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APPLICATION OF A GLOBAL IMPLICIT FUNCTION THEOREM TO A GENERAL FRACTIONAL INTEGRO-DIFFERENTIAL SYSTEM OF VOLTERRA TYPE

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ABSTRACT. In this paper, we use a global implicit function theorem for the investigation of the existence and uniqueness of a solution as well as the sensitivity of a Cauchy problem for a general integro-differential system of order $\alpha \in (0,1)$ of Volterra type, involving two functional parameters nonlinearly.

1. Introduction. Integro-differential systems have recently been studied by Aghajani et al. [**1**], Ahmad and Nieto [**2**], Bushnaq et al. [**6**], Gayathri et al. [**8**], Matar [**19**], Nazari and Shahmorad [**20**], Sudsutad and Tariboon [**22**], Wang and Wei [**23**] and Yan [**26**]. These systems are investigated in finite and infinite dimensional spaces, with Riemann-Liouville and Caputo derivatives as well as with different types of initial and boundary conditions: local, nonlocal, involving values of solutions or their fractional integrals, delay. Tools used in studies of such systems also are of different types: Banach, Brouwer, Schauder, Schaefer, Krasnoselskii fixed point theorems, nonlinear alternative Leray-Schauder type, fractional differential transform method, strongly continuous operator semigroups, and the reproducing kernel Hilbert space method.

In [**15**], a global inverse function theorem obtained in [**14**] has been applied to the Cauchy problem:

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$$(1.1) \quad \begin{cases} D_{a+}^{\alpha}x(t) + \int_0^t \Phi(t,\tau,x(\tau))\,d\tau = v(t), & t \in J := [a,b] \text{ a.e.} \\ I_{a+}^{1-\alpha}x(a) = 0, \end{cases}$$

where $v \in L_n^2 = L^2(J,\mathbb{R}^n)$ is a functional parameter and $x \in I_{a+}^{\alpha}(L_n^2)$ is an unknown function. More precisely, sufficient conditions for the existence, uniqueness and continuous differentiability of the mapping

$$L_n^2 \ni v \longmapsto x_v \in I_{a+}^{\alpha}(L_n^2)$$

where x_v is a unique solution to problem (1.1), corresponding to v, have been formulated. The continuous differentiability of the above mapping is often called *sensitivity* of the problem.

The aim of the present paper is to derive results of such a type for a more general problem, namely,

$$(1.2) \quad \begin{cases} D_{a+}^{\alpha}x(t) + \int_a^t \Phi(t,\tau,x(\tau),u(\tau))\,d\tau = f(t,x(t),v(t)), & t \in J \text{ a.e.,} \\ I_{a+}^{1-\alpha}x(a) = 0, \end{cases}$$

where $u \in L_m^{\infty} = L^{\infty}(J,\mathbb{R}^m)$ and $v \in L_r^{\infty} = L^{\infty}(J,\mathbb{R}^r)$ are functional parameters, involved nonlinearly, and $x \in I_{a+}^{\alpha}(L_n^2)$ is an unknown function. Equation (1.2) can be the basic system for an integro-differential fractional games theory.

It is worth noting (cf., **[3]**) that, if $\alpha \in (1/2,1)$, then any function $x \in I_{a+}^{\alpha}(L_n^2)$ satisfies the condition $x(a) = 0$. Consequently, in such a case, each solution of problem (1.2) satisfies the additional condition $x(a) = 0$.

To study the existence, uniqueness and continuous differentiability of the mapping

$$L_m^{\infty} \times L_r^{\infty} \ni (u,v) \longmapsto x_{u,v} \in I_{a+}^{\alpha}(L_n^2),$$

where $x_{u,v}$ is a unique solution to (1.2), corresponding to the pair of functional parameters (u,v), we apply a new method, based on a global implicit function theorem derived in **[10]**. Such a method has been applied in **[10]** to an integro-differential equation of the first order, involving parameter u nonlinearly and parameter v linearly, and to the classical differential equation of the first order, containing one parameter u nonlinearly.

In **[11]**, we obtained some strengthening of the global implicit function theorem derived in **[10]**. Shortly speaking, we replaced the "bijectivity" condition by a "nonorthogonality" one. An open problem is to formulate assumptions on Φ and f such that the strengthened version of the global implicit function theorem is applicable to problem (1.2) whereas the theorem from **[10]** is not.

An extension of the global inverse function theorem, obtained in **[14]**, to the case of the Banach range space, has been obtained in **[7]** and applied to the problem

$$\begin{cases} x'(t) + \int_a^t \Phi(t,\tau,x(\tau))\,d\tau = v(t), & t \in J \text{ a.e.,} \\ x(a) = 0. \end{cases}$$

To our best knowledge, global sensitivity of the fractional integro-differential problem (1.2) is a new result, and it has not been studied by other authors. Also, the global implicit function theorem was not applied to fractional problems of such a type until now.

2. Preliminaries. Let $\alpha > 0$ and $h \in L^1(J,\mathbb{R}^n)$. By the left Riemann-Liouville fractional integral of h on the interval J we mean (cf., **[21]**) a function $I^{\alpha}_{a+}h$ given by:

$$(I^{\alpha}_{a+}h)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{h(\tau)}{(t-\tau)^{1-\alpha}}\,d\tau, \quad t \in J \text{ almost everywhere,} \tag{2.1}$$

where Γ is the Euler function. One can show that the above integral exists and is finite almost everywhere on J. Moreover, we also have:

Theorem 2.1. *If $\alpha > 0$ and $1 \le p < \infty$, then $I^{\alpha}_{a+}h \in L^p(J,\mathbb{R}^n)$ for any $h \in L^p(J,\mathbb{R}^n)$, and*

$$\left\| I^{\alpha}_{a+}h \right\|_{L^p(J,\mathbb{R}^n)} \le \gamma_\alpha \left\| h \right\|_{L^p(J,\mathbb{R}^n)} \tag{2.2}$$

where $\gamma_\alpha = (b-a)^\alpha/\Gamma(\alpha+1)$. If, additionally, $0 < \alpha < 1$ and $1 < p < 1/\alpha$, then $I^{\alpha}_{a+}h \in L^q(J,\mathbb{R}^n)$ for any $h \in L^p(J,\mathbb{R}^n)$, where $q = p/(1-\alpha p)$, and

$$\left\| I^{\alpha}_{a+}h \right\|_{L^q(J,\mathbb{R}^n)} \le \gamma_{\alpha,p,q} \left\| h \right\|_{L^p(J,\mathbb{R}^n)}$$

for some constant $\gamma_{\alpha,p,q} \ge 0$. If $\alpha > 0$, $1 \le p < \infty$ and $p > 1/\alpha$, then the function $I^{\alpha}_{a+}h$ is continuous on J for any $h \in L^p(J,\mathbb{R}^n)$.

Remark 2.2. The first part of the above theorem can be found in [**21**]. For the second part see [**9**] and also [**18**], [**21**]. The last part can be deduced from the results of [**9**] (cf. also [**21**]).

Now, let $\alpha \in (0,1)$. We say that (cf., [**21**]) $x \in L^1(J,\mathbb{R}^n)$ possesses the left Riemann-Liouville derivative $D^{\alpha}_{a+}x$ of order $\alpha \in (0,1)$ on the interval J if the integral $I^{1-\alpha}_{a+}x$ is absolutely continuous on J (more precisely, if there exists an absolutely continuous function g on J such that $g = I^{1-\alpha}_{a+}x$ almost everywhere on J). In such a case, we identify $I^{1-\alpha}_{a+}x$ with its absolutely continuous representant g and, by the derivative $D^{\alpha}_{a+}x$, we mean the classical derivative $D^1 g$, i.e., (after identifying $I^{1-\alpha}_{a+}x$ with g)

$$(D^{\alpha}_{a+}x)(t) = \frac{1}{\Gamma(1-\alpha)} D^1\left(\int_a^t \frac{x(\tau)}{(t-\tau)^{\alpha}} d\tau\right),$$
$$t \in J \text{ almost everywhere,}$$

(cf., [**12**] for the case of any $\alpha > 0$). By the value $I^{1-\alpha}_{a+}x(a)$ of the function $I^{1-\alpha}_{a+}x$ at the point a, we mean the value $g(a)$.

In [**4**], it is proved that a function $x \in L^1(J,\mathbb{R}^n)$ has the left Riemann-Liouville derivative $D^{\alpha}_{a+}x$ of order $\alpha \in (0,1)$ if and only if there exist a constant $c \in \mathbb{R}^n$ and a function $\varphi \in L^1(J,\mathbb{R}^n)$ such that:

$$x(t) = \frac{1}{\Gamma(\alpha)} \frac{c}{(t-a)^{1-\alpha}} + \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\varphi(\tau)}{(t-\tau)^{1-\alpha}}\, d\tau, \tag{2.3}$$

for $t \in J$ almost everywhere. In such a case $I^{1-\alpha}_{a+}x(a) = c$ (after identifying $I^{1-\alpha}_{a+}x$ with its absolutely continuous representant) and $D^{\alpha}_{a+}x = \varphi$ almost everywhere on J. The above formula is a generalization of the well-known integral formula for absolutely continuous functions.

The set of all functions x possessing Riemann-Liouville derivative $D^{\alpha}_{a+}x \in L^2_n$ is denoted by $AC^{\alpha,2}_{a+}(J,\mathbb{R}^n)$. It consists of all functions x possessing the representation (2.3) with $\varphi \in L^2_n$. So, if $x \in I^{\alpha}_{a+}(L^2_n)$, then $x = I^{\alpha}_{a+}D^{\alpha}_{a+}x$. Of course, the range $I^{\alpha}_{a+}(L^2_n)$ of the space L^2_n under the operator $I^{\alpha}_{a+} : L^2_n \to L^2_n$ is contained in $AC^{\alpha,2}_{a+}(J,\mathbb{R}^n)$. It is easy to see that $I^{\alpha}_{a+}(L^2_n)$ with the scalar product

$$\langle x, y\rangle_{I^{\alpha}_{a+}(L^2_n)} = \left\langle D^{\alpha}_{a+}x, D^{\alpha}_{a+}y\right\rangle_{L^2_n}$$

is complete, i.e., it is the Hilbert space. The corresponding norm in $I^{\alpha}_{a+}(L^2_n)$ is given by:

$$\|x\|_{I^{\alpha}_{a+}(L^2_n)} = \left\|D^{\alpha}_{a+}x\right\|_{L^2_n}.$$

3. A global implicit function theorem. Let X be a real Banach space and $\varphi : X \to \mathbb{R}$ a Frechet differentiable functional. We say that φ satisfies the *Palais-Smale* (PS) *condition* if any sequence (x_m) such that

- $|\varphi(x_m)| \le M$ for all $m \in \mathbb{N}$ and some $M > 0$,
- $\varphi'(x_m) \longrightarrow 0$,

admits a convergent subsequence ($\varphi'(x_m)$ denotes the Frechet differential of φ at x_m). A sequence (x_m) satisfying the above conditions is called the (PS) *sequence*.

In **[10]**, the following global implicit function theorem has been derived. It is a generalization of the global inverse function theorem obtained in **[14]**,

Theorem 3.1. *Let X and Y be real Banach spaces, H a real Hilbert space. If $F : X \times Y \to H$ is continuous differentiable with respect to $(x, y) \in X \times Y$ and*

- *for any $y \in Y$, the functional $\varphi_y : X \ni x \longmapsto \frac{1}{2}\|F(x, y)\|^2 \in \mathbb{R}$ satisfies the* (PS) *condition*
- *$F'_x(x, y) : X \longrightarrow H$ is bijective for any $(x, y) \in X \times Y$,*

then there exists a unique function $\lambda : Y \to X$ such that $F(\lambda(y), y) = 0$ for any $y \in Y$, and this function is continuous differentiable with the differential $\lambda'(y)$ at y given by:

$$\lambda'(y) = -[F'_x(\lambda(y), y)]^{-1} \circ F'_y(\lambda(y), y). \tag{3.1}$$

4. The integro-differential problem. Let us consider the control system (1.2) with functions $\Phi : P_\Delta \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ ($P_\Delta = \{(t, \tau) \in J \times J; \tau \le t\}$), $f : J \times \mathbb{R}^n \times \mathbb{R}^r \to \mathbb{R}^n$ such that

(*A*1) $\Phi(\cdot, \cdot, x, u)$ is measurable on P_Δ for any $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$; $\Phi(t, \tau, \cdot, \cdot)$ is continuous differentiable on $\mathbb{R}^n \times \mathbb{R}^m$ for $(t, \tau) \in P_\Delta$ almost everywhere;

($A2$) there exist functions a_Φ, $b_\Phi \in L^2(P_\Delta, \mathbb{R}_0^+)$, $\omega_\Phi \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$ and a constant $C_\Phi > 0$ such that:

$$|\Phi(t,\tau,x,u)| \le a_\Phi(t,\tau)\,|x| + b_\Phi(t,\tau)\omega_\Phi(|u|)$$
$$|\Phi_x(t,\tau,x,u)| \le C_\Phi\,|x|\,\omega_\Phi(|u|),$$
$$|\Phi_u(t,\tau,x,u)| \le a_\Phi(t,\tau)\,|x| + b_\Phi(t,\tau)\omega_\Phi(|u|),$$

for $(t,\tau) \in P_\Delta$ almost everywhere, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$;

($B1$) $f(\cdot, x, u)$ is measurable on J for any $x \in \mathbb{R}^n$, $v \in \mathbb{R}^r$; $f(t,\cdot,\cdot)$ is continuous differentiable on $\mathbb{R}^n \times \mathbb{R}^r$ for $t \in J$ almost everywhere;

($B2$) there exist functions $b_f \in L^2(J, \mathbb{R}_0^+)$, $\omega_f \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$ and constants a_f, $d_f > 0$ such that:

$$|f(t,x,v)| \le a_f\,|x| + b_f(t)\omega_f(|v|)$$
$$|f_x(t,x,v)| \le d_f \omega_f(|v|)$$
$$|f_v(t,x,v)| \le a_f\,|x| + b_f(t)\omega_f(|v|)$$

for $t \in J$ almost everywhere, $x \in \mathbb{R}^n$, $v \in \mathbb{R}^r$;

(AB) the inequality

$$\gamma_\alpha(\|a_\Phi\|_{L^2(P_\Delta,\mathbb{R}^n)} + a_f) < \frac{1}{\sqrt{2}} \tag{4.1}$$

is satisfied.

We shall check that the mapping

$$F : I^\alpha_{a+}(L^2_n) \times L^\infty_m \times L^\infty_r \longrightarrow L^2_n,$$

$$F(x,u,v) = D^\alpha_{a+}x(t) + \int_a^t \Phi(t,\tau,x(\tau),u(\tau))\,d\tau$$
$$- f(t,x(t),v(t)),$$

satisfies assumptions of Theorem 3.1 with $X = I^\alpha_{a+}(L^2_n)$, $Y = L^\infty_m \times L^\infty_r$ and $H = L^2_n$.

The well-posedness of F can be checked in the same way as in [**14,** Proof of Lemma 5]. More precisely, in the same way as in [**14,** Proof of Lemma 5] one can check that the mapping

$$I^\alpha_{a+}(L^2_n) \times L^\infty_m \times L^\infty_r \ni (x,u,v) \longmapsto \int_a^{\cdot} \Phi(\cdot,\tau,x(\tau),u(\tau))d\tau \in L^2_n$$

is well-posed; well-posedeness of the mapping

$$I^{\alpha}_{a+}(L^2_n) \times L^{\infty}_m \times L^{\infty}_r \ni (x, u, v) \longmapsto D^{\alpha}_{a+}x(t) - f(t, x(t), v(t)) \in L^2_n$$

is obvious.

Now, we shall prove:

Lemma 4.1. *The operator F is continuous differentiable in the Gateaux (equivalently, in the Frechet) sense on $I^{\alpha}_{a+}(L^2_n) \times L^{\infty}_m \times L^{\infty}_r$ and the mappings*

$$F'_x(x, u, v) : I^{\alpha}_{a+}(L^2_n) \longrightarrow L^2_n,$$

$$\begin{aligned}F'_x(x, u, v)h = D^{\alpha}_{a+}h(t) + \int_a^t \Phi_x(t, \tau, x(\tau), u(\tau))h(\tau)\, d\tau \\ - f_x(t, x(t), v(t))h(t)\end{aligned} \tag{4.2}$$

$$F'_{u,v}(x, u, v) : L^{\infty}_m \times L^{\infty}(J, \mathbb{R}^n) \to L^2_n,$$

$$\begin{aligned}F'_{u,v}(x, u, v)(f, g) = \int_a^t \Phi_u(t, \tau, x(\tau), u(\tau))f(\tau)\, d\tau \\ - f_v(t, x(t), v(t))g(t)\end{aligned}$$

are the differentials of F at (x, u, v) in x and (u, v), respectively.

Proof. Let us define the auxiliary operators

$$P : I^{\alpha}_{a+}(L^2_n) \times L^{\infty}_m \times L^{\infty}_r \ni (x, u, v) \longmapsto D^{\alpha}_{a+}x(t) \in L^2_n,$$

$$Q : I^{\alpha}_{a+}(L^2_n) \times L^{\infty}_m \times L^{\infty}_r \ni (x, u, v) \longmapsto \int_a^t \Phi(t, \tau, x(\tau), u(\tau))d\tau \in L^2_n,$$

$$R : I^{\alpha}_{a+}(L^2_n) \times L^{\infty}_m \times L^{\infty}_r \ni (x, u, v) \longmapsto f(t, x(t), v(t)) \in L^2_n.$$

Of course, $F = P + Q + R$.

Differentiability of P. Operator P is linear and continuous:

$$\begin{aligned}\|P(x, u, v)\|_{L^2_n} = \left\|D^{\alpha}_{a+}x(t)\right\|_{L^2_n} = \|x\|_{I^{\alpha}_{a+}(L^2_n)} \\ \leq \|(x, u, v)\|_{I^{\alpha}_{a+}(L^2_n) \times L^{\infty}_m \times L^{\infty}_r} .\end{aligned}$$

Thus, it is of class C^1.

Differentiability of Q with respect to x on $I^{\alpha}_{a+}(L^2_n) \times L^{\infty}_m \times L^{\infty}_r$. Let us fix a point $(x, u, v) \in I^{\alpha}_{a+}(L^2_n) \times L^{\infty}_m \times L^{\infty}_r$. We shall show that the mapping

$$Q'_x(x, u, v) : I^{\alpha}_{a+}(L^2_n) \ni h \longmapsto \int_a^{\cdot} \Phi_x(\cdot, \tau, x(\tau), u(\tau))h(\tau)d\tau \in L^2_n$$

is a partial Frechet differential of Q with respect to x at (x, u, v) and Q'_x is continuous on $I^{\alpha}_{a+}(L^2_n) \times L^{\infty}_m \times L^{\infty}_r$. We proceed in the same way as in [**15,** Proof of Lemma 5]. We give here the reasoning for the convenience of the reader and because of the change of the growth condition on Φ_x. So, if $h \in I^{\alpha}_{a+}(L^2_n) \subset L^2_n$, then the function

$$P_{\Delta} \ni (t, \tau) \longmapsto \Phi_x(t, \tau, x(\tau), u(\tau))h(\tau) \in \mathbb{R}^n$$

is measurable and integrable by $(A2)$. From the Fubini theorem, it follows that the function

$$J \ni t \longmapsto \int_a^t \Phi_x(t, \tau, x(\tau), u(\tau))h(\tau)\, d\tau \in \mathbb{R}^n \tag{4.3}$$

is integrable. In fact,

$$\begin{aligned}
&\left| \int_a^t \Phi_x(t, \tau, x(\tau), u(\tau))h(\tau)\, d\tau \right|^2 \\
&\qquad\qquad \le \left(\int_a^t C_{\Phi} \left|x(\tau)\right| \omega_{\Phi}(\left|u(\tau)\right|) \left|h(\tau)\right|\, d\tau \right)^2 \\
&\qquad \le C^2_{\Phi} (\max\{\omega_{\Phi}(|u(\tau)|);\ \tau \in J\})^2 \left(\int_a^t \left|x(\tau)\right| \left|h(\tau)\right|\, d\tau \right)^2 \\
&\qquad \le C^2_{\Phi} (\max\{\omega_{\Phi}(|u(\tau)|);\ \tau \in J\})^2 \int_a^b \left|x(\tau)\right|^2\, d\tau \int_a^b \left|h(\tau)\right|^2 d\tau.
\end{aligned}$$

So, function (4.3) belongs to L^2_n. It means that the operator $Q'_x(x, u, v)$ is well-defined.

Clearly, $Q'_x(x, u, v)$ is linear. Continuity of it follows from the following estimation:

$$\|h\|_{L^2_n} = \left\|I^{\alpha}_{a+} D^{\alpha}_{a+} h\right\|_{L^2_n} \le \gamma_{\alpha} \left\|D^{\alpha}_{a+} h\right\|_{L^2_n} = \gamma_{\alpha} \|h\|_{I^{\alpha}_{a+}(L^2_n)}\, .$$

Now, we shall check that

$$\lim_{\lambda\to 0}\frac{Q(x+\lambda h,u,v)-Q(x,u,v)}{\lambda}=Q'_x(x,u,v)h$$

in L^2_n. So, let us fix a sequence (λ_n) converging to 0 in $\mathbb{R}$, and consider the limit

$$\lim_{k\to\infty}\int_a^b\left|\int_a^t\left(\frac{\Phi(t,\tau,x(\tau)+\lambda_k h(\tau),u(\tau))-\Phi(t,\tau,x(\tau),u(\tau))}{\lambda_k}\right.\right.$$
$$\left.\left.-\,\Phi_x(t,\tau,x(\tau),u(\tau))h(\tau)\right)d\tau\right|^2 dt.$$

From the differentiability of Φ with respect to x we obtain, for $t\in J$ almost everywhere, the convergence of the sequence of functions

$$[a,t]\ni\tau\longmapsto\frac{\Phi(t,\tau,x(\tau)+\lambda_k h(\tau),u(\tau))-\Phi(t,\tau,x(\tau))}{\lambda_k}$$
$$-\,\Phi_x(t,\tau,x(\tau),u(\tau))h(\tau)\in\mathbb{R}^n \tag{4.4}$$

almost everywhere on $[a,t]$ to the zero function. Moreover, from the mean value theorem applied to the coordinate functions

$$[0,1]\ni\vartheta\longmapsto\Phi_j(t,\tau,x(\tau)+\vartheta\lambda_k h(\tau))\in\mathbb{R},$$

where $j=1,\dots,n$, it follows that the absolute values of functions (4.4) are commonly bounded by an integrable function. Indeed, since

$$\Phi_j(t,\tau,x(\tau)+\lambda_k h(\tau),u(\tau))-\Phi_j(t,\tau,x(\tau))$$
$$=(\Phi_j)_x(t,\tau,x(\tau)+\vartheta_j(t,\tau)\lambda_k h(\tau)x(\tau),u(\tau))\lambda_k h(\tau)$$

for some $\vartheta_j(t,\tau)\in(0,1)$; therefore,

$$\left|\frac{\Phi_j(t,\tau,x(\tau)+\lambda_k h(\tau),u(\tau))-\Phi_j(t,\tau,x(\tau),u(\tau))}{\lambda_k}\right|$$
$$\le\left|(\Phi_j)_x(t,\tau,x(\tau)+\vartheta_j(t,\tau)\lambda_k h(\tau),u(\tau))\right|\left|h(\tau)\right|$$
$$\le C_\Phi\left|x(\tau)+\vartheta_j(t,\tau)\lambda_k h(\tau)\right|\omega_\Phi(|u(\tau)|)\left|h(\tau)\right|$$
$$\le C_\Phi(|x(\tau)|+|h(\tau)|)\omega_\Phi(|u(\tau)|)\left|h(\tau)\right|$$

for sufficiently large k (such that $|\lambda_k|<1$). Also,

$$\left|\Phi_x(t,\tau,x(\tau),u(\tau))h(\tau)\right|\le C_\Phi\left|x(\tau)\right|\omega_\Phi(|u(\tau)|)\left|h(\tau)\right|.$$

So, using the Lebesgue dominated convergence theorem (cf., [**24**]) we state that the sequence of functions

$$J \ni t \longmapsto \int_a^t \Bigg(\frac{\Phi(t,\tau,x(\tau)+\lambda_k h(\tau),u(\tau)) - \Phi(t,\tau,x(\tau),u(\tau))}{\lambda_k} \\ - \Phi_x(t,\tau,x(\tau),u(\tau))h(\tau) \Bigg)\, d\tau \in \mathbb{R}^n$$

converges almost everywhere on J to the zero function. Moreover,

$$\begin{aligned} &\Bigg| \int_a^t \Bigg(\frac{\Phi(t,\tau,x(\tau)+\lambda_k h(\tau),u(\tau)) - \Phi(t,\tau,x(\tau),u(\tau))}{\lambda_k} \\ &\qquad\qquad - \Phi_x(t,\tau,x(\tau),u(\tau))h(\tau) \Bigg)\, d\tau \Bigg|^2 \\ &\le \Bigg(\int_a^t \Bigg| \frac{\Phi(t,\tau,x(\tau)+\lambda_k h(\tau),u(\tau)) - \Phi(t,\tau,x(\tau),u(\tau))}{\lambda_k} \\ &\qquad\qquad - \Phi_x(t,\tau,x(\tau),u(\tau))h(\tau) \Bigg|\, d\tau \Bigg)^2 \\ &\le \Bigg(n \int_a^t C_\Phi(|x(\tau)|+|h(\tau)|)\omega_\Phi(|u(\tau)|)\,|h(\tau)|\, d\tau \\ &\quad + \int_a^t C_\Phi\,|x(\tau)|\,\omega_\Phi(|u(\tau)|)\,|h(\tau)|\, d\tau \Bigg)^2 \\ &\le (n+1) \int_a^b C_\Phi(|x(\tau)|+|h(\tau)|)\omega_\Phi(|u(\tau)|)\,|h(\tau)|\, d\tau. \end{aligned}$$

Consequently, using once again the Lebesgue dominated convergence theorem we obtain

$$\lim_{k\to\infty} \int_a^b \Bigg| \int_a^t \Bigg(\frac{\Phi(t,\tau,x(\tau)+\lambda_k h(\tau),u(\tau)) - \Phi(t,\tau,x(\tau),u(\tau))}{\lambda_k} \\ - \Phi_x(t,\tau,x(\tau),u(\tau))h(\tau) \Bigg)\, d\tau \Bigg|^2 dt = 0.$$

To finish this part of the proof we shall show that Q'_x is continuous on the space $I^\alpha_{a+}(L^2_n) \times L^\infty_m \times L^\infty_r$. Indeed, let (x_j,u_j,v_j) be a sequence converging in this space to a point (x_0,u_0,v_0). We have:

$$\|(Q'_x(x_j,u_j,v_j) - Q'_x(x_0,u_0,v_0))h\|^2_{L^2_n}$$

$$\leq \int_a^b \left(\int_a^t |\Phi_x(t,\tau,x_j(\tau),u_j(\tau)) - \Phi_x(t,\tau,x_0(\tau),u_0(\tau))| \, |h(\tau)| \, d\tau \right)^2 dt$$
$$\leq \int_a^b \left(\int_a^t |\Phi_x(t,\tau,x_j(\tau),u_j(\tau)) - \Phi_x(t,\tau,x_0(\tau),u_0(\tau))|^2 \, d\tau \times \int_a^t |h(\tau)|^2 \, d\tau \right) dt$$
$$\leq \|h\|^2_{L^2_n} \int_a^b \int_a^t |\Phi_x(t,\tau,x_j(\tau),u_j(\tau)) - \Phi_x(t,\tau,x_0(\tau),u_0(\tau))|^2 \, d\tau \, dt$$
$$\leq (\gamma_\alpha)^2 \|h\|^2_{I^\alpha_{a+}(L^2_n)} \int_a^b \times \int_a^t |\Phi_x(t,\tau,x_j(\tau),u_j(\tau)) - \Phi_x(t,\tau,x_0(\tau),u_0(\tau))|^2 \, d\tau \, dt.$$

So,

$$\|(Q'_x(x_j,u_j,v_j) - Q'_x(x_0,u_0,v_0))\|_{\mathcal{L}(L^2_n,L^2_n)}$$
$$\leq \gamma_\alpha \left(\int_a^b \int_a^t |\Phi_x(t,\tau,x_j(\tau),u_j(\tau)) - \Phi_x(t,\tau,x_0(\tau),u_0(\tau))|^2 \, d\tau \, dt \right)^{1/2}$$

where $\mathcal{L}(L^2_n, L^2_n)$ is the space of linear continuous operators acting from L^2_n to L^2_n, considered with the classical operator norm. From the generalized Krasnoselskii's theorem (cf., **[13]** with $\Omega = P_\Delta$) and the theorem on the majorized subsequence (cf., [**5,** Theoreme IV.9]) it follows that the above integral converges to 0.

Differentiability of Q with respect to u on $I^\alpha_{a+}(L^2_n) \times L^\infty_m \times L^\infty_r$. Let us fix a point $(x,u,v) \in I^\alpha_{a+}(L^2_n) \times L^\infty_m \times L^\infty_r$ and consider the mapping

$$Q'_u(x,u,v) : L^\infty_m \ni f \longmapsto \int_a^{\cdot} \Phi_u(\cdot,\tau,x(\tau),u(\tau)) f(\tau) \, d\tau \in L^2_n. \tag{4.5}$$

The function

$$P_\Delta \ni (t,\tau) \longmapsto \Phi_u(t,\tau,x(\tau),u(\tau)) f(\tau)$$

is measurable and integrable by $(A2)$. So, the Fubini theorem implies integrability of the function

$$J \ni t \longmapsto \int_a^t \Phi_u(t,\tau,x(\tau),u(\tau))f(\tau)\,d\tau \in \mathbb{R}^n. \tag{4.6}$$

Moreover,

$$\begin{aligned}
&\left|\int_a^t \Phi_u(t,\tau,x(\tau),u(\tau))f(\tau)\,d\tau\right|^2 \\
&\leq \left(\|f\|_{L_m^\infty}\int_a^t (a_\Phi(t,\tau)\,|x(\tau)| + b_\Phi(t,\tau)\omega_\Phi(|u(\tau)|))\,d\tau\right)^2 \\
&\leq 2\,\|f\|_{L_m^\infty}^2\left(\int_a^t a_\Phi^2(t,\tau)\,d\tau \int_a^t |x(\tau)|^2\,d\tau + K_1^2\left(\int_a^t b_\Phi(t,\tau)d\tau\right)^2\right) \\
&\leq 2\,\|f\|_{L_m^\infty}^2\left(\int_a^t a_\Phi^2(t,\tau)\,d\tau\,\|x\|_{L_n^2} + K_1^2(b-a)\int_a^t b_\Phi^2(t,\tau)\,d\tau\right),
\end{aligned}$$

where $K_1 = \max\{\omega_\Phi(|u(\tau)|);\ \tau \in J\}$. So, in fact, the function (4.6) belongs to L_n^2. This means that mapping (4.5) is well defined. Of course, it is linear and continuous:

$$\begin{aligned}
&\left(\int_a^b\left|\int_a^t \Phi_u(t,\tau,x(\tau),u(\tau))f(\tau)\,d\tau\right|^2 dt\right)^{1/2} \\
&\leq\sqrt{2}\,\|f\|_{L_m^\infty}\left(\|x\|_{L_n^2}^2\int_a^b\int_a^t a_\Phi^2(t,\tau)\,d\tau dt + K_1^2(b-a)\int_a^b\int_a^t b_\Phi^2(t,\tau)\,d\tau dt\right)^{1/2} \\
&= \sqrt{2}\,\|f\|_{L_m^\infty}\,(\|x\|_{L_n^2}^2\,\|a_\Phi\|_{L^2(P_\Delta,\mathbb{R})}^2 + K_1^2(b-a)\,\|b_\Phi\|_{L^2(P_\Delta,\mathbb{R})}^2)^{1/2}.
\end{aligned}$$

Similarly, as in the above,

$$\lim_{\lambda\to 0}\frac{Q(x,u+\lambda f,v) - Q(x,u,v)}{\lambda} = Q'_u(x,u,v)h$$

in L_n^2. Indeed, let us fix a sequence (λ_n) such that $\lambda_n \to 0$, and consider the limit

$$\lim_{k\to\infty}\int_a^b\left|\int_a^t\left(\frac{\Phi(t,\tau,x(\tau),u(\tau)+\lambda_k f(\tau)) - \Phi(t,\tau,x(\tau),u(\tau))}{\lambda_k} - \Phi_u(t,\tau,x(\tau),u(\tau))f(\tau)\right)d\tau\right|^2 dt.$$

From the differentiability of Φ with respect to u we obtain the convergence of the sequence of functions

$$(4.7)\quad [a,t] \ni \tau \longmapsto \frac{\Phi(t,\tau,x(\tau),u(\tau)+\lambda_k f(\tau)) - \Phi(t,\tau,x(\tau),u(\tau))}{\lambda_k} - \Phi_u(t,\tau,x(\tau),u(\tau))f(\tau) \in \mathbb{R}^n$$

almost everywhere on $[0,t]$ to the zero function, for $t \in J$ almost everywhere. Moreover, from the mean value theorem applied to the coordinate functions

$$[0,1] \ni \vartheta \longmapsto \Phi_j(t,\tau,x(\tau),u(\tau)+\vartheta\lambda_k f(\tau)) \in \mathbb{R},$$

it follows that the absolute values of functions (4.7) are commonly bounded by an integrable function. Indeed,

$$\begin{aligned}
&\left|\frac{\Phi_j(t,\tau,x(\tau),u(\tau)+\lambda_k f(\tau)) - \Phi_j(t,\tau,x(\tau),u(\tau))}{\lambda_k}\right| \\
&\quad\le |(\Phi_j)_u(t,\tau,x(\tau),u(\tau)+\vartheta_j(t,\tau)\lambda_k f(\tau))|\,|f(\tau)| \\
&\quad\le (a_\Phi(t,\tau)\,|x(\tau)| + b_\Phi(t,\tau)\omega_\Phi(|u(\tau)+\vartheta_j(t,\tau)\lambda_k f(\tau)|))\,|f(\tau)| \\
&\quad\le (a_\Phi(t,\tau)\,|x(\tau)| + b_\Phi(t,\tau)K_2)\,|f(\tau)|
\end{aligned}$$

for some $\vartheta_j(t,\tau) \in (0,1)$ and sufficiently large k (such that $|\lambda_k| < 1$), where $K_2 = \max\{\omega_\Phi(\eta);\ 0 \le \eta \le \|u\|_{L^\infty_m} + \|f\|_{L^\infty_m}\}$. Also,

$$\begin{aligned}
|\Phi_u(t,\tau,x(\tau),u(\tau))f(\tau)| &\le (a_\Phi(t,\tau)\,|x(\tau)| + b_\Phi(t,\tau)\omega_\Phi(|u(\tau)|))\,|f(\tau)| \\
&\le (a_\Phi(t,\tau)\,|x(\tau)| + b_\Phi(t,\tau)K_2)\,|f(\tau)|\,.
\end{aligned}$$

So, from the Lebesgue dominated convergence theorem, it follows that, for $t \in [a,b]$ almost everywhere, the integral

$$\int_a^t \left(\frac{\Phi(t,\tau,x(\tau),u(\tau)+\lambda_k f(\tau)) - \Phi(t,\tau,x(\tau),u(\tau))}{\lambda_k} - \Phi_u(t,\tau,x(\tau),u(\tau))f(\tau)\right) d\tau$$

converges to 0, i.e., the sequence of functions

$$J \ni t \longmapsto \int_a^t \bigg(\frac{\Phi(t,\tau,x(\tau),u(\tau)+\lambda_k f(\tau)) - \Phi(t,\tau,x(\tau),u(\tau))}{\lambda_k} \\ - \Phi_u(t,\tau,x(\tau),u(\tau))f(\tau) \bigg) \, d\tau \in \mathbb{R}^n$$

converges almost everywhere on J to the zero function. Moreover,

$$\begin{aligned}
&\bigg| \int_a^t \bigg(\frac{\Phi(t,\tau,x(\tau),u(\tau)+\lambda_k f(\tau)) - \Phi(t,\tau,x(\tau),u(\tau))}{\lambda_k} \\
&\quad - \Phi_u(t,\tau,x(\tau),u(\tau))f(\tau) \bigg) \, d\tau \bigg|^2 \\
&\le \bigg(\int_a^t \bigg| \frac{\Phi(t,\tau,x(\tau),u(\tau)+\lambda_k f(\tau)) - \Phi(t,\tau,x(\tau),u(\tau))}{\lambda_k} \\
&\quad - \Phi_u(t,\tau,x(\tau),u(\tau))f(\tau) \bigg| \, d\tau \bigg)^2 \\
&\le \bigg((n+1) \int_a^t (a_\Phi(t,\tau)\,|x(\tau)| + b_\Phi(t,\tau)K_2)\,|f(\tau)| \, d\tau \bigg)^2 \\
&\le (n+1)^2 \, \|f\|^2_{L^\infty_m} \\
&\quad \times 2 \bigg(\int_a^t a_\Phi^2(t,\tau)\, d\tau \int_a^b |x(\tau)|^2 \, d\tau + K_2^2(b-a) \int_a^t b_\Phi^2(t,\tau)\, d\tau \bigg).
\end{aligned}$$

Consequently, from the Lebesgue dominated convergence theorem, we obtain

$$\lim_{k\to\infty} \int_a^b \bigg| \int_a^t \bigg(\frac{\Phi_j(t,\tau,x(\tau),u(\tau)+\lambda_k f(\tau)) - \Phi(t,\tau,x(\tau),u(\tau))}{\lambda_k} \\ - \Phi_x(t,\tau,x(\tau),u(\tau))f(\tau) \bigg) \, d\tau \bigg|^2 dt = 0.$$

Now, we shall show that Q'_u is continuous on the space

$$I^\alpha_{a+}(L^2_n) \times L^\infty_m \times L^\infty_r.$$

Indeed, let (x_j, u_j, v_j) be a sequence converging in this space to a point (x_0, u_0, v_0). We have:

$$\|(Q'_u(x_j,u_j,v_j) - Q'_u(x_0,u_0,v_0))f\|^2_{L^2_n}$$

$$\leq \int_a^b \left(\int_a^t |\Phi_u(t,\tau,x_j(\tau),u_j(\tau)) - \Phi_u(t,\tau,x_0(\tau),u_0(\tau))|\,|f(\tau)|\,d\tau \right)^2 dt$$
$$\leq \|f\|^2_{L^\infty_m} \int_a^b \left(\int_a^t |\Phi_u(t,\tau,x_j(\tau),u_j(\tau)) - \Phi_u(t,\tau,x_0(\tau),u_0(\tau))|\,d\tau \right)^2 dt.$$

Thus,

$$\|Q'_u(x_j,u_j,v_j) - Q'_u(x_0,u_0,v_0)\|_{\mathcal{L}(L^\infty_m, L^2_n)}$$
$$\leq \left(\int_a^b \left(\int_a^t |\Phi_u(t,\tau,x_j(\tau),u_j(\tau)) - \Phi_u(t,\tau,x_0(\tau),u_0(\tau))|\,d\tau \right)^2 dt \right)^{1/2}.$$

Of course, there exists a subsequence $(x_{j_i}, u_{j_i}, v_{j_i})$ such that $x_{j_i} \to x_0$ and $u_{j_i} \to u_0$ pointwise almost everywhere on J. So, for $t \in J$ almost everywhere, the sequence of functions

$$[a,t] \ni \tau \longmapsto |\Phi_u(t,\tau,x_{j_i}(\tau),u_j(\tau)) - \Phi_u(t,\tau,x_0(\tau),u_0(\tau))| \in \mathbb{R}$$

is pointwise converging to zero function and majorized by an integrable function:

$$\begin{aligned}
&|\Phi_u(t,\tau,x_{j_i}(\tau),u_{j_i}(\tau)) - \Phi_u(t,\tau,x_0(\tau),u_0(\tau))| \\
&\leq a_\Phi(t,\tau)\,|x_{j_i}(\tau)| + b_\Phi(t,\tau)\omega_\Phi(|u_{j_i}(\tau)|) \\
&\quad + a_\Phi(t,\tau)\,|x_0(\tau)| + b_\Phi(t,\tau)\omega_\Phi(|u_0(\tau)|) \\
&\leq a_\Phi(t,\tau)\,|x_{j_i}(\tau)| + 2b_\Phi(t,\tau)K_3 + a_\Phi(t,\tau)\,|x_0(\tau)|,
\end{aligned}$$

where $K_3 = \max\{\omega_\Phi(|u_j(\tau)|);\ \tau \in J,\ j = 0,1,\dots\}$. So, the sequence of functions

$$J \ni t \longmapsto \int_a^t |\Phi_u(t,\tau,x_{j_i}(\tau),u_{j_i}(\tau)) - \Phi_u(t,\tau,x_0(\tau),u_0(\tau))|\,d\tau \in \mathbb{R}$$

converges pointwise to the zero function. Moreover,

$$\left(\int_a^t |\Phi_u(t,\tau,x_{j_i}(\tau),u_j(\tau)) - \Phi_u(t,\tau,x_0(\tau),u_0(\tau))|\,d\tau \right)^2$$
$$\leq \left(\int_a^t (|\Phi_u(t,\tau,x_{j_i}(\tau),u_j(\tau))| + |\Phi_u(t,\tau,x_0(\tau),u_0(\tau))|)\,d\tau \right)^2$$
$$\leq \left(\int_a^t (a_\Phi(t,\tau)\,|x_{j_i}(\tau)| + b_\Phi(t,\tau)\omega_\Phi(|u_{j_i}(\tau)|) \right.$$

$$+ a_\Phi(t,\tau)\,|x_0(\tau)| + b_\Phi(t,\tau)\omega_\Phi(|u_0(\tau)|))\,d\tau\Bigg)^2$$
$$\le \Bigg(\Bigg(\int_a^t a_\Phi^2(t,\tau)\,d\tau\Bigg)^{1/2}\Bigg(\int_a^t |x_{j_i}(\tau)|^2\,d\tau\Bigg)^{1/2} + 2K_3\int_a^t b_\Phi(t,\tau)\,d\tau$$
$$+ \Bigg(\int_a^t a_\Phi^2(t,\tau)\,d\tau\Bigg)^{1/2}\Bigg(\int_a^t |x_0(\tau)|^2\,d\tau\Bigg)^{1/2}\Bigg)^2$$
$$\le \Bigg(\Bigg(2\int_a^t a_\Phi^2(t,\tau)\,d\tau\Bigg)^{1/2}\gamma_\alpha M + 2K_3\sqrt{b-a}\Bigg(\int_a^t b_\Phi^2(t,\tau)\,d\tau\Bigg)^{1/2}\Bigg)^2,$$

where $M > 0$ is such that $\|x_j\|_{I^\alpha_{a+}(L^2_n)} \le M$ for $j = 0, 1, \ldots$. From the classical Lebesgue theorem on the dominated convergence it follows that

$$\|Q'_u(x_{j_i}, u_{j_i}, v_{j_i}) - Q'_u(x_0, u_0, v_0)\|_{\mathcal{L}(L^\infty_m, L^2_n)} \longrightarrow 0.$$

In fact this means that

$$\|Q'_u(x_j, u_j, v_j) - Q'_u(x_0, u_0, v_0)\|_{\mathcal{L}(L^\infty_m, L^2_n)} \longrightarrow 0.$$

Indeed, in the opposite case, we could choose a subsequence such that

$$0 < \varepsilon < \|Q'_u(x_{j_k}, u_{j_k}, v_{j_k}) - Q'_u(x_0, u_0, v_0)\|_{\mathcal{L}(L^\infty_m, L^2_n)} \tag{4.8}$$

for $k \in \mathbb{N}$ and some $\varepsilon > 0$. Repeating the above reasoning we could choose a subsequence $(x_{j_{k_i}}, u_{j_{k_i}}, v_{j_{k_i}})$ such that

$$\left\|Q'_u(x_{j_{k_i}}, u_{j_{k_i}}, v_{j_{k_i}}) - Q'_u(x_0, u_0, v_0)\right\|_{\mathcal{L}(L^\infty_m, L^2_n)} \longrightarrow 0,$$

in contrast to (4.8).

Differentiability of R with respect to x on $I^\alpha_{a+}(L^2_n) \times L^\infty_m \times L^\infty_r$. Let us fix a point $(x, u, v) \in I^\alpha_{a+}(L^2_n) \times L^\infty_m \times L^\infty_r$. It is easy to see that the mapping

$$R'_x(x, u, v) : I^\alpha_{a+}(L^2_n) \ni h \longmapsto f_x(\cdot, x(\cdot), v(\cdot))h(\cdot) \in L^2_n$$

is well-defined, linear and continuous:

$$\int_a^b |f_x(t, x(t), v(t))h(t)|^2\,dt \le \int_a^b |f_x(t, x(t), v(t))|^2\,|h(t)|^2\,dt$$
$$\le \int_a^b (d_f^2(\omega_f(|v(t)|))^2\,|h(t)|^2\,dt$$

$$\leq d_f^2 K_4^2 \left\| h \right\|_{L_n^2}^2 \leq d_f^2 K_4^2 (\gamma_\alpha)^2 \left\| h \right\|_{I_{a+}^\alpha (L_n^2)}^2 ,$$

where $K_4 = \max\{\omega_f(|v(t)|;\ t \in J\}$. In the same way as in the previous cases we check that $R'_x(x,u,v)$ is the differential of R at (x,u,v) with respect to x. First we check that

$$\lim_{k\to\infty} \int_a^b \left| \frac{f(t,x(t)+\lambda_k h(t), v(t)) - f(t,x(t),v(t))}{\lambda_k} \right.$$
$$\left. - f_x(t,x(t),v(t))h(t)) \right|^2 dt = 0.$$

Indeed, it is clear that the sequence of the above integrands converges pointwise to the zero function. Basing this on the mean value theorem we assert that this sequence is majorized by an integrable function, namely, $(n+1)^2 d_f^2 K_4^2 |h(\cdot)|^2$. So, it is sufficient to use the Lebesgue dominated convergence theorem.

To finish this part of the proof we shall demonstrate that the mapping

$$R' : I_{a+}^\alpha (L_n^2) \times L_m^\infty \times L_r^\infty \ni (x,u,v) \longmapsto R'_x(x,u,v) \in \mathcal{L}(I_{a+}^\alpha (L_n^2), L_n^2)$$

is continuous. So, let (x_j,u_j,v_j) be a sequence converging in $I_{a+}^\alpha (L_n^2) \times L_m^\infty \times L_r^\infty$ to a point (x_0,u_0,v_0). Similarly, as in the previous case, we may assume that $x_j \to x_0$ and $v_j \to v_0$ pointwise almost everywhere on J. We have:

$$\begin{aligned} (4.9)\qquad & \left\| (R'_x(x_j,u_j,v_j) - R'_x(x_0,u_0,v_0))h \right\|_{L_n^2}^2 \\ &= \int_a^b \left| (f_x(t,x_j(t),v_j(t)) - f_x(t,x_0(t),v_0(t)))h(t) \right|^2 dt \\ &\leq \int_a^b \left| f_x(t,x_j(t),v_j(t)) - f_x(t,x_0(t),v_0(t)) \right|^2 \left| h(t) \right|^2 dt. \end{aligned}$$

Now, we shall consider three cases. The reasoning that we present below is due to Kamocki, and it is contained in **[17]**.

Case 1^0. Assume that $1/2 < \alpha$. Then

$$|h(t)| = \left| I_{a+}^\alpha (D_{a+}^\alpha h)(t) \right| \leq \frac{1}{\Gamma(\alpha)} \int_a^t \frac{\left| D_{a+}^\alpha h(\tau) \right|}{(t-\tau)^{1-\alpha}} \, d\tau$$

$$\begin{aligned}
&\le \frac{1}{\Gamma(\alpha)} \left\| D^{\alpha}_{a+} h \right\|_{L^2_n} \int_a^t \frac{d\tau}{(t-\tau)^{(1-\alpha)2}} \\
&= \frac{1}{\Gamma(\alpha)} \left\| D^{\alpha}_{a+} h \right\|_{L^2_n} \frac{1}{2\alpha - 1} (t-a)^{2\alpha-1} \\
&\le \frac{1}{\Gamma(\alpha)} \left\| D^{\alpha}_{a+} h \right\|_{L^2_n} \frac{1}{2\alpha - 1} (b-a)^{2\alpha-1} \\
&= \frac{1}{\Gamma(\alpha)} \frac{1}{2\alpha - 1} (b-a)^{2\alpha-1} \left\| h \right\|_{I^{\alpha}_{a+}(L^2_n)}
\end{aligned}$$

for $t \in J$ almost everywhere. So, continuing (4.9), we obtain

$$\begin{aligned}
&\int_a^b \left| f_x(t, x_j(t), v_j(t)) - f_x(t, x_0(t), v_0(t)) \right|^2 \left| h(t) \right|^2 dt \\
&\quad \le \left(\frac{1}{\Gamma(\alpha)} \frac{1}{2\alpha - 1} (b-a)^{2\alpha-1} \right)^2 \left\| h \right\|^2_{I^{\alpha}_{a+}(L^2_n)} \\
&\qquad \times \int_a^b \left| f_x(t, x_j(t), v_j(t)) - f_x(t, x_0(t), v_0(t)) \right|^2 dt.
\end{aligned}$$

Thus, in this case,

$$\begin{aligned}
&\left\| R'_x(x_j, u_j, v_j) - R'_x(x_0, u_0, v_0) \right\|_{\mathcal{L}(I^{\alpha}_{a+}(L^2_n), L^2_n)} \\
&\le \frac{1}{\Gamma(\alpha)} \frac{1}{2\alpha - 1} (b-a)^{2\alpha-1} \\
&\quad \times \left(\int_a^b \left| f_x(t, x_j(t), v_j(t)) - f_x(t, x_0(t), v_0(t)) \right|^2 dt \right)^{1/2}.
\end{aligned}$$

Convergence of the integral $\int_a^b |f_x(t, x_j(t), v_j(t)) - f_x(t, x_0(t), v_0(t))|^2 dt$ to 0 follows from the Lebesgue dominated convergence theorem (the integrands are bounded almost everywhere on $[a, b]$ by a constant (cf., $(B2)$)).

Case 2^0. Assume that $0 < \alpha < 1/2$. Then, from the second part of Theorem 2.1, it follows that if $h \in I^{\alpha}_{a+}(L^2_n)$, then $h \in L^{2/(1-2\alpha)}_n$ and, consequently, $|h|^2 \in L^q_1$ with $q = 1/(1-2\alpha)$ (of course, $1 < 1/(1-2\alpha) < \infty$). Function $|f_x(\cdot, x_j(\cdot), v_j(\cdot)) - f_x(\cdot, x_0(\cdot), v_0(\cdot))|^2$ being essentially bounded belongs to $L^{q'}_1$ with $q' = q/(q-1)$. Consequently,

$$\int_a^b \left| f_x(t, x_j(t), v_j(t)) - f_x(t, x_0(t), v_0(t)) \right|^2 \left| h(t) \right|^2 dt$$

$$
\begin{aligned}
&\leq \left(\int_a^b |f_x(t,x_j(t),v_j(t)) - f_x(t,x_0(t),v_0(t))|^{2q'} \, dt \right)^{1/q'} \\
&\quad \times \left(\int_a^b |h(t)|^{2q} \, dt \right)^{1/q} \\
&= \left\| |f_x(\cdot,x_j(\cdot),v_j(\cdot)) - f_x(\cdot,x_0(\cdot),v_0(\cdot))| \right\|^2_{L_1^{2q'}} \|h\|^2_{L_n^{2q}} \\
&= \left\| |f_x(\cdot,x_j(\cdot),v_j(\cdot)) - f_x(\cdot,x_0(\cdot),v_0(\cdot))| \right\|^2_{L_1^{2q'}} \left\| I^\alpha_{a+} D^\alpha_{a+} h \right\|^2_{L_n^{2q}} \\
&\leq \left\| |f_x(\cdot,x_j(\cdot),v_j(\cdot)) - f_x(\cdot,x_0(\cdot),v_0(\cdot))| \right\|^2_{L_1^{2q'}} \gamma^2_{\alpha,2,2q} \left\| D^\alpha_{a+} h \right\|^2_{L_n^2} \\
&= \left\| |f_x(\cdot,x_j(\cdot),v_j(\cdot)) - f_x(\cdot,x_0(\cdot),v_0(\cdot))| \right\|^2_{L_1^{2q'}} \gamma^2_{\alpha,2,2q} \|h\|^2_{I^\alpha_{a+}(L^2_n)} .
\end{aligned}
$$

So,

$$
\begin{aligned}
&\|R'_x(x_j,u_j,v_j) - R'_x(x_0,u_0,v_0)\|_{\mathcal{L}(I^\alpha_{a+}(L^2_n),L^2_n)} \\
&\qquad\qquad \leq \left\| |f_x(\cdot,x_j(\cdot),v_j(\cdot)) - f_x(\cdot,x_0(\cdot),v_0(\cdot))| \right\|_{L_1^{2q'}} \gamma_{\alpha,2,2q}.
\end{aligned}
$$

In the same way as in case 1^0, convergence

$$
\int_a^b |f_x(t,x_j(t),v_j(t)) - f_x(t,x_0(t),v_0(t))|^{2q'} \, dt \longrightarrow 0
$$

follows from the Lebesgue dominated convergence theorem.

Case 3^0. Assume that $\alpha = 1/2$. Let us observe that, if $z \in L^2_n$, then $z \in L^p(J,\mathbb{R}^n)$ for any $p \in (1,2)$ and (cf., the second part of Theorem 2.1) $I^\alpha_{a+} z \in L^q(J,\mathbb{R}^n)$ where

$$
q = \frac{p}{1-\alpha p} = \frac{2p}{2-p} > 2
$$

as well as

$$
\begin{aligned}
\left\| I^\alpha_{a+} z \right\|_{L^q(J,\mathbb{R}^n)} &\leq c_{\alpha,p,q} \|z\|_{L^p(J,\mathbb{R}^n)} \\
&\leq \gamma_{\alpha,p,q} \left(\left(\int_a^b 1 \, dt \right)^{1-p/2} \left(\int_a^b (|z(t)|^p)^{2/p} dt \right)^{p/2} \right)^{1/p} \\
&= \gamma_{\alpha,p,q} (b-a)^{(2-p)/2p} \|z\|_{L^2_n} .
\end{aligned}
$$

So, for arbitrary fixed $p \in (1,2)$,

$$
q = \frac{p}{1-\alpha p}, \qquad q' = \frac{q}{q-2},
$$

we obtain

$$\begin{aligned}
&\int_a^b |f_x(t, x_j(t), v_j(t)) - f_x(t, x_0(t), v_0(t))|^2 \, |h(t)|^2 \, dt \\
&= \int_a^b |f_x(t, x_j(t), v_j(t)) - f_x(t, x_0(t), v_0(t))|^2 \left| I_{a+}^{\alpha} D_{a+}^{\alpha} h(t) \right|^2 dt \\
&\leq \Big(\int_a^b |f_x(t, x_j(t), v_j(t)) - f_x(t, x_0(t), v_0(t))|^{2q'} \, dt)^{1/q'} \\
&\quad \times \Big(\int_a^b (\left| I_{a+}^{\alpha} D_{a+}^{\alpha} h(t) \right|^2)^{q/2} dt)^{2/q} \\
&= \| |f_x(\cdot, x_j(\cdot), v_j(\cdot)) - f_x(\cdot, x_0(\cdot), v_0(\cdot))| \|_{L_1^{2q'}}^2 \left\| I_{a+}^{\alpha} D_{a+}^{\alpha} h \right\|_{L_n^q}^2 \\
&\leq \| |f_x(\cdot, x_j(\cdot), v_j(\cdot)) - f_x(\cdot, x_0(\cdot), v_0(\cdot))| \|_{L_1^{2q'}}^2 \\
&\quad \times \gamma_{\alpha,p,q}^2 (b-a)^{(2-p)/p} \left\| D_{a+}^{\alpha} h \right\|_{L_n^2}^2 \\
&= \| |f_x(\cdot, x_j(\cdot), v_j(\cdot)) - f_x(\cdot, x_0(\cdot), v_0(\cdot))| \|_{L_1^{2q'}}^2 \\
&\quad \times \gamma_{\alpha,p,q}^2 (b-a)^{(2-p)/p} \|h\|_{I_{a+}^{\alpha}(L_n^2)}^2 .
\end{aligned}$$

Similarly, as in the previous cases, convergence

$$\int_a^b |f_x(t, x_j(t), v_j(t)) - f_x(t, x_0(t), v_0(t))|^{2q'} \, dt \longrightarrow 0$$

follows from the Lebesgue dominated convergence theorem.

Differentiability of R with respect to v on $I_{a+}^{\alpha}(L_n^2) \times L_m^{\infty} \times L_r^{\infty}$. Let us fix a point $(x, u, v) \in I_{a+}^{\alpha}(L_n^2) \times L_m^{\infty} \times L_r^{\infty}$. Clearly, the mapping

$$R_v'(x, u, v) : L_r^{\infty} \ni g \longmapsto f_v(\cdot, x(\cdot), v(\cdot)) g(\cdot) \in L_n^2$$

is well defined, linear and continuous:

$$\begin{aligned}
&\int_a^b |f_v(t, x(t), v(t)) g(t)|^2 \, dt \\
&\leq \int_a^b |f_v(t, x(t), v(t))|^2 \, |g(t)|^2 \, dt \\
&\leq \int_a^b (a_f \, |x(t)| + b_f(t) \omega_f(|v(t)|))^2 \, |g(t)|^2 \, dt
\end{aligned}$$

$$\leq \|g\|_{L_r^\infty}^2 \int_a^b (a_f\,|x(t)| + b_f(t)\omega_f(|v(t)|))^2 dt.$$

In the same way as in the previous cases, using the Lebesgue dominated convergence theorem, we check that

$$\lim_{k\to\infty} \int_a^b \left| \frac{f(t,x(t),v(t)+\lambda_k g(t)) - f(t,x(t),v(t))}{\lambda_k} \right.$$
$$\left. - f_v(t,x(t),v(t))g(t)) \right|^2 dt = 0.$$

Moreover, the mapping

$$R' : I_{a+}^{\alpha}(L_n^2) \times L_m^\infty \times L_r^\infty \ni (x,u,v) \longmapsto R'_x(x,u,v) \in \mathcal{L}(L_r^\infty, L_n^2)$$

is continuous. Indeed, let (x_j, u_j, v_j) be a sequence converging in $I_{a+}^{\alpha}(L_n^2) \times L_m^\infty \times L_r^\infty$ to a point (x_0, u_0, v_0). As in the previous cases, we may assume that $x_j \to x_0$ and $v_j \to v_0$ pointwise almost everywhere on J. We have

$$\begin{aligned}
&\|(R'_v(x_j,u_j,v_j) - R'_v(x_0,u_0,v_0))\|_{L_n^2}^2 \\
&= \int_a^b |(f_v(t,x_j(t),v_j(t)) - f_v(t,x_0(t),v_0(t)))g(t)|^2\,dt \\
&\leq \int_a^b |f_v(t,x_j(t),v_j(t)) - f_v(t,x_0(t),v_0(t))|^2\,|g(t)|^2\,dt \\
&\leq \|g\|_{L_r^\infty}^2 \int_a^b |f_v(t,x_j(t),v_j(t)) - f_v(t,x_0(t),v_0(t))|^2\,dt.
\end{aligned}$$

So,

$$\begin{aligned}
&\|(R'_v(x_j,u_j,v_j) - R'_v(x_0,u_0,v_0))g\|_{\mathcal{L}(L_r^\infty, L_n^2)} \\
&\qquad \leq \left(\int_a^b |f_v(t,x_j(t),v_j(t)) - f_v(t,x_0(t),v_0(t))|^2\,dt \right)^{1/2}.
\end{aligned}$$

Using once again the Lebesgue dominated convergence theorem, we assert that

$$\int_a^b |f_v(t,x_j(t),v_j(t)) - f_v(t,x_0(t),v_0(t))|^2\,dt \longrightarrow 0.$$

The proof is completed. □

Lemma 4.2. *For any fixed* $(x,u,v) \in I^{\alpha}_{a+}(L^2_n) \times L^{\infty}_m \times L^{\infty}_r$*, the operator* $F'_x(x,u,v) : I^{\alpha}_{a+}(L^2_n) \to L^2_n$ *given by* (4.2) *is* "one to one" *and* "onto."

Proof. Let us fix a function $z \in L^2_n$ and consider in $I^{\alpha}_{a+}(L^2_n)$ the equation

$$D^{\alpha}_{a+}h(t) + \int_a^t \Phi_x(t,\tau,x(\tau),u(\tau))h(\tau)\,d\tau - f_x(t,x(t),v(t))h(t) = z(t). \tag{4.10}$$

Putting

$$\begin{aligned} \Psi(t,\tau,h) &= \Phi_x(t,\tau,x(\tau),u(\tau))h, \\ g(t,h) &= f_x(t,x(t),v(t))h + z(t), \end{aligned}$$

we see that Ψ and g satisfy assumptions from the Appendix with

$$d(t,\tau) = C_{\Phi}|x(\tau)|\omega_{\Phi}(|u(\tau)|)$$

and

$$L = d_f \max\{\omega_f(|v(\tau)|);\ \tau \in J\}.$$

So, from Lemma 6.1 and the observation formulated before this lemma it follows that equation (4.10) has a unique solution in $I^{\alpha}_{a+}(L^2_n)$. The proof is completed. □

Now, let us fix a function $(u,v) \in L^{\infty}_m \times L^{\infty}_r$ and consider the functional

$$\begin{aligned} &\varphi : I^{\alpha}_{a+}(L^2_n) \ni x \longmapsto \frac{1}{2}\,\|F(x,u,v)\|^2 \\ &= \frac{1}{2}\int_a^b \left| D^{\alpha}_{a+}x(t) + \int_a^t \Phi(t,\tau,x(\tau),u(\tau))\,d\tau - f(t,x(t),v(t)) \right|^2 dt \in \mathbb{R}. \end{aligned}$$

It is easy to see that, for any $x \in I^{\alpha}_{a+}(L^2_n)$,

$$\begin{aligned} |\varphi(x)|^{1/2} \ge \frac{1}{\sqrt{2}}\bigg(&\|x\|_{I^{\alpha}_{a+}(L^2_n)} \\ &- \left\| \int_a^{\cdot} \Phi(\cdot,\tau,x(\tau),u(\tau))\,d\tau \right\|_{L^2_n} - \|f(\cdot,x(\cdot),u(\cdot))\|_{L^2_n} \bigg). \end{aligned}$$

Moreover,

$$
\begin{aligned}
&\left(\int_a^b \left| \int_a^t \Phi(t,\tau,x(\tau),u(\tau))\, d\tau \right|^2 dt \right)^{1/2} \\
&\le \left(\int_a^b \left(\int_a^t (a_\Phi(t,\tau)\,|x(\tau)| + b_\Phi(t,\tau)\omega_\Phi(|u(\tau)|)\,)d\tau \right)^2 dt \right)^{1/2} \\
&\le \left(\int_a^b \left(\left(\int_a^t a_\Phi^2(t,\tau)\, d\tau \right)^{1/2} \left(\int_a^t |x(\tau)|^2\, d\tau \right)^{1/2} \right.\right. \\
&\qquad\qquad \left.\left. + K_1\sqrt{b-a} \left(\int_a^t b_\Phi^2(t,\tau)\, d\tau \right)^{1/2} \right)^2 dt \right)^{1/2} \\
&\le \sqrt{2} \left(\int_a^b \left(\left(\int_a^t a_\Phi^2(t,\tau)\, d\tau \right) \left(\int_a^t |x(\tau)|^2\, d\tau \right) + K_1^2(b-a) \right.\right. \\
&\quad \left.\left. \times \left(\int_a^t b_\Phi^2(t,\tau)\, d\tau \right) \right) dt \right)^{1/2} \\
&\le \sqrt{2}(\|x\|_{L_n^2}^2 \, \|a_\Phi\|_{L^2(P_\Delta,\mathbb{R})}^2 + K_1^2(b-a)\, \|b_\Phi\|_{L^2(P_\Delta,\mathbb{R})}^2)^{1/2} \\
&\le \sqrt{2}(\|x\|_{L_n^2} \, \|a_\Phi\|_{L^2(P_\Delta,\mathbb{R})} + K_1\sqrt{b-a}\, \|b_\Phi\|_{L^2(P_\Delta,\mathbb{R})}) \\
&\le \sqrt{2}(\gamma_\alpha \, \|x\|_{I_{a+}^\alpha(L_n^2)} \, \|a_\Phi\|_{L^2(P_\Delta,\mathbb{R})} + K_1\sqrt{b-a}\, \|b_\Phi\|_{L^2(P_\Delta,\mathbb{R})})
\end{aligned}
$$

where

$$K_1 = \max\{\omega_\Phi(|u(t)|);\ t \in [a,b]\},$$

and

$$
\begin{aligned}
&\left(\int_a^b |f(t,x(t),v(t))|^2\, dt \right)^{1/2} \\
&\le \left(\int_a^b (a_f\,|x(t)| + b_f(t)\omega_f(|v(t)|))^2\, dt \right)^{1/2} \\
&\le \sqrt{2} \left(\int_a^b \left(a_f^2\,|x(t)|^2 + (b_f(t))^2(\omega_f(|v(t)|))^2 \right) dt \right)^{1/2} \\
&\le \sqrt{2}(a_f^2(\gamma_\alpha)^2\, \|x\|_{I_{a+}^\alpha(L_n^2)}^2 + K_4^2\, \|b_f\|_{L_1^2}^2)^{1/2} \\
&\le \sqrt{2}(a_f\gamma_\alpha\, \|x\|_{I_{a+}^\alpha(L_n^2)} + K_4\, \|b_f\|_{L_1^2}),
\end{aligned}
$$

where

$$K_4 = \max\{\omega_f(|v(t)|);\ t \in [a,b]\}.$$

So,

$$
\begin{aligned}
|\varphi(x)|^{1/2} \geq \frac{1}{\sqrt{2}}(\|x\|_{I^{\alpha}_{a+}(L^2_n)} &- \sqrt{2}\gamma_\alpha \|x\|_{I^{\alpha}_{a+}(L^2_n)} \|a_\Phi\|_{L^2(P_\Delta,\mathbb{R})} \\
&- \sqrt{2}K_1\sqrt{b-a}\,\|b_\Phi\|_{L^2(P_\Delta,\mathbb{R})} \\
&\quad - \sqrt{2}a_f\gamma_\alpha \|x\|_{I^{\alpha}_{a+}(L^2_n)} - \sqrt{2}K_4\|b_f\|_{L^2_1}) \\
&= \left(\frac{1}{\sqrt{2}} - \|a_\Phi\|_{L^2(P_\Delta,\mathbb{R})}\gamma_\alpha - a_f\gamma_\alpha\right)\|x\|_{I^{\alpha}_{a+}(L^2_n)} \\
&\quad - K_1\sqrt{b-a}\,\|b_\Phi\|_{L^2(P_\Delta,\mathbb{R})} - K_4\|b_f\|_{L^2_1}
\end{aligned}
$$

for $x \in I^{\alpha}_{a+}(L^2_n)$. It means that φ is coercive (cf., (4.1)).

In a standard way, we check that the differential $\varphi'(x)$ of φ at x is given by

$$
\begin{aligned}
\varphi'(x)h = \int_a^b &\Big(D^{\alpha}_{a+}x(t) + \int_a^t \Phi(t,\tau,x(\tau),u(\tau))\,d\tau - f(t,x(t),v(t))\Big) \\
&\times \Big(D^{\alpha}_{a+}h(t) + \int_a^t \Phi_x(t,\tau,x(\tau),u(\tau))h(\tau)\,d\tau \\
&\qquad - f_x(t,x(t),v(t))h(t)\Big)\,dt
\end{aligned}
$$

for $h \in I^{\alpha}_{a+}(L^2)$. Consequently, for any x_m, $x_0 \in I^{\alpha}_{a+}(L^2_n)$, we have

$$
\begin{aligned}
\varphi'(x_m)(x_m - x_0) = \int_a^b &\Big(D^{\alpha}_{a+}x_m(t) + \int_a^t \Phi(t,\tau,x_m(\tau),u(\tau))\,d\tau \\
&\qquad - f(t,x_m(t),v(t))\Big) \\
&\times \Big((D^{\alpha}_{a+}x_m(t) - D^{\alpha}_{a+}x_0(t)) \\
&\quad + \int_a^t \Phi_x(t,\tau,x_m(\tau),u(\tau))(x_m(\tau) - x_0(\tau))\,d\tau \\
&\qquad - f_x(t,x_m(t),v(t))(x_m(t) - x_0(t))\Big)\,dt,
\end{aligned}
$$

$$
\begin{aligned}
\varphi'(x_0)(x_m - x_0) = \int_a^b &\Big(D^{\alpha}_{a+}x_0(t) \\
&+ \int_a^t \Phi(t,\tau,x_0(\tau),u(\tau))\,d\tau - f(t,x_0(t),v(t))\Big)
\end{aligned}
$$

$$\times \Big((D^{\alpha}_{a+}x_m(t) - D^{\alpha}_{a+}x_0(t)) + \int_a^t \Phi_x(t,\tau,x_0(\tau),u(\tau))(x_m(\tau)-x_0(\tau))\,d\tau - f_x(t,x_0(t),v(t))(x_m(t)-x_0(t)) \Big)\,dt$$

and

$$\varphi'(x_m)(x_m-x_0) - \varphi'(x_0)(x_m-x_0) = \|x_m-x_0\|^2_{I^{\alpha}_{a+}(L^2_n)} + \sum_{i=1}^{14}\psi_i(x_m),$$

where

$$\psi_1(x_m) = \int_a^b \Big(\int_a^t \Phi(t,\tau,x_m(\tau),u(\tau))\,d\tau - \int_a^t \Phi(t,\tau,x_0(\tau),u(\tau))\,d\tau \Big)(D^{\alpha}_{a+}x_m(t) - D^{\alpha}_{a+}x_0(t))\,dt,$$

$$\psi_2(x_m) = \int_a^b (f(t,x_0(t),v(t)) - f(t,x_m(t),v(t))) \times (D^{\alpha}_{a+}x_m(t) - D^{\alpha}_{a+}x_0(t))\,dt,$$

$$\psi_3(x_m) = \int_a^b D^{\alpha}_{a+}x_m(t) \int_a^t \Phi_x(t,\tau,x_m(\tau),u(\tau))(x_m(\tau)-x_0(\tau))\,d\tau\,dt,$$

$$\psi_4(x_m) = -\int_a^b D^{\alpha}_{a+}x_0(t) \times \int_a^t \Phi_x(t,\tau,x_0(\tau),u(\tau))(x_m(\tau)-x_0(\tau))\,d\tau\,dt,$$

$$\psi_5(x_m) = \int_a^b \Big(\int_a^t \Phi(t,\tau,x_m(\tau),u(\tau))\,d\tau \times \int_a^t \Phi_x(t,\tau,x_m(\tau),u(\tau))(x_m(\tau)-x_0(\tau))\,d\tau \Big)\,dt,$$

$$\psi_6(x_m) = -\int_a^b \Big(\int_a^t \Phi(t,\tau,x_0(\tau),u(\tau))\,d\tau \times \int_a^t \Phi_x(t,\tau,x_0(\tau),u(\tau))(x_m(\tau)-x_0(\tau))\,d\tau \Big)\,dt,$$

$$\psi_7(x_m) = -\int_a^b \Big(f(t, x_m(t), v(t)) \times \int_a^t \Phi_x(t, \tau, x_m(\tau), u(\tau))(x_m(\tau) - x_0(\tau))\, d\tau \Big)\, dt,$$

$$\psi_8(x_m) = \int_a^b \Big(f(t, x_0(t), v(t)) \times \int_a^t \Phi_x(t, \tau, x_0(\tau), u(\tau))(x_m(\tau) - x_0(\tau))\, d\tau \Big)\, dt,$$

$$\psi_9(x_m) = -\int_a^b D_{a+}^{\alpha} x_m(t) f_x(t, x_m(t), v(t))(x_m(t) - x_0(t))\, dt,$$

$$\psi_{10}(x_m) = \int_a^b D_{a+}^{\alpha} x_0(t) f_x(t, x_0(t), v(t))(x_m(t) - x_0(t))\, dt,$$

$$\psi_{11}(x_m) = -\int_a^b \int_a^t \Phi(t, \tau, x_m(\tau), u(\tau))\, d\tau \times f_x(t, x_m(t), v(t))(x_m(t) - x_0(t))\, dt,$$

$$\psi_{12}(x_m) = \int_a^b \int_a^t \Phi(t, \tau, x_0(\tau), u(\tau))\, d\tau \times f_x(t, x_0(t), v(t))(x_m(t) - x_0(t))\, dt,$$

$$\psi_{13}(x_m) = \int_a^b f(t, x_m(t), v(t)) \times f_x(t, x_m(t), v(t))(x_m(t) - x_0(t))\, dt,$$

$$\psi_{14}(x_m) = -\int_a^b f(t, x_0(t), v(t)) \times f_x(t, x_0(t), v(t))(x_m(t) - x_0(t))\, dt.$$

We shall show that φ satisfies the (PS) condition. Indeed, if (x_m) is a (PS) sequence for φ, then the coercivity of φ implies its boundedness. Consequently, there exists a subsequence (x_{m_k}) which is weakly convergent in $I_{a+}^{\alpha}(L_n^2)$ to some x_0. Since (cf., [**15,** Lemma 3]) for any sequence (x_n) weakly convergent in $I_{a+}^{\alpha}(L_n^2)$ to x_0, (x_n) is strongly convergent to x_0 in L_n^2 and the sequence of derivatives $(D_{a+}^{\alpha} x_n)$ is weakly convergent to $D_{a+}^{\alpha} x_0$ in L_n^2, therefore $x_{m_k} \to x_0$ in L_n^2 with respect to the norm and $D_{a+}^{\alpha} x_{m_k} \rightharpoonup D_{a+}^{\alpha} x_0$ weakly in L_n^2.

First, we shall show that $\psi_i(x_{m_k}) \underset{k\to\infty}{\to} 0$ for $i = 1, \ldots, 14$.

Let us consider the first term $\psi_1(x_{m_k})$. In the same way as in the proof of Lemma 4.1, we check that

$$\int_a^{\cdot} (\Phi(\cdot, \tau, x_{m_k}(\tau)) - \Phi(\cdot, \tau, x_0(\tau)))\, d\tau \underset{m\to\infty}{\longrightarrow} 0$$

in L_n^2. Consequently, $\psi_1(x_{m_k})$ as a scalar product in L_n^2 of the functions $D_{a+}^{\alpha} x_m(\cdot) - D_{a+}^{\alpha} x_0(\cdot)$ and $\int_a^{\cdot} (\Phi(\cdot, \tau, x_{m_k}(\tau), u(\tau)) - \Phi(\cdot, \tau, x_0(\tau), u(\tau)))\, d\tau$ tends to 0 as $k \to \infty$. Similarly, using the growth condition on f, we assert that $\psi_2(x_{m_k}) \to 0$. Convergence of $\psi_i(x_{m_k})$ to 0 for $i = 3, \ldots, 14$ follows from the convergence of $x_{m_k}(\cdot)$ to $x_0(\cdot)$ in L^2.

Since $\varphi'(x_0)$ is linear and continuous functional on $I_{a+}^{\alpha}(L_n^2)$, convergence of $\varphi'(x_0)(x_{m_k} - x_0)$ to 0 follows directly from the weak convergence $x_{m_k} \rightharpoonup x_0$ in $I_{a+}^{\alpha}(L_n^2)$.

Convergence of $\varphi'(x_{m_k})(x_{m_k} - x_0)$ to 0 follows from the estimation

$$|\varphi'(x_{m_k})(x_{m_k} - x_0)| \le \|\varphi'(x_{m_k})\|_{\mathcal{L}(I_{a+}^{\alpha}(L_n^2), \mathbb{R})} \|x_{m_k} - x_0\|_{I_{a+}^{\alpha}(L_n^2)},$$

boundedness of the sequence (x_{m_k}) in $I_{a+}^{\alpha}(L_n^2)$ and convergence of $\varphi'(x_{m_k})$ to 0.

Thus, $x_{m_k} \to x_0$ in $I_{a+}^{\alpha}(L_n^2)$ with respect to the norm.

So, all assumptions of the global implicit function theorem are satisfied. Consequently, for any $(u, v) \in L_m^{\infty} \times L_r^{\infty}$, there exists a unique solution $x_{u,v} \in I_{a+}^{\alpha}(L_n^2)$ of the problem (1.2), the mapping

$$\lambda : L_m^{\infty} \times L_r^{\infty} \ni (u, v) \longmapsto x_{u,v} \in I_{a+}^{\alpha}(L_n^2)$$

is continuous differentiable in the Frechet sense on $L_m^{\infty} \times L_r^{\infty}$ and the differential $\lambda'(u, v)$ at a point $(u, v) \in L_m^{\infty} \times L_r^{\infty}$ is the following

$$\lambda'(u, v) : L_m^{\infty} \times L_r^{\infty} \ni (f, g) \longmapsto z_{f,g} \in I_{a+}^{\alpha}(L_n^2),$$

where $z_{f,g}$ is such that

$$\begin{aligned} D_{a+}^{\alpha} z_{f,g}(t) + \int_a^t & \Phi_x(t, \tau, x_{u,v}(\tau), u(\tau)) z_{f,g}(\tau)\, d\tau \\ & - f_x(t, x_{u,v}(t), v(t)) z_{f,g}(t) \\ = -\int_a^t & \Phi_u(t, \tau, x_{u,v}(\tau), u(\tau)) f(\tau)\, d\tau + f_v(t, x_{u,v}(t), v(t)) g(t) \end{aligned}$$

almost everywhere on J.

Remark 4.3. Let us point out that, if we replace the growth condition on Φ_x from $(A2)$ with the following one

$$|\Phi_x(t,\tau,x,u)| \leq c_\Phi(t,\tau)\omega_\Phi(|u|)$$

for $(t,\tau) \in P_\Delta$ almost everywhere, $x \in \mathbb{R}^n$, $u \in \mathbb{R}^m$, where $c_\Phi \in L^2(P_\Delta, \mathbb{R}_0^+)$, $\omega_\Phi \in C(\mathbb{R}_0^+, \mathbb{R}_0^+)$, assuming additionally that

$$\int_a^t c_\Phi^2(t,\tau)\, d\tau \leq C$$

for $t \in J$ almost everywhere and some $C > 0$, then the existence of a unique solution $x_{u,v}$ of (1.2), for any fixed (u,v), can be deduced from the results contained in the Appendix (with the aid of the mean value theorem and without a condition of type (4.1)). Applying, additionally, the local implicit function theorem, one can obtain continuous differentiability in Frechet sense of the mapping $L_m^\infty \times L_r^\infty \ni (u,v) \mapsto x_{u,v} \in I_{a+}^\alpha(L_n^2)$.

5. Example. Let us fix $\alpha \in (0,1)$, $C > 0$, $E > 0$, and consider the following problem

(5.1)
$$\begin{cases} D_{a+}^\alpha x(t) + \int_a^t C\sqrt{t-\tau}\ln(\sqrt{(x(\tau))^2+1} + (u(\tau))^2)\, d\tau \\ \quad = E\sqrt[3]{t}\sin(x(t)+1+v(t)), \ t \in [0,1] \text{ almost everywhere}, \\ I_{a+}^{1-\alpha}x(a) = 0, \end{cases}$$

where $x \in I_{a+}^\alpha(L^2([0,1],\mathbb{R}))$, $u \in L^\infty([0,1],\mathbb{R})$ and $v \in L^\infty([0,1],\mathbb{R})$. So, it is a particular case of problem (1.2) with the functions

$$\Phi : P_\Delta \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

$$\Phi(t,\tau,x,u) = C\sqrt{t-\tau}\ln(\sqrt{x^2+1} + u^2)$$

where $P_\Delta = \{(t,\tau) \in [0,1] \times 0,1]; \tau \leq t\}$ and

$$f : [0,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

$$f(t,x,v) = E\sqrt[3]{t}\sin(x+1+v).$$

Of course, these functions are measurable in $(t,\tau) \in P_\Delta$, $t \in [0,1]$, respectively, and continuous differentiable with respect to $(x,u) \in \mathbb{R} \times \mathbb{R}$, $(x,v) \in \mathbb{R} \times \mathbb{R}$, respectively. Moreover, using the inequalities

$\ln s \le \sqrt{s}$ and $\sqrt{a+b} \le \sqrt{a} + \sqrt{b}$, we obtain

$$\begin{aligned}
|\Phi(t,\tau,x,u)| &= \left| C\sqrt{t-\tau}\ln(\sqrt{x^2+1}+u^2) \right| \\
&\le C\sqrt{t-\tau}\ln(\sqrt{x^2}+1+u^2) \\
&\le C\sqrt{t-\tau}\sqrt{\sqrt{x^2}+1+u^2} \\
&\le C\sqrt{t-\tau}(\sqrt{\sqrt{x^2}}+\sqrt{1+u^2}) \\
&\le C\sqrt{t-\tau}(\sqrt{|x|}+\sqrt{1+u^2}) \\
&\le C\sqrt{t-\tau}(|x|+1+\sqrt{1+u^2}) \\
&= C\sqrt{t-\tau}\,|x| + C\sqrt{t-\tau}(\sqrt{1+|u|^2}+1) \\
&\le C\sqrt{t-\tau}\,|x| + C\sqrt{t-\tau}\max\{(\sqrt{1+|u|^2}+1), 2\,|u|\}
\end{aligned}$$

and

$$\begin{aligned}
|\Phi_x(t,\tau,x,u)| &= C\sqrt{t-\tau}\frac{|x|}{(\sqrt{x^2+1}+u^2)\sqrt{x^2+1}} \\
&\le C\,|x|\,\frac{1}{(\sqrt{x^2+1}+u^2)\sqrt{x^2+1}} \\
&\le C\,|x| \le C\,|x|\max\{(\sqrt{1+|u|^2}+1), 2\,|u|\}\,|\Phi_u(t,\tau,x,u)| \\
&= C\sqrt{t-\tau}\frac{2\,|u|}{\sqrt{x^2+1}+u^2} \le C\sqrt{t-\tau}2\,|u| \\
&\le C\sqrt{t-\tau}\max\{(\sqrt{1+|u|^2}+1), 2\,|u|\} \\
&\le C\sqrt{t-\tau}\,|x| \\
&\quad + C\sqrt{t-\tau}\max\{(\sqrt{1+|u|^2}+1), 2\,|u|\}
\end{aligned}$$

for $(t,\tau) \in P_\Delta$, $(x,u) \in \mathbb{R} \times \mathbb{R}$. So, the growth conditions concerning Φ are satisfied with

$$a_\Phi(t,\tau) = b_\Phi(t,\tau) = C\sqrt{t-\tau}, \quad C_\Phi = C,$$

$$\omega_\Phi(r) = \max\{\sqrt{1+r^2}+1, 2r\}.$$

Let us observe that

$$\|a_\Phi\|_{L^2(P_\Delta,\mathbb{R})} = \frac{C}{\sqrt{6}}.$$

Similarly,

$$\begin{aligned} |f(t,x,v)| &= E\sqrt[3]{t}\,|\sin(x+1+v)| \\ &\le E\sqrt[3]{t}\,|x+1+v)| \\ &\le E\sqrt[3]{t}\,|x| + E\sqrt[3]{t}(|v|+1) \\ &\le E\,|x| + E\sqrt[3]{t}(|v|+1) \end{aligned}$$

and

$$\begin{aligned} |f_x(t,x,v)| &= E\sqrt[3]{t}\,|\cos(x+1+v)| \\ &\le E \le E(|v|+1), \\ |f_v(t,x,v)| &= E\sqrt[3]{t}\,|\cos(x+1+v)| \\ &\le E\sqrt[3]{t} \le E\sqrt[3]{t}(|v|+1) \\ &\le E\,|x| + E\sqrt[3]{t}(|v|+1) \end{aligned}$$

for $t \in [0,1]$, $(x,u) \in \mathbb{R}\times\mathbb{R}$. Thus, the growth conditions concerning f are satisfied with

$$a_f = d_f = E, \qquad b_f(t) = E\sqrt[3]{t},$$
$$\omega_f(r) = r+1.$$

Choosing the constants C and E such that

$$\frac{1}{\Gamma(\alpha+1)}\left(\frac{C}{\sqrt{6}} + E\right) < \frac{1}{\sqrt{2}}$$

we see that (4.1) is satisfied (using MAPLE one can check that, for example, constants $\alpha = 1/2$ and $C = E = 1/3$ satisfy the above inequality).

Consequently, for any pair of functional parameters $u \in L^\infty([0,1],\mathbb{R})$, $v \in L^\infty([0,1],\mathbb{R})$, there exists a unique solution $x_{u,v}$ to problem (5.1), in the space $I^\alpha_{a+}(L^2_n)$, and the mapping

$$L^\infty([0,1],\mathbb{R}) \times L^\infty([0,1],\mathbb{R}) \ni (u,v) \longrightarrow x_{u,v} \in I^\alpha_{a+}(L^2_n)$$

is continuous differentiable.

6. Appendix. Let us consider the following system:

$$D^{\alpha}_{a+}x(t) + \int_a^t \Psi(t,\tau,x(\tau))\,d\tau = g(t,x(t)), \quad t \in J \text{ a.e.,} \tag{6.1}$$

where $\Psi : P_\Delta \times \mathbb{R}^n \to \mathbb{R}^n$, $g : J \times \mathbb{R}^n \to \mathbb{R}^n$ and $x \in I^{\alpha}_{a+}(L^2_n)$. On the functions Ψ and g we assume that

- $\Psi(\cdot,\cdot,x)$ is measurable on P_Δ for any $x \in \mathbb{R}^n$, and
$$|\Psi(t,\tau,x_1) - \Psi(t,\tau,x_2)| \le d(t,\tau)\,|x_1 - x_2|$$
for $(t,\tau) \in P_\Delta$ almost everywhere, x_1, $x_2 \in \mathbb{R}^n$, where $d \in L^2(P_\Delta,\mathbb{R}^n)$ and
$$\int_a^t d^2(t,s)\,ds \le D$$
for $t \in J$ almost everywhere and some $D > 0$;
- $\Psi(\cdot,\cdot,0) \in L^2(P_\Delta,\mathbb{R}^n)$;
- $g(\cdot,x)$ is measurable on J for any $x \in \mathbb{R}^n$, and
$$|g(t,x_1) - g(t,x_2)| \le L\,|x_1 - x_2|$$
for $t \in J$ almost everywhere, x_1, $x_2 \in \mathbb{R}^n$, where $L > 0$ is some constant;
- $g(\cdot,0) \in L^2_n$.

It is easy to see that the existence of a unique solution x to system (6.1) in the space $I^{\alpha}_{a+}(L^2_n)$ is equivalent to the existence of a unique solution l to system

$$l(t) + \int_a^t \Psi(t,\tau,I^{\alpha}_{a+}l(\tau))\,d\tau = g(t,I^{\alpha}_{a+}l(t)), \quad t \in J \text{ a.e.,} \tag{6.2}$$

in the space L^2_n; in such a case, $D^{\alpha}_{a+}x = l$. We have:

Lemma 6.1. *There exists a unique fixed point of the operator*

$$T : L^2_n \ni l \longmapsto g(t,I^{\alpha}_{a+}l(t)) - \int_a^t \Psi(t,\tau,I^{\alpha}_{a+}l(\tau))\,d\tau \in L^2_n. \tag{6.3}$$

Proof. Operator T is well-posed. Indeed, the squared integrability of the first term follows from the Lipschitz condition posed on g. The fact that the second term belongs to L^2_n can be checked as in the proof of Lemma 4.1. We shall show that there exists a positive integer k such that the operators

$$T_g : L_n^2 \ni l \longmapsto g(t, I_{a+}^{\alpha} l(t)) \in L_n^2$$
$$T_\Psi : L_n^2 \ni l \longmapsto \int_a^t \Psi(t, \tau, I_{a+}^{\alpha} l(\tau))\, d\tau \in L_n^2$$

are contracting with constants ξ_1, $\xi_2 \in (0, 1/2)$, respectively, provided that L_n^2 is considered with the Bielecki norm

$$\|l\|_k = \left(\int_a^1 e^{-kt} \, |l(t)|^2 \, dt \right)^{1/2}, \quad l \in L_n^2.$$

Indeed, let us fix $k \in \mathbb{N}$. In **[17]**, it is shown that

$$\begin{aligned} \|T_g(l_1) - T_g(l_2)\|_k^2 &= \int_a^b e^{-kt} \left| g(t, I_{a+}^{\alpha} l_1(t)) - g(t, I_{a+}^{\alpha} l_2(t)) \right|^2 dt \\ &\leq \gamma_\alpha (L^2/(2k)^\alpha) \, \|l_1 - l_2\|_k^2 \end{aligned}$$

for l_1, $l_2 \in L_n^2$. In **[15,** Proof of Lemma 7], it is shown that

$$\|T_\Psi(l_1) - T_\Psi(l_2)\|_k^2 \leq (\gamma_\alpha)^2 (D/k) \, \|l_1 - l_2\|_k^2$$

for l_1, $l_2 \in L_n^2$. So, it is sufficient to choose k such that

$$\max\{L\sqrt{\gamma_\alpha/(2k)^\alpha}, \gamma_\alpha \sqrt{(D/k)}\} < \frac{1}{2},$$

and the proof is completed. □

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