APPROXIMATION OF SOLUTIONS TO A DELAY EQUATION WITH A RANDOM FORCING TERM AND NON LOCAL CONDITIONS

RENU CHAUDHARY AND DWIJENDRA N. PANDEY

Communicated by William McLean

ABSTRACT. The existence and approximation of a solution to a delay equation with a random forcing term and non local conditions is studied by using a stochastic version of the well-known Banach fixed point theorem and semigroup theory. Moreover, the convergence of Faedo-Galerkin approximations of the solution is shown. An example is given which illustrates the results.

1. Introduction. The differential equations that involve randomness in the mathematical description of a given phenomenon are known as *stochastic differential equations*. Due to randomness, these differential equations can provide more accurate descriptions than the deterministic differential equations. In recent years, stochastic differential equations in both finite and infinite dimensions have attracted much attention in many areas such as physics, population dynamics, electrical engineering, ecology, medicine, biology and other areas of science and engineering, because of their practical applications in these areas. In fact, real phenomena in different fields of science and engineering, and especially in finance, involving stochastic excitations of a Gaussian white noise type have been extensively investigated, both theoretically and experimentally, over a long period of time. Remember that Gaussian white noise–mathematically described as the formal derivative of a Brownian motion process–is a tolerable abstraction and is never a completely faithful representation of a physical noise source.

For various stochastic differential systems, qualitative properties, such as existence, uniqueness, controllability and stability have been studied by many researchers (see [6, 29, 30] and references therein). Although theoretical procedures for determining the solution of a stochastic integro-differential equation have been widely studied by several authors, in most cases these equations are still not solvable in closed form. Therefore, it is important to find approximate solutions in an explicit form or in a form suitable for applications of numerical methods. For more details on stochastic differential equations and applications, we refer to [13, 15, 16, 23, 27].

Non local conditions for an unknown function are given by an expression involving the value(s) of this function at regular intervals rather than continuously over the history period. The nonlocal Cauchy problem was first introduced by Byszewski [11]. He has done pioneering work on nonlocal condition problems [8, 9, 10]. These types of conditions are of great importance because they are usually more precise for physical measurements than the classical ones. For instance, Deng [14] showed that the diffusion of a small amount of gas in a transparent tube can be described using the nonlocal conditions better than using local conditions. Therefore, non local conditions are useful in the applied sciences and engineering. For more details on non local conditions we refer to [1, 2, 3] and the references therein.

The theory of the approximation of solutions to stochastic differential equations has been studied by many authors. Maruyama [24] introduced the Cauchy-Maruyama approximation. Barbu [7] studied Picard type methods to find an approximate solution of a stochastic semilinear equation in a Hilbert space. Balasubramaniam and Dauer [6] gave a Caratheodory successive approximate solution to a semilinear stochastic evolution equation with time delays in Hilbert space. The Taylor series approximation has been studied by Jankovic and Ilic [18] for obtaining an approximate solution of a stochastic differential equation. Kovács and Printems [19, 20] studied the weak and strong order convergence of a fully discrete approximation of a linear stochastic evolution equation. Balasubramaniam [5] studied the Faedo-Galerkin approximate solution of a stochastic semilinear integrodifferential equation in Hilbert space.

Our main concern in this paper is to treat the Faedo-Galerkin approximate solution to a delay equation with a random forcing term

and non local conditions. Faedo-Galerkin approximate solutions for deterministic differential equations have been studied by many authors [**4, 25, 26**], but this approach is new to the literature on the study of delay equations with a random forcing term. For recent work on the Faedo-Galerkin approximations of solutions to different differential equations see [**12, 21, 22**] and the references therein.

Let H and K be two separable Hilbert spaces, and let $L(K, H)$ be the space of bounded linear operators from K into H. For convenience, we will use the same notation $\| \cdot \|$ to denote the norms in H, K and $L(K, H)$, and use $(\cdot, \cdot)$ to denote inner products in H and K. To the best of our knowledge, existence and approximation of solutions to delay equations with a random forcing term and non local conditions have not been considered in the literature. Motivated by the above consideration, and based on the semigroup theory of operators and the Banach fixed point theorem, in the present work we shall use Faedo-Galerkin approximations to prove the existence of solutions to the following delay equation with a random forcing term and non local conditions:

$$(1.1) \quad \frac{du(t)}{dt} + Au(t) = f\big(t, u(t), u_t\big)$$
$$+ \int_0^t a(t-s)g\big(s, u(s), u_s\big) \, d\omega(s), \quad t \in (0, T],$$

subject to the non local condition

$$(1.2) \qquad\qquad \kappa(u_0) = \phi \quad \text{on } [-\tau, 0],$$

where $u(\cdot)$ is a random variable with values in a separable Hilbert space H. The linear operator $A : D(A) \subset H \to H$ is closed, densely defined, positive definite and self-adjoint, and is assumed to have a pure point spectrum $0 < \lambda_0 \leq \lambda_1 \leq \cdots$ and corresponding complete orthonormal eigensystem $\{\psi_i\}$, so that $A\psi_i = \lambda_i \psi_i$ and $(\psi_i, \psi_j) = \delta_{ij}$, where δ_{ij} is the Kronecker delta function. These assumptions on A guarantee that $-A$ is the infinitesimal generator of an analytic semigroup $\{e^{-tA} : t \geq 0\}$ in H. The history u_t is defined by

$$u_t(\theta) = u(t + \theta) \quad \text{for } \theta \in [-\tau, 0].$$

The functions f and g depend on the history of u over the time interval $(t - \tau, t)$ and are required to meet the hypotheses (H4) and

(H5) in Section 2. The given K-valued Wiener process $\{\omega(t) : t \geq 0\}$ is assumed to possess a finite trace, nuclear covariance operator $Q \geq 0$ defined on a complete probability space $(\Omega, \mu, \{\mu_t\}_{t\geq 0}, \mathbb{P})$. In the nonlocal condition, κ is a map defined from C_0 into C_0, where C_0 is the space of all continuous functions from $[-\tau, 0]$ into H endowed with the supremum norm

$$\|\varphi\|_0 = \sup_{-\tau \leq \nu \leq 0} \|\varphi(\nu)\|.$$

The rest of the paper is organized as follows. In Section 2, we provide some necessary notation, spaces and preliminaries which will be used throughout the paper, and reformulate (1.1) as a stochastic integral equation. In Section 3, we use a sequence of projection operators to define a corresponding sequence of approximate integral equations. The existence and uniqueness of the solution to each approximate integral equation is established. Then, the convergence of these solutions to the desired solution of the original integral equation is shown. In Section 4, we consider the Faedo-Galerkin approximate solution and prove the main result concerning the convergence of such an approximation. In Section 5, we give an example as an application of the proposed theory.

2. Preliminaries and assumptions. Let $(\Omega, \mu, \{\mu_t\}_{t\geq 0}, \mathbb{P})$ be a filtered complete probability space satisfying the usual conditions (i.e., the filtration is a right-continuous increasing family and μ_0 contains all $\mathbb{P}$-null sets) and $\{\mu_t\}_{t\geq 0} \subset \mu$. A *random variable* $u(t)$ is a μ-measurable function from Ω into H, and a collection of random variables

$$S = \{u(t) : \Omega \longrightarrow H\}_{t\in[0,T]}$$

is called a *stochastic process*. Let $B_n(t)$, $n = 1, 2, 3, \ldots$, be a sequence of one-dimensional Brownian motions defined on the probability space, and set

$$\omega(t) = \sum_{n=1}^{\infty} \sqrt{\widehat{\lambda}_n} B_n(t) \varsigma_n, \quad t \geq 0,$$

where $\widehat{\lambda}_n \geq 0$, $n = 1, 2, \ldots$, are nonnegative real numbers and $\{\varsigma_n\}$ is a complete orthonormal basis for K. Let $Q \in L(K, K)$ be the operator

defined by $Q\varsigma_n = \widehat{\lambda}_n\varsigma_n$ with finite trace

$$\text{Tr}(Q) = \sum_{n=1}^{\infty} \widehat{\lambda}_n < \infty.$$

Then, the above K-valued stochastic process $\omega(t)$ is called a Q-*Wiener process*. We assume that $\mu_t = \sigma(\omega(s) : 0 \leq s \leq t)$ is the σ-algebra generated by ω and that $\mu_t = \mu$. Let $\phi \in L(K, H)$, and define

$$\|\phi\|_Q^2 = \text{Tr}(\phi Q \phi^*) = \sum_{n=1}^{\infty} \left\| \sqrt{\widehat{\lambda}_n}\, \phi\, \varsigma_n \right\|^2.$$

If $\|\phi\|_Q < \infty$, then ϕ is called a Q-Hilbert-Schmidt operator. Let $L_Q(K, H)$ denote the space of all Q-Hilbert-Schmidt operators $\phi : K \to H$. The completion $L_Q(K, H)$ of $L(K, H)$ with respect to the topology induced by the norm $\| \cdot \|_Q$, where $\|\phi\|_Q^2 = \langle \phi, \phi \rangle$, is a Hilbert space with the above norm topology. Let $L_2(\Omega, \mu, \mathbb{P}; H) \equiv L_2(\Omega; H)$ denote the space of strongly-measurable, square integrable random variables equipped with the norm

$$\|u\|_{L_2(\Omega, H)} = (E\|u\|_H^2)^{1/2},$$

where E is defined as integration with respect to the probability measure $\mathbb{P}$. We recall that, for a stochastic process $\{F(t) : \Omega \to L_Q(K, H)\}$,

$$E\left\| \int_0^t F(s)\,d\omega(s) \right\|^2 \leq \int_0^t E\|F(s)\|_Q^2\,ds$$

$$\leq \text{Tr}(Q) \int_0^t E\|F(s)\|^2\,ds;$$

see Da Prato and Zabczyk [**13**, Lemma 7.2].

A subspace of $L_2(\Omega, H)$ is given by $L_2^0(\Omega, H) = \{f \in L_2(\Omega, H) : f$ is μ_0-measurable$\}$. Our assumptions on A imply that $0 \in \rho(-A)$, the resolvent set of $-A$. Then, for $0 < \alpha \leq 1$, it is possible to define the fractional power A^α as a closed linear operator on its domain $D(A^\alpha)$, being dense in H, and we denote the Banach space $D(A^\alpha)$ by H_α endowed with the norm

$$\|u\|_\alpha = \|A^\alpha u\|, \quad u \in D(A^\alpha),$$

which is equivalent to the graph norm of A^α. For more details on fractional power operators, see Pazy [**28**, page 69]. For all $t \in [-\tau, T]$, define the Banach space $C_T^\alpha = C([-\tau, T], H_\alpha)$ endowed with the supremum norm, as

$$\|u\|_{T,\alpha} = \sup_{t \in [-\tau, T]} \|A^\alpha u(t)\|.$$

Recall that the time history u_t is defined by $u_t(\theta) = u(t + \theta)$ for $\theta \in [-\tau, 0]$ and belongs to the phase space C_0^α. The norm $\|u_t\|_{0,\alpha}$ is given by:

$$\|u_t\|_{0,\alpha} = \sup_{\theta \in [-\tau, 0]} \|A^\alpha u_t(\theta)\|, \quad u_t \in C_0^\alpha.$$

To prove our main results, we assume the following hypotheses.

(H1) $a \in L_{\text{loc}}^{2p}(0, \infty)$ for some $1 < p < \infty$.

(H2) There exists a function $h \in C_T^\alpha$ such that $\kappa(h_0) = \phi$ for all $t \in [-\tau, 0]$.

(H3) The function $h(t) \in D(A^\alpha)$ for all $t \in [-\tau, 0]$ and h is locally Hölder continuous on $[-\tau, 0]$.

(H4) The map $f : [0, T] \times D(A^\alpha) \times C_0^\alpha \to H$ is continuous, and there exists a non-decreasing function $L_f : [0, \infty) \to [0, \infty)$, dependent on $R > 0$, such that

$$\|f(t, u, v)\| \leq L_f(t)$$

and

$$\|f(t, u_1, v_1) - f(t, u_2, v_2)\| \leq L_f(t)\big[\|u_1 - u_2\|_\alpha + \|v_1 - v_2\|_{0,\alpha}\big],$$

for all $t \in [0, T]$, $u_1, u_2 \in B_R(D(A^\alpha), h(0))$ and $v_1, v_2 \in B_R(C_0^\alpha, h)$. Here,

$$B_R(Z, z_0) = \{z \in Z : \|z - z_0\|_Z \leq R\}$$

for any Banach space $(Z, \|\cdot\|_Z)$ and $z_0 \in Z$.

(H5) The map $g : [0, T] \times D(A^\alpha) \times C_0^\alpha \to L_Q(K, H)$ is continuous, and there exists a non-decreasing function $L_g \in L_{\text{loc}}^{2q}(0, \infty)$, dependent on $R > 0$, where $1 < q < \infty$ and $1/p + 1/q = 1$, such that

$$\|g(t, u, v)\|_Q \leq L_g(t)$$

and

$$\|g(t, u_1, v_1) - g(t, u_2, v_2)\|_Q \le L_g(t)\big[\|u_1 - u_2\|_\alpha + \|v_1 - v_2\|_{0,\alpha}\big],$$

for all $t \in [0, T]$, $u_1, u_2 \in B_R(D(A^\alpha), h(0))$ and $v_1, v_2 \in B_R(C_0^\alpha, h)$.

Definition 2.1. [5] A stochastic process $\{u(t)\}_{[-\tau, T_0]}$, $0 < T_0 \le T$, is called a *mild solution* of (1.1)–(1.2) if

  (i) u(t) is measurable and μ_t-adapted.
  (ii) u(t) is continuous on $[0, T_0]$ almost surely and satisfies the following stochastic integral equation

$$(2.1) \quad u(t) = e^{-tA}h(0) + \int_0^t e^{-(t-s)A} f\big(s, u(s), u_s\big)\, ds$$

$$+ \int_0^t e^{-(t-s)A}\left(\int_0^s a(s-\tau)g\big(\tau, u(\tau), u_\tau\big)\, d\omega(\tau)\right) ds,$$

for $t \in [0, T_0]$, with $u(t) = h(t)$ for $t \in [-\tau, 0]$.

The existence of solutions of (1.1) is associated with integral equation (2.1), see Da Prato [**13**, Theorem 5.4].

3. Existence of approximate solutions. In this section, we consider approximate integral equations to equation (2.1) and establish the existence and uniqueness of their solutions. A solution u of equation (2.1) on $[-\tau, T_0]$ is a function u in the Banach space $C_\alpha(T_0) = C([-\tau, T_0], L_2(\Omega, D(A^\alpha)))$ of all continuous functions from $[-\tau, T_0]$ into $L_2(\Omega, D(A^\alpha))$ endowed with the supremum norm

$$\|u\|_{C_\alpha(T_0)} = \sup_{t \in [-\tau, T_0]} \|A^\alpha u(t)\|.$$

Lemma 3.1. *Suppose $0 < \alpha \le 1$, let $-A$ be the infinitesimal generator of an analytic semigroup $\{e^{-tA} : t \ge 0\}$ such that $\|e^{-tA}\| \le C$ for $t \ge 0$, and assume $0 \in \rho(-A)$. Then,*

  (i) *$D(A^\alpha)$ is a Hilbert space.*
  (ii) *For every $t > 0$, the operator $A^\alpha e^{-tA}$ is bounded and satisfies*

$$\|Ae^{-tA}\| \le Ct^{-1} \quad and \quad \|A^\alpha e^{-tA}\| \le C_\alpha t^{-\alpha}.$$

Proof. See Pazy [**28**].   □

We define

$$\bar{h}(t) = \begin{cases} h(t), & t \in [-\tau, 0], \\ h(0), & t \in [0, T_0], \end{cases}$$

and let H_n denote the finite-dimensional subspace of H spanned by the first $n+1$ orthonormal eigenfunctions $\psi_0, \psi_1, \ldots, \psi_n$ of A. The corresponding orthoprojector is denoted by by $P^n : H \to H_n$ for $n = 0, 1, 2, \ldots$.

We choose $T_0 \in (0, T]$ and $R > 0$ in such a way that

$$(3.1) \qquad \sup_{0 \le t \le T_0} E\|A^\alpha(e^{-tA} - I)h(0)\|^2 \le \frac{R}{6}.$$

Moreover, for $0 < \alpha < 1/2$,

$$(3.2) \qquad T_0 < \left(\frac{R(1 - 2\alpha)}{6C(R)C_\alpha^2} \right)^{1/(1-2\alpha)},$$

where

$$(3.3) \quad C(R) = (1 + R)\big(TL_f^2(T_0) + \mathrm{Tr}(Q)\|a^2\|_{L^p(0,T_0)}\|L_g^2\|_{L^q(0,T_0)}\big).$$

We define

$$f_n : [0, T] \times D(A^\alpha) \times C_0^\alpha \longrightarrow H$$

and

$$g_n : [0, T] \times D(A^\alpha) \times C_0^\alpha \longrightarrow L_Q(K, H),$$

respectively, by

$$f_n(t, u, v) = f(t, P^n u, P^n v)$$

and

$$g_n(t, u, v) = g(t, P^n u, P^n v),$$

for all $t \in [0, T_0]$, where $(P^n v)(s) = P^n(v(s))$ for all $s \in [-\tau, 0]$. Now, define a map S_n on $B_R(C_\alpha(T_0), \bar{h})$ by

$$(3.4)$$
$$(S_n u)(t) = e^{-tA}h(0) + \int_0^t e^{-(t-s)A} f_n\big(s, u(s), u_s\big)\, ds$$

$$+ \int_0^t e^{-(t-s)A} \left(\int_0^s a(s-\tau) g_n\big(\tau, u(\tau), u_\tau\big) \, d\omega(\tau) \right) ds,$$

for $t \in [0, T_0]$, with $(S_n u)(t) = h(t)$ for $t \in [-\tau, 0]$.

Theorem 3.2. *Let $\alpha \in (0, 1/2)$ and the hypotheses* (H1)–(H5) *hold. If $h(t) \in L_2^0(\Omega, D(A^\alpha))$ for all $t \in [-\tau, 0]$, then there exists a unique $u_n \in B_R$ such that $S_n u_n = u_n$ for all $n = 0, 1, 2, \ldots$.*

Proof. First, we show that $S_n : B_R(C_\alpha(T_0), \overline{h}) \to B_R(C_\alpha(T_0), \overline{h})$. For this, we need to show that the map $t \to (S_n u)(t)$ is continuous from $[-\tau, T_0]$ into $L_2(\Omega, D(A^\alpha))$ with respect to the $\| \cdot \|$-norm. Thus, for any $u \in B_R(C_\alpha(T_0), \overline{h})$ and $t_1, t_2 \in [-\tau, 0]$, we have

(3.5) $\qquad (S_n u)(t_2) - (S_n u)(t_1) = h(t_2) - h(t_1).$

Since the function h satisfies hypothesis (H2), we get the required result. Now, for $t_1, t_2 \in (0, T_0]$ with $t_1 < t_2$,

$$(S_n u)(t_2) - (S_n u)(t_1) = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V},$$

where the five terms on the right are

$$\mathrm{I} = (e^{-t_2 A} - e^{-t_1 A}) h(0),$$

$$\mathrm{II} = \int_{t_1}^{t_2} e^{-(t_2 - s)A} f_n\big(s, u(s), u_s\big) \, ds,$$

$$\mathrm{III} = \int_0^{t_1} \big(e^{-(t_2 - s)A} - e^{-(t_1 - s)A}\big) f_n\big(s, u(s), u_s\big) \, ds$$

$$\mathrm{IV} = \int_0^{t_1} \big(e^{-(t_2 - s)A} - e^{-(t_1 - s)A}\big)$$

$$\times \int_0^s a(s-\tau) g_n\big(\tau, u(\tau), u_\tau\big) \, d\omega(\tau) \, ds,$$

and

$$\mathrm{V} = \int_{t_1}^{t_2} e^{-(t_2 - s)A} \int_0^s a(s-\tau) g_n\big(\tau, u(\tau), u_\tau\big) \, d\omega(\tau) \, ds.$$

Thus,

$$E \| (S_n u)(t_2) - (S_n u)(t_1) \|_\alpha^2$$

$$\leq 5\Big(\|\,\mathrm{I}\,\|_\alpha^2 + E\|\,\mathrm{II}\,\|_\alpha^2 + E\|\,\mathrm{III}\,\|_\alpha^2 + E\|\,\mathrm{IV}\,\|_\alpha^2 + E\|\,\mathrm{V}\,\|_\alpha^2\Big).$$

Now, Pazy [**28,** Theorem 6.13 (d)] states that, if $0 < \beta \leq 1$ and $u \in D(A^\beta)$, then

$$(3.6) \qquad \|(e^{-tA} - I)u\| \leq C_\beta t^\beta \|A^\beta u\|.$$

Thus, if $\beta \in (0,1)$ is such that $0 < \alpha + \beta < 1$, then for $s, t \in (0, T_0]$, we have

$$(3.7) \qquad \|(e^{-tA} - I)A^\alpha e^{-sA}u\| \leq C_\beta t^\beta \|A^{\alpha+\beta} e^{-sA}u\|$$
$$\leq C_\beta C_{\alpha+\beta} t^\beta s^{-(\alpha+\beta)}\|u\|,$$

so

$$\|\,\mathrm{I}\,\|_\alpha = \|(I - e^{-(t_2-t_1)A})A^\alpha e^{-t_1 A}h(0)\| \leq C_\alpha (t_2 - t_1)^\alpha \|h(0)\|_\alpha$$

and

$$E\|\,\mathrm{II}\,\|_\alpha^2 \leq \left(\int_{t_1}^{t_2} \|A^\alpha e^{-(t_2-s)A}\|\,\|E\|\,\big\|f_n\big(s, u(s), u_s\big)\big\|\, ds\right)^2$$
$$\leq (t_2 - t_1)\int_{t_1}^{t_2} C_\alpha^2 (t_2 - s)^{-2\alpha} L_f(s)^2\, ds$$
$$\leq C_\alpha^2 L_f (t_2 - t_1)^2\, \frac{(t_2 - t_1)^{2(1-\alpha)}}{1 - 2\alpha}.$$

Next, provided $\alpha + \beta < 1/2$,

$$E\|\,\mathrm{III}\,\|_\alpha^2 \leq t_1 \int_0^{t_1} \big\|(I - e^{-(t_2-t_1)A})A^\alpha e^{-(t_1-s)A}\big\|^2 L_f(s)^2\, ds$$
$$\leq C_\beta^2 C_{\alpha+\beta}^2 (t_2 - t_1)^{2\beta} L_f(t_1)^2 \int_0^{t_1} (t_1 - s)^{-2(\alpha+\beta)}\, ds$$
$$= C_\beta^2 C_{\alpha+\beta}^2 L_f(t_1)^2 (t_2 - t_1)^{2\beta}\, \frac{t_1^{1-2(\alpha+\beta)}}{1 - 2(\alpha+\beta)},$$

with

$$E\|\,\mathrm{IV}\,\|_\alpha^2 \leq \int_0^{t_1} \big\|A^\alpha\big(e^{-(t_2-s)A} - e^{-(t_1-s)A}\big)\big\|^2$$
$$\mathrm{Tr}(Q)\int_0^s a(s - \tau)^2 E\big\|g_n\big(\tau, u(\tau), u_\tau\big)\big\|^2\, d\tau\, ds$$
$$\leq \mathrm{Tr}(Q)\|a^2\|_{L^p(0,T_0)}\|L_g^2\|_{L^q(0,T_0)}$$

$$\int_0^{t_1} \left\| \left(e^{-(t_2-t_1)A} - I \right) A^\alpha e^{-(t_1-s)A} \right\|^2 ds$$

$$\leq \operatorname{Tr}(Q) \|a^2\|_{L^p(0,T_0)} \|L_g^2\|_{L^q(0,T_0)}$$

$$C_\beta^2 C_{\alpha+\beta}^2 (t_2 - t_1)^{2\beta} \frac{t_1^{1-2(\alpha+\beta)}}{1 - 2(\alpha+\beta)},$$

and

$$E\|V\|_\alpha^2 \leq \int_{t_1}^{t_2} \|A^\alpha e^{-(t_2-s)A}\|^2$$

$$\operatorname{Tr}(Q) \int_0^s a(s-\tau)^2 E\left\| g_n\big(\tau, u(\tau), u_\tau\big) \right\|^2 d\tau \, ds$$

$$\leq \operatorname{Tr}(Q) \|a^2\|_{L_p(0,T_0)} \|L_g^2\|_{L_q(0,T_0)} C_\alpha^2 \frac{(t_2 - t_1)^{1-2\alpha}}{1 - 2\alpha}.$$

The preceding estimates imply that

$$E\|(S_n u)(t_2) - (S_n u)(t_1)\|_\alpha^2 = O\big((t_2 - t_1)^\gamma\big)$$

for $\gamma = \min(2\alpha, 2(1-\alpha), 2\beta) > 0$, so the map $t \mapsto (S_n u)(t)$ is continuous from $[-\tau, T_0]$ into $L_2(\Omega, D(A^\alpha))$ with respect to the $\|\cdot\|$ norm.

Now, for $t \in [-\tau, 0]$, we have

$$E\|(S_n u)(t) - \overline{h}(t)\|_\alpha^2 = 0,$$

and for $t \in (0, T_0]$, we have

$$(3.8) \quad E\|(S_n u)(t) - \overline{h}(t)\|_\alpha^2 \leq 3\|A^\alpha(e^{-tA} - I)h(0)\|^2$$

$$+ 3 \int_0^t \|A^\alpha e^{-(t-s)A}\|^2 \left(TE\left\| f_n\big(s, u(s), u_s\big) \right\|^2 \right.$$

$$\left. + \operatorname{Tr}(Q) \int_0^s |a(s-\tau)|^2 E\left\| g_n\big(\tau, u(\tau), u_\tau\big) \right\|^2 d\tau \right) ds.$$

Using Lemma 3.1 and equations (3.1)–(3.3), we get

$$E\|(S_n u)(t) - \overline{h}(t)\|_\alpha^2 \leq 3\left(\frac{R}{6} + C(R) C_\alpha^2 \frac{T_0^{1-2\alpha}}{1 - 2\alpha} \right) \leq R,$$

showing that $S_n : B_R(C_\alpha(T_0), \overline{h}) \to B_R(C_\alpha(T_0), \overline{h})$.

Let $u, v \in B_R(C_\alpha(T_0), \overline{h})$. If $t \in [-\tau, 0]$, then

$$E\|S_n u(t) - S_n v(t)\|^2 = 0,$$

whereas, if $t \in (0, T_0]$, then

$$E\|(S_n u)(t) - (S_n v)(t)\|_\alpha^2 \leq 2 \int_0^t \|A^\alpha e^{-(t-s)A}\|^2$$

$$\left(E\|f_n(s, u(s), u_s) - f_n(s, v(s), v_s)\|^2 \right.$$

$$\left. + \mathrm{Tr}(Q) \int_0^s a(s-\tau)^2 E\|g_n(\tau, u(\tau), u_\tau) - g_n(\tau, v(\tau), v_\tau)\|^2 d\tau \right) ds$$

$$\leq 2\big(T L_f^2(T_0) + Tr(Q)\|a^2\|_{L^p(0,T_0)}\|L_g^2\|_{L^q(0,T_0)}\big)$$

$$\big(E\|u - v\|_\alpha^2 + E\|u_\tau - v_\tau\|_{0,\alpha}^2\big)C_\alpha^2 \frac{T_0^{1-2\alpha}}{1 - 2\alpha}$$

$$\leq \frac{2}{R} C(R)C_\alpha^2 \frac{T_0^{1-2\alpha}}{1 - 2\alpha} \big(E\|u - v\|_\alpha^2 + E\|u_\tau - v_\tau\|_{0,\alpha}^2\big)$$

$$\leq \frac{1}{3} E\|u - v\|_{C_\alpha(T)}^2.$$

Thus, taking the supremum over $t \in [-\tau, T_0]$, we get

$$\|S_n u - S_n v\|_{C_\alpha(T)}^2 \leq \frac{1}{3}\|u - v\|_{C_\alpha(T)}^2.$$

Hence, S_n is a strict contraction on $B_R(C_\alpha(T_0), \overline{h})$, and, by the Banach fixed point theorem, there exists a unique $u_n \in B_R(C_\alpha(T_0), \overline{h})$ such that $S_n u_n = u_n$ or, in other words, u_n satisfies the approximate integral equation,

(3.9)

$$u_n(t) = \begin{cases} h(t), & t \in [-\tau, 0]; \\ e^{-tA}h(0) + \int_0^t e^{-(t-s)A} f_n(s, u_n(s), (u_n)_s) \, ds \\ \quad + \int_0^t e^{-(t-s)A} \int_0^s a(s-\tau)g_n(\tau, u_n(\tau), (u_n)_\tau) \, d\omega(\tau) \, ds, & t \in [0, T_0]. \end{cases}$$

This completes the proof.   □

Lemma 3.3. *Let* $h(t) \in L_2(\Omega, D(A))$ *for all* $t \in [-\tau, 0]$. *Then* $u_n(t) \in L_2(\Omega, D(A^\beta))$ *for all* $t \in [-\tau, T_0]$ *and* $0 \leq \beta < 1/2$.

Proof. For $t \in [-\tau, 0]$, it is obvious. Therefore, let $t \in (0, T]$. From Theorem 3.2, we have the existence of a unique $u_n \in B_R(C_\alpha(T_0), \overline{h})$ satisfying (3.9), and a result of Pazy **[28**, Theorem 2.6.13 (a)] implies that $e^{-tA} : H \to L_2(\Omega, D(A^\beta))$ for $t > 0$ and $0 \le \beta < 1/2$, and that $D(A^\beta) \subseteq D(A^\alpha)$ for $0 \le \alpha \le \beta < 1$. By another result of Pazy **[28**, Theorem 1.2.4], we have $e^{-tA}u \in L_2(\Omega, D(A))$ if $u \in L_2(\Omega, D(A))$. The required result follows from these facts and the fact that $L_2(\Omega, D(A)) \subseteq L_2(\Omega, D(A^\beta))$ for $0 \le \beta < 1/2$. $\qquad\square$

Lemma 3.4. *Let $h(t) \in L_2^0(\Omega, D(A^\alpha))$ for all $t \in [-\tau, 0]$. Then, for any $t \in [-\tau, T_0]$, there exists a constant C_{t_0}, independent of n, such that*

$$E\|u_n(t)\|_\beta^2 \le C_{t_0} \quad \text{for } 0 \le \alpha < \beta < \frac{1}{2} \text{ and } 0 \le t_0 \le T_0.$$

Proof. Applying A^β to both sides of approximate integral equation (3.9), it follows **[28**, Theorem 6.13 (c)] that, for any $t \in [-\tau, 0]$,

$$E\|u_n(t)\|_\beta^2 \le E\|h(t)\|_\beta^2 \le \|h\|_{0,\beta},$$

and, again for $t \in (0, T_0]$,

$$
\begin{aligned}
E\|u_n(t)\|_\beta^2 &\le 3\bigg(\|A^\beta e^{-tA}h(0)\|^2 + E\bigg\|\int_0^t A^\beta e^{-(t-s)A} f_n\big(s, u(s), u_s\big)\, ds\bigg\|^2 \\
&\quad + E\bigg\|\int_0^t A^\beta e^{-(t-s)A}\int_0^s a(s-\tau) g_n\big(\tau, u(\tau), u_\tau\big)\, d\omega(\tau)\, ds\bigg\|_Q^2\bigg) \\
&\le 3\|A^\beta e^{-tA}h(0)\|^2 \\
&\quad + 3T\int_0^t \|A^\beta e^{-(t-s)A}\|^2 E\big\|f_n\big(s, u(s), u_s\big)\big\|^2 ds \\
&\quad + 3\operatorname{Tr}(Q)\int_0^t \|A^\beta e^{-(t-s)A}\|^2 \\
&\qquad \bigg(\int_0^s a(s-\tau)^2 E\big\|g_n\big(\tau, u(\tau), u_\tau\big)\big\|^2 d\tau\bigg)\, ds \\
&\le 3\bigg(C_\beta^2 t_0^{-2\beta} E\|h(0)\|^2 + C_\beta^2 C(R)\frac{T_0^{1-2\beta}}{1-2\beta}\bigg) \le C_{t_0}. \qquad\square
\end{aligned}
$$

Theorem 3.5. *If the hypotheses* (H1)–(H5) *hold, and if* $h(t) \in L_2(\Omega, D(A))$, *then* $\{u_n\} \subset C_\alpha(T_0)$ *is a Cauchy sequence and therefore converges to a function* $u \in C_\alpha(T_0)$ *satisfying equation* (2.1).

Proof. We divide the proof into two steps.

Step 1. Let $m, n, n_0 \in \mathbb{N}$ be such that $n \geq m \geq n_0$ and n_0 is sufficiently large. For $t \in [-\tau, 0]$, we have

$$E\|u_n(t) - u_m(t)\|_\alpha^2 = E\|h(t) - h(t)\|_\alpha^2 = 0,$$

and, for $t \in (0, T_0]$,

$$E\|u_n(t) - u_m(t)\|_\alpha^2 \leq 2E\left\|\int_0^t A^\alpha e^{-(t-s)A}\big[f_n\big(s, u_n(s), (u_n)_s\big)\right.$$
$$\left. - f_m\big(s, u_m(s), (u_m)_s\big)\big]\,ds\right\|^2$$
$$+ 2E\left\|\int_0^t A^\alpha e^{-(t-s)A}\left(\int_0^s a(s-\tau)\right.\right.$$
$$\big[g_n\big(\tau, u_n(\tau), (u_n)_\tau\big) - g_m\big(\tau, u_m(\tau), (u_m)_\tau\big)\big]d\omega(\tau)\bigg)ds\bigg\|_Q^2.$$

Now, for $0 < t_0' < t$, we have

$$E\|u_n(t) - u_m(t)\|_\alpha^2 \leq 2\left(\int_0^{t_0'} + \int_{t_0'}^t\right)\|A^\alpha e^{-(t-s)A}\|^2$$
(3.10)
$$\left(TE\|f_n\big(s, u_n(s), (u_n)_s\big) - f_m\big(s, u_m(s), (u_m)_s\big)\|^2\right.$$
$$+ \operatorname{Tr}(Q)\int_0^s a(s-\tau)^2$$
$$E\|g_n\big(\tau, u_n(\tau), (u_n)_\tau\big) - g_m\big(\tau, u_m(\tau), (u_m)_\tau\big)\|^2 d\tau\bigg)ds.$$

The first integral of equation (3.10) is bounded by

$$\int_0^{t_0'}\|A^\alpha e^{-(t-s)A}\|^2\left(TE\|f_n\big(s, u_n(s), (u_n)_s\big)\right.$$
$$\left. - f_m\big(s, u_m(s), (u_m)_s\big)\|^2 + \operatorname{Tr}(Q)\int_0^s a(s-\tau)^2\right.$$

(3.11)
$$E\big\|g_n\big(\tau, u_n(\tau), (u_n)_\tau\big) - g_m\big(\tau, u_m(\tau), (u_m)_\tau\big)\big\|^2 d\tau\bigg) ds$$

$$\leq 2C(R)C_\alpha^2(t_0 - t_0')^{-2\alpha}t_0'.$$

Now,

$$E\big\|f_n\big(s, u_n(s), (u_n)_s\big) - f_m\big(s, u_m(s), (u_m)_s\big)\big\|^2$$
$$\leq E\big\|f_n\big(s, u_n(s), (u_n)_s\big) - f_n\big(s, u_m(s), (u_m)_s\big)\big\|^2$$
$$\qquad + E\big\|f_n\big(s, u_m(s), (u_m)_s\big) - f_m\big(s, u_m(s), (u_m)_s\big)\big\|^2$$
$$\leq 2L_f^2(T_0)\big(E\|u_n - u_m\|_{s,\alpha}^2 + E\|(P^n - P^m)u_m(t)\|_\alpha^2\big),$$

and, for $0 < \alpha < \beta < 1/2$, we have

$$E\|A^\alpha(P^n - P^m)u_m(t)\|^2 \leq E\|A^{\alpha-\beta}(P^n - P^m)A^\beta u_m(t)\|^2$$
$$\leq \lambda_m^{-2(\beta-\alpha)}E\|A^\beta u_m(t)\|^2.$$

Therefore,

$$E\|f_n(s, u_n(s), (u_n)_s) - f_m(s, u_m(s), (u_m)_s)\|^2$$
$$\leq 2L_f^2(T_0)\big(E\|u_n - u_m\|_{s,\alpha}^2 + C_{t_0}\lambda_m^{-2(\beta-\alpha)}\big).$$

Similarly,

$$E\|g_n(s, u_n(s), (u_n)_s) - g_m(s, u_m(s), (u_m)_s)\|^2$$
$$\leq 2L_g^2(T_0)\big(E\|u_n - u_m\|_{s,\alpha}^2 + C_{t_0}\lambda_m^{-2(\beta-\alpha)}\big).$$

The second integral of (3.10) is bounded by

(3.12)
$$\int_{t_0'}^t \|A^\alpha e^{-(t-s)A}\|^2\bigg(TE\big\|f_n\big(s, u_n(s), (u_n)_s\big) - f_m\big(s, u_m(s), (u_m)_s\big)\big\|^2$$

$$\qquad + \mathrm{Tr}(Q)\int_0^s a(s-\tau)^2$$

$$\qquad\qquad E\big\|g_n\big(\tau, u_n(\tau), (u_n)_\tau\big) - g_m\big(\tau, u_m(\tau), (u_m)_\tau\big)\big\|^2 d\tau\bigg) ds$$

$$\leq 2\int_{t_0'}^t C_\alpha^2(t-s)^{-2\alpha}\bigg(TL_f^2(T_0)E\|u_n - u_m\|_{s,\alpha}^2$$

$$+ L_f^2(T_0) C_{t_0} \lambda_m^{-2(\beta-\alpha)} + \text{Tr}(Q) \int_0^s a(s-\tau)^2$$

$$\left(L_g^2(T_0) E\|u_n - u_m\|_{s,\alpha}^2 + L_g^2(T_0) C_{t_0} \lambda_m^{-2(\beta-\alpha)} \right) \right) ds$$

$$\leq 2C(R) C_\alpha^2 \left(\frac{C_{t_0}}{\lambda_m^{2(\beta-\alpha)}} \frac{T_0^{1-2\alpha}}{(1-2\alpha)} + \int_{t_0'}^t (t-s)^{-2\alpha} E\|u_n - u_m\|_\alpha^2 \, ds \right).$$

Using (3.11) and (3.12) in (3.10), we get

(3.13) $E\|u_n(t) - u_m(t)\|_\alpha^2$

$$\leq M_1 t_0' + M_2 \lambda_m^{-2(\beta-\alpha)} + M_3 \int_{t_0'}^t (t-s)^{-2\alpha} E\|u_n - u_m\|_\alpha^2 \, ds,$$

where

$$M_1 = \frac{4C(R) C_\alpha^2}{(t_0 - t_0')^{2\alpha}},$$

$$M_2 = 4C(R) C_\alpha^2 C_{t_0} \frac{T_0^{1-2\alpha}}{1-2\alpha},$$

$$M_3 = 4C(R) C_\alpha^2.$$

Now replace t by $t + \theta$ in inequality (3.13), where $\theta \in [t_0' - t, 0]$, to get

(3.14) $E\|u_n(t+\theta) - u_m(t+\theta)\|_\alpha^2 \leq M_1 t_0' + M_2 \lambda_m^{-2(\beta-\alpha)}$

$$+ M_3 \int_{t_0'}^{t+\theta} \frac{E\|u_n - u_m\|_{s,C_{\tilde{h}}}^2}{(t+\theta-s)^{2\alpha}} \, ds,$$

and then put $s - \theta = \nu$ in (3.14) to obtain

$E\|u_n(t+\theta) - u_m(t+\theta)\|_\alpha^2$

$$\leq M_1 t_0' + M_2 \lambda_m^{-2(\beta-\alpha)} + M_3 \int_{t_0'}^t \frac{E\|u_n - u_m\|_{\nu,C_{\tilde{h}}}^2}{(t-\nu)^{2\alpha}} \, ds.$$

Since

$$\sup_{t_0' - t \leq \theta \leq 0} E\|u_n(t+\theta) - u_m(t+\theta)\|_\alpha^2$$

$$\leq M_1 t_0' + M_2 \lambda_m^{-2(\beta-\alpha)} + M_3 \int_{t_0'}^t \frac{E\|u_n - u_m\|_{\nu,\alpha}^2}{(t-\nu)^{2\alpha}} \, ds,$$

we have

$$\sup_{-\tau-t\leq\theta\leq0} E\|u_n(t+\theta)-u_m(t+\theta)\|_\alpha^2$$

$$\leq \sup_{0\leq\theta+t\leq t_0'} E\|u_n(t+\theta)-u_m(t+\theta)\|_\alpha^2$$

$$+ \sup_{t_0'-t\leq\theta\leq0} E\|u_n(t+\theta)-u_m(t+\theta)\|_\alpha^2.$$

Using the inequalities above, it follows that

$$\sup_{-\tau\leq t+\theta\leq t} E\|u_n(t+\theta)-u_m(t+\theta)\|_\alpha^2$$

$$\leq (M_1+M_4)t_0' + (M_2+M_5)\lambda_m^{-2(\beta-\alpha)}$$

$$+ M_3 \int_{t_0'}^t \frac{E\|u_n-u_m\|_{\nu,\alpha}^2}{(t-\nu)^{2\alpha}}\, ds,$$

where M_4 and M_5 are some constants. Now, applying a generalized Gronwall inequality [**17**] to the above inequality, and as t_0' is arbitrarily small, we conclude that u_n is a Cauchy sequence.

Step 2. The sequence u_n converges to u in $C_\alpha(T_0)$.

We have $h(t) \in D(A)$ for all $t \in [-\tau,0]$. Using Step 1,

$$\lim_{m\to\infty} \sup_{\substack{n\geq m \\ 0\leq t\leq T_0}} E\|u_n(t)-u_m(t)\|_\alpha^2 = 0,$$

i.e., there exists $u \in C_\alpha(T_0)$ such that u_n converges to u in the mean square sense. For $u_n \in B_R(C_\alpha(T_0), \overline{h})$,

$$E\|u_n(t)-u(t)\|_\alpha^2 \leq 2\int_0^t \|A^\alpha e^{-(t-s)A}\|^2$$

$$\left(TE\|f_n(s,u_n(s),(u_n)_s) - f(s,u(s),u_s)\|^2\right.$$

$$+ \mathrm{Tr}(Q)\int_0^s |a(s-\tau)|^2$$

$$E\|g_n(\tau,u_n(\tau),(u_n)_\tau) - g(\tau,u(\tau),u_\tau)\|^2 d\tau\Bigg) ds$$

$$\leq 2C_\alpha^2\Big(TE\|f(t,P^n u_n(t),P^n(u_n)_t) - f(t,u(t),u_t)\|^2$$

$$+ \operatorname{Tr}(Q) \int_0^t |a(t - \tau)|^2$$

$$E\big\|g\big(\tau, P^n u_n(\tau), P^n (u_n)_\tau\big) - g\big(\tau, u(\tau), u_\tau\big)\big\|^2 \, d\tau\bigg) \frac{T_0^{1-2\alpha}}{1 - 2\alpha}$$

$$\leq \frac{2C_\alpha^2 T_0^{1-2\alpha}}{1 - 2\alpha} \bigg(TE\big\|f\big(t, P^n u_n(t), P^n (u_n)_t\big) - f\big(t, P^n u(t), P^n u_t\big)\big\|_\alpha^2$$

$$+ TE\big\|f\big(t, P^n u(t), P^n u_t\big) - f\big(t, u(t), u_t\big)\big\|_\alpha^2$$

$$+ \operatorname{Tr}(Q) \int_0^t |a(t - \tau)|^2$$

$$\big[E\big\|g\big(\tau, P^n u_n(\tau), P^n (u_n)_\tau\big) - g\big(\tau, P^n u(\tau), P^n u_\tau\big)\big\|_\alpha^2$$

$$+ E\big\|g\big(\tau, P^n u(\tau), P^n u_\tau\big) - g\big(\tau, u(\tau), u_\tau\big)\big\|_\alpha^2\big] \, d\tau\bigg)$$

$$\leq \frac{2C_\alpha^2 T_0^{1-2\alpha}}{1 - 2\alpha} C(R) \Big(E\|u_n(t) - u_t\|_\alpha^2 + E\|(P^n - I)u(t)\|_\alpha^2 \Big).$$

Taking the supremum over $t \in [0, T_0]$, we get

$$\lim_{n \to 0} \sup_{0 \leq t \leq T_0} E\|u_n(t) - u(t)\|_\alpha^2 = 0$$

in the mean square sense. Let $h(t) \in L_2(\Omega, D(A^\alpha))$ for $t \in [-\tau, 0]$. Since, for $0 < t \leq T_0$, the sequence $A^\alpha u_n(t)$ converges to $A^\alpha u(t)$ as $n \to \infty$, and since $u_n(t) = u(t) = h(t)$ for all n when $t \in [-\tau, 0]$, if $-\tau = t = T$, then $A^\alpha u_n(t)$ converges to $A^\alpha u(t)$ in the mean square sense in $L_2(\Omega, H)$. Since $u_n \in B_R(C_\alpha(T_0), \widetilde{h})$, it follows that $u \in B_R(C_\alpha(T_0), \widetilde{\phi})$ and, for any $0 < t_0 \leq T$,

$$\lim_{n \to \infty} \sup_{t_0 \leq t \leq T_0} E\|u_n(t) - u(t)\|_\alpha^2 = 0$$

in the mean square sense. Also, we have

$$\sup_{t_0 \leq t \leq T_0} \big\|f_n\big(t, (u_n)(t), (u_n)_t\big) - f\big(t, u(t), u_t\big)\big\|^2$$

$$\leq L_f^2(T_0) \Big(E\|u_n - u\|_{T_0, \alpha}^2 + E\|(P^n - I)u\|_{T_0, \alpha}^2 \Big) \longrightarrow 0,$$

as $n \to \infty$, and

$$\sup_{t_0 \leq t \leq T_0} \big\|g_n\big(t, (u_n)(t), (u_n)_t\big) - g\big(t, u(t), u_t\big)\big\|^2$$

$$\leq L_g^2(T_0)\Big(E\|u_n - u\|_{T_0,\alpha}^2 + E\|(P^n - I)u\|_{T_0,\alpha}^2\Big) \longrightarrow 0.$$

Now we will show that u satisfies the integral equation (2.1). For $0 < t_0 < t$, (3.9) can be written as

$$u_n(t) = e^{-tA}h(0) + \left(\int_0^{t_0} + \int_{t_0}^t\right)e^{-(t-s)A}\left(f_n\big(s, u_n(s), (u_n)_s\big)\right.$$
$$\left. + \int_0^s a(s-\tau)g_n\big(\tau, u_n(\tau), (u_n)_\tau\big)\,d\omega(\tau)\right)ds.$$

Thus,

$$E\Big\|u_n(t) - e^{-tA}h(0) - \int_{t_0}^t e^{-(t-s)A}\left(f_n\big(s, u_n(s), (u_n)_s\big)\right.$$
$$\left. + \int_0^s a(s-\tau)g_n\big(\tau, u_n(\tau), (u_n)_\tau\big)\,d\omega(\tau)\right)ds\Big\|^2$$
$$= E\Big\|\int_0^{t_0} e^{-(t-s)A}\left(f_n\big(s, u_n(s), (u_n)_s\big)\right.$$
$$\left. + \int_0^s a(s-\tau)g_n\big(\tau, u_n(\tau), (u_n)_\tau\big)\,d\omega(\tau)\right)ds\Big\|^2$$
$$\leq \int_0^{t_0} \|e^{-(t-s)A}\|^2\left(TE\|f_n\big(s, u_n(s), (u_n)_s\big)\|^2\right.$$
$$\left. + \operatorname{Tr}(Q)\int_0^s |a(s-\tau)|^2 E\|g_n\big(\tau, u_n(\tau), (u_n)_\tau\big)\|^2\,d\tau\right)ds$$
$$\leq C^2 C(R)t_0.$$

Sending $n \to \infty$ in this estimate, we get

$$E\Big\|u(t) - e^{-tA}h(0) - \int_{t_0}^t e^{-(t-s)A}\left(f\big(s, u(s), u_s\big)\right.$$
$$\left. + \int_0^s a(s-\tau)g\big(\tau, u(\tau), u_\tau\big)\,d\omega(\tau)\right)ds\Big\|^2 \leq C^2 C(R)t_0.$$

Since t_0 is arbitrary, it follows that u satisfies the integral equation (2.1).

Now we will show the uniqueness of u. Let u_1 and u_2 be two functions satisfying (2.1), and put $u = u_1 - u_2$. Then $u = 0$ on $[-\tau, 0]$, and,

for $t \in [0, t_0]$, we have

$$E\|u(t)\|_\alpha^2 \leq 2C(R)C_\alpha^2 \int_0^t (t-s)^{-2\alpha} \sup_{-\tau \leq \theta \leq 0} E\|u(s+\theta)\|_\alpha^2 \, ds,$$

where $R := \max\{E\|u_1 - \overline{h}\|_{C_\alpha(T)}^2, E\|u_2 - \overline{h}\|_{C_\alpha(T)}^2\}$. Let $\overline{\theta} \in [-t, 0]$ and $t \in [0, t_0]$, and assume that $t_0 \leq \tau$; hence, $0 \leq t \leq \tau$. For $t \leq -\overline{\theta}$, we have $u(t + \overline{\theta}) = 0$, and, for $t \geq -\overline{\theta}$, we have

$$E\|u(t+\overline{\theta})\|_\alpha^2 \leq 2C(R)C_\alpha^2 \int_0^{t+\overline{\theta}} (t + \overline{\theta} - s)^{-2\alpha}$$
$$\cdot \sup_{-\tau \leq \theta \leq 0} E\|u(s+\theta)\|_\alpha^2 \, ds.$$

We put $\overline{\theta} - s = -\gamma$ in the above inequality to obtain

$$E\|u(t+\overline{\theta})\|_\alpha^2 \leq 2C(R)C_\alpha^2 \int_{-\overline{\theta}}^t (t - \gamma)^{-2\alpha}$$
$$\cdot \sup_{-\tau \leq \theta \leq 0} E\|u(\gamma + \theta + \overline{\theta})\|_\alpha^2 \, d\gamma,$$

and then $\theta = \eta - \overline{\theta}$ to obtain

$$E\|u(t+\overline{\theta})\|_\alpha^2 \leq 2C(R)C_\alpha^2 \int_{-\overline{\theta}}^t (t - \gamma)^{-2\alpha}$$
$$\cdot \sup_{-\tau + \overline{\theta} \leq \eta \leq 0} E\|u(\gamma + \eta)\|_\alpha^2 \, d\gamma.$$

Since $u(\gamma + \eta) = 0$ on $[-\tau + \overline{\theta}, -\tau]$, the above inequality can be written as

$$E\|u_t(\overline{\theta})\|_\alpha^2 \leq 2C(R)C_\alpha^2 \int_{-\overline{\theta}}^t (t - \gamma)^{-2\alpha} \sup_{-\tau \leq \eta \leq 0} E\|u_\gamma(\eta)\|_\alpha^2 \, d\gamma$$
$$\leq 2C(R)C_\alpha^2 \int_0^t (t - \gamma)^{-2\alpha} \sup_{-\tau \leq \eta \leq 0} E\|u_\gamma(\eta)\|_\alpha^2 \, d\gamma.$$

Taking the supremum over $\overline{\theta} \in [-\tau, 0]$, we get

$$E\|u_t\|_{0,\alpha}^2 \leq 2C(R)C_\alpha^2 \int_0^t (t - \gamma)^{-2\alpha} E\|u_\gamma\|_{0,\alpha}^2 \, d\gamma,$$

and the required result follows by applying the generalized Gronwall's inequality [**17**]. □

4. Faedo-Galerkin approximations. From the previous sections we know that, for any $0 \le T_0 \le T$, we have a unique $u \in C_\alpha(T_0)$ satisfying the integral equation

(4.1)
$$u(t) = e^{-tA}h(0) + \int_0^t e^{-(t-s)A}f\big(s, u(s), u_s\big)\, ds$$
$$+ \int_0^t e^{-(t-s)A}\left(\int_0^s a(s-\tau)g\big(\tau, u(\tau), u_\tau\big)\, d\omega(\tau) \right) ds,$$

for $t \in [0, T_0]$, with $u(t) = h(t)$ for $t \in [-\tau, 0]$. Also, there is a unique solution $u_n \in C_\alpha(T_0)$ of the approximate integral equation

(4.2) $\quad u_n(t) = e^{-tA}h(0) + \int_0^t e^{-(t-s)A}f\big(s, P^n u_n(s), P^n(u_n)_s\big)\, ds$
$$+ \int_0^t e^{-(t-s)A}\left(\int_0^s a(s-\tau)g\big(\tau, P^n u_n(\tau), P^n(u_n)\tau\big)\, d\omega(\tau) \right) ds$$

for $t \in [0, T_0]$, with $u_n(t) = h(t)$ for $t \in [-\tau, 0]$. Now, if we project (4.2) onto H_n, we get the Faedo-Galerkin approximations $\widehat{u}_n = P^n u_n$ satisfying

(4.3) $\quad \widehat{u}_n(t) = e^{-tA}P^n h(0) + \int_0^t e^{-(t-s)A}f\big(s, P^n\widehat{u}_n(s), P^n(\widehat{u}_n)_s\big)\, ds$
$$+ \int_0^t e^{-(t-s)A}\Bigg(\int_0^s a(s-\tau)g\big(\tau, P^n\widehat{u}_n(\tau),$$
$$P^n(\widehat{u}_n)\tau\big)\, d\omega(\tau) \Bigg) ds$$

for $t \in [0, T_0]$, with $\widehat{u}_n(t) = P^n h(t)$ for $t \in [-\tau, 0]$. The solutions $u(t)$ to (4.1) and $\widehat{u}_n(t)$ to (4.3) have the representations

(4.4)
$$u(t) = \sum_{i=0}^{\infty} \alpha_i(t)u_i, \quad \alpha_i(t) = \big(u(t), u_i\big),$$
$$\widehat{u}_n(t) = \sum_{i=0}^{n} \alpha_i^n(t)u_i, \quad \alpha_i^n(t) = \big(\widehat{u}_n(t), u_i\big).$$

As a consequence of Theorems 3.2 and 3.5, we have the following result.

Theorem 4.1. *Let the hypotheses* (H1)–(H5) *hold and assume that* $h(t) \in L_2(\Omega, D(A))$ *for all* $t \in [-\tau, 0]$. *Then there exists a unique func-*

tion $\widehat{u}_n \in C([-\tau, T_0]; H_n)$ and a unique $u \in C([-\tau, T_0]; H)$ satisfying (4.3) and (4.1), respectively, such that $\widehat{u}_n \to u$ as $n \to \infty$.

Theorem 4.2. *Let the hypotheses* (H1)–(H5) *hold. Then we have the following*:

(i) *If $h(t) \in L_2^0(\Omega, D(A^\alpha))$, then, for any $0 < t_0 \leq T_0$,*

$$\lim_{n \to \infty} \sup_{t_0 \leq t \leq T_0} \sum_{i=0}^{n} \lambda_i^{2\alpha} E \|\alpha_i(t) - \alpha_i^n(t)\|^2 = 0.$$

(ii) *If $h(t) \in L_2^0(\Omega, D(A))$, then*

$$\lim_{n \to \infty} \sup_{0 \leq t \leq T_0} \sum_{i=0}^{n} \lambda_i^{2\alpha} E \|\alpha_i(t) - \alpha_i^n(t)\|^2 = 0.$$

Proof. Since

$$E\|A^\alpha(u(t) - \widehat{u}_n(t))\|^2 = \sum_{i=0}^{\infty} E\|A^\alpha(\alpha_i(t) - \alpha_i^n(t))u_i\|^2$$

$$= \sum_{i=0}^{\infty} \lambda_i^{2\alpha} E\|(\alpha_i(t) - \alpha_i^n(t))u_i\|^2,$$

it follows that

$$E\|A^\alpha(u(t) - \widehat{u}_n(t))\|^2 \geq \sum_{i=0}^{n} \lambda_i^{2\alpha} E\|(\alpha_i(t) - \alpha_i^n(t))u_i\|^2.$$

Hence, as a consequence of Theorem 4.1, we get the required result. □

5. An example. Let $H = L^2(0,1)$, $0 < T < \infty$ and $\tau > 0$. Consider the partial differential equation

$$\frac{\partial u}{\partial t}(x, t) = \frac{\partial^2}{\partial x^2} u(t, x) + F(t, x, u)$$

(5.1)
$$+ \int_0^t a(t - s)G(s, x, u) \, d\omega(s),$$

$$\int_{-\tau}^0 r_3(-\theta)h_5\big(u(\theta, x), \partial_x u(\theta, x)\big) \, d\theta = \phi(x),$$

$u(t,0) = u(t,1) = 0$, where $0 < x < 1$ and $0 < t < T$, with

$$F(t, x, u) = f_1(t, x) + \int_0^1 h_1\big(u(t, y), \partial_y u(t, y)\big)\, dy$$
$$\int_{-\tau}^0 r_1(-\theta) h_2\big(u(t + \theta, x), \partial_x u(t + \theta, x)\big)\, d\theta,$$

and

$$G(t, x, u) = f_2(t, x) + \int_0^1 h_3\big(u(t, y), \partial_y u(t, y)\big)\, dy$$
$$\int_{-\tau}^0 r_2(-\theta) h_4\big(u(t + \theta, x), \partial_x u(t + \theta, x)\big)\, d\theta.$$

Here, h_1, h_2, h_3, h_4, h_5, f_1 and f_2 are smooth, real-valued functions. Furthermore, $r_1, r_2, r_3 \in L^2(0,1)$, $a \in L^{2q}_{loc}(0, \infty)$, and ω is a standard $L^2(0,1)$-valued Wiener process.

We define the operator A as

(5.2)
$$Au = -\frac{\partial^2 u}{\partial x^2},$$

with domain $D(A) = H^2(0,1) \cap H^1_0(0,1)$. Clearly, A is self-adjoint with compact resolvent and is the infinitesimal generator of an analytic semigroup e^{-tA}. Moreover, A has a discrete spectrum with the eigenvalues $k^2\pi^2$ for $k \in \mathbb{N}$, whose corresponding (normalized) eigenfunctions are $e_k(x) = \sqrt{2}\sin k\pi x$. Therefore, for $u \in D(A)$,

$$u(x) = \sum_{k \in \mathbb{N}} (u, e_k) e_k(x).$$

Now, for $\alpha = 1/4$, $D(A^{1/4})$ (denoted by $H_{1/4}$) is a Banach space endowed with the norm

$$\|u\|_{1/4} = \|A^{1/4}u\| \quad \text{for } u \in D(A^{1/4}).$$

Also, we define the space

$$C_t^{1/4} = C\big([-\tau, t], D(A^{1/4})\big), \quad t \in [0, T]$$

endowed with the supremum norm

$$\|\varphi\|_{t,1/4} = \sup_{-\tau \le \nu \le t} \|\varphi(\nu)\|_\alpha, \quad \varphi \in C_t^{1/4};$$

then,

$$A^{1/4}u(x) = \sum_{k \in \mathbb{N}} (k^2\pi^2)^{1/4} \langle u(x), e_k(x) \rangle e_k(x)$$

with $u \in D(A^{1/4})$. Differential equation (5.1) can be reformulated as

$$\frac{du(t)}{dt} + Au(t) = F\big(t, u(t), u_t\big)$$

$$+ \int_0^1 a(t-s)G\big(s, u(s), u_s\big)\, d\omega(s),$$

$$\kappa(u_0) = \phi \quad \text{on } [-\tau, 0],$$

where $u(t)(x) = u(t, x)$ and $u_t(\theta)(x) = u(t+\theta, x)$ for $t \in [0, T]$, $\theta \in [-\tau, 0]$ and $x \in (0, 1)$, and where the operator A is defined as in equation (5.2). The functions F and G are defined on $[0, T] \times D(A^\alpha) \times C_0^{1/2}$ by

$$F(t, \varphi, \xi)(x) = f_1(t, x) + \int_0^1 h_1\big(\varphi(x), \varphi'(x)\big)\, dx$$

$$\times \int_{-\tau}^0 r_1(-\theta)h_2\big(\xi(\theta)(x), \partial_x\xi(\theta)(x)\big)\, d\theta$$

and

$$G(t, \varphi, \xi)(x) = f_2(t, x) + \int_0^1 h_3\big(\varphi(x), \varphi'(x)\big)\, dx$$

$$\int_{-\tau}^0 r_2(-\theta)h_4\big(\xi(\theta)(x), \partial_x\xi(\theta)(x)\big)\, d\theta,$$

where h_1, h_2, h_3 and h_4 are non-decreasing, Lipschitz continuous functions such that

$$\|h_i(x_i, y_i) - h_i(x_{i+1}, y_{i+1})\| \le L_i\big(\|x_i - y_i\| + \|x_{i+1} - y_{i+1}\|\big)$$

for $i = 1, 3$, and

$$\|h_i(x_i, y_i) - h_i(x_{i+1}, y_{i+1})\| \le L_i\big(\|x_i - y_i\|_{[0,\alpha]} + \|x_{i+1} - y_{i+1}\|_{[0,\alpha]}\big)$$

for $i = 2, 4$. The mapping $\kappa : C_0 \to C_0$ is defined by

$$\kappa(\varphi)(x) = \int_{-\tau}^0 r_3(-\theta)h_5\big(\varphi(\theta, x), \partial_x\varphi(\theta, x)\big)\, d\theta,$$

where the functions f_1, $f_2 : [0, T] \times (0, 1) \to \mathbb{R}$ satisfy $f_1(0, \cdot)$, $f_2(0, \cdot) \in L^2(0, 1)$, and there exists $\ell_i \in L^2(0, 1)$ such that

$$\|f_i(t, x) - f_i(s, x)\| \leq \ell_i(x)\|t - s\|^{\theta} \quad \text{for } i = 1, 2.$$

Therefore, the functions F and G satisfy

$$\|F(t, u_1, v_1) - F(t, u_2, v_2)\| \leq L_f\big(\|u_1 - u_2\|_\alpha + \|v_1 - v_2\|_{0,\alpha}\big),$$
$$\|G(t, u_1, v_1) - G(t, u_2, v_2)\| \leq L_g\big(\|u_1 - u_2\|_\alpha + \|v_1 - v_2\|_{0,\alpha}\big).$$

Thus assumptions (H1)–(H5) are satisfied and therefore, by the results established in the earlier sections, one can obtain approximate solutions and their convergence.

REFERENCES

1. D. Bahuguna and S. Agarwal, *Approximations of solutions to neutral functional differential equations with nonlocal history conditions*, J. Math. Anal. Appl. **317** (2006), 583–602.

2. D. Bahuguna and M. Muslim, *Approximation of solutions to non-local history-valued retarded differential equations*, Appl. Math. Comp. **174** (2006), 165–179.

3. ———, *A study of nonlocal history-valued retarded differential equations using analytic semigroups*, Nonlin. Dynam. Syst. Th. **6** (2006), 63–75.

4. D. Bahuguna, S.K. Srivastava and S. Singh, *Approximations of solutions to semilinear integrodifferential equations*, Numer. Funct. Anal. Optim. **22** (2001), 487–504.

5. P. Balasubramaniam, M. Syed Ali and J.H. Kim, *Faedo-Galerkin approximate solutions for stochastic semilinear integrodifferential equations*, Comp. Math. Appl. **58** (2009), 48–57.

6. P. Balasubramaniam and J.P. Dauer, *Controllability of semilinear stochastic delay evolution equations in Hilbert spaces*, Int. J. Math. Math. Sci. **31** (2002), 157–166.

7. D. Barbu and G. Bocsan, *Approximations to mild solutions of stochastic semilinear equations with non-Lipschitz coefficient*, Czechoslovak Math. J. **52** (2002), 87–95.

8. L. Byszewski, *Theorems about the existence and uniquness of solutions of a semilinear evolution nonlocal Cauchy problem*, J. Math. Anal. Appl. **162** (1991), 494–505.

9. ———, *Existence and uniqueness of solutions of semilinear evolution nonlocal Cauchy problem*, Z. Nauk. Pol. **18** (1993), 109–112.

10. L. Byszewski and H. Akca, *Existence of solutions of a nonlinear functional differential evolution nonlocal problem*, Nonlin. Anal. **4** (1998), 65–72.

11. L. Byszewski and V. Lakshmikantham, *Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space*, Appl. Anal. **40** (1990), 11–19.

12. A. Chaddha and D.N. Pandey, *Approximations of solutions for a Sobolev type fractional order differential equation*, Nonlin. Dynam. Syst. Th. **14** (2014), 11–29.

13. G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, in *Encyclopedia of mathematics and its applications* **44**, Cambridge University Press, Cambridge, 1992.

14. K. Deng, *Exponential decay of solutions of semilinear parabolic equations with nonlocal initial conditions*, J. Math. Anal. Appl. **179** (1993), 630–637.

15. L.C. Evans, *An introduction to stochastic differential equations*, American Mathematical Society, Philadelphia, 2013.

16. T.C. Gard, *Introduction to stochastic differential equations*, in Mono. Text. Pure Appl. Math. **114**, Dekker, New York, 1988.

17. Y. Haiping, G. Jianming and D. Yongsheng, *A generalized Gronwall inequality and its application to a fractional differential equation*, J. Math. Anal. Appl. **328** (2007), 1075–1081.

18. S. Jankovic and D. Ilic, *An analytic approximation of solutions of stochastic differential equations*, Comp. Math. Appl. **47** (2004), 903–912.

19. M. Kovács and J. Printems, *Weak convergence of a fully discrete of a linear stochastic evolution equation with a positive-type memory term*, J. Math. Anal. Appl. **413** (2014), 939–952.

20. _____ , *Strong order of convergence of a fully discrete aproximation of a linear stochastic Volterra type evolution equation*, Math. Comp. **83** (2014), 2325–2346.

21. P. Kumar, D.N. Pandey and D. Bahuguna, *Existence of piecewise continuous mild solutions for impulsive functional differential equations with iterated deviating arguments*, Elec. J. Diff. Eqn. **241** (2013), 1–15.

22. _____ , *Approximation of solutions to a fractional differential equation with a deviating argument*, Diff. Eqn. Dynam. Syst. **22** (2014), 333–352.

23. X.R. Mao, *Stochastic differential equations and applications*, Horwood, Chichester, UK, 1997.

24. G. Maruyama, *Continuous Markov processes and stochastic equations*, Rend. Circ. Mat. Palermo **4** (1955), 48–90.

25. P.D. Miletta, *Approximation of solutions to evolution equations*, Math. Meth. Appl. Sci. **17** (1994), 753–763.

26. M. Muslim and A.K. Nandakumaran, *Existence and approximations of solutions to some fractional order functional integral equations*, J. Integral Equations Appl. **22** (2010), 95–114.

27. B. Oksendal, *Stochastic differential equations*, Springer, Berlin, 2002.

28. A. Pazy, *Semigroups of linear operators and applications to partial differential equations*, Appl. Math. Sci. **44**, Springer, New York, 1983.

29. Z. Yan and F. Lu, *On approximate controllability of fractional stochastic neutral integro-differential inclusions with infinite delay*, Appl. Anal. **2014** (2014), 23 pages.

30. Z. Yan and H. Zhang, *Asymptotic stability of fractional impulsive neutral stochastic partial integro-differential equations with state-dependent delay*, Elec. J. Diff. Eqn. **206** (2013), 1–29.

Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee 247667, Uttarakhand, India
Email address: **rrenu94@gmail.com**

Department of Mathematics, Indian Institute of Technology Roorkee, Roorkee 247667, Uttarakhand, India
Email address: **dwij.iitk@gmail.com**