alpha-\beta})J_1(x'(t_1^-))
\end{aligned}
$$

$$+ W[0, t_f]W^{-1}[0, t_f]\Big[y_{t_f} - E_{\alpha-\beta}(A(t_f - t_1)^{\alpha-\beta})I_1(x(t_1^-))$$

$$+ A(t_f - t_1)^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(A(t_f - t_1)^{\alpha-\beta})I_1(x(t_1^-))$$

$$- (t_f - t_1)E_{\alpha-\beta,2}(A(t_f - t_1)^{\alpha-\beta})J_1(x'(t_1^-))\Big]$$

$$= x_{t_f}.$$

Thus system (1.1) is controllable on $[0, t_f]$, if $t_f \in (t_1, t_2]$.

Moreover, for $t_f \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots, k$, define the control function as

(4.4)

$$u(t) = (t_f - t)^{\alpha-1}B^* E_{\alpha-\beta,\alpha}(A^*(t_f - t)^{\alpha-\beta})W^{-1}[0, t_f]$$

$$\Big[y_{t_f} - \sum_{j=1}^{i} E_{\alpha-\beta}(A(t_f - t_j)^{\alpha-\beta})I_j(x(t_j^-))$$

$$+ A\sum_{j=1}^{i}(t_f - t_j)^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(A(t_f - t_j)^{\alpha-\beta})I_j(x(t_j^-))$$

$$- \sum_{j=1}^{i}(t_f - t_j)E_{\alpha-\beta,2}(A(t_f - t_j)^{\alpha-\beta})J_j(x'(t_j^-))\Big].$$

Substituting $t = t_f$ into (3.2) and inserting (4.2) and (4.4), we have

$$x(t_f) = E_{\alpha-\beta}(At_f^{\alpha-\beta})x_0 - At_f^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(At_f^{\alpha-\beta})x_0$$

$$+ t_f E_{\alpha-\beta,2}(At_f^{\alpha-\beta})x_0' + \sum_{j=1}^{i} E_{\alpha-\beta}(A(t_f - t_j)^{\alpha-\beta})I_j(x(t_j^-))$$

$$+ \sum_{j=1}^{i}(t_f - t_j)E_{\alpha-\beta,2}(A(t_f - t_j)^{\alpha-\beta})J_j(x'(t_j^-))$$

$$- A\sum_{j=1}^{i}(t_f - t_j)^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(A(t_f - t_j)^{\alpha-\beta})I_j(x(t_j^-))$$

$$+ \int_0^{t_f}(t_f - s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t_f - s)^{\alpha-\beta})B(t_f - s)^{\alpha-1}$$

$$\times B^* E_{\alpha-\beta,\alpha}(A^*(t_f - s)^{\alpha-\beta})W^{-1}[0, t_f]$$

$$\left[y_{t_f} - \sum_{j=1}^{i} E_{\alpha-\beta}(A(t_f - t_j)^{\alpha-\beta}) I_j(x(t_j^-)) \right.$$

$$- \sum_{j=1}^{i} (t_f - t_j) E_{\alpha-\beta,2}(A(t_f - t_j)^{\alpha-\beta}) J_j(x'(t_j^-))$$

$$\left. + A \sum_{j=1}^{i} (t_f - t_j)^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(A(t_f - t_j)^{\alpha-\beta}) I_j(x(t_j^-)) \right] ds$$

$$= x_{t_f} - y_{t_f} + W[0, t_f] W^{-1}[0, t_f] y_{t_f}$$

$$= x_{t_f}.$$

Thus, system (1.1) is controllable on $[0, t_f]$ if $t_f \in (0, T]$.

Necessity. Suppose that system (1.1) is controllable. Now we prove that $W[0, t_f]$ is nonsingular. In fact, if $W[0, t_f]$ is singular, without loss of generality, for $t_f \in (t_i, t_{i+1}]$, $i = 0, 1, \ldots, k$, there exists a nonzero z_0 such that

$$z_0^* W[0, t_f] z_0 = 0,$$

that is,

$$\int_0^{t_f} z_0^* (t_f - s)^{2\alpha-2} E_{\alpha-\beta,\alpha}(A(t_f - s)^{\alpha-\beta}) BB^*$$

$$\times E_{\alpha-\beta,\alpha}(A^*(t_f - s)^{\alpha-\beta}) z_0 \, ds = 0.$$

Then, it follows that

$$z_0^* (t_f - s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(t_f - s)^{\alpha-\beta}) B = 0,$$

on $s \in [0, t_f]$. Since system (1.1) is controllable, then there exists a control $u_1(t)$ such that it steers x_0, $x_0' \in \mathbb{R}^n$ to the origin $x_{t_f} = 0$, i.e.,

$$(4.5) \quad 0 = x(t_f)$$

$$= E_{\alpha-\beta}(At_f^{\alpha-\beta}) x_0 + t_f E_{\alpha-\beta,2}(At_f^{\alpha-\beta}) x_0'$$

$$- At_f^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(At_f^{\alpha-\beta}) x_0$$

$$+ \sum_{j=1}^{i} E_{\alpha-\beta}(A(t_f - t_j)^{\alpha-\beta}) I_j(x(t_j^-))$$

$$+ \sum_{j=1}^{i} (t_f - t_j) E_{\alpha-\beta,2}(A(t_f - t_j)^{\alpha-\beta}) J_j(x'(t_j^-))$$

$$- A \sum_{j=1}^{i} (t_f - t_j)^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(A(t_f - t_j)^{\alpha-\beta}) I_j(x(t_j^-))$$

$$+ \int_0^{t_f} (t_f - s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(t_f - s)^{\alpha-\beta}) B u_1(s) \, ds.$$

Similarly, for the above vectors $x_0, x_0' \in \mathbb{R}^n$ and the final state $x_{t_f}^* = z_0$, there is also a control $u_2(t)$ such that

$$(4.6) \quad z_0 = x(t_f)$$

$$= E_{\alpha-\beta}(At_f^{\alpha-\beta})x_0 + t_f E_{\alpha-\beta,2}(At_f^{\alpha-\beta})x_0'$$

$$- At_f^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(At_f^{\alpha-\beta})x_0$$

$$+ \sum_{j=1}^{i} E_{\alpha-\beta}(A(t_f - t_j)^{\alpha-\beta}) I_j(x(t_j^-))$$

$$+ \sum_{j=1}^{i} (t_f - t_j) E_{\alpha-\beta,2}(A(t_f - t_j)^{\alpha-\beta}) J_j(x'(t_j^-))$$

$$- A \sum_{j=1}^{i} (t_f - t_j)^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(A(t_f - t_j)^{\alpha-\beta}) I_j(x(t_j^-))$$

$$+ \int_0^{t_f} (t_f - s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(t_f - s)^{\alpha-\beta}) B u_2(s) \, ds.$$

We obtain the following from (4.5) and (4.6).

$$(4.7) \quad z_0 + \int_0^{t_f} (t_f - s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(t_f - s)^{\alpha-\beta}) B(u_1(s) - u_2(s)) \, ds = 0.$$

Multiplying z_0^* on both sides of (4.7), we obtain

$$z_0^* z_0 + \int_0^{t_f} (t_f - s)^{\alpha-1} z_0^* E_{\alpha-\beta,\alpha}(A(t_f - s)^{\alpha-\beta}) B(u_1(s) - u_2(s)) \, ds = 0.$$

By the fact that $z_0^* W[0, t_f] z_0 = 0$, we have $z_0^* z_0 = 0$. Thus, $z_0 = 0$. This is a contraction. The proof is completed. $\qquad\square$

Theorem 4.2 is a geometric type condition. By simple transformation, we can get an algebraic type condition.

Theorem 4.3. *System* (1.1) *is controllable on* $[0, t_f]$, $t_f \in (0, T]$, *if and only if*

$$\mathrm{rank}\, Q = \mathrm{rank}(B|AB|\cdots|A^{n-1}B) = n;$$

here $(B|AB|\cdots|A^{n-1}B)$ *is an* $n \times nm$ *matrix composed by the column vectors of* $B, \ldots, A^{n-1}B$ *and the mark* $|$ *denotes the block of the matrix.*

Proof. By the Cayley-Hamilton theorem, we have

$$t^{\alpha-1} E_{\alpha-\beta,\alpha}(At^{\alpha-\beta}) = \sum_{m=0}^{\infty} \frac{t^{(\alpha-\beta)m+\alpha-1}}{\Gamma(m(\alpha-\beta)+\alpha)} A^m = \sum_{m=0}^{n-1} c_m(t) A^m,$$

where $c_m(t)$ is a polynomial in t, and it follows that

$$(t_f - s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(t_f - s)^{\alpha-\beta}) = \sum_{m=0}^{n-1} c_m(t_f - s) A^m.$$

By Theorem 4.2, for $t_f \in [0, t_1]$, we have

$$(4.8) \quad x_{t_f} = E_{\alpha-\beta}(At_f^{\alpha-\beta}) x_0$$
$$+ t_f E_{\alpha-\beta,2}(At_f^{\alpha-\beta}) x_0' - At_f^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(At_f^{\alpha-\beta}) x_0$$
$$+ \int_0^{t_f} (t_f - s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(t_f - s)^{\alpha-\beta}) Bu(s) \, ds.$$

For convenience, we denote

$$(4.9) \quad \eta_1 = E_{\alpha-\beta}(At_f^{\alpha-\beta}) x_0 + t_f E_{\alpha-\beta,2}(At_f^{\alpha-\beta}) x_0'$$
$$- At_f^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(At_f^{\alpha-\beta}) x_0.$$

Combining (4.8) and (4.9) yields

$$x_{t_f} - \eta_1 = \int_0^{t_f} (t_f - s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(t_f - s)^{\alpha-\beta}) Bu(s) \, ds$$
$$= \sum_{m=0}^{n-1} A^m B \int_0^{t_f} c_m(t_f - s) u(s) \, ds$$

$$= (B|AB|\cdots|A^{n-1}B) \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{n-1} \end{pmatrix},$$

where

$$d_m = \int_0^{t_f} c_m(t_f - s)u(s)\, ds.$$

Note that x_{t_f}, x_0 and x_0' are arbitrary; to have a unique solution of $u(t)$, the necessary and sufficient condition is clearly that

$$\operatorname{rank} Q = \operatorname{rank}(B|AB|\cdots|A^{n-1}B) = n.$$

For $t_f \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots, k$, we obtain

(4.10)

$$
\begin{aligned}
x_{t_f} ={}& E_{\alpha-\beta}(At_f^{\alpha-\beta})x_0 + t_f E_{\alpha-\beta,2}(At_f^{\alpha-\beta})x_0' \\
& - At_f^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(At_f^{\alpha-\beta})x_0 \\
& + \sum_{j=1}^{i} E_{\alpha-\beta}(A(t_f - t_j)^{\alpha-\beta})I_j(x(t_j^-)) \\
& + \sum_{j=1}^{i}(t_f - t_j)E_{\alpha-\beta,2}(A(t_f - t_j)^{\alpha-\beta})J_j(x'(t_j^-)) \\
& - A\sum_{j=1}^{i}(t_f - t_j)^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(A(t_f - t_j)^{\alpha-\beta})I_j(x(t_j^-)) \\
& + \int_0^{t_f}(t_f - s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t_f - s)^{\alpha-\beta})Bu(s)\, ds.
\end{aligned}
$$

Denote

(4.11)

$$
\begin{aligned}
\eta_i ={}& E_{\alpha-\beta}(At_f^{\alpha-\beta})x_0 + t_f E_{\alpha-\beta,2}(At_f^{\alpha-\beta})x_0' \\
& - At_f^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(At_f^{\alpha-\beta})x_0 \\
& + \sum_{j=1}^{i} E_{\alpha-\beta}(A(t_f - t_j)^{\alpha-\beta})I_j(x(t_j^-))
\end{aligned}
$$

$$+ \sum_{j=1}^{i} (t_f - t_j) E_{\alpha-\beta,2}(A(t_f - t_j)^{\alpha-\beta}) J_j(x'(t_j^-))$$

$$- A \sum_{j=1}^{i} (t_f - t_j)^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(A(t_f - t_j)^{\alpha-\beta}) I_j(x(t_j^-)).$$

Combining equations (4.10) and (4.11), we have

$$x_{t_f} - \eta_i = \int_0^{t_f} (t_f - s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(t_f - s)^{\alpha-\beta}) Bu(s)\, ds$$

$$= \sum_{m=0}^{n-1} A^m B \int_0^{t_f} c_m(t_f - s) u(s)\, ds$$

$$= (B|AB|\cdots|A^{n-1}B) \begin{pmatrix} d_0 \\ d_1 \\ \vdots \\ d_{n-1} \end{pmatrix}.$$

Note that x_{t_f}, x_0 and x_0' are arbitrary. In order to have a unique solution of $u(t)$, the necessary and sufficient condition is clear that

$$\operatorname{rank} Q = \operatorname{rank}(B|AB|\cdots|A^{n-1}B) = n.$$

The proof is completed. □

4.2. Nonlinear system case. Here, we study the controllability of the nonlinear system (1.2) by Schauder's fixed point theorem. In order to obtain the main result we make the following assumptions:

$H(1)$: The function $f : J \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies the following:

(i) f is measurable for all $t \in J$ and for every fixed $x \in \mathbb{R}^n$.
(ii) There exists a constant $L_f > 0$ such that, for almost every $t \in J$,

$$\|f(t,x) - f(t,y)\| \le L_f \|x - y\|, \quad \text{for all } x, y \in \mathbb{R}^n.$$

(iii) There exist a function $\Psi \in L^{1/\gamma}(J, \mathbb{R}_+), \gamma \in (0, \alpha - 1)$ and a constant $\theta > 0$, such that

$$\|f(t,x)\| \le \Psi(t) + \theta \|x\|, \quad \text{for almost every } t \in J.$$

$H(2)$: $I_i, J_i : \mathbb{R}^n \to \mathbb{R}^n$, $i = 1, 2, \ldots, k$, satisfy the following: I_i and J_i are continuous, and there exist constants $h_i, d_i > 0$, $i = 1, \ldots, k$,

such that

$$\|I_i(x) - I_i(y)\| \le h_i \|x - y\|, \quad \text{for all } x, y \in \mathbb{R}^n,$$
$$\|J_i(x) - J_i(y)\| \le d_i \|x - y\|, \quad \text{for all } x, y \in \mathbb{R}^n.$$

In the sequel, for convenience, let us set

$$\|I(0)\| = \sup\{\|I_i(0)\|\}, \qquad \|J(0)\| = \sup\{\|J_i(0)\|\} \quad i = 1, 2, \ldots, k;$$

$$a_1 = \sup_{t \in J} \|E_{\alpha-\beta}(At^{\alpha-\beta})\|,$$

$$a_2 = \sup_{t \in J} \|AE_{\alpha-\beta,\alpha-\beta+1}(At^{\alpha-\beta})t^{\alpha-\beta}\|;$$

$$a_3 = \sup_{t \in J} \|E_{\alpha-\beta,2}(At^{\alpha-\beta})t\|,$$

$$a_4 = \sup_{t \in J} \|(T-t)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(T-t)^{\alpha-\beta})\|;$$

$$a_5 = \sup_{t \in J} \|E_{\alpha-\beta,\alpha-1}(A(T-t)^{\alpha-\beta})\|,$$

$$b = \max\{a_1, a_2, a_3, a_4, a_5\};$$

$$a_6 = \sup_{t \in J} \|(T-t)^{\alpha-1}B^*E_{\alpha-\beta,\alpha}(A^*(T-t)^{\alpha-\beta})W^{-1}[0,T]\|;$$

$$a_7 = \|y_T\|,$$

$$\delta_1 = \left(\frac{1-\gamma}{\alpha-1-\gamma}T^{(\alpha-1-\gamma)/(1-\gamma)}\right)^{1-\gamma}, \qquad \delta_2 = \max\left\{T, \frac{T^{\alpha-1}}{\alpha-1}\right\};$$

$$\|B\| = M_B;$$

$$c_1 = 1 + bM_B T a_6, \qquad c_2 = 1 + bM_B \frac{T^{\alpha-1}}{\alpha-1}a_6,$$

$$c = \max\{c_1, c_2\};$$

$$\rho_1 = T^{1-\gamma} + bM_B T^{2-\gamma}a_6, \qquad \rho_2 = \delta_1 + bM_B \frac{T^{\alpha-\gamma}}{\alpha-1}a_6,$$

$$\rho = \max\{\rho_1, \rho_2\}.$$

Now, we are in a position to provide the main result of this part.

Theorem 4.4. *If $H(1)$ and $H(2)$ are satisfied and the linear system* (1.1) *is controllable on $[0, t_f]$ for some $t_f \in (0, T]$, then system* (1.2) *is*

also controllable on $[0, t_f]$, *provided that*

$$\left(\sum_{i=1}^{k} (2h_i + d_i) + \theta T \right) bc < 1.$$

Proof. From the controllability of linear system (1.1) on $[0, t_f]$ for $t_f \in (t_i, t_{i+1}] \subset (0, T]$, we can define the control as follows:

$$u(t) = \begin{cases} (T-t)^{\alpha-1} B^* E_{\alpha-\beta,\alpha}(A^*(T-t)^{\alpha-\beta}) W^{-1}[0,T] \\ \quad \times [y_{t_f} - \int\limits_0^T (T-s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta}) f(s, x(s))\, ds], \\ \quad t \in [0, t_1], \\ (T-t)^{\alpha-1} B^* E_{\alpha-\beta,\alpha}(A^*(T-t)^{\alpha-\beta}) W^{-1}[0,T] \\ \quad \times \Big[y_{t_f} - \sum\limits_{j=1}^{i} E_{\alpha-\beta}(A(T-t_j)^{\alpha-\beta}) I_j(x(t_j^-)) \\ \quad - \sum\limits_{j=1}^{i} (T-t_j) E_{\alpha-\beta,2}(A(T-t_j)^{\alpha-\beta}) J_j(x'(t_j^-)) \\ \quad + A \sum\limits_{j=1}^{i} (T-t_j)^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(A(T-t_j)^{\alpha-\beta}) I_j(x(t_j^-)) \\ \quad - \int\limits_0^T (T-s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(T-s)^{\alpha-\beta}) f(s, x(s))\, ds \Big], \\ \quad t \in (t_i, t_{i+1}], \quad i = 1, 2, \ldots, k, \end{cases}$$

where

$$\begin{aligned} y_{t_f} &= x_{t_f} - E_{\alpha-\beta}(AT^{\alpha-\beta}) x_0 - AT^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(AT^{\alpha-\beta}) x_0 \\ &\quad + T E_{\alpha-\beta,2}(AT^{\alpha-\beta}) x_0'. \end{aligned}$$

According to this control, the operator $F : PC^1(J, \mathbb{R}^n) \to PC^1(J, \mathbb{R}^n)$ is well defined by

$$(Fx)(t) = \begin{cases} E_{\alpha-\beta}(At^{\alpha-\beta})x_0 - At^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(At^{\alpha-\beta})x_0 \\ \quad + tE_{\alpha-\beta,2}(At^{\alpha-\beta})x_0' \\ \quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})[Bu(s)+f(s,x(s))]ds, \\ \quad t \in [0,t_1], \\ E_{\alpha-\beta}(At^{\alpha-\beta})x_0 - At^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(At^{\alpha-\beta})x_0 \\ \quad + tE_{\alpha-\beta,2}(At^{\alpha-\beta})x_0' + \sum_{j=1}^i E_{\alpha-\beta}(A(t-t_j)^{\alpha-\beta})I_j(x(t_j^-)) \\ \quad + \sum_{j=1}^i (t-t_j)E_{\alpha-\beta,2}(A(t-t_j)^{\alpha-\beta})J_j(x'(t_j^-)) \\ \quad - A\sum_{j=1}^i (t-t_j)^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(A(t-t_j)^{\alpha-\beta})I_j(x(t_j^-)) \\ \quad + \int_0^t (t-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})[Bu(s)+f(s,x(s))]ds, \\ \quad t \in (t_i,t_{i+1}], \quad i=1,2,\ldots,k. \end{cases}$$

We easily observe that $(Fx)(t_f) = x_{t_f}$. Hence, in order to prove the controllability of system (1.2), we only need to show that F has a fixed point.

For $t_f \in (0,t_1]$, we know that system (1.2) is controllable on $[0,t_f]$ from [**2**]. Now we show system (1.2) is controllable on $[0,t_f]$ for $t_f \in (t_i,t_{i+1}]$, $i=1,\ldots,k$.

First, we choose

$$b[(2\|x_0\| + \|x_0'\|) + kc(2\|I(0)\| + \|J(0)\|) + \rho\|\Psi\|_{L^{1/\gamma}} + M_B\delta_2 a_6 a_7]$$

$$\times \left(1 - bc\sum_{i=1}^k (2h_i + d_i) - bcT\theta\right)^{-1} \le r,$$

and consider the bounded set $B_r = \{x \in PC^1(J,\mathbb{R}^n) : \|x\|_{PC^1} \le r\}$. Next, we divide the proof into three steps.

Step 1: We prove that $FB_r \subseteq B_r$. For all $x \in B_r$ and for all $t \in (t_i,t_{i+1}]$, $i=1,2,\ldots,k$, we have

$$\|u(t)\| \le \|(T-t)^{\alpha-1}B^*E_{\alpha-\beta,\alpha}(A^*(T-t)^{\alpha-\beta})W^{-1}[0,T]\|$$

$$\times \left[\|y_{t_f}\| + \left\|\sum_{j=1}^i E_{\alpha-\beta}(A(T-t_j)^{\alpha-\beta})I_j(x(t_j^-))\right\|\right.$$

$$\left. + \left\|\sum_{j=1}^i (T-t_j)E_{\alpha-\beta,2}(A(T-t_j)^{\alpha-\beta})J_j(x'(t_j^-))\right\|\right.$$

$$+ \left\| A \sum_{j=1}^{i} (T - t_j)^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(A(T - t_j)^{\alpha-\beta}) I_j(x(t_j^-)) \right\|$$

$$+ \left\| \int_0^T (T - s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(T - s)^{\alpha-\beta}) f(s, x(s)) \, ds \right\| \Bigg]$$

$$\leq a_6 \times \Bigg[a_7 + b \sum_{i=1}^{k} h_i \|x(t_i^-)\| + bk\|I(0)\|$$

$$+ b \sum_{i=1}^{k} d_i \|x'(t_i^-)\| + bk\|J(0)\| + b \sum_{i=1}^{k} h_i \|x(t_i^-)\| + bk\|I(0)\|$$

$$+ b \int_0^T [\Psi(s) + \theta\|x(s)\|] \, ds \Bigg]$$

$$\leq a_6 \times \Bigg[a_7 + bk(2\|I(0)\| + \|J(0)\|) + b \sum_{i=1}^{k} (2h_i + d_i) r$$

$$+ bT^{1-\gamma} \|\Psi\|_{L^{1/\gamma}} + bT\theta r \Bigg].$$

From the above estimate, we have

$$\|(Fx)(t)\| \leq \|E_{\alpha-\beta}(At^{\alpha-\beta})x_0\|$$

$$+ \|tE_{\alpha-\beta,2}(At^{\alpha-\beta})x_0'\|$$

$$+ \|At^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(At^{\alpha-\beta})x_0\|$$

$$+ \left\| \sum_{j=1}^{i} E_{\alpha-\beta}(A(t - t_j)^{\alpha-\beta}) I_j(x(t_j^-)) \right\|$$

$$+ \left\| \sum_{j=1}^{i} (t - t_j) E_{\alpha-\beta,2}(A(t - t_j)^{\alpha-\beta}) J_j(x'(t_j^-)) \right\|$$

$$+ \left\| A \sum_{j=1}^{i} (t - t_j)^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(A(t - t_j)^{\alpha-\beta}) I_j(x(t_j^-)) \right\|$$

$$+ \left\| \int_0^t (t-s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(t-s)^{\alpha-\beta})[Bu(s)+f(s, x(s))]ds \right\|$$

$$\leq b\left(2\|x_0\| + \|x_0'\|\right) + b \sum_{i=1}^{k} h_i \|x(t_i^-)\| + bk\|I(0)\|$$

$$+ b \sum_{i=1}^{k} d_i \|x'(t_i^-)\| + bk\|J(0)\| + b \sum_{i=1}^{k} h_i \|x(t_i^-)\| + bk\|I(0)\|$$

$$+ b \left\| \int_0^t Bu(s)\,ds \right\| + b \int_0^t [\Psi(s) + \theta\|x(s)\|]\,ds$$

$$\leq b(2\|x_0\| + \|x_0'\|) + bk(2\|I(0)\| + \|J(0)\|) + b\sum_{i=1}^{k}(2h_i + d_i)\|x\|$$

$$+ bM_B T a_6 \times \left[a_7 + bk(2\|I(0)\| + \|J(0)\|) \right.$$

$$+ b\sum_{i=1}^{k}(2h_i + d_i)r + bT^{1-\gamma}\|\Psi\|_{L^{1/\gamma}} + bT\theta r \Bigg]$$

$$+ bT^{1-\gamma}\|\Psi\|_{L^{1/\gamma}} + b\int_0^t \theta\|x(s)\|\,ds$$

$$\leq b(2\|x_0\| + \|x_0'\|) + bk(1 + bM_B T a_6)(2\|I(0)\| + \|J(0)\|)$$

$$+ bM_B T a_6 a_7 + b(T^{1-\gamma} + bM_B T^{2-\gamma} a_6)\|\Psi\|_{L^{1/\gamma}}$$

$$+ b(1 + bM_B T a_6)\left(\sum_{i=1}^{k}(2h_i + d_i) + T\theta \right) r$$

$$\leq b(2\|x_0\| + \|x_0'\|) + bkc(2\|I(0)\| + \|J(0)\|) + b\rho\|\Psi\|_{L^{1/\gamma}}$$

$$+ bM_B \delta_2 a_6 a_7 + bc\left(\sum_{i=1}^{k}(2h_i + d_i) + T\theta \right) r \leq r.$$

Next, from the definition of F and the Mittag-Leffer function, we obtain:

$$\|(Fx)'(t)\| \leq \|E_{\alpha-\beta}(At^{\alpha-\beta})x_0'\|$$

$$+ \left\| \sum_{j=1}^{i} E_{\alpha-\beta}(A(t-t_j)^{\alpha-\beta})J_j(x'(t_j^-)) \right\|$$

$$+ \left\| \int_0^t (t-s)^{\alpha-2}E_{\alpha-\beta,\alpha-1}(A(t-s)^{\alpha-\beta})[Bu(s)+f(s,x(s))]ds \right\|$$

$$\leq b\|x_0'\| + b\sum_{i=1}^{k} d_i\|x'(t_i^-)\| + bk\|J(0)\|$$

$$+ bM_B \left\| \int_0^t (t-s)^{\alpha-2} u(s)\, ds \right\|$$

$$+ b \int_0^t (t-s)^{\alpha-2} [\Psi(s) + \theta\|x(s)\|]\, ds$$

$$\leq b\|x_0'\| + b\sum_{i=1}^k d_i \|x'\| + bk\|J(0)\| + \frac{bM_B\|u\|T^{\alpha-1}}{\alpha-1}$$

$$+ b\|\Psi\|_{L^{1/\gamma}} \left(\frac{1-\gamma}{\alpha-1-\gamma} T^{(\alpha-1-\gamma)/(1-\gamma)} \right)^{1-\gamma} + bT\theta\|x\|$$

$$\leq b\|x_0'\| + bk\|J(0)\| + \frac{bM_B T^{\alpha-1} a_6}{\alpha-1}$$

$$\left[a_7 + bk(2\|I(0)\| + \|J(0)\|) \right.$$

$$\left. + b\sum_{i=1}^k (2h_i + d_i)r + bT^{1-\gamma}\|\Psi\|_{L^{1/\gamma}} + bT\theta r \right]$$

$$+ b\|\Psi\|_{L^{1/\gamma}} \delta_1 + b\left(\sum_{i=1}^k d_i + \theta T \right) r$$

$$\leq b\|x_0'\| + kb\left(1 + \frac{bM_B T^{\alpha-1} a_6}{\alpha-1} \right)(2\|I(0)\| + \|J(0)\|)$$

$$+ b\left(\delta_1 + \frac{bM_B T^{\alpha-\gamma} a_6}{\alpha-1} \right)\|\Psi\|_{L^{1/\gamma}} + \frac{bM_B T^{\alpha-1} a_6 a_7}{\alpha-1}$$

$$+ b\left(1 + \frac{bM_B T^{\alpha-1} a_6}{\alpha-1} \right)\left(\sum_{i=1}^k (2h_i + d_i) + \theta T \right) r$$

$$\leq b\|x_0'\| + kbc(2\|I(0)\| + \|J(0)\|) + b\rho\|\Psi\|_{L^{1/\gamma}} + bM_B \delta_2 a_6 a_7$$

$$+ bc\left(\sum_{i=1}^k (2h_i + d_i) + \theta T \right) r$$

$$\leq r.$$

Hence, according to the above, we obtain $\|Fx\|_{PC^1} \leq r$, which means that $FB_r \subseteq B_r$.

Step 2: We show that F is continuous. Let $\{x_n\}$ be a sequence such that $x_n \to x$, $x_n' \to x'$ in $PC^1(J, \mathbb{R}^n)$ as $n \to \infty$. Then, for each $t \in (t_i, t_{i+1}]$, $i = 1, 2, \ldots, k$, we obtain:

$$\|(Fx_n)(t) - (Fx)(t)\|$$

$$\leq \left\| \sum_{j=1}^{i} E_{\alpha-\beta}(A(t-t_j)^{\alpha-\beta})[I_j(x_n(t_j^-)) - I_j(x(t_j^-))] \right\|$$

$$+ \left\| \sum_{j=1}^{i} (t-t_j)E_{\alpha-\beta,2}(A(t-t_j)^{\alpha-\beta})[J_j(x'_n(t_j^-)) - J_j(x'(t_j^-))] \right\|$$

$$+ \left\| A \sum_{j=1}^{i} (t - t_j)^{\alpha-\beta} E_{\alpha-\beta,\alpha-\beta+1}(A(t - t_j)^{\alpha-\beta}) \right.$$

$$\left. [I_j(x_n(t_j^-)) - I_j(x(t_j^-))] \right\|$$

$$+ \left\| \int_0^t (t - s)^{\alpha-1} E_{\alpha-\beta,\alpha}(A(t - s)^{\alpha-\beta})[f(s, x_n(s)) - f(s, x(s))]) \, ds \right\|$$

$$\leq b \sum_{i=1}^{k} h_i \|x_n(t_i^-) - x(t_i^-)\| + b \sum_{i=1}^{k} d_i \|x'_n(t_i^-) - x'(t_i^-)\|$$

$$+ b \sum_{i=1}^{k} h_i \|x_n(t_i^-) - x(t_i^-)\| + b L_f \int_0^t \|x_n(s) - x(s)\| \, ds$$

$$\leq b \left(\sum_{i=1}^{k} 2h_i + L_f T \right) \|x_n - x\| + b \sum_{i=1}^{k} d_i \|x'_n - x'\|,$$

$$\|(Fx_n)'(t) - (Fx)'(t)\|$$

$$\leq \left\| \sum_{j=1}^{i} (t - t_j)E_{\alpha-\beta}(A(t - t_j^{\alpha-\beta})[J_j(x'_n(t_j^-)) - J_j(x'(t_j^-))] \right\|$$

$$+ \left\| \int_0^t (t-s)^{\alpha-2} E_{\alpha-\beta,\alpha-1}(A(t-s)^{\alpha-\beta})[f(s, x_n(s)) - f(s, x(s))] ds \right\|$$

$$\leq b \sum_{i=1}^{k} d_i \|x'_n(t_i^-) - x'(t_i^-)\| + b L_f \int_0^t (t - s)^{\alpha-2} \|x_n(s) - x(s)\| \, ds$$

$$\leq b \sum_{i=1}^{k} d_i \|x'_n - x'\| + b L_f \frac{T^{\alpha-1}}{\alpha - 1} \|x_n - x\|.$$

As $x_n \to x$, $x_n' \to x'$, it is easy to see that

$$\|(Fx_n)(t) - (Fx)(t)\| \longrightarrow 0, \quad \|(Fx_n)'(t) - (Fx)'(t)\| \longrightarrow 0, \text{ as } n \to \infty.$$

Hence, $\|Fx_n - Fx\|_{PC^1} \to 0$, as $x_n \to x$, $x_n' \to x'$. Then F is continuous on interval B_r.

Step 3: We prove F is equicontinuous on B_r. Let $0 \leq t_i \leq \tau_1 < \tau_2 \leq t_{i+1} \leq T$, then for each $x \in B_r$, we obtain:

$$
\begin{aligned}
\|(Fx)&(\tau_2) - (Fx)(\tau_1)\| \\
\leq & \|E_{\alpha-\beta}(A\tau_2^{\alpha-\beta})x_0 - E_{\alpha-\beta}(A\tau_1^{\alpha-\beta})x_0\| \\
& + \|\tau_2 E_{\alpha-\beta,2}(A\tau_2^{\alpha-\beta})x_0' - \tau_1 E_{\alpha-\beta,2}(A\tau_1^{\alpha-\beta})x_0'\| \\
& + \|A\tau_1^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(A\tau_1^{\alpha-\beta})x_0 - A\tau_2^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(A\tau_2^{\alpha-\beta})x_0\| \\
& + \sum_{j=1}^{i} \|E_{\alpha-\beta}(A(\tau_2-t_j)^{\alpha-\beta}) - E_{\alpha-\beta}(A(\tau_1-t_j)^{\alpha-\beta})\|\|I_j(x(t_j^-))\| \\
& + \sum_{j=1}^{i} \|(\tau_2-t_j)E_{\alpha-\beta,2}(A(\tau_2-t_j)^{\alpha-\beta}) \\
& \quad - (\tau_1-t_j)E_{\alpha-\beta,2}(A(\tau_1-t_j)^{\alpha-\beta})\|\|J_j(x'(t_j^-))\| \\
& + \sum_{j=1}^{i} \|(\tau_1-t_j)^{\alpha-\beta}AE_{\alpha-\beta,\alpha-\beta+1}(A(\tau_1-t_j)^{\alpha-\beta}) \\
& \quad - (\tau_2-t_j)^{\alpha-\beta}AE_{\alpha-\beta,\alpha-\beta+1}(A(\tau_2-t_j)^{\alpha-\beta})\|\|I_j(x(t_j^-))\| \\
& + \Big\| \int_0^{\tau_2} (\tau_2-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(\tau_2-s)^{\alpha-\beta})[Bu(s) + f(s,x(s))]\,ds \\
& \quad - \int_0^{\tau_1} (\tau_1-s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(\tau_1-s)^{\alpha-\beta})[Bu(s) + f(s,x(s))]\,ds \Big\| \\
\leq & \|E_{\alpha-\beta}(A\tau_2^{\alpha-\beta}) - E_{\alpha-\beta}(A\tau_1^{\alpha-\beta})\|\|x_0\| \\
& + \|\tau_2 E_{\alpha-\beta,2}(A\tau_2^{\alpha-\beta}) - \tau_1 E_{\alpha-\beta,2}(A\tau_1^{\alpha-\beta})\|\|x_0'\| \\
& + \|A\tau_1^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(A\tau_1^{\alpha-\beta}) - A\tau_2^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(A\tau_2^{\alpha-\beta})\|\|x_0\| \\
& + \sum_{j=1}^{i} \|E_{\alpha-\beta}(A(\tau_2-t_j)^{\alpha-\beta}) - E_{\alpha-\beta}(A(\tau_1-t_j)^{\alpha-\beta})\|
\end{aligned}
$$

$$\times (h_j\|x(t_j^-)\| + \|I(0)\|) + \sum_{j=1}^{i} \|(\tau_2 - t_j)E_{\alpha-\beta,2}(A(\tau_2 - t_j)^{\alpha-\beta})$$

$$- (\tau_1 - t_j)E_{\alpha-\beta,2}(A(\tau_1 - t_j)^{\alpha-\beta})\|(d_j\|x'(t_j^-)\| + \|J(0)\|)$$

$$+ \sum_{j=1}^{i} A\|(\tau_1 - t_j)^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(A(\tau_1 - t_j)^{\alpha-\beta})$$

$$- (\tau_2 - t_j)^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(A(\tau_2 - t_j)^{\alpha-\beta})\|(h_j\|x(t_j^-)\| + \|I(0)\|)$$

$$+ \int_0^{\tau_1} \|(\tau_2 - s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(\tau_2 - s)^{\alpha-\beta})$$

$$- (\tau_1 - s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(\tau_1 - s)^{\alpha-\beta})\|\|f(s,x(s)) + Bu(s)\| \, ds$$

$$+ \left\| \int_{\tau_1}^{\tau_2} (\tau_2 - s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(\tau_2 - s)^{\alpha-\beta})[f(s,x(s)) + Bu(s)] \, ds \right\|.$$

By Hölder's inequality and $H(1)$ and $H(3)$, we obtain:

$$\|(Fx)(\tau_2) - (Fx)(\tau_1)\|$$

$$\leq \|E_{\alpha-\beta}(A\tau_2^{\alpha-\beta}) - E_{\alpha-\beta}(A\tau_1^{\alpha-\beta})\|\|x_0\|$$

$$+ \|\tau_2 E_{\alpha-\beta,2}(A\tau_2^{\alpha-\beta}) - \tau_1 E_{\alpha-\beta,2}(A\tau_1^{\alpha-\beta})\|\|x_0'\|$$

$$+ \|A\tau_1^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(A\tau_1^{\alpha-\beta}) - A\tau_2^{\alpha-\beta}E_{\alpha-\beta,\alpha-\beta+1}(A\tau_2^{\alpha-\beta})\|\|x_0\|$$

$$+ \sum_{j=1}^{i} \|E_{\alpha-\beta}(A(\tau_2 - t_j)^{\alpha-\beta}) - E_{\alpha-\beta}(A(\tau_1 - t_j)^{\alpha-\beta})\|$$

$$\times (h_j\|x(t_j^-)\| + \|I(0)\|) + \sum_{j=1}^{i} \|(\tau_2 - t_j)E_{\alpha-\beta,2}(A(\tau_2 - t_j)^{\alpha-\beta})$$

$$- (\tau_1 - t_j)E_{\alpha-\beta,2}(A(\tau_1 - t_j)^{\alpha-\beta})\|(d_j\|x'(t_j^-)\| + \|J(0)\|)$$

$$+ \sum_{j=1}^{i} \|(\tau_1 - t_j)^{\alpha-\beta}AE_{\alpha-\beta,\alpha-\beta+1}(A(\tau_1 - t_j)^{\alpha-\beta})$$

$$- (\tau_2 - t_j)^{\alpha-\beta}AE_{\alpha-\beta,\alpha-\beta+1}(A(\tau_2 - t_j)^{\alpha-\beta})\|(h_j\|x(t_j^-)\| + \|I(0)\|)$$

$$+ \sup_{s\in[0,\tau_1]} \|(\tau_2 - s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(\tau_2 - s)^{\alpha-\beta})$$

$$- (\tau_1 - s)^{\alpha-1}E_{\alpha-\beta,\alpha}(A(\tau_1 - s)^{\alpha-\beta})\|(\|\Psi\|_{L^{1/\gamma}}T^{1-\gamma} + \theta Tr + M_B\varrho T)$$

$$+ b\|\Psi\|_{L^{1/\gamma}}(\tau_2 - \tau_1)^{1-\gamma} + b(\theta r + M_B\varrho)(\tau_2 - \tau_1),$$

where

$$\varrho := a_6 \times [a_7 + bk(2\|I(0)\| + \|J(0)\|) + b\sum_{i=1}^{k}(2h_i + d_i)r$$

$$+ bT^{1-\gamma}\|\Psi\|_{L^{1/\gamma}} + bT\theta r.$$

Similarly, we also have

$$\|(Fx)'(\tau_2) - (Fx)'(\tau_1)\|$$

$$\leq \|E_{\alpha-\beta}(A\tau_2^{\alpha-\beta}) - E_{\alpha-\beta}(A\tau_1^{\alpha-\beta})\|\|x_0'\| + \sum_{j=1}^{i}\|E_{\alpha-\beta}(A(\tau_2-t_j)^{\alpha-\beta})$$

$$- E_{\alpha-\beta}(A(\tau_1 - t_j)^{\alpha-\beta})\|(d_j\|x'(t_j^-)\| + \|J(0)\|)$$

$$+ \sup_{s\in[0,\tau_1]}\|(\tau_2 - s)^{\alpha-2}E_{\alpha-\beta,\alpha-1}(A(\tau_2 - s)^{\alpha-\beta})$$

$$- (\tau_1 - s)^{\alpha-2}E_{\alpha-\beta,\alpha-1}(A(\tau_1 - s)^{\alpha-\beta})\|$$

$$(\|\Psi\|_{L^{1/\gamma}}T^{1-\gamma} + T\theta r + M_B\varrho T) + \omega(\tau_2 - \tau_1)^{\alpha-1-\gamma}$$

$$+ \frac{b(\theta r + M_B\varrho)}{\alpha - 1}(\tau_2 - \tau_1)^{\alpha-1},$$

where

$$\omega := b\|\Psi\|_{L^{1/\gamma}}\left(\frac{1-\gamma}{\alpha - 1 - \gamma}\right)^{1-\gamma}.$$

From the above arguments, let $\tau_2 \to \tau_1$. Then we have

$$\|(Fx)(\tau_2) - (Fx)(\tau_1)\| \longrightarrow 0,$$

and

$$\|(Fx)'(\tau_2) - (Fx)'(\tau_1)\| \longrightarrow 0.$$

Thus, F is equicontinuous on interval B_r.

By Arzela-Ascoli's theorem, we obtain that the operator F is completely continuous and by the Schauder fixed-point theorem, we obtain that F has a fixed point x on B_r. This further implies that system (1.2) has a solution x in $PC^1(J, \mathbb{R}^n)$. Hence, system (1.2) is controllable on J. The proof is completed. $\square$

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