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PERTURBED HAMMERSTEIN INTEGRAL EQUATIONS WITH SIGN-CHANGING KERNELS AND APPLICATIONS TO NONLOCAL BOUNDARY VALUE PROBLEMS AND ELLIPTIC PDES

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ABSTRACT. We demonstrate the existence of at least one positive solution to the perturbed Hammerstein integral equation

$$y(t) = \gamma_1(t)H_1\big(\varphi_1(y)\big) + \gamma_2(t)H_2\big(\varphi_2(y)\big) + \lambda\int_0^1 G(t,s)f\big(s,y(s)\big)\,ds,$$

where certain asymptotic growth properties are imposed on the functions f, H_1 and H_2. Moreover, the functionals φ_1 and φ_2 are realizable as Stieltjes integrals with signed measures, which means that the nonlocal elements in the Hammerstein equation are possibly of a very general, sign-changing form. We focus here on the case where the kernel $(t,s) \mapsto G(t,s)$ is allowed to change sign and demonstrate the existence of at least one positive solution to the integral equation. As applications, we demonstrate that, by choosing γ_1 and γ_2 in particular ways, we obtain positive solutions to boundary value problems, both in the ODEs and elliptic PDEs setting, even when the Green's function is sign-changing, and, moreover, we are able to localize the range of admissible values of the parameter λ. Finally, we also provide a result that for each $\lambda > 0$ yields the existence of at least one positive solution.

1. Introduction. In this paper, we consider the existence of at least one positive solution of the perturbed Hammerstein integral equation

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$$y(t) = \gamma_1(t)H_1\big(\varphi_1(y)\big) + \gamma_2(t)H_2\big(\varphi_2(y)\big) + \lambda\int_0^1 G(t,s)f\big(s,y(s)\big)\,ds. \tag{1.1}$$

In equation (1.1), the functions $\gamma_i : [0,1] \to [0,+\infty)$, for $i = 1,2$, and $H_i : [0,+\infty) \to [0,+\infty)$ are continuous maps, whereas the functionals $\varphi_i : \mathcal{C}([0,1]) \to \mathbb{R}$, $1 \le i \le 2$, are linear, and are realized as Stieltjes integrals with (possibly) signed measures. We would like to point out (see Section 3) that, although H_1 and H_2 could be nonlinear, they need not be; in particular, they can be everywhere affine or linear away from 0. The main contribution of this work is that we allow for the case where the kernel $G : [0,1] \times [0,1] \to \mathbb{R}$ is sign-changing, and, as one of the key novelties within this program, we must develop a novel cone in order to deduce the existence results presented herein. We will discuss the construction of this new cone later in this section.

As described momentarily, in general, it is difficult to obtain the existence of positive solutions in this setting, and so, one often settles for the existence of a nontrivial, possibly sign-changing solution; see, for example, the recent work by Infante and Pietramala [**27**]. However, by extending some techniques introduced in [**7, 14**], we are able to demonstrate the existence of a positive solution to problem (1.1), and we do this without having to assume, say, that f is monotone in its arguments or other stronger assumptions. And we will demonstrate, in particular, that the method utilized here allows us to apply our results in some cases in which other techniques fail to apply. Moreover, our methods will also allow us to provide an easily computable localization of the admissible range of the parameter λ appearing in equation (1.1). We also provide a dual result, see Corollary 3.7, that applies no matter the value of $\lambda > 0$, so long as we are willing to impose a condition on the magnitude of the map $(t,y) \mapsto f(t,y)$ when its second argument is "small." Thus, we are able to obtain existence of positive solutions even in the case where $\lambda > 0$ is large. Finally, the allowance of the nonlocal, (possibly) nonlinear boundary conditions requires some special care here, as we shall describe in the sequel, and, in fact, dealing with this allowance is one of the key contributions of this article.

To demonstrate the applicability of the abstract integral equation result obtained in this paper, we will illustrate the use of our theory by means of some examples. In particular, we shall show that, by choosing the maps $t \mapsto \gamma_1(t)$, $\gamma_2(t)$ in special ways, it follows that a positive solution of problem (1.1) is in turn a positive solution of a specific boundary value problem. For example, letting $\beta_1 < 0$ and $0 < \eta < 1$, we show that the boundary value problems

$$\begin{aligned} &-y'' = \lambda f\big(t, y(t)\big), \quad 0 < t < 1 \\ &y'(0) = H_2\left(\varphi_2(y)\right) - H_1\left(\varphi_1(y)\right) \\ &y(1) = \beta_1 y(\eta) + (1-\beta_1\eta) H_2\big(\varphi_2(y)\big) - \beta_1(1-\eta) H_1\big(\varphi_1(y)\big) \end{aligned} \tag{1.2}$$

and

$$\begin{aligned} &-y'' = \lambda f\big(t, y(t)\big), \quad 0 < t < 1 \\ &y'(0) = 0 \\ &y(1) = \beta_1 y(\eta) + 2\left(1-\beta_1\right) H_1\big(\varphi_1(y)\big) + 3\left(1-\beta_1\right) H_2\big(\varphi_2(y)\big) \end{aligned} \tag{1.3}$$

have positive solutions even though the Green's function associated to the linear boundary condition problem

$$\begin{aligned} -y'' &= \lambda f\big(t, y(t)\big), \quad 0 < t < 1 \\ y'(0) &= 0 \\ y(1) &= \beta_1 y(\eta) \end{aligned} \tag{1.4}$$

is sign-changing. Besides equations (1.2) and (1.3), we demonstrate several other specific problems to which our results apply.

The main point of the specific examples is to demonstrate that, by choosing γ_1, γ_2, and G in particular ways, we can obtain many different collections of boundary value problems with nonlocal, (possibly) nonlinear boundary conditions that admit positive solutions in spite of sign-changing Green's functions; keeping in mind that $\beta_1 < 0$ and that the map $(t, y) \mapsto f(t, y)$ is nonnegative, obviously, any solution of equation (1.4) cannot both be nonnegative and nontrivial. Consequently, by introducing the nonlocal elements, we are able to exploit them to demonstrate that the problem must possess a positive solution, not merely a nontrivial solution.

In addition to the ordinary differential equation setting, we can also apply our results very easily to the setting of radially symmetric solutions of elliptic partial differential equations. In particular, as a specific example we consider later, see equation (3.26), a modification of the problem

$$
\begin{aligned}
-\Delta u(x) &= \lambda f\big(u(x)\big),\ x \in \Omega \subseteq \mathbb{R}^n,\ |x| \in [r_1, r_2] \\
\frac{\partial}{\partial r} u(x)\bigg|_{x \in \partial \mathcal{B}_{r_2}} &= 0 \\
\big(u\left(r_1 x\right) - \beta_1 u(\eta x)\big)\Big|_{x \in \partial \mathcal{B}_1} &= 0,
\end{aligned} \tag{1.5}
$$

where $r_1 < \eta < r_2$, in which we replace the second boundary condition, which is linear nonlocal, with a nonlocal, (potentially) nonlinear version. Recall that in the case of radially symmetric solutions the PDE reduces, as is well known, to a particular ordinary differential equation of second order. Thus, in addition to the ODEs setting, we shall give a brief demonstration of how the abstract integral equation result may also be transplanted to the elliptic PDEs setting, wherein our PDE is equipped with some sort of radially symmetric nonlocal boundary condition in the sense of equation (1.5) above. As we point out in Section 3, these examples are motivated by some recent investigations of Infante and Pietramala [**27,** Sections 6, 7].

In recent years many papers on Hammerstein integral equations have appeared, see, for example, [**3, 30, 35, 36, 37, 41, 45, 56, 61, 63, 64**]. Part of the interest in such problems is the fact that they can be used as a means to study the existence of solutions to boundary value problems, and in the specific setting of problems that admit either vanishing or sign-changing Green's functions several works in recent years such as [**3, 17, 40, 42, 45, 47, 64**] have appeared. In addition, nonlocal boundary value problems have received considerable attention over the past many years. Of particular note is the paper by Infante and Webb [**53**], which provided a unified theory for studying such problems in the case where the nonlocal elements were linear. Several subsequent works by those authors [**52, 54**] as well as Webb [**48, 50, 51**] have continued to extend and refine their theory. Many other authors have also studied linear, nonlocal boundary conditions, such as Graef and Webb [**18**], Infante, et al. [**23, 24, 27, 28**], Jankowski [**29**], Karakostas and Tsamatos [**32, 33**], and Yang [**59, 60, 61**]. In addition, the study of boundary value problems equipped with nonlinear conditions has also been studied by several authors such as Anderson [**1**], Goodrich [**4, 5, 6, 7, 9**], Infante, et al. [**20, 21, 22, 25, 26**], Karakostas [**31**] and Yang [**57, 58**]. Finally, the classical papers by Picone [**44**] and Whyburn [**55**] are recommended for their historical value. So far as we aware, however, and as discussed earlier in this section, the results on problems that couple sign-changing Green's function with nonlocal boundary conditions are few.

With these contexts in mind, then, our contributions here are twofold. First, while there have been recent works that deduce the existence of positive solutions to a given ODE even in the case of an associated sign-changing Green's function, obtaining such results appear generally to have required imposing more stringent hypotheses on the nonlinearity f. For example, in [**45**], the authors obtained just such a result in the case of a third-order differential equation. To obtain their results, they supposed that the map $(t, y) \mapsto f(t, y)$ is decreasing in t and increasing in y. Similarly, in [**3, 64**] the nonlinearity is supposed to assume slightly restrictive growth hypotheses. Ma [**42**] also produced some interesting existence and nonexistence results for a boundary value problem with sign-changing Green's function and achieved this with minimal assumptions on the nonlinearity f; however, in that work, the parameter λ was not localized. By contrast, here, we are able to fully localize the parameter λ and, furthermore, just require a mild, asymptotic sublinearity condition at $+\infty$ on the nonlinearity f; in fact, we are also able to achieve existence results for any $\lambda > 0$. We are able to achieve this by means of suitably exploiting the nonlocal elements appearing in equation (1.1). Moreover, because we treat a general Hammerstein equation here and do not rely on very specific properties of the kernel, our results apply to many boundary value problems rather than just one particular problem.

A second and more substantial contribution is to demonstrate how to incorporate the nonlocal elements while requiring minimal growth conditions on the maps H_1 and H_2 appearing in equation (1.1). This turns out to be a somewhat technical problem. In particular, we only wish to assume asymptotic growth of the maps $z \mapsto H_1(z)$, $H_2(z)$ at $+\infty$. However, this requires having some sort of lower control over the map $y \mapsto \varphi_i(y)$. In particular, a sort of coercivity condition, say $\varphi_1(y) \geq C_0\|y\|$ and $\varphi_2(y) \geq D_0\|y\|$ for suitable constants C_0 and $D_0 > 0$, would suffice, for then, if we can control the size of $\|y\|$, we can thus control the size of $\varphi_i(y)$ itself. Furthermore, in addition to the coercivity being used to relax the growth conditions imposed on H_1

and H_2, that the functionals possess some sort of coercivity also plays an important role in demonstrating that the integral equation can have a positive solution in spite of the sign-changing nature of the kernel; see, for example, the proofs of Lemma 2.7 and of Theorem 3.1. Thus, it is essential that we equip φ_1 and φ_2 with such properties.

We have essentially utilized this technique in various ways in several recent papers; see, for example, **[7]**, where this idea was introduced, as well as **[8, 11, 12, 13, 14]**. In those papers, however, the Green's function was *nonnegative* and did *not* vanish on the interior of its domain, which allowed us to utilize a well known Harnack-like inequality of the form $\min_{t\in[a,b]} y(t) \geq \gamma\|y\|$ for some $\gamma \in (0,1)$ and $0 < a < b < 1$. Due to the availability of this inequality, we then obtain the desired coercivity of φ_i by means of suitable decomposition of the functional φ_i; again, see, for example, **[7]**.

In this work, however, and by considerable contrast, we find that an approach that relies on the existence of a suitable Harnack-like inequality would be severely limited in application. Since in this paper the Green's function (or kernel) can both vanish and change sign, one minimally would have to suppose the existence of $0 < a < b < 1$ such that

$$\min_{t\in[a,b]} G(t,s) > 0, \quad \text{for each } s \in [0,1].$$

Consequently, insofar as the abstract integral equation (1.1) is concerned, we could not treat problems in which this condition were to fail, e.g., if the kernel $(t,s) \mapsto G(t,s)$ were, say, to vanish along the diagonal $t = s$. And, in turn, this would severely limit the sorts of problems we could treat. In addition, the types of nonlocal boundary conditions that we could treat would also be potentially limited.

In consideration of the preceding limitations of an approach relying on suitable Harnack inequalities, it is highly mathematically desirable that we adapt the methods we have previously introduced. So, in order to accomplish this, we have to construct a new cone that is a type of amalgamation of several well-known cones utilized in the literature. In particular, here we use the cone, a version of which we introduced recently in **[15, 16]**,

$$\begin{aligned}(1.6)\quad \mathcal{K} := \Bigg\{ & y \in \mathcal{C}([0,1]) : y(t) \geq 0, \\ & \varphi_1(y) \geq \left(\inf_{s \in S_0} \frac{1}{\mathcal{G}(s)} \int_0^1 G(t,s)\, d\alpha_1(t) \right) \|y\|, \\ & \qquad \varphi_2(y) \geq \left(\inf_{s \in S_0} \frac{1}{\mathcal{G}(s)} \int_0^1 G(t,s)\, d\alpha_2(t) \right) \|y\| \Bigg\}.\end{aligned}$$

Note that, in equation (1.6), the set $S_0 \subseteq [0,1]$ one of full measure on which, by assumption, the ratio of $\int_0^1 G(t,s)\, d\alpha_i(t)$ to $\mathcal{G}(s)$ exists, where $\mathcal{G}(s) := \sup_{t \in [0,1]} |G(t,s)|$. Furthermore, the integrator $t \mapsto \alpha_i(t)$ is associated to the functional $y \mapsto \varphi_i(y)$, for $i = 1, 2$. See Section 2 for more details.

Essentially, the idea behind the construction in (1.6) is to introduce a controlled blow-up by means of the map $s \mapsto (\mathcal{G}(s))^{-1}$ so that if for some $i \in \{1, 2\}$ we obtain

$$\inf_{s \in S_0} \int_0^1 G(t,s)\ d\alpha_i(t) = 0,$$

which can, in fact, occur in our examples, we instead consider

$$\inf_{s \in S_0} \frac{1}{\mathcal{G}(s)} \int_0^1 G(t,s)\, d\alpha_i(t),$$

where $s \mapsto (\mathcal{G}(s))^{-1}$ provides just enough blow-up to cause the infimum to be a positive, finite number, see Examples 3.9 and 3.11, for instance. This then yields the desired coercivity condition, which, in this case and in great contrast to **[8, 11, 12, 13, 14]**, is incorporated directly into the cone itself. It should also be mentioned, in closing, that, by introducing and utilizing this novel cone, we are able to avoid having to select some interval $(a, b) \Subset (0, 1)$ such that a Harnack-like inequality can be applied. In fact, such an interval *never* arises in any of our calculations. Thus, in addition to the benefits already detailed, this new cone also simplifies the treatment of the nonlocal elements since, in some sense, we identify here an *internal* constant that produces the desired coercivity rather than an *external* constant that must be derived and deduced from other inequalities thus leading to potentially cumbersome and non-optimal calculations.

All in all, we believe that the introduction of a new cone is an interesting aspect of this study. It may be that the cone introduced in equation (1.6) is useful in other contexts involving nonlocal terms in perturbed Hammerstein integral equations and their application to boundary value problems.

2. Preliminary lemmata and notation. We begin by stating some of the structural and regularity assumptions that we make regarding problem (1.1). The first four assumptions concern regularity and growth on the constituent parts of (1.1). We note that, here and throughout this work, we denote by $\|\cdot\|$ the usual supremum norm on the space $\mathcal{C}([0,1])$.

(H1) The functionals $\varphi_1(y)$ and $\varphi_2(y)$ have the form

$$\varphi_1(y) := \int_{[0,1]} y(t)\, d\alpha_1(t), \qquad \varphi_2(y) := \int_{[0,1]} y(t)\, d\alpha_2(t),$$

where $\alpha_1,\ \alpha_2 : [0,1] \to \mathbb{R}$ satisfy $\alpha_1,\ \alpha_2 \in BV([0,1])$. Moreover, we denote by $C_1,\ D_1 > 0$ some finite constants such that

$$|\varphi_1(y)| \leq C_1\|y\| \quad \text{and} \quad |\varphi_2(y)| \leq D_1\|y\|,$$

for each $y \in \mathcal{C}([0,1])$.

(H2) The functions $H_1,\ H_2 : [0,+\infty) \to [0,+\infty)$ are continuous, and there exist $A_1,\ A_2 \in (0,+\infty)$ such that

$$+\infty > \lim_{z\to+\infty} \frac{H_1(z)}{z} > A_1 \quad \text{and} \quad +\infty > \lim_{z\to+\infty} \frac{H_2(z)}{z} > A_2,$$

where it is assumed that these limits exist. In addition, the functions γ_1, $\gamma_2 : [0,1] \to [0,+\infty)$ are continuous and thus satisfy $\|\gamma_1\|,\ \|\gamma_2\| < +\infty$.

(H3) The function $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ is continuous and satisfies

$$\lim_{y\to+\infty} \frac{f(t,y)}{y} = 0, \text{ uniformly for } t \in [0,1].$$

(H4) The function $G : [0,1] \times [0,1] \to \mathbb{R}$ satisfies:

(1) $G \in L^1([0,1] \times [0,1])$;

(2) for each $\tau \in [0,1]$ it holds that

$$\lim_{t\to\tau} |G(t,s) - G(\tau,s)| = 0, \text{almost everywhere } s \in [0,1];$$

and

(3) $\sup_{t\in[0,1]} |G(t,s)| < +\infty$ for each $s \in [0,1]$.

Remark 2.1. As was briefly mentioned in Section 1, we wish to point out straightaway that condition (H2) allows for the nonlinear elements appearing in equation (1.1) to be linear or affine. In particular, as the results of Section 3 demonstrate, the functions H_1 and H_2 may be linear on neighborhoods that do not intersect 0, and on neighborhoods that do intersect 0, they may be affine. Thus, while our theory admits a nonlinear perturbation of the Hammerstein equation, it is not required that the perturbation be nonlinear. The importance of this is primarily that, in the application of our result to boundary value problems, the boundary conditions can be linear or affine in the manner described in the previous paragraph. The examples provided in Section 3 should explicate these points.

We next define an operator, fixed points of which will determine solutions to problem (1.1). To this end, let $T : \mathcal{K} \to \mathcal{K}$ be the operator defined by

$$\begin{aligned}(Ty)(t) := \gamma_1(t)H_1\big(\varphi_1(y)\big) + \gamma_2(t)H_2\big(\varphi_2(y)\big)\\ + \lambda\int_0^1 G(t,s)f\big(s,y(s)\big)\,ds.\end{aligned} \tag{2.1}$$

The main interest in operator (2.1) is that a fixed point of the operator will be a solution of the Hammerstein equation (1.1). This will also allow us to relate solutions of equation (1.1) to solutions of various boundary value problems. In Lemma 2.7 in the sequel, we will demonstrate that, under appropriate hypotheses, $T(\mathcal{K}) \subseteq \mathcal{K}$ so that the operator T is well defined on the set $\mathcal{K}$.

Notation 2.2. For use in the sequel, we now make some notational conventions. These will be used throughout the remainder of the paper.
• Define the map $\mathcal{G} : [0,1] \to [0,+\infty)$ by

$$\mathcal{G}(s) := \sup_{t\in[0,1]} \big|G(t,s)\big|.$$

Obviously, for each fixed $s \in [0,1]$, it holds that $G(t,s) \leq \mathcal{G}(s)$, for all $t \in [0,1]$, and that $0 \leq \mathcal{G}(s) < +\infty$.

• Given functions a, $A : X \subseteq \mathbb{R} \to \mathbb{R}$, we define the numbers a_X^m, a_X^M, A_X^m, and A_X^M to be, respectively,

$$a_X^m := \inf_{t\in X} a(t), \qquad A_X^m := \inf_{t\in X} A(t)$$

and

$$a_X^M := \sup_{t\in X} a(t), \qquad A_X^M := \sup_{t\in X} A(t).$$

If the function name is subscripted, as in a_1, then we shall write $a_{1,X}^m$ or $a_{1,X}^M$, as the case may be, instead.

• Given a function $f : [0,1] \times [0,+\infty) \to [0,+\infty)$ and a set $X \subseteq [0,+\infty)$, define f_X^m and f_X^M by, respectively,

$$f_X^m := \inf_{(t,y)\in[0,1]\times X} f(t,y) \quad \text{and} \quad f_X^M := \sup_{(t,y)\in[0,1]\times X} f(t,y).$$

• Given a function $b : X \subseteq \mathbb{R} \to \mathbb{R}$, we denote by $b^{\pm} : X \to [0,+\infty)$ the functions defined by

$$b^-(t) := \max\{-b(t), 0\} \quad \text{and} \quad b^+(t) := \max\{b(t), 0\}.$$

Thus, b^- and b^+ represent, respectively, the negative and positive parts of the function b, and, importantly, these functions are nonnegative maps.

• We denote by Γ_0 the number

$$\Gamma_0 := \min_{t\in[0,1]} \big(\gamma_1(t) + \gamma_2(t)\big). \tag{2.2}$$

With this notation in mind, we list some additional conditions that we impose; these conditions ensure that the cone theoretic argument can be carried out successfully. The first, condition (H5), will be very important in defining the cone we use in this work, as shall be described in the sequel. On the other hand, the third condition, (H7), helps to define the range of admissible values for the parameter λ appearing in problem (1.1). Finally, conditions (H6) and (H8) are essentially technical and ensure that the cone theoretic argument we provide is accurate.

(H5) Assume that, for each $i = 1, 2$, the map

$$s \longmapsto \frac{1}{\mathcal{G}(s)} \int_0^1 G(t,s)\, d\alpha_i(t)$$

is defined for $s \in S_0$, where $|S_0| = 1$, i.e., S_0 has full measure, and the constants defined by

$$C_0 := \inf_{s \in S_0} \frac{1}{\mathcal{G}(s)} \int_0^1 G(t,s)\, d\alpha_1(t)$$

and

$$D_0 := \inf_{s \in S_0} \frac{1}{\mathcal{G}(s)} \int_0^1 G(t,s)\, d\alpha_2(t)$$

are well defined and, in addition, satisfy both $+\infty > C_0 > 0$ and $+\infty > D_0 > 0$.

(H6) The number Γ_0 defined in equation (2.2) satisfies

$$\Gamma_0 > 0.$$

(H7) Let ρ_1 be the number defined by

(2.3)
$$\begin{aligned} \rho_1 := \inf \Bigg\{ \rho_0 \in (0, +\infty) : \frac{H_1(z)}{z} > A_1, \ \frac{H_2(z)}{z} > A_2, \\ \frac{f(t,y)}{y} < 1 \text{ for all } t \in [0,1], \text{ whenever } y, z \in [\rho_0, +\infty) \Bigg\}, \end{aligned}$$

and then put

$$\rho_1^* := \max \left\{ 1, \frac{\rho_1}{\min\{C_0, D_0\}} \right\}. \tag{2.4}$$

Assume that the quantity

$$\begin{aligned} \lambda_0 := \min \Bigg\{ \Gamma_0 \min \left\{ H^m_{1,[0,C_1\rho_1^*]}, H^m_{2,[0,D_1\rho_1^*]} \right\} \\ \times \left(f^M_{[0,\rho_1^*]} \sup_{t \in [0,1]} \int_0^1 G^-(t,s)\, ds \right)^{-1}, \\ \left(\Gamma_0 \min\{A_1 C_0, A_2 D_0\} - \varepsilon \right) \left(\sup_{t \in [0,1]} \int_0^1 G^-(t,s)\, ds \right)^{-1}, \end{aligned}$$

$$\varepsilon\left(f^M_{[0,\rho_1]}\sup_{t\in[0,1]}\int_0^1 G^-(t,s)\,ds\right)^{-1}\Bigg\} \tag{2.5}$$

is well defined and satisfies $\lambda_0 > 0$, where $\varepsilon > 0$ is a fixed number satisfying

$$\Gamma_0 \min\{A_1C_0, A_2D_0\} - \varepsilon > 0,$$

which, by means of conditions (H5) and (H6), is well defined.

(H8) The following holds

$$\varphi_1(\gamma_i) \geq C_0\|\gamma_i\| \quad \text{and} \quad \varphi_2(\gamma_i) \geq D_0\|\gamma_i\|, \tag{2.6}$$

for each $i = 1, 2$.

Remark 2.3. We note that the definitions of the constants C_0 and D_0 are not entirely dissimilar in their purpose from the ratio $\frac{\beta}{M}$ utilized in Graef, et al. [**17,** page 178]; see also Goodrich [**10,** Lemma 2.11]. It should also be noted, as the proofs in Section 3 will reveal, that, in the definition of ρ_1, we could impose, for any $\eta > 0$, the condition $f(t,y)/y < \eta$, for all $t \in [0,1]$, not just for $\eta = 1$.

Remark 2.4. The number ε, which appears in condition (H7), could be replaced, of course, by a specific number. However, to state the result more generally we elect to let ε be arbitrary, as in (H7).

Remark 2.5. As the examples toward the end of Section 3 demonstrate, the number λ_0 is easily computed, and so, the result is *not* merely "abstract" and nonapplicable. Furthermore, as the examples also demonstrate, each of the conditions is easy to check.

As alluded to in Section 1, we must use a modified cone in this paper, relative to the approach taken in, for example, [**7, 14**]. In particular, and as noted in equation (1.6), here we introduce and use the cone

$$\mathcal{K} := \left\{y \in \mathcal{C}([0,1]) : y(t) \geq 0,\ \varphi_1(y) \geq C_0\|y\|,\ \varphi_2(y) \geq D_0\|y\|\right\}.$$

Note that $\mathcal{K}$ is both nonempty and nontrivial since, by assumption, we have that γ_1, $\gamma_2 \in \mathcal{K}$, which follows from conditions (H2) and (H8) above. We note that the definition of this cone is essentially an amalgamation of cones introduced and utilized by Goodrich [**7**], Graef, et al. [**17**], Infante and Webb [**53**], Ma and Zhang [**43**] and Webb [**49**].

It is easy to demonstrate that $\mathcal{K}$ is actually a cone, and so, we omit the proof of this fact.

Remark 2.6. If the quantities

$$\inf_{s\in[0,1]}\int_0^1 G(t,s)\,d\alpha_i(t),\quad i=1,2, \tag{2.7}$$

do not vanish, then, technically, we do not require the factor $1/\mathcal{G}(s)$ in the definitions of C_0 and D_0 in (H5). However, there is no harm in including this factor. As explained in Section 1 the reason for including this factor is to allow for a controlled blow-up to offset the vanishing of the quantity in equation ((2.7)).

We begin by showing that $T:\mathcal{K}\to\mathcal{K}$. Normally, such a verification is essentially trivial. Here, one must be much more careful due to the fact that the kernel G is allowed to change sign as well due to the inclusion of the coercivity-type conditions in the cone and the definitions of C_0 and D_0, and so, the verification follows that $T(\mathcal{K})\subseteq\mathcal{K}$ is more delicate than normal.

Lemma 2.7. *Let the operator T be defined as in equation* (2.1)*, and assume that conditions* (H1)–(H8) *hold. Furthermore, let λ_0 be as in equation* (2.5)*. Then, whenever $\lambda\in(0,\lambda_0)$, it holds that $T(\mathcal{K})\subseteq\mathcal{K}$.*

Proof. We first demonstrate that $(Ty)(t)\geq 0$ whenever $y\in\mathcal{K}$ and $t\in[0,1]$. In order to accomplish this we shall consider cases which depend upon the size of $\|y\|$. For this purpose, fix $\rho_1\in(0,+\infty)$ as in condition (H7) above. Select $y\in\mathcal{K}$, and let it be henceforth be fixed but arbitrary.

The case in which $\|y\|\geq\rho_1^*$ is considered first; one may recall the definition of ρ_1^* from equation (2.4). We first observe, keeping in mind that $(t,y)\mapsto f(t,y)$ and $(t,s)\mapsto G^-(t,s)$ are nonnegative maps, that, if $f(t,y)\leq M_0$ for almost every $(t,y)\in E\times[0,R_0]$, where $E\subseteq[0,1]$ is some measurable set, and if $0\leq y(s)\leq R_0$ for almost every $s\in E$, then

$$\begin{aligned}\int_E G(t,s)f\big(s,y(s)\big)\,ds &= \int_E \left(G^+(t,s)-G^-(t,s)\right) f\big(s,y(s)\big)\,ds\\ &\geq -\int_E G^-(t,s)f\big(s,y(s)\big)\,ds\\ &\geq -M_0\int_E G^-(t,s)\,ds\\ &\geq -M_0\sup_{t\in[0,1]}\int_E G^-(t,s)\,ds.\end{aligned} \tag{2.8}$$

We will use an estimate like equation (2.8) in the sequel.

Now, by hypothesis, we have that $\Gamma_0 \min\{A_1C_0, A_2D_0\} > 0$. From condition (H7), there exists a number $\varepsilon > 0$ such that

$$\Gamma_0 \min\{A_1C_0, A_2D_0\} - \varepsilon > 0.$$

Furthermore, it holds that $H_i(\varphi_i(y)) \geq A_i\varphi_i(y)$, for since $\|y\| \geq \rho_1^*$, we calculate

$$\varphi_1(y) \geq C_0\|y\| \geq C_0\rho_1^* \geq \rho_1 \text{ and } \varphi_2(y) \geq D_0\|y\| \geq D_0\rho_1^* \geq \rho_1.$$

Thus, in light of these observations and the fact that

$$\lambda < \frac{\varepsilon}{f^M_{[0,\rho_1]}}\left(\sup_{t\in[0,1]}\int_0^1 G^-(t,s)\,ds\right)^{-1}, \tag{2.9}$$

we may write, keeping equations (2.8) and (2.9) in mind,

(2.10)

$$\begin{aligned}(Ty)(t) &= \gamma_1(t)H_1\big(\varphi_1(y)\big)+\gamma_2(t)H_2\big(\varphi_2(y)\big)+\lambda\int_0^1 G(t,s)f\big(s,y(s)\big)\,ds\\ &\geq A_1\gamma_1(t)\varphi_1(y)+A_2\gamma_2(t)\varphi_2(y)+\lambda\int_0^1 G(t,s)f\big(s,y(s)\big)\,ds\\ &\geq A_1C_0\|y\|\gamma_1(t)+A_2D_0\|y\|\gamma_2(t)+\lambda\int_0^1 G(t,s)f\big(s,y(s)\big)\,ds\\ &\geq \min\{A_1C_0,A_2D_0\}\|y\|\big(\gamma_1(t)+\gamma_2(t)\big)-\lambda\int_0^1 G^-(t,s)f\big(s,y(s)\big)ds\\ &\geq \Gamma_0\min\{A_1C_0,A_2D_0\}\|y\|-\lambda\int_{\{s:y(s)\geq\rho_1\}} G^-(t,s)f\big(s,y(s)\big)\,ds\end{aligned}$$

$$
\begin{aligned}
&\quad - \lambda \int_{\{s:y(s)<\rho_1\}} G^-(t,s) f\big(s, y(s)\big)\, ds \\
&\geq \Gamma_0 \min\{A_1C_0, A_2D_0\} \|y\| - \lambda \int_{\{s:y(s)\geq\rho_1\}} G^-(t,s) y(s)\, ds \\
&\quad - \lambda \int_{\{s:y(s)<\rho_1\}} G^-(t,s) f^M_{[0,\rho_1]}\, ds \\
&\geq \Gamma_0 \min\{A_1C_0, A_2D_0\} \|y\| - \lambda \|y\| \int_0^1 G^-(t,s)\, ds \\
&\quad - \lambda f^M_{[0,\rho_1]} \int_0^1 G^-(t,s)\, ds \\
&\geq \Gamma_0 \min\{A_1C_0, A_2D_0\} \|y\| - \lambda \|y\| \sup_{t\in[0,1]} \int_0^1 G^-(t,s)\, ds \\
&\quad - \lambda f^M_{[0,\rho_1]} \sup_{t\in[0,1]} \int_0^1 G^-(t,s)\, ds \\
&= \Gamma_0 \min\{A_1C_0, A_2, D_0\} \|y\| - \lambda \|y\| \sup_{t\in[0,1]} \int_0^1 G^-(t,s)\, ds - \varepsilon \\
&= \left(\Gamma_0 \min\{A_1C_0, A_2D_0\} - \lambda \sup_{t\in[0,1]} \int_0^1 G^-(t,s)\, ds\right) \|y\| - \varepsilon,
\end{aligned}
$$

for any $t \in [0,1]$. Since, by the choice of λ specified in the statement of this lemma, it also holds that

$$
\lambda < \big(\Gamma_0 \min\{A_1C_0, A_2D_0\} - \varepsilon\big) \left(\sup_{t\in[0,1]} \int_0^1 G^-(t,s)\, ds \right)^{-1},
$$

we obtain from equation (2.10) that

$$
\begin{aligned}
(Ty)(t) &\geq \left(\Gamma_0 \min\{A_1C_0, A_2D_0\} - \lambda \sup_{t\in[0,1]} \int_0^1 G^-(t,s)\, ds\right) \|y\| - \varepsilon \\
&\geq \varepsilon\big(\|y\| - 1\big) \geq \varepsilon\,(\rho_1^* - 1) \geq 0,
\end{aligned}
$$

where we also use the assumption that $\rho_1^* \geq 1$. Thus, we conclude that $(Ty)(t) \geq 0$, for each $t \in [0,1]$, whenever $\|y\| \geq \rho_1^*$.

On the other hand, in the case of $\|y\| < \rho_1^*$, we write

$$
\begin{aligned}
(Ty)(t) &= \gamma_1(t)H_1\big(\varphi_1(y)\big)+\gamma_2(t)H_2\big(\varphi_2(y)\big)+\lambda\int_0^1 G(t,s)f\big(s,y(s)\big)\,ds \\
&\ge \gamma_1(t)H_{1,[0,C_1\rho_1^*]}^m + \gamma_2(t)H_{2,[0,D_1\rho_1^*]}^m + \lambda\int_0^1 G(t,s)f\big(s,y(s)\big)\,ds \\
&\ge \Gamma_0 \min\Big\{H_{1,[0,C_1\rho_1^*]}^m, H_{2,[0,D_1\rho_1^*]}^m\Big\} + \lambda\int_0^1 G(t,s)f\big(s,y(s)\big)\,ds \\
&\ge \Gamma_0 \min\Big\{H_{1,[0,C_1\rho_1^*]}^m, H_{2,[0,D_1\rho_1^*]}^m\Big\} - \lambda\int_0^1 G^-(t,s)f\big(s,y(s)\big)\,ds \\
&\ge \Gamma_0 \min\Big\{H_{1,[0,C_1\rho_1^*]}^m, H_{2,[0,D_1\rho_1^*]}^m\Big\} - \lambda\int_0^1 G^-(t,s)f_{[0,\rho_1^*]}^M\,ds \\
&\ge \Gamma_0 \min\Big\{H_{1,[0,C_1\rho_1^*]}^m, H_{2,[0,D_1\rho_1^*]}^m\Big\} - \lambda f_{[0,\rho_1^*]}^M \sup_{t\in[0,1]}\int_0^1 G^-(t,s)\,ds,
\end{aligned} \tag{2.11}
$$

for each $t \in [0,1]$, where, to obtain the lower bound in equation (2.11), we have used the facts that

$$0 \le \varphi_1(y) \le C_1\|y\| \le C_1\rho_1^*$$

and that, similarly, $0 \le \varphi_2(y) \le D_1\rho_1^*$. Then, using the fact that λ satisfies

$$\lambda < \frac{\Gamma_0 \min\{H_{1,[0,C_1\rho_1^*]}^m, H_{2,[0,D_1\rho_1^*]}^m\}}{f_{[0,\rho_1^*]}^M}\left(\sup_{t\in[0,1]}\int_0^1 G^-(t,s)\,ds\right)^{-1},$$

we deduce from equation (2.11) that $(Ty)(t) \ge 0$, for each $t \in [0,1]$, whenever $\|y\| < \rho_1^*$. Since these two cases are exhaustive and, by the arbitrariness of y, we deduce that $(Ty)(t) \ge 0$, for each $t \in [0,1]$, whenever $y \in \mathcal{K}$.

We next argue that $\varphi_1(Ty) \ge C_0\|Ty\|$ for each $y \in \mathcal{K}$. To this end, begin by noting that

$$
\begin{aligned}
\|Ty\| &\le \|\gamma_1\|H_1\big(\varphi_1(y)\big) + \|\gamma_2\|H_2\big(\varphi_2(y)\big) \\
&\qquad + \lambda \sup_{t\in[0,1]}\int_0^1 \big|G(t,s)\big|f\big(s,y(s)\big)\,ds \\
&\le \|\gamma_1\|H_1\big(\varphi_1(y)\big) + \|\gamma_2\|H_2\big(\varphi_2(y)\big)
\end{aligned} \tag{2.12}
$$

$$
\begin{aligned}
&\qquad\qquad + \lambda \int_0^1 \sup_{t\in[0,1]} \big|G(t,s)\big| f\big(s,y(s)\big)\, ds\\
&= \|\gamma_1\| H_1\big(\varphi_1(y)\big) + \|\gamma_2\| H_2\big(\varphi_2(y)\big)\\
&\qquad\qquad + \lambda \int_0^1 \mathcal{G}(s) f\big(s,y(s)\big)\, ds.
\end{aligned}
$$

Moreover, we compute

(2.13)

$$
\begin{aligned}
\varphi_1(Ty) &= H_1\big(\varphi_1(y)\big)\int_0^1 \gamma_1(t)\, d\alpha_1(t) + H_2\big(\varphi_2(y)\big)\int_0^1 \gamma_2(t)\, d\alpha_1(t)\\
&\qquad + \lambda \int_0^1 \int_0^1 G(t,s) f\big(s,y(s)\big)\, ds\, d\alpha_1(t)\\
&\geq C_0 H_1\big(\varphi_1(y)\big)\|\gamma_1\| + C_0 H_2\big(\varphi_2(y)\big)\|\gamma_2\|\\
&\qquad + \lambda \int_0^1 \left[\int_0^1 G(t,s)\ d\alpha_1(t)\right] f\big(s,y(s)\big)\, ds,
\end{aligned}
$$

where we have used the fact that $\gamma_1,\ \gamma_2 \in \mathcal{K}$, by assumption. Using the assumption that the map $s \mapsto 1/\mathcal{G}(s)\int_0^1 G(t,s)\, d\alpha_i(t)$ is defined for $s \in S_0$, for each $i = 1, 2$, observe that

$$
\begin{aligned}
(2.14)\qquad & \lambda \int_0^1 \left[\int_0^1 G(t,s)\, d\alpha_1(t)\right] f\big(s,y(s)\big)\, ds\\
&= \lambda \int_{S_0} \left[\frac{1}{\mathcal{G}(s)}\int_0^1 G(t,s)\, d\alpha_1(t)\right] \mathcal{G}(s) f\big(s,y(s)\big)\, ds\\
&\geq \lambda \int_{S_0} \left(\inf_{s\in S_0} \frac{1}{\mathcal{G}(s)}\int_0^1 G(t,s)\, d\alpha_1(t)\right) \mathcal{G}(s) f\big(s,y(s)\big)\, ds\\
&= C_0 \lambda \int_0^1 \mathcal{G}(s) f\big(s,y(s)\big)\, ds,
\end{aligned}
$$

where we have used the fact that both S_0 and $[0,1]$ are sets of equal and full measure. Consequently, putting (2.12)–(2.14) together we estimate

$$
\begin{aligned}
\varphi_1(Ty) &\geq C_0 H_1\big(\varphi_1(y)\big)\|\gamma_1\| + C_0 H_2\big(\varphi_2(y)\big)\|\gamma_2\|\\
&\qquad + \lambda \int_0^1 \left[\int_0^1 G(t,s)\, d\alpha_1(t)\right] f\big(s,y(s)\big)\, ds
\end{aligned}
$$

$$
\begin{aligned}
&\geq C_0\bigg[H_1\big(\varphi_1(y)\big)\|\gamma_1\| + H_2\big(\varphi_2(y)\big)\|\gamma_2\| \\
&\qquad\qquad + \lambda\int_0^1 \mathcal{G}(s)f\big(s,y(s)\big)\,ds\bigg] \\
&\geq C_0\|Ty\|,
\end{aligned}
$$

which confirms that, whenever $y \in \mathcal{K}$, it follows that $\varphi_1(Ty) \geq C_0\|Ty\|$.

Finally, an essentially symmetric argument demonstrates that, whenever $y \in \mathcal{K}$, it follows that $\varphi_2(Ty) \geq D_0\|Ty\|$. Indeed, we merely observe that

$$
\begin{aligned}
\varphi_2(Ty) &= H_1\big(\varphi_1(y)\big)\int_0^1 \gamma_1(t)\,d\alpha_2(t) + H_2\big(\varphi_2(y)\big)\int_0^1 \gamma_2(t)\,d\alpha_2(t) \\
&\qquad + \lambda\int_0^1\int_0^1 G(t,s)f\big(s,y(s)\big)\,ds\,d\alpha_2(t) \\
&\geq D_0H_1\big(\varphi_1(y)\big)\|\gamma_1\| + D_0H_2\big(\varphi_2(y)\big)\|\gamma_2\| \\
&\qquad + \lambda\int_{S_0}\left(\inf_{s\in S_0}\frac{1}{\mathcal{G}(s)}\int_0^1 G(t,s)\,d\alpha_2(t)\right)\mathcal{G}(s)f\big(s,y(s)\big)ds \\
&\geq D_0\|Ty\|.
\end{aligned}
$$

This completes the proof that $T(\mathcal{K}) \subseteq \mathcal{K}$. $\square$

Next, we state and prove a technical lemma that will be used in the proof of Theorem 3.1. Essentially this lemma was stated and proved (in a slightly simpler version) in [**14,** Lemma 3.2]; see also, Wang [**46,** Lemma 2.8].

Lemma 2.8. *Let* $f : [0,1]\times\mathbb{R} \to [0,+\infty)$ *satisfy condition* (H3). *Define the map* $\mathcal{F} : [0,+\infty) \to [0,+\infty)$ *by*

$$\mathcal{F}(\rho) := \max_{(t,y)\in[0,1]\times[0,\rho]} f(t,y).$$

Then, it holds that

$$\lim_{\rho\to+\infty}\frac{\mathcal{F}(\rho)}{\rho} = 0. \tag{2.15}$$

Proof. For contradiction, suppose that equation (2.15) fails. Then there is a sequence $\{t_n, y_n, \rho_n\}_{n=1}^{\infty} \subseteq [0,1]\times[0,+\infty)\times[0,+\infty)$, with

$\rho_n \to +\infty$, together with a number $\eta > 0$ such that

$$\frac{\mathcal{F}(\rho_n)}{\rho_n} = \frac{1}{\rho_n} \cdot \max_{(t,y)\in[0,1]\times[0,\rho_n]} f(t,y) = \frac{1}{\rho_n} f(t_n, y_n) \geq \eta > 0,$$

for each $n \in \mathbb{N}$.

Now, we consider two cases. If f is bounded, then the result is immediate. On the other hand, if f is not bounded, then $y_n \to +\infty$. But then, recalling that the limit is uniform with respect to t, we obtain

$$\frac{f(t_n, y_n)}{y_n} \geq \frac{f(t_n, y_n)}{\rho_n} \geq \eta > 0,$$

which contradicts condition (H3) since $\{y_n\}_{n=1}^{\infty}$ is a sequence satisfying $y_n \to +\infty$. Therefore, equality (2.15) holds, as desired. $\square$

We conclude by stating the fixed point result, which we utilize. This may be found in **[62,** subsection 7.9]; see also **[19]**.

Definition 2.9 (**[62,** Definition 7.32.b])**.** Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces over $\mathbb{R}$. Suppose that $\mathcal{X}$ has an order cone $\mathcal{K}$ and that $T : \mathcal{K} \to \mathcal{K}$ is an operator. Then the operator $T'(+\infty) \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, where $\mathcal{L}(\mathcal{X}, \mathcal{Y})$ is the collection of all linear transformations between $\mathcal{X}$ and $\mathcal{Y}$, is called the *positive Fréchet derivative of* T *at* $+\infty$ *along the cone* $\mathcal{K}$ if and only if

$$\frac{\|Tx - T'(+\infty)x\|}{\|x\|} \longrightarrow 0 \quad \text{as } \|x\| \to +\infty \text{ for } x \in \mathcal{K}.$$

Lemma 2.10 (**[62,** Corollary 7.34])**.** *Suppose that*

(i) $T : \mathcal{K} \subseteq \mathcal{X} \to \mathcal{K}$ *is a compact operator on the Banach space* $\mathcal{X}$ *with order cone* $\mathcal{K}$; *and*
(ii) $T'(+\infty) : \mathcal{K} \to \mathcal{K}$ *exists as a positive Fréchet derivative of* T *at* $+\infty$ *along the cone* $\mathcal{K}$, *and if* μ *is an eigenvalue for* $T'(+\infty)$, *then* $|\mu| < 1$.

Then the operator T *has a fixed point in the cone* $\mathcal{K}$.

Remark 2.11. Due to the hypotheses in force the operator T in equation (2.1) can be shown to be completely continuous, see for example, **[53]**. Since it is a routine argument, we omit it in the proof of Theorem 3.1 in the sequel.

3. Main results. We begin by stating the existence theorem for the perturbed Hammerstein equation (1.1), and we subsequently provide a number of corollaries that follow rather directly from the main existence theorem. After this, and as preliminarily discussed in Section 1, we shall provide a few specific examples and applications of the abstract result embodied by Theorem 3.1 and its corollaries. The proof of the existence theorem is similar in spirit to [**14,** Theorem 3.3], but with a significantly improved conclusion since the hypotheses here are much simpler and the range of application much greater. Furthermore, while in Theorem 3.1 we restrict the range of λ insofar as λ must be small for the existence result to hold, we provide a corollary, namely Corollary 3.7, that, in fact, allows any value of $\lambda \in (0, +\infty)$ as an admissible value of the parameter.

Theorem 3.1. *Suppose that conditions* (H1)–(H8) *hold. Moreover, let* λ_0 *be defined as in equation* (2.5)*. Let* $\widetilde{A}_1$ *and* $\widetilde{A}_2$ *be the finite, positive values of two limits, respectively, in condition* (H2)*. Finally, assume both that*

$$\widetilde{A}_1 C_1 \|\gamma_1\| + \widetilde{A}_2 D_1 \|\gamma_2\| < 1, \tag{3.1}$$

and that there exists a number $t_0 \in [0,1]$ *such that*

$$\int_0^1 G(t_0, s)\, f(s, 0)\, ds > 0. \tag{3.2}$$

Then, for each $\lambda \in (0, \lambda_0)$*, problem (1.1) has at least one positive solution.*

Proof. Define the map $L : \mathcal{K} \to \mathcal{K}$ by

$$(Ly)(t) := \widetilde{A}_1 \gamma_1(t) \varphi_1(y) + \widetilde{A}_2 \gamma_2(t) \varphi_2(y),$$

and observe that L is linear in y. We shall first demonstrate that L is, in fact, the Fréchet derivative of T at $+\infty$ along the cone $\mathcal{K}$. In particular, we demonstrate that, for each $\varepsilon > 0$, there is a $\rho_\varepsilon > 0$ sufficiently large such that, whenever $\|y\| \geq \rho_\varepsilon$, it holds that $|(Ty)(t) - (Ly)(t)| \leq \varepsilon \|y\|$, for each $t \in [0,1]$. Recall that we have already demonstrated that $T(\mathcal{K}) \subseteq \mathcal{K}$ as a consequence of Lemma 2.7.

To this end, first observe that we can write

$$\begin{aligned}\frac{\big|(Ty)(t)-(Ly)(t)\big|}{\|y\|} &\le \frac{\gamma_1(t)|H_1\big(\varphi_1(y)\big)-\widetilde{A}_1\varphi_1(y)|}{\|y\|}\\ &\quad+\frac{\gamma_2(t)|H_2\big(\varphi_2(y)\big)-\widetilde{A}_2\varphi_2(y)|}{\|y\|}\\ &\quad+\frac{1}{\|y\|}\lambda\int_0^1\big|G(t,s)\big|f\big(s,y(s)\big)\,ds\\ &=: I_1+I_2+I_3,\end{aligned}\tag{3.3}$$

where I_i, for $1\le i\le 3$, are defined in the obvious way. Henceforth, let $\lambda\in(0,\lambda_0)$ be fixed but otherwise arbitrary. Furthermore, let $\eta>0$ be fixed but otherwise arbitrary. Then, there is a $\rho_\eta>0$ such that, whenever $z\ge\rho_\eta$,it holds that $|H_1(z)-\widetilde{A}_1z|\le\eta z$ and $|H_2(z)-\widetilde{A}_2z|\le\eta z$. Consequently, whenever $y\in\mathcal{K}$ satisfies $\|y\|\ge\rho_\eta/\min\{C_0,D_0\}$, it holds that

$$\Big|H_1\big(\varphi_1(y)\big)-\widetilde{A}_1\varphi_1(y)\Big|<\eta\varphi_1(y)\le\eta C_1\|y\|\tag{3.4}$$

and that

$$\Big|H_2\big(\varphi_2(y)\big)-\widetilde{A}_2\varphi_2(y)\Big|<\eta\varphi_2(y)\le\eta D_1\|y\|,\tag{3.5}$$

where we use the fact that $\varphi_1(y)\ge C_0\|y\|\ge\rho_\eta$, and similarly with respect to φ_2. By choosing ρ_η even larger, if necessary, and then henceforth fixing $\rho_\eta>0$, we may also obtain both that

$$\frac{f(t,y)}{y}<\eta,\tag{3.6}$$

whenever $t\in[0,1]$ and $y\ge\rho_\eta$, and that

$$\frac{\mathcal{F}(\rho_\eta)}{\rho_\eta}<\eta,\tag{3.7}$$

where the latter inequality invokes Lemma 2.8.

Consequently, from equations (3.4) and (3.5), it follows that whenever $y\in\mathcal{K}$ with $\|y\|\ge\rho_\eta/\min\{C_0,D_0\}$, we may estimate

$$I_1\le\left(\max_{t\in[0,1]}\gamma_1(t)\right)\eta C_1\tag{3.8}$$

and that

$$I_2 \leq \left(\max_{t\in[0,1]} \gamma_2(t)\right) \eta D_1. \tag{3.9}$$

On the other hand, and again using that $\|y\| \geq \rho_\eta/\min\{C_0, D_0\}$, we may write

(3.10)

$$\begin{aligned}
I_3 &= \frac{1}{\|y\|}\left(\lambda \int_{\{s:y(s)<\rho_\eta\}} \big|G(t,s)\big| f\big(s, y(s)\big)\, ds \right.\\
&\qquad\qquad \left. + \lambda \int_{\{s:y(s)\geq\rho_\eta\}} \big|G(t,s)\big| f\big(s, y(s)\big)\, ds\right)\\
&\leq \frac{1}{\|y\|}\left(\lambda \max_{(\tau,\sigma)\in[0,1]\times[0,\rho_\eta]} f(\tau,\sigma) \int_0^1 \big|G(t,s)\big|\, ds \right.\\
&\qquad\qquad \left. + \lambda\eta\|y\| \int_0^1 \big|G(t,s)\big|\, ds\right)\\
&\leq \lambda\left(\frac{\max_{(\tau,\sigma)\in[0,1]\times[0,\rho_\eta]} f(\tau,\sigma)}{\|y\|} + \eta\right) \int_0^1 \big|G(t,s)\big|\, ds\\
&\leq \lambda\left(\frac{\max_{(\tau,\sigma)\in[0,1]\times[0,\rho_\eta]} f(\tau,\sigma)}{\|y\|} + \eta\right)\left(\sup_{t\in[0,1]} \int_0^1 \big|G(t,s)\big|\, ds\right)\\
&= \lambda\left(\frac{\mathcal{F}(\rho_\eta)}{\|y\|} + \eta\right)\left(\sup_{t\in[0,1]} \int_0^1 \big|G(t,s)\big|\, ds\right)\\
&\leq \lambda\left(\frac{\mathcal{F}(\rho_\eta)}{(\rho_\eta/\min\{C_0, D_0\})} + \eta\right)\left(\sup_{t\in[0,1]} \int_0^1 \big|G(t,s)\big|\, ds\right)\\
&\leq 2\eta\lambda \max\big\{1, \min\{C_0, D_0\}\big\}\left(\sup_{t\in[0,1]} \int_0^1 \big|G(t,s)\big|\, ds\right),
\end{aligned}$$

where we have utilized estimates (3.6) and (3.7). Thus, we deduce that

$$I_3 \leq 2\eta\lambda \max\big\{1, \min\{C_0, D_0\}\big\}\left(\sup_{t\in[0,1]} \int_0^1 \big|G(t,s)\big|\, ds\right). \tag{3.11}$$

Finally, putting together estimates (3.3) and (3.8)–(3.11), we arrive at

$$\frac{\big|(Ty)(t)-(Ly)(t)\big|}{\|y\|} \le \eta\bigg[\left(\max_{t\in[0,1]}\gamma_1(t)\right)C_1+\left(\max_{t\in[0,1]}\gamma_2(t)\right)D_1 + 2\lambda\max\big\{1,\min\{C_0,D_0\}\big\} \left(\sup_{t\in[0,1]}\int_0^1\big|G(t,s)\big|\,ds\right)\bigg] := \eta\widetilde{C}, \tag{3.12}$$

for each $t\in[0,1]$. But, since $\widetilde{C}:=\widetilde{C}(\gamma_1,\gamma_2,C_0,C_1,D_0,D_1,G,\lambda)<+\infty$ is a constant dependent only upon initial data (recall that λ is fixed and itself depends only upon initial data), it follows that the ratio on the left-hand side of equation (3.12) may be made as small as desired, for all $t\in[0,1]$, by requiring that $\|y\|$ be sufficiently large. And, by the arbitrariness of $\eta>0$, this proves the desired claim that L is the Fréchet derivative of T at $+\infty$ along the cone $\mathcal{K}$. This completes the first part of the proof.

We next argue that the map L has no eigenvalue greater than or equal to unity. To prove this claim, suppose for contradiction that there exist an eigenvalue $\mu\ge 1$ and an associated eigenvector $y\in\mathcal{K}$ for the operator L so that $\|y\|\ne 0$. Then, $(Ly)(t)=\mu y(t)$, for each $t\in[0,1]$. Let $t_0\in[0,1]$ be a point at which $y(t_0)=\|y\|$. Then we compute

$$\begin{aligned}0<\|y\|=y(t_0)&\le \mu y(t_0)\\ &=\widetilde{A}_1\gamma_1(t_0)\varphi_1(y)+\widetilde{A}_2\gamma_2(t_0)\varphi_2(y)\\ &\le \widetilde{A}_1C_1\|\gamma_1\|\|y\|+\widetilde{A}_2D_1\|\gamma_2\|\|y\|,\end{aligned}$$

from which it follows that

$$\widetilde{A}_1C_1\|\gamma_1\|+\widetilde{A}_2D_1\|\gamma_2\|\ge 1. \tag{3.13}$$

But, since inequality (3.13) violates assumption (3.1), we have a contradiction, and so, we conclude that the operator L cannot have an eigenvalue of magnitude unity or greater. All in all, then, by Lemma 2.10, we conclude the existence of $y_0\in\mathcal{K}$ such that $Ty_0=y_0$. Thus, problem (1.1) has at least one solution.

Finally, we demonstrate that the fixed point y_0 so obtained is nontrivial and thus is a positive solution. To this end, suppose, for contradiction, that $y_0\equiv 0$. Then, for the point $t_0\in[0,1]$ as given in the statement of the theorem, we compute

(3.14)

$$\begin{aligned}0 &= y_0(t_0) = (T0)(t_0)\\ &= \gamma_1(t_0)H_1\big(\varphi_1(0)\big) + \gamma_2(t_0)H_2\big(\varphi_2(0)\big) + \lambda\int_0^1 G(t_0,s)f(s,0)\,ds\\ &= \gamma_1(t_0)H_1(0) + \gamma_2(t_0)H_2(0) + \lambda\int_0^1 G(t_0,s)f(s,0)\,ds\\ &\geq \lambda\int_0^1 G(t_0,s)f(s,0)\,ds > 0,\end{aligned}$$

where the final inequality follows from assumption (3.2) in the statement of the theorem and the fact that $H_1(0)$, $H_2(0) \geq 0$, see Remark 3.2. As equation (3.14) is a contradiction, we conclude that y_0 cannot be identically zero, and so, the previously obtained fixed point is nontrivial, as asserted. And this concludes the proof of the theorem. $\square$

Remark 3.2. As regards the application of Theorem 3.1, we note that, by the definition of λ_0 in (2.5) and the fact that we require $\lambda_0 > 0$, it follows that the condition

$$\min\left\{H^m_{1,[0,C_1\rho_1^*]}, H^m_{2,[0,D_1^*\rho_1]}\right\} > 0$$

must be satisfied. In particular, this means that $H_1(0)$, $H_2(0) \neq 0$. Consequently, while H_1 and H_2 can be linear away from 0, they cannot be linear on any neighborhood of 0. This may be compared with Corollary 3.6.

Obviously, one can immediately obtain the following two corollaries from Theorem 3.1. In particular, Corollary 3.3 replaces condition (3.2) by a slightly different condition, whereas Corollary 3.4 points out that, if at least one of $\gamma_1(t_0)$ and $\gamma_2(t_0)$ does not vanish, then the strict inequality of condition (3.2) can be relaxed to a non-strict inequality, which thus provides for a more general result.

Corollary 3.3. *Suppose that conditions* (H1)–(H8) *hold. Fix* $\lambda \in (0, \lambda_0)$, *with* λ_0 *as in equation* (2.5), *and assume both that equation* (3.1) *holds and that, in addition, there exists a number* $t_0 \in [0,1]$ *such that*

$$\gamma_1(t_0)H_1(0) + \gamma_2(t_0)H_2(0) + \lambda \int_0^1 G(t_0,s)f(s,0)\,ds > 0.$$

Then problem (1.1) *has at least one positive solution.*

Corollary 3.4. *Suppose that conditions* (H1)–(H8) *hold. Assume both that equation* (3.1) *holds and that, in addition, there exists a number* $t_0 \in [0,1]$ *such that*:

(i) *condition* (3.2) *holds but with the strict inequality replaced by non-strict inequality*; *and*
(ii) *for* t_0 *of condition* (3.2), *it also holds that*

$$\max\{\gamma_1(t_0), \gamma_2(t_0)\} > 0.$$

Then, for each $\lambda \in (0,\lambda_0)$, *problem* (1.1) *has at least one positive solution, with* λ_0 *as in* (2.5).

It is also possible to obtain a third corollary by slightly altering condition (3.2) in the statement of Theorem 3.1.

Corollary 3.5. *Suppose that conditions* (H1)–(H8) *hold. Assume that there exists a number* $t_0 \in [0,1]$ *such that*

$$\gamma_1(t_0)H_1(0) + \gamma_2(t_0)H_2(0) + \lambda \int_0^1 G(t_0,s)f(s,0)\,ds < 0. \tag{3.15}$$

Finally, assume that (3.1) *holds. Then, for each* $\lambda \in (0,\lambda_0)$, *problem* (1.1) *has at least one positive solution.*

Proof. The first part of the proof, i.e., the demonstration that the operator L is the Fréchet derivative of T at $+\infty$ along the cone $\mathcal{K}$, remains unaltered. The proof that L has no eigenvalue greater than or equal to unity does not change either. Finally, the proof that the fixed point $y_0 \in \mathcal{K}$ is not identically zero proceeds by simply noting, similar to (3.14), that assumption (3.15) implies the estimate

$$\begin{aligned} 0 = (T0)(t_0) &= \gamma_1(t_0)H_1\big(\varphi_1(0)\big) \\ &\quad + \gamma_2(t_0)H_2\big(\varphi_2(0)\big) + \lambda \int_0^1 G(t_0,s)f(s,0)\,ds \end{aligned} \tag{3.16}$$

$$= \gamma_1(t_0)H_1(0) + \gamma_2(t_0)H_2(0) + \lambda \int_0^1 G(t_0, s)f(s, 0)\, ds < 0,$$

and so we obtain that $0 < 0$, which is a contradiction. Thus, y_0 is not identically zero, and this completes the proof. $\square$

The next corollary demonstrates that if we are willing to make stricter hypotheses regarding the maps γ_1 and γ_2, then it is possible, in fact, to require only that *one* of the maps H_1 and H_2 is not linear on neighborhoods intersecting zero. Consequently, in this case, it is possible to have one of the two maps, H_1 or H_2, be purely linear. Thus, in particular, under the conclusion of Corollary 3.6 it is allowable for exactly one of H_1 and H_2 to have the form $z \mapsto \omega_0 z$, for some $\omega_0 > 0$, for each $z \in [0, +\infty)$.

Corollary 3.6. *Assume that inequality* (3.1) *holds. Let* $E_0 \subseteq [0, 1]$ *be defined by*

$$E_0 := \big\{t \in [0, 1] \ : \ G(t, s) < 0, \text{ for some } s \in [0, 1]\big\}.$$

Suppose that

$$\widetilde{\Gamma}_0 := \min_{t\in E_0} \big\{\gamma_1(t), \gamma_2(t)\big\} > 0,$$

and let $\widetilde{\varepsilon} > 0$ *be selected so that*

$$\widetilde{\Gamma}_0 \left(A_1C_0 + A_2D_0\right) - \widetilde{\varepsilon} > 0$$

holds. Define λ_0^{**} *by*

$$\begin{aligned}\lambda_0^{**} := \min\Bigg\{&\widetilde{\Gamma}_0 \left(H^m_{1,[0,C_1\rho_1^*]} + H^m_{2,[0,D_1\rho_1^*]}\right)\\ &\times \left(f^M_{[0,\rho_1^*]} \sup_{t\in[0,1]} \int_0^1 G^-(t, s)\, ds\right)^{-1},\\ &\left(\widetilde{\Gamma}_0 \left(A_1C_0 + A_2D_0\right) - \widetilde{\varepsilon}\right) \left(\sup_{t\in[0,1]} \int_0^1 G^-(t, s)\, ds\right)^{-1},\\ &\widetilde{\varepsilon} \left(f^M_{[0,\rho_1]} \sup_{t\in[0,1]} \int_0^1 G^-(t, s)\, ds\right)^{-1}\Bigg\},\end{aligned}$$

where ρ_1 *and* ρ_1^* *are defined exactly as in condition* (H7). *If conditions* (H1)–(H5) *and* (H7)–(H8) *hold, where in condition* (H7), λ_0 *is replaced* *by* λ_0^{**} *as above, then for each* $\lambda \in (0, \lambda_0^{**})$*, problem* (1.1) *has at least one positive solution.*

Proof. Omitted since the proof of Theorem 3.1 carries over without change so that essentially the only change is the obvious one in inequality (2.11). $\square$

We present a final corollary of Theorem 3.1. This fifth corollary demonstrates that if we are willing to make a stronger assumption regarding the size of the quantity $f^M_{[0,\rho_1^*]}$, then we can obtain an extension of Theorem 3.1 that is applicable no matter the value of the parameter $\lambda > 0$; thus, our theory can be extended to the case where λ is free. This is thus more directly related to [**14,** Theorem 3.3], see also, [**2,** Theorem 3.1].

Corollary 3.7. *Assume that inequality* (3.1) *holds. Suppose that conditions* (H1)–(H6) *and* (H8) *hold. Fix a number* $\lambda > 0$*, and suppose that*

$$\min\left\{f^M_{[0,\rho_1]}, f^M_{[0,\rho_1^*]}\right\} \leq \frac{\Gamma_0}{\lambda}\left(\sup_{t\in[0,1]}\int_0^1 G^-(t,s)\,ds\right)^{-1} \times \min\left\{\frac{1}{2}\min\left\{A_1C_0, A_2D_0\right\}, \min\left\{H^m_{1,[0,C_1\rho_1^*]}, H^m_{2,[0,D_1\rho_1^*]}\right\}\right\},$$

where the number ρ_1^* *is defined in the proof of this corollary. Then problem* (1.1) *has at least one positive solution for the value of* λ *so fixed.*

Proof. Notice that in the proof of Theorem 3.1 the number λ is not restricted, other than $\lambda > 0$ must hold. This is seen by observing that in the final estimate, namely (3.12), the number λ is absorbed into the constant $\widetilde{C}$. Since this constant is entirely composed of initial data, we may select $\eta > 0$ as small as we like to ensure that the product $\eta\widetilde{C}$ is as small as we like. Furthermore, the parts of the proof regarding the spectral radius of L and the nontriviality of the obtained fixed point of T also go through without change.

Thus, it only remains to argue that $T(\mathcal{K}) \subseteq \mathcal{K}$ when we do not require that $\lambda < \lambda_0$ hold. We first note that the second part of the proof, namely the proofs of the coercivity of the maps φ_1 and φ_2, go through without change since estimates (2.12)–(2.14) do not require that λ have any particular restriction on its magnitude.

On the other hand, to demonstrate that $(Ty)(t) \geq 0$, for each $t \in [0,1]$, whenever $y \in \mathcal{K}$, we consider cases, as before. Define the number ρ_1^* as in (2.4), except here put

$$\begin{aligned}
\text{(3.17)} \quad \rho_1 := \inf\Bigg\{\rho_0 \in (0,+\infty) : \frac{H_1(z)}{z} > A_1,\ \frac{H_2(z)}{z} > A_2, \\
\frac{f(t,y)}{y} < \frac{\Gamma_0 \min\{A_1C_0, A_2D_0\}}{2\lambda}\left(\sup_{t\in[0,1]}\int_0^1 G^-(t,s)\,ds\right)^{-1} \\
\text{for all } t \in [0,1],\ \text{whenever } y, z \in [\rho_0,+\infty)\Bigg\}.
\end{aligned}$$

With the modified selection of ρ_1 above, it follows that, if $\|y\| \geq \rho_1^* \geq 1$, then, similar to equation (2.10), we compute

$$\begin{aligned}
(Ty)(t) &\geq \Gamma_0 \min\{A_1C_0, A_2D_0\}\|y\| - \lambda\int_{\{s:y(s)\geq\rho_1\}} G^-(t,s) f\big(s,y(s)\big)\,ds \\
&\quad - \lambda\int_{\{s:y(s)<\rho_1\}} G^-(t,s) f\big(s,y(s)\big)\,ds \\
&\geq \Gamma_0 \min\{A_1C_0, A_2D_0\}\,\|y\| \\
&\quad - \lambda\cdot\frac{\Gamma_0 \min\{A_1C_0, A_2D_0\}}{2\lambda}\left(\sup_{t\in[0,1]} G^-(t,s)\,ds\right)^{-1} \\
&\quad \times \left(\int_0^1 G^-(t,s)\,ds\right)\|y\| - \lambda f_{[0,\rho_1]}^M \sup_{t\in[0,1]}\int_0^1 G^-(t,s)\,ds \\
&\geq \frac{1}{2}\Gamma_0 \min\{A_1C_0, A_2D_0\}\,\|y\| - \lambda f_{[0,\rho_1]}^M \sup_{t\in[0,1]}\int_0^1 G^-(t,s)\,ds \\
&\geq \frac{1}{2}\Gamma_0 \min\{A_1C_0, A_2D_0\} - \lambda f_{[0,\rho_1]}^M \sup_{t\in[0,1]}\int_0^1 G^-(t,s)\,ds \geq 0,
\end{aligned}$$

for each $t \in [0,1]$ where, to obtain the final inequality, we utilize the smallness condition imposed on $f_{[0,\rho_1]}^M$ in the statement of this corollary.

Finally, if instead it holds that $\|y\| < \rho_1^*$, then, similar to estimate (2.11), we write

$$\begin{aligned}(Ty)(t) &\geq \Gamma_0 \min\left\{H^m_{1,[0,C_1\rho_1^*]}, H^m_{2,[0,D_1\rho_1^*]}\right\} - \lambda\int_0^1 G^-(t,s)f\big(s,y(s)\big)\,ds \\ &\geq \Gamma_0 \min\left\{H^m_{1,[0,C_1\rho_1^*]}, H^m_{2,[0,D_1\rho_1^*]}\right\} - \lambda f^M_{[0,\rho_1^*]}\sup_{t\in[0,1]}\int_0^1 G^-(t,s)\,ds \\ &\geq 0,\end{aligned}$$

for each $t \in [0,1]$. Thus, whether $\|y\| \geq \rho_1^*$ or $\|y\| < \rho_1^*$ holds, it follows that $(Ty)(t) \geq 0$ for each $t \in [0,1]$, and so, we conclude that $T(\mathcal{K}) \subseteq \mathcal{K}$. This completes the proof. □

Remark 3.8. A simple modification of the proof of Corollary 3.7 allows us to replace the inequality in the statement of the corollary with the following pair of inequalities:

$$f^M_{[0,\rho_1]} \leq \frac{1}{2}\Gamma_0 \min\{A_1C_0, A_2D_0\}\left(\sup_{t\in[0,1]}\int_0^1 G^-(t,s)\,ds\right)^{-1};$$

and

$$f^M_{[0,\rho_1^*]} \leq \frac{\Gamma_0}{\lambda}\min\left\{H^m_{1,[0,C_1\rho_1^*]}, H^m_{2,[0,D_1\rho_1^*]}\right\}\left(\sup_{t\in[0,1]}\int_0^1 G^-(t,s)\,ds\right)^{-1}.$$

We conclude this paper with some applications to explicate the use of Theorem 3.1. We begin with an application to an ordinary differential equation, equipped with boundary conditions that, without the nonlocal terms, obviate the possibility of a positive solution. In the first example, we compute all the constants carefully to illustrate that these computations are practical and that the result is not simply "abstract." In the succeeding examples, we mostly omit the computations and focus more on the relationship between an abstract solution of the perturbed Hammerstein integral equation (1.1) and a particular boundary value problem. As was mentioned in Section 1, these examples are motivated, in part, by some of the examples given by Infante and Pietramala [**27,** Sections 6, 7]. Finally, our third example, i.e., Example 3.12, illustrates how our theory can yield results for elliptic PDEs with nonlocal boundary conditions in the context of radially symmetric solutions.

Example 3.9. Consider the choice $\gamma_1(t) := 1 - t$ and $\gamma_2(t) := t$. Let $G : [0,1] \times [0,1] \to \mathbb{R}$ be the map defined by

(3.18)
$$G(t,s) := \begin{cases} \frac{1}{2}(1-s) + \frac{1}{2}\left(\frac{1}{2} - s\right) - (t-s), & 0 \leq s \leq \min\left\{t, \frac{1}{2}\right\} \\ \frac{1}{2}(1-s) + \frac{1}{2}\left(\frac{1}{2} - s\right), & 0 \leq t \leq s \leq \frac{1}{2} \\ \frac{1}{2}(1-s) - (t-s), & 0 < \frac{1}{2} \leq s \leq t \\ \frac{1}{2}(1-s), & 0 \leq \max\left\{t, \frac{1}{2}\right\} < s. \end{cases}$$

Then the perturbed Hammerstein integral equation (1.1) becomes
(3.19)
$$y(t) = (1-t)H_1\big(\varphi_1(y)\big) + tH_2\big(\varphi_2(y)\big) + \lambda \int_0^1 G(t,s) f\big(s, y(s)\big)\, ds,$$

and an elementary exercise, which we omit, demonstrates that if y is a solution of the perturbed Hammerstein integral equation (3.19), then y is also a solution of problem (1.2) in the special case where $\beta_1 = -1$ and $\eta = 1/2$, i.e., any solution of (3.19) solves the boundary value problem

$$\begin{aligned} -y'' &= \lambda f\big(t, y(t)\big), \quad 0 < t < 1 \\ y'(0) &= H_2\big(\varphi_2(y)\big) - H_1\big(\varphi_1(y)\big) \\ y(1) &= -y\left(\frac{1}{2}\right) + \frac{1}{2}H_1\big(\varphi_1(y)\big) + \frac{3}{2}H_2\big(\varphi_2(y)\big). \end{aligned}$$

For definiteness and to demonstrate that all constants and conditions in Theorem 3.1 can be explicitly checked and computed, let us define the functionals φ_1 and φ_2 as follows.

$$\begin{aligned} \varphi_1(y) &:= \frac{1}{2}y\left(\frac{1}{2}\right) - \frac{1}{50}y\left(\frac{1}{6}\right) \\ \varphi_2(y) &:= \frac{3}{4}y\left(\frac{1}{2}\right) - \frac{1}{50}y\left(\frac{1}{5}\right). \end{aligned}$$

Let us notice from the outset that, for this particular choice of the map $(t,s) \mapsto G(t,s)$, it holds that

$$\inf_{s\in[0,1]} \int_0^1 G(t,s)\, d\alpha_i(t) = 0,$$

for each $i = 1, 2$. This is due to the fact that

$$\inf_{s\in[1/2,1]} \int_0^1 G(t,s)\,d\alpha_1(t) = \inf_{s\in[1/2,1]} \frac{6}{25}(1-s) = 0.$$

and that

$$\inf_{s\in[1/2,1]} \int_0^1 G(t,s)\,d\alpha_2(t) = \inf_{s\in[1/2,1]} \frac{73}{200}(1-s) = 0.$$

Thus, the blow-up strategy outlined in Section 1 is necessary for this problem. We would also like to point out that the functions α_1 and α_2, which are the integrators, are defined by

$$\alpha_1(t) := \begin{cases} 0, & t < \frac{1}{6} \\ -\frac{1}{50}, & \frac{1}{6} \le t < \frac{1}{2} \\ \frac{12}{25}, & t \ge \frac{1}{2} \end{cases}$$

and

$$\alpha_2(t) := \begin{cases} 0, & t < \frac{1}{5} \\ -\frac{1}{50}, & \frac{1}{5} \le t < \frac{1}{2} \\ \frac{73}{100}, & t \ge \frac{1}{2} \end{cases}.$$

So, for the Green's function defined by (3.18), one can show that the map $s \mapsto \mathcal{G}(s)$ assumes the form

$$\mathcal{G}(s) := \sup_{t\in[0,1]} \left|G(t,s)\right| = \begin{cases} \frac{3}{4} - s, & s \le \frac{1}{2} \\ \frac{1}{2}(1-s), & s > \frac{1}{2}. \end{cases}$$

Consequently, we compute

$$\frac{1}{\mathcal{G}(s)} \int_0^1 G(t,s)\,d\alpha_1(t) = \begin{cases} \frac{17}{150}\left(\frac{3}{4} - s\right)^{-1}, & 0 \le s \le \frac{1}{6} \\ \left(\frac{1}{50}s + \frac{11}{100}\right)\left(\frac{3}{4} - s\right)^{-1}, & \frac{1}{6} < s \le \frac{1}{2} \\ \frac{12}{25}, & \frac{1}{2} < s < 1. \end{cases}$$

Note that $S_0 = [0, 1)$ so that, in particular, then we compute

$$\begin{aligned}C_0 &:= \inf_{s\in[0,1)} \frac{1}{\mathcal{G}(s)} \int_0^1 G(t,s)\, d\alpha_1(t)\\ &= \min\left\{\frac{34}{225}, \frac{34}{175}, \frac{12}{25}\right\} = \frac{34}{225} > 0.\end{aligned}$$

Similarly, we find that

$$\frac{1}{\mathcal{G}(s)} \int_0^1 G(t,s)\, d\alpha_2(t) = \begin{cases} \frac{353}{2000}\left(\frac{3}{4}-s\right)^{-1}, & 0 \le s \le \frac{1}{5}\\ \left(\frac{1}{50}s + \frac{69}{400}\right)\left(\frac{3}{4}-s\right)^{-1}, & \frac{1}{5} < s \le \frac{1}{2}\\ \frac{73}{100}, & \frac{1}{2} < s < 1,\end{cases}$$

so that

$$D_0 := \inf_{s\in[0,1)} \frac{1}{\mathcal{G}(s)} \int_0^1 G(t,s)\, d\alpha_2(t) = \min\left\{\frac{353}{1500}, \frac{353}{1100}, \frac{73}{100}\right\} = \frac{353}{1500} > 0.$$

It can also easily be shown that

$$\begin{aligned}&G^-(t,s)\\ &:= \begin{cases} -\frac{1}{2}(1-s) - \frac{1}{2}\left(\frac{1}{2}-s\right) + (t-s), & 0 \le s \le t, s < \frac{1}{2}, t \ge \frac{3}{4}\\ -\frac{1}{2}(1-s) + (t-s), & \frac{1}{2} \le s \le 1, \frac{1}{2} + \frac{1}{2}s \le t \le 1\\ 0, & \text{otherwise},\end{cases}\end{aligned}$$

so that

$$\sup_{t\in[0,1]} \int_0^1 G^-(t,s)\, ds = \sup_{t\in[3/4,1]} \int_0^1 G^-(t,s)\, ds = \max_{t\in[3/4,1]} \left(t^2 - t + \frac{3}{16}\right) = \frac{3}{16} > 0.$$

Finally, using the fact that $\|\gamma_1\| = \|\gamma_2\| = 1$ one can check that the following inequalities hold.

$$\begin{aligned}\varphi_1(\gamma_1) &= \frac{7}{30} \ge \frac{34}{225} = C_0\|\gamma_1\|\\ \varphi_1(\gamma_2) &= \frac{37}{150} \ge \frac{34}{225} = C_0\|\gamma_2\|\\ \varphi_2(\gamma_1) &= \frac{359}{1000} \ge \frac{353}{1500} = D_0\|\gamma_1\|\\ \varphi_2(\gamma_2) &= \frac{371}{1000} \ge \frac{353}{1500} = D_0\|\gamma_2\|.\end{aligned}$$

Consequently, condition (H8) is satisfied.

So, with all of the preliminary computations performed, we can explicitly demonstrate the form that the auxiliary conditions take in this specific problem. Condition (3.1) takes the form:

$$\frac{13}{25}\widetilde{A}_1 + \frac{77}{100}\widetilde{A}_2 < 1, \tag{3.20}$$

since here we may take $C_1 := 13/25$ and $D_1 := 77/100$. For definiteness, let us select H_1, H_2, and f as follows.

$$H_1(z) := \begin{cases} \frac{1}{10}z + \frac{1}{10}, & z \le 1 \\ \frac{1}{5}z, & z > 1 \end{cases}$$

$$H_2(z) := \begin{cases} \frac{1}{45}z + \frac{2}{45}, & z \le 1 \\ \frac{1}{15}z, & z > 1 \end{cases}$$

$$f(t,y) := 2t\sqrt{y}.$$

We may thus put $A_1 := 1/6$ and $A_2 := 1/16$; obviously, f is uniformly sublinear at $+\infty$. Since we see that $\Gamma_0 = 1$, it follows that

$$\Gamma_0 \min\{A_1 C_0, A_2 D_0\} = \min\left\{\frac{1}{6}\cdot\frac{34}{225}, \frac{1}{16}\cdot\frac{353}{1500}\right\} = \frac{353}{24000}.$$

Thus, we may put, for example,

$$\varepsilon := \frac{352}{24000} = \frac{11}{750}.$$

Since we calculate $\rho_1 = 4$ so that $\rho_1^* = 450/17 > 1$, we obtain that

$$\begin{aligned}
\lambda_0 &= \frac{16}{3}\min\left\{\frac{\min\{H^m_{1,[0,C_1\rho_1^*]}, H^m_{2,[0,D_1\rho_1^*]}\}}{f^M_{[0,\rho_1^*]}}, \frac{1}{24000}, \frac{\varepsilon}{f^M_{[0,4]}}\right\} \\
&= \frac{16}{3}\min\left\{\frac{\min\{H^m_{1,[0,234/17]}, H^m_{2,[0,693/34]}\}}{f^M_{[0,450/17]}}, \frac{1}{24000}, \frac{\varepsilon}{f^M_{[0,4]}}\right\} \\
&= \frac{16}{3}\min\left\{\frac{\min\{1/10, 2/45\}}{2\sqrt{450/17}}, \frac{1}{24000}, \frac{11/750}{4}\right\} \\
&= \frac{16}{3}\min\left\{\frac{1}{45\sqrt{450/17}}, \frac{1}{24000}, \frac{11}{3000}\right\} = \frac{1}{4500}.
\end{aligned}$$

Condition (3.20) is also satisfied since

$$\frac{13}{25}\cdot\frac{1}{5} + \frac{77}{100}\cdot\frac{1}{15} < 1.$$

All in all, by means of Corollary 3.4 (since $f(t,0)\equiv 0$), we conclude that the problem

$$\begin{aligned}
-y'' &= 2t\sqrt{y}, \quad 0<t<1\\
y'(0) &= H_1\left(\frac{1}{2}y\left(\frac{1}{2}\right)-\frac{1}{50}y\left(\frac{1}{6}\right)\right)\\
&\quad - H_2\left(\frac{3}{4}y\left(\frac{1}{2}\right)-\frac{1}{50}y\left(\frac{1}{5}\right)\right)\\
y(1)+y\left(\frac{1}{2}\right) &= \frac{1}{2}H_1\left(\frac{1}{2}y\left(\frac{1}{2}\right)-\frac{1}{50}y\left(\frac{1}{6}\right)\right)\\
&\quad + \frac{3}{2}H_2\left(\frac{3}{4}y\left(\frac{1}{2}\right)-\frac{1}{50}y\left(\frac{1}{5}\right)\right)
\end{aligned}$$

has at least one positive solution whenever the parameter λ satisfies

$$\lambda\in\left(0,\frac{1}{4500}\right).$$

Finally, due to the choice of the functions H_1 and H_2 in this example, the boundary condition both at $t=0$ and at $t=1$ can be realized in a multipoint-form. For instance, if it holds that $\varphi_1(y)$, $\varphi_2(y)\in[0,1]$, then it follows that

$$y(1)=-\frac{19}{20}y\left(\frac{1}{2}\right)-\frac{1}{1000}y\left(\frac{1}{6}\right)-\frac{3}{1500}y\left(\frac{1}{5}\right)+\frac{7}{60}.$$

The other cases proceed similarly.

Remark 3.10. Observe, as Example 3.9 demonstrates, that it *need not* be the case that the maps $z\mapsto H_1(z)$, $H_2(z)$ are actually nonlinear. In fact, they may be affine on a neighborhood of 0 and linear on any neighborhood that misses 0. Thus, our results demonstrate that the nonlocal elements, *even when affine or linear*, may be used to obtain existence of positive solutions in ways that have not been previously observed.

Example 3.11. Consider the choices $\gamma_1(t)\equiv 2$ and $\gamma_2(t)\equiv 3$. In this case, let us choose the kernel $G:[0,1]\times[0,1]\to\mathbb{R}$ defined by

$$G(t,s) := \begin{cases} \frac{3}{4} - t, & 0 \le s \le \min\left\{\frac{1}{2}, t\right\} \\ \frac{3}{4} - s, & t < s \le \frac{1}{2} \\ 1 - t, & \frac{1}{2} < s \le t \\ 1 - s, & \max\left\{\frac{1}{2}, t\right\} < s \le 1. \end{cases} \tag{3.21}$$

It can be shown that

$$\mathcal{G}(s) := \max_{t \in [0,1]} \left|G(t,s)\right| = \begin{cases} \frac{3}{4} - s, & 0 \le s \le \frac{1}{2} \\ 1 - s, & \frac{1}{2} < s \le 1. \end{cases} \tag{3.22}$$

Due to equation (3.21), it can be shown that any solution of the perturbed Hammerstein integral equation (1.1) is a solution of the boundary value problem

$$-y'' = \lambda f\big(t, y(t)\big), \quad 0 < t < 1 \tag{3.23}$$

$$y'(0) = 0 \tag{3.24}$$

$$y(1) = \frac{1}{4} y'\left(\frac{1}{2}\right) + 2H_1\big(\varphi_1(y)\big) + 3H_2\big(\varphi_2(y)\big). \tag{3.25}$$

Finally, let us now select φ_1 and φ_2 as follows.

$$\varphi_1(y) := \frac{1}{2} y\left(\frac{1}{2}\right) - \frac{1}{500} y\left(\frac{1}{6}\right)$$
$$\varphi_2(y) := \frac{3}{4} y\left(\frac{1}{2}\right) - \frac{1}{500} y\left(\frac{1}{5}\right).$$

As in Example 3.9, it is necessary here to use the “blow up” idea since

$$\inf_{s \in [0,1]} \int_0^1 G(t,s)\, d\alpha_i(t) = 0,$$

for each $i = 1, 2$, as one can easily calculate.

Since we already demonstrated the calculation of the constants in our theory in detail in Example 3.9, we will omit most of the details here. Thus, it can be shown that

$$\frac{1}{\mathcal{G}(s)} \int_0^1 G(t,s)\, d\alpha_1(t) = \begin{cases} \frac{743}{6000}\left(\frac{3}{4} - s\right)^{-1}, & 0 \le s \le \frac{1}{6} \\ \frac{4s+247}{1500-2000s}, & \frac{1}{6} < s \le \frac{1}{2} \\ \frac{249}{500}, & \frac{1}{2} < s < 1, \end{cases}$$

so that $S_0 = [0,1)$, and

$$C_0 := \inf_{s\in[0,1)} \frac{1}{\mathcal{G}(s)} \int_0^1 G(t,s)\, d\alpha_1(t) = \frac{743}{4500} > 0.$$

Similarly, one can show that

$$\frac{1}{\mathcal{G}(s)} \int_0^1 G(t,s)\, d\alpha_2(t) = \begin{cases} \frac{233}{1250}\left(\frac{3}{4}-s\right)^{-1}, & 0 \le s \le \frac{1}{5} \\ \frac{s+93}{375-500s}, & \frac{1}{5} < s \le \frac{1}{2} \\ \frac{187}{250}, & \frac{1}{2} < s < 1, \end{cases}$$

so that

$$D_0 := \inf_{s\in[0,1)} \frac{1}{\mathcal{G}(s)} \int_0^1 G(t,s)\, d\alpha_2(t) = \frac{466}{1875} > 0.$$

We then discover that condition (H8) holds. Consequently, provided that the functions H_1, H_2, and f satisfy the necessary structural and growth conditions, we obtain that problem (3.23) has at least one positive solution for $\lambda \in (0, \lambda_0)$ with $\lambda_0 > 0$ sufficiently small. Since we have already demonstrated in detail how to calculate the value of λ_0 in Example 3.9, we will not repeat a similar calculation for this example.

Example 3.12. As is well known, once one obtains existence-type results for second-order boundary value problems, it is then easy to transfer such results to the setting of radially symmetric solutions for certain elliptic PDEs. In a recent paper by Infante and Pietramala **[27]**, the PDE

$$\begin{aligned} \Delta w + h\big(|x|\big) f(w) = 0, &\quad |x| \in [R_1, R_0] \\ \left.\frac{\partial w}{\partial r}\right|_{\partial B_{R_0}} &= 0 \\ \Big(w\left(R_1 x\right) - \beta_1 w\left(R_\eta x\right)\Big)\Big|_{x\in\partial\mathcal{B}_1} &= 0, \end{aligned}$$

where $R_1 < R_\eta < R_0$ and $x \in \mathbb{R}^2$, was considered as an application of their results; here, they took $\beta_1 < 0$. They then demonstrated the existence of a nontrivial but possibly *sign-changing* radially symmetric solution of this PDE by means of the second-order ODE

$$w''(t) + \phi(t) h(r(t)) f(w(t)) = 0, \quad \text{almost everywhere } t \in [0,1]$$

$$\begin{aligned} w'(0) &= 0 \\ \beta_1 w(\eta) &= w(1), \end{aligned}$$

where

$$\phi(t) = \left(R_0^{1-t} R_1^t \ln\left(\frac{R_0}{R_1}\right)\right)^2$$

and

$$r(t) := R_0^{1-t} R_1^t \quad \text{for } t \in [0,1].$$

Note that the derivation of the above ODE from the original elliptic PDE can be found, among other places, in **[27, 34, 38, 39]**.

So, motivated by the preceding example, we briefly demonstrate how our theory allows us to deduce the existence of a *positive* solution to a similar problem, namely, the problem

$$\begin{aligned} -\Delta u(x) &= \lambda g(u), \quad |x| \in [1, e] \\ \frac{\partial}{\partial r} u(x)\bigg|_{x\in\partial\mathcal{B}_e} &= 0 \\ \big(u(x) + u\left(x\sqrt{e}\right)\big)\Big|_{x\in\partial\mathcal{B}_1} &= 4H_1\left(\frac{1}{2}u\left(x\sqrt{e}\right) - \frac{1}{50}u\left(xe^{5/6}\right)\right)\bigg|_{x\in\partial\mathcal{B}_1} \\ &\quad + 6H_2\left(\frac{3}{4}u\left(x\sqrt{e}\right) - \frac{1}{50}u\left(xe^{4/5}\right)\right)\bigg|_{x\in\partial\mathcal{B}_1}, \end{aligned} \tag{3.26}$$

where $x \in \mathbb{R}^2$; note that, here, for convenience, we take $h(r) \equiv 1$. By following the standard transformation described earlier, we obtain that a radially symmetric solution of equation (3.26), i.e., a solution of the form $w = w(|x|)$, solves the boundary value problem:

(3.27)

$$\begin{aligned} w'' + \lambda e^{2(1-t)} g\big(w(t)\big) &= 0, \quad t \in [0,1] \\ w'(0) &= 0 \\ w(1) &= -w\left(\frac{1}{2}\right) + 4H_1\left(\frac{1}{2}w\left(\frac{1}{2}\right) - \frac{1}{50}w\left(\frac{1}{6}\right)\right) \\ &\quad + 6H_2\left(\frac{3}{4}w\left(\frac{1}{2}\right) - \frac{1}{50}w\left(\frac{1}{5}\right)\right). \end{aligned}$$

Observe that problem (3.27) is precisely the situation considered in Example 3.9, but in this case, with $\gamma_1(t) \equiv 2$ and $\gamma_2(t) \equiv 3$. It can be shown that condition (H8) is satisfied in this case. Consequently, so long as the functions H_1, H_2 and $f(t,w) := e^{2(1-t)}g(w)$ satisfy the necessary conditions, we find that equation (3.27) and thus the PDE given by equation (3.26) have at least one positive solution for all λ sufficiently small. Moreover, if Corollary 3.7 may be applied (depending upon the form of f), then for each $\lambda > 0$, the PDE has at least one positive solution.

Finally, the same comments as before apply here, namely, that H_1 and H_2 need not be nonlinear. Thus, in particular, as in Example 3.9, it is possible for the (radially symmetric) solution of equation (3.26), and thus equation (3.27), to satisfy a multipoint-type condition in the second boundary condition.

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