JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 27, Number 4, Winter 2015

EXISTENCE OF SOLUTION OF IMPULSIVE SECOND ORDER NEUTRAL INTEGRO-DIFFERENTIAL EQUATIONS WITH STATE DELAY

SANJUKTA DAS, D.N. PANDEY AND N. SUKAVANAM

Communicated by Terry Herdman

ABSTRACT. This paper consists of two parts. The first part deals with the existence of a mild solution of a class of instantaneous impulsive second order partial neutral differential equations with state dependent delay. The second part studies the non-instantaneous impulsive conditions on the same problem. The Kuratowski measure of noncompactness and Mónch fixed point theorem are used to prove the existence of the mild solution. We remove the restrictive conditions on the priori estimation available in literature. The compactness assumption on the associated cosine or sine family of operators, nonlinear terms and associated impulsive term are also not required in this paper. The noncompactness measure estimation, the Lipschitz conditions and compactness on the nonlinear functions are replaced by simple and natural assumptions. We introduce new non-instantaneous impulses with fixed delays. In the last section, we study examples to illustrate the result presented.

1. Introduction. *Neutral differential equations* are functional differential equations in which the highest order derivative of the unknown function appears both with and without deviations. Neutral differential equations with unbounded delay appear abundantly as mathematical models in mechanics, electrical engineering, medicine, biology, ecology, etc. Hence, it is a widely studied topic in several papers and monographs, for instance, the partial neutral differential equation with unbounded delay arises in the theory of heat conduction

---

2010 AMS *Mathematics subject classification.* Primary 34K40, 34K45, 47D09, 47D60.

Keywords and phrases. Cosine family, State dependent delay, Non-instantaneous Impulses, Second-order neutral differential equation.

Ministry of Human Resource and Development, India. The first author would like to thank the Ministry of Human Resource and Development for the funding of Grant no. MHR-02-23-200-429/304.

Received by the editors on February 19, 2015.

DOI:10.1216/JIE-2015-27-4-489

of materials with fading memory in **[24]**. In addition, one may see **[10, 12, 15, 21, 27, 28]** and the references cited therein. Second order neutral differential equations model variational problems in the calculus of variation and in the study of vibrating masses attached to an electric bar. For more details, we refer our readers to **[13, 14, 18, 32, 34]**.

In recent times, much attention is paid to functional differential equations with state dependent delay. We refer to **[1, 2, 3, 9, 17, 20]** for details. The literature related to the state dependent delay mostly deals with functional differential equations in which the state belongs to a finite dimensional space. As a consequence, the study of partial functional differential equations with state dependent delay is neglected. This is one of the motivations of our paper.

Impulsive differential equations are known for their utility in simulating processes and phenomena subject to short term perturbations during their evolution. Discrete perturbations are negligible to the total duration of the process. We refer to **[8, 11, 16, 25, 33, 37]** regarding discrete impulses. However, in these papers, the compactness condition on the impulsive terms, restrictive conditions on a priori estimation and the restrictive condition on measure of noncompactness estimation are used.

Here in this paper we study the second order partial neutral differential equation with state dependent delay modeled in the form

$$\frac{d^2}{dt^2}x(t) = A(x(t) - g(t, x_t)) + \int_0^t f(t, x_{\rho(t,x_t)}, x'_t)\, dt, \\ t \in [0, b],\ t \neq t_i,\ i = 1, \dots, n,$$

$$\begin{aligned} x_0 = \phi \in \mathfrak{B}, \qquad & x'(0) = \xi \in X, \\ \Delta x(t_i) = I_i^1(x_{t_i}, x'_{t_i}), \quad & i = 1, 2, \dots, n \\ \Delta x'(t) = I_i^2(x_{t_i}, x'_{t_i}), \quad & i = 1, 2, \dots, n. \end{aligned} \tag{1.1}$$

Here $0 = t_0 < t_1 \leq t_2, \dots, < t_n \leq t_{n+1} = b$ are prefixed numbers.

We also study the second order partial neutral differential equation with state dependent delay modeled in the form

$$\frac{d^2}{dt^2}x(t) = A(x(t) - g(t, x_t)) + \int_0^t f(t, x_{\rho(t,x_t)}, x'(t))\, dt,$$

$$t \in (s_i, t_{i+1}], \quad i = 0, \dots, n,$$

$$\begin{aligned} x_0 &= \phi \in \mathfrak{B}, \\ x'(0) &= \xi \in X, \\ x(t) &= J_i^1(t, x(t - t_1)), \quad t \in (t_i, s_i],\ i = 1, 2, \dots, n \\ x'(t) &= J_i^2(t, x(t - t_1)), \quad t \in (t_i, s_i],\ i = 1, 2, \dots, n \end{aligned} \tag{1.2}$$

Here $0 = t_0 = s_0 < t_1 \le s_1 \le t_2, \dots, < t_n \le s_n \le t_{n+1} = b$ are prefixed numbers.

In (1.1) and (1.2), A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathsf{R}\}$ of bounded linear operators on a Banach space X and $t \in [0, b] = J$. The history-valued function $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t + \theta)$ belongs to some abstract phase space $\mathfrak{B}$ defined axiomatically and $g, f, I_i^1, I_i^2, J_i^1, J_i^2$, $i = 1, \dots, n$, are appropriate functions which are defined in Section 2 in the hypotheses (Hf), (Hg), (HI) and (HJ), respectively.

The second order abstract partial neutral differential equation similar to (1.1) is extensively studied in **[4, 5, 7, 35]**. As a matter of fact, in these papers, the authors assume severe conditions on the operator family generated by A, which imply that the underlying space X has finite dimension. Thus, the equations treated in these works are really ordinary and not partial differential equations. Hence, motivated by this fact and the results in **[30]** and their various applications we study the existence of the mild solution of the partial neutral differential equation of second order with state delay and non-instantaneous impulses. The contribution of this paper lies in the removal of the compactness assumption on the associated cosine or sine family of operators and the associated impulsive term. The noncompactness measure estimation and the Lipschitz conditions on the nonlinear functions are replaced by simple and natural assumptions.

2. Preliminaries. In this section, some definitions, notation and lemmata that are used throughout this paper are stated. The family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(X)$ is a strongly continuous cosine family if the following are satisfied:

(a) $C(0) = I$ (I is the identity operator in X);
(b) $C(t + s) + C(t - s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$;

(c) The map $t \to C(t)x$ is strongly continuous for each $x \in X$.

The one parameter family of operators $\{S(t) : t \in \mathbb{R}\}$ is the sine family associated to the strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$, and it is defined as

$$S(t)x = \int_0^t C(s)x\,ds, \quad x \in X,\ t \in \mathbb{R}.$$

The operator A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in R}$, and $S(t)$ is the associated sine function. Let $N, \widetilde{N}, \widetilde{N_1}, \widetilde{N_2}$ be certain constants such that $\|C(t)\| \le N$, $\|S(t)\| \le \widetilde{N}$, $\|AS\| \le \widetilde{N_1}\, \|AC\| \le \widetilde{N_2}$ for every $t \in J = [0, b]$. For more details, see the books by Goldstein **[22]** and Fattorini **[19]**. In this work, we use the axiomatic definition of phase space $\mathfrak{B}$, introduced by Hale and Kato **[26]**.

$PC([0, b], X)$ is the space formed by normalized piecewise continuous function from $[0, b]$ into X. In particular, it is the space PC formed by all functions $u : [0, b] \to X$ such that u is continuous at $t \neq t_i$, $u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists for all $i = 1, 2, \ldots, n$. It is clear that PC endowed with the norm $\|x\|_{PC} = \sup_{t\in J} \|x(t)\|$ is a Banach space. For any $x \in PC$,

$$\widetilde{x}_i(t) = \begin{cases} x(t), & t \in (t_i, t_{i+1}]; \\ x(t_i^+), & t = t_i,\ i = 1, 2, ..., n. \end{cases} \tag{2.1}$$

So, $\widetilde{x} \in C([t_i, t_{i+1}], X)$.

$PC^1([0, b], X) = \{u \in PC([0, b], X) : u' \in PC([0, b], X)\}$ is a Banach space with respect to the norm $\|u\|_1 = \|u\|_\infty + \|u'\|_\infty$.

Definition 2.1 (**[26]**)**.** The phase space $\mathfrak{B}$ is a linear space of functions mapping $(-\infty, 0]$ into X endowed with seminorm $\|.\|_{\mathfrak{B}}$ and satisfies the following conditions:

(A) If $x : (-\infty, \sigma + b] \to X$, $b > 0$, such that $x_t \in \mathfrak{B}$ and $x|_{[\sigma,\sigma+b]} \in C([\sigma, \sigma + b] : X)$, then, for every $t \in [\sigma, \sigma + b)$ the following conditions hold:
  (i) x_t is in $\mathfrak{B}$,
  (ii) $\|x(t)\| \le H\|x_t\|_{\mathfrak{B}}$,
  (iii) $\|x_t\|_{\mathfrak{B}} \le K(t - \sigma) \sup\{\|x(s)\| : \sigma \le s \le t\} + M(t - \sigma)\|x_\sigma\|_{\mathfrak{B}}$,

where $H > 0$ is a constant $K, M : [0, \infty) \to [1, \infty)$. K is a continuous function, M is locally bounded and the functions H, K, M are independent of $x(.)$.

(B) The space $\mathfrak{B}$ is complete.

Definition 2.2 ([**6**])**.** For a bounded set B in any Banach space Y the Kuratowski measure of noncompactness α_Y is defined by

$$\alpha_Y(B) = \inf\{r > 0, B \text{ can be covered by } \text{a finite number of balls with diameter } r\}.$$

Lemma 2.3 ([**6**])**.** *Let Y be a Banach space and $B, C \subset Y$ be bounded. Then the following properties hold*:

(i) *B is pre-compact if and only if $\alpha_Y(B) = 0$*;
(ii) *$\alpha_Y(B) = \alpha_Y(\overline{B}) = \alpha_Y(\operatorname{conv} B)$, where $\overline{B}$ and $\operatorname{conv} B$ are the closure and convex hull of B, respectively*;
(iii) *$\alpha_Y(B) \le \alpha_Y(C)$ when $B \subset C$*;
(iv) *$\alpha_Y(B+C) \le \alpha_Y(B)+\alpha_Y(C)$ where $B+C = \{x+y; x \in B, y \in C\}$*;
(v) *$\alpha_Y(B \cup C) = \max\{\alpha_Y(B), \alpha_Y(C)\}$.*
(vi) *$\alpha_Y(\lambda B) = |\lambda|\alpha_Y(B)$ for any $\lambda \in \mathrm{R}$*;
(vii) *If the map $Q : D(Q) \subset Y \to Z$ is Lipschitz continuous with constant k, then $\alpha_Z(Q(B)) \le k\alpha_Y(B)$ for any bounded subset $B \subset D(Q)$, where Z is a Banach space.*
(viii) *If $\{W_n\}_{n=1}^{+\infty}$ is a decreasing sequence of a bounded, closed, nonempty subset of Y and $lim_{n\to\infty}\alpha_Y(W_n) = 0$, then $\cap_{n=1}^{+\infty} W_n$ is nonempty and compact in Y.*

Lemma 2.4 ([**6**])**.**

(i) *If $W \subset PC([0, b]; X)$ is bounded, then $\alpha(W(t)) \le \alpha_{PC}(W)$ for any $t \in [0, b]$ where $W(t) = \{u(t) : u \in W\} \subset X$.*
(ii) *If W is piecewise equicontinuous on $[0, b]$, then $\alpha(W(t))$ is piecewise continuous for $t \in [0, b]$, and*

$$\alpha_{PC}(W) = \sup\{\chi(W(t)),\ t \in [0, b]\}.$$

(iii) *If $W \subset PC([0, b]; X)$ is bounded and piecewise equicontinuous,* *then $\alpha(W(t))$ is piecewise continuous for $t \in [0,b]$ and*

$$\alpha\bigg(\int_a^t W(s)\,ds\bigg) \leq 2\int_a^t \alpha(W(s))\,ds \quad t \in [0,b].$$

(iv) *If $W \subset PC^1([0,b],X)$ is bounded and the elements of W' are equicontinuous on each $J_i = (t_i, t_{i+1}]$, $(i = 0,1,\dots,n)$, then*

$$\alpha_{PC^1}(W) = \max\{\sup_{t\in J} \alpha W(t), \sup_{t\in J} \alpha(W'(t))\},$$

where α_{PC^1} denotes the Kuratowski measure of noncompactness in the space $PC^1(J,X)$.

Lemma 2.5 ([**6**])**.** *If the semigroup $S(t)$ is equicontinuous and $\eta \in L([0,b];R^+)$, then the set*

$$\bigg\{\int_0^t S(t-s)u(s)\,ds : \|u(s)\| \leq \eta(s), \quad \text{for almost every } s \in [0,b]\bigg\}$$

is equicontinuous for $t \in [0,b]$.

Lemma 2.6 ([**6**])**.** *Let $h : [0,b] \to E$ be an integrable function such that $h \in PC$. Then the function $v(t) = \int_0^t C(t-s)h(s)\,ds$ belongs to PC^1, the function $s \to AS(t-s)h(s)$ is integrable on $[0,t]$ for $t \in [0,b]$, and*

$$\begin{aligned} v'(t) &= h(t) + A\int_0^t S(t-s)h(s)\,ds \\ &= h(t) + \int_0^t AS(t-s)h(s)\,ds, \quad t \in [0,b]. \end{aligned}$$

Lemma 2.7 ([**6**])**.** *Let $H = h_n \subset L^1([0,b],X)$. If there exists $\varrho \in L^1([0,b],[0,+\infty))$ such that $\|h_n(t)\| \leq \varrho(t)$ for $h_n \in H$ and almost every $t \in [0,b]$, then $\alpha(H(t)) \in L^1([0,b],[0,+\infty))$ and*

$$\alpha\bigg(\bigg\{\int_0^t h_n(s)\,ds : n \in \mathbb{N}\bigg\}\bigg) \leq 2\int_0^t \alpha(H(s))\,ds, \quad t \in [0,b].$$

Lemma 2.8 (Mónch [**6**])**.** *Let X be a Banach space, Ω a bounded open subset in X and $0 \in \Omega$. Assume that the operator $F : \Omega \to X$ is continuous and satisfies the following conditions*:

(i) *$x \neq \lambda F x$, for all $\lambda \in (0,1)$, $x \in \partial\Omega$.*
(ii) *D is relatively compact if $D \subset \overline{\mathrm{co}}(0 \cup F(D))$ for any countable set $D \subset \overline{\Omega}$. Then F has a fixed point in $\overline{\Omega}$.*

3. Main result. We define the mild solution of problem (1.1) as follows.

Definition 3.1. A function $x : (-\infty, a] \to X$ is a mild solution of problem (1.1) if $x_0 = \phi$, $x'(0) = \xi$, $x(.)|_{[0,b]} \in PC^1(X)$, and

$$\begin{aligned} x(t) &= C(t)\phi(0) + S(t)\xi + g(t, x_t) - \int_0^t AS(t-s)g(s, x_s)\, ds \\ &\quad + \int_0^t S(t-s) \int_0^s f(r, x_{\rho(r,x_r)}, x'_r)\, dr\, ds \\ &\quad + \sum_{0<t_i<t} C(t-t_i) I_i^1(x_{t_i}, x'_{t_i}) + \sum_{0<t_i<t} S(t-t_i) I_i^2(x_{t_i}, x'_{t_i}). \end{aligned} \tag{3.1}$$

To prove our result, we always assume $\rho : J \times \mathfrak{B} \to (-\infty, a]$ is a continuous function. Let $y : (-\infty, a] \to X$ be the function defined by $y_0 = \phi$ and $y(t) = C(t)(\phi(0)) + S(t)(\xi)$ on $[0, t_1]$. Clearly, $\|y_t\|_{\mathfrak{B}} \leq M_1 := K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}}$, where $\|y\|_b = \sup_{0\leq t\leq b} \|y(t)\|$. Let $\overline{x} = x + y$

$$\|\overline{x}_{\rho(s,\overline{x}_s)}\|_{\mathfrak{B}} \leq M_2^* := (M_a + \widetilde{J^\phi})\|\phi\|_{\mathfrak{B}} + K_a\|y\|_a + K_a\|x\|_a.$$

Taking the supremum of M_1, M_2 as $\overline{M}$ and the supremum of y' as M' we define the space $S(b)$ as

$$S(b) = \{x : (-\infty, b] \to X : x_0 = 0,\ x'(0) = 0,\ x|_J \in PC^1\}$$

endowed with norm $\|u\|_1 = \|u\|_\infty + \|u'\|_\infty$.

The following hypotheses are required to prove our result.

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathbb{R}(\rho^-) = \{\rho(s,\psi) : \rho(s,\psi) \leq 0\}$ into $\mathfrak{B}$, and there exists a continuous bounded function $J^\phi : \mathbb{R}(\rho^-) \to (0, \infty)$ such that $\|\phi_t\|_{\mathfrak{B}} \leq J^\phi(t)\|\phi\|_{\mathfrak{B}}$ for every $t \in \mathbb{R}(\rho^-)$.

(Hf) The function $f : J \times \mathfrak{B} \times \mathfrak{B} \to X$ satisfies the following:

(1) For every $x : (-\infty, a] \to X, x_0 = 0$, $x'(0) = 0$, $x|_J \in PC^1$ the function $f(., x_t, x'_t) : J \to X$ is strongly measurable and $f(t, ., .)$ is continuous for almost every $t \in J$.
(2) There exists an integrable function $p : J \to [0, +\infty)$ such that $\|f(t, u, v)\| \leq p(t)(\|u\|_{\mathfrak{B}} + \|v\|_{\mathfrak{B}})$ for all $t \in J$ and $u, v \in \mathfrak{B}$.
(3) There exists an integrable function $\mu : J \to [0, \infty)$ such that $\alpha(f(t, D_t, D'_t)) \leq \mu(t)(\alpha(D_t) + \alpha(D'_t))$ for almost every $t \in J$, where $D_t = \{v_t : v \in D\}$. $D'_t = \{v'_t : v' \in D'\} \subset \mathfrak{B}(t \in J)$, $V' \subset PC^1$.

(Hg) The function $g : J \times \mathfrak{B}$ satisfies the following.
(1) $g(t, .)$ is continuous for all $t \in J$.
(2) For every bounded $V \subset S(b)$ the set $\{\widetilde{(v_x)}_i(t) : x \in V\}$ is uniformly equicontinuous on $[t_i, t_{i+1}]$ for all $i = 0, \ldots, n$, where $v_x(t) = g(t, x_t)$.
(3) For any bounded set $Q \subset PC^1$, $\alpha(g(t, Q_t)) < c\alpha(Q_t)$, $t \in J$, where c is a positive constant.

(HI) For the maps $I_i^1 : \mathfrak{B} \times \mathfrak{B} \to E$, $I_i^2 : \mathfrak{B} \times \mathfrak{B} \to X$, there exist positive constants c_i^1, c_i^2, d_i^1, d_i^2 such that $\|I_i^j(t, v)\| \leq c_i^j \|v\|_{\mathfrak{B}} + d_i^j$, for all $j = 1, 2$.

(H1) There exists a Banach space $(Y, \|.\|_Y)$ continuously included in X such that $AS(t) \in \mathcal{L}(Y, X)$, for all $t \in J$ and $AS(.)x \in C(J; X)$ for every $x \in Y$. There exist constants $N_Y, \widetilde{N_1}$ such that $\|y\| \leq N_Y \|y\|_Y$, for all $y \in Y$ and $\|AS(t)\|_{\mathcal{L}(Y,X)} \leq \widetilde{N_1}$, for all $t \in J$.

(H2) $\mathcal{R}(C(t) - I)$ is closed and dim Ker$\,(C(t) - I) < \infty$, for all $0 < t \leq b$.

(HJ) (1) For the maps $J_i^1(t, \phi) : J \times \mathfrak{B} \to X$, there exist positive constants c_i^1, c_i^2, d_i^1, d_i^2 such that

$$\|J_i^j(t, v)\| \leq c_i^j \|v\|_{\mathfrak{B}} + d_i^j, \quad \text{for all } j = 1, 2.$$

(2) The maps $J_i^1(., \psi), J_i^2(., \psi)$ are continuous for all $(., \psi) \in (t_i, s_i] \times \mathfrak{B}$, $i = 1, \ldots, n$.

Lemma 3.2 ([**36**]). *If $y : (-\infty, b] \to X$ is a function such that $y_0 = \phi$ and $y|_J \in PC(X)$, then*

$$\|y_{\rho(s,y_s)}\|_{\mathfrak{B}} \leq (M_b + \widetilde{J^\phi})\|\phi\|_{\mathfrak{B}} + K_b \sup\{\|y(\theta)\|;\ \theta \in [0, \max\{0, s\}]\},$$
$$s \in \mathbb{R}(\rho^-) \cup [0, b],$$

where

$$\widetilde{J^\phi} = \sup_{t \in \mathbb{R}(\rho^-)} J^\phi(t), \qquad M_b = \sup_{t \in J} M(t)$$

and

$$K_b = \max_{t \in J} K(t).$$

Lemma 3.3 ([**29**]). *Let condition* (H2) *be satisfied and* $B \subset Y$. *If* B *is bounded in* X *and the set* $\{AS(t)y : t \in [0, b], y \in B\}$ *is relatively compact in* X, *then* B *is relatively compact in* X.

Proof. Since, for $y \in B$,

$$C(t)y - y = A\int_0^t S(s)\, y\, dy = \int_0^t AS(s)y\, dy,$$

it follows from the mean value theorem for the Bochner integral that

$$C(t)y - y \in t \times \overline{\mathrm{co}\,(\mathrm{AS}\,(s)y : 0 \leq s \leq t, y \in B)},$$

where co is the convex hull. Then, by hypothesis (H2), the result follows. $\square$

Lemma 3.4 ([**29**]). *A set* $B \subset PC^1$ *is relatively compact in* PC^1 *if and only if each set* $\widetilde{B_i}$, $i = 1, \ldots, n$, *is relatively compact in* $C^1([t_i, t_{i+1}], X)$.

Theorem 3.5. *If the hypotheses* (Hϕ), (Hf), (Hg), (HI), (H1) *and* (H2) *hold and the cosine family is equicontinuous, then there exists a mild solution of the problem* (1.1).

Proof. Let us define the function $z : (-\infty, 0] \to X$ as $z_0 = x_0'$, $z(t) = x'(t)$, $t \in J$, $S(b) = \{x : (-\infty, b] \to X : x_0 = 0, x'(0) = 0,$

$x(.)|_J \in PC^1\}$. Let $\Gamma = (\Gamma_1, \Gamma_2) : S(b) \times S(b) \to S(b)$ be defined as:

(3.2)
$$\Gamma_1(x,z)(t) = \begin{cases} 0, & t \leq 0; \\ +\int_0^t AS(t-s)g(s, x_s + y_s)\,ds & \\ +\int_0^t S(t-s)\int_0^s f(r, \overline{x}_{rho(r,x_r)}, x'_r + y'_r)\,dr & \\ +\sum_{0<t_i<t} C(t-t_i)I_i^1(x_{t_i} + y_{t_i}, z_{t_i} + y'_{t_i}) & \\ +\sum_{0<t_i<t} S(t-t_i)I_i^2(x_{t_i} + y_{t_i}, z_{t_i} + y'_{t_i}), & t \in J, \end{cases}$$

and $\Gamma_2(x,z)(t) = \Gamma_1(x,z)'(t)$. Therefore,

(3.3)
$$\Gamma_2(x,z)(t) = \begin{cases} 0, & t \leq 0; \\ +\int_0^t AC(t-s)g(s, x_s + y_s)\,ds & \\ +\int_0^t C(t-s)\int_0^s f(r, \overline{x}_{\rho(r,x_r)} + y_r, x'_r + y'_r)\,dr & \\ +\sum_{0<t_i<t} AS(t-t_i)I_i^1(x_{t_i} + y_{t_i}, z_{t_i} + y'_{t_i}) & \\ +\sum_{0<t_i<t} C(t-t_i)I_i^2(x_{t_i} + y_{t_i}, z_{t_i} + y'_{t_i}), & t \in J. \end{cases}$$

Γ is seen to be continuous by the Lebesgue dominated convergence theorem, axioms of phase space and the hypotheses (Hϕ), (Hf), (Hg) and (HI).

Step 1. It is shown that

$$\Omega_0 = \{(x,z) \in S(b) \times S(b) : (x,z) = \lambda\Gamma(x,z) \quad \text{for some } \lambda \in (0,1)\}$$

is bounded. If $t \in J_0 = [0, t_1]$, then

$$\begin{aligned} \|x(t)\| &= \|\Gamma_1(x,z)(t)\| \\ &\leq \widetilde{N_1}\int_0^t \|[c(\|x\|_{\mathfrak{B}} + \overline{M}) + d]\,ds \\ &\quad + \widetilde{N}\int_0^t\int_0^s p(r)(\|x_r\|_{\mathfrak{B}} + \|z_r\|_{\mathfrak{B}} + M' + \overline{M})\,dr\,ds \\ &\leq \overline{M}\int_0^t \left(\widetilde{N_1}c + \widetilde{N}\int_0^s p(r)\,dr\right)ds + \widetilde{N_1}bd \\ &\quad + K_b\int_0^t \left(\widetilde{N_1}c + \widetilde{N}\int_0^s p(r)\,dr\right)(\|x\|_s + \|z\|_s)\,ds \\ &\quad + M'\widetilde{N}\int_0^t\int_0^s p(r)\,dr\,ds, \end{aligned} \tag{3.4}$$

$$\begin{aligned}\|z(t)\| \le \|\Gamma_2(x,z)(t)\| &\le \widetilde{N_2}\int_0^t [c(\|x\|_{\mathfrak{B}} + \overline{M}) + d]\,ds \\ &\quad + N\int_0^t\int_0^s p(r)(\|x_r\|_{\mathfrak{B}} + \|z_r\|_{\mathfrak{B}} + M' + \overline{M})\,dr\,ds \\ &\le \overline{M}\int_0^t \left(\widetilde{N_2}c + N\int_0^s p(r)\,dr\right) ds + \widetilde{N_2}bd \\ &\quad + K_b\int_0^t \left(\widetilde{N_2}c + N\int_0^s p(r)\,dr\right)(\|x\|_s + \|z\|_s)\,ds \\ &\quad + M'\widetilde{N}\int_0^t\int_0^s p(r)\,dr\,ds. \end{aligned} \tag{3.5}$$

Therefore,

$$\begin{aligned}\|x\|_t + \|z\|_t \le (\widetilde{N_1} + \widetilde{N_2})\,bd & \\ + \overline{M}\Bigg[\int_0^t \left[c(\widetilde{N_1} + \widetilde{N_2}) + (N + \widetilde{N})\int_0^s p(r)\,dr\right] ds & \\ + M'(\widetilde{N} + N)\int_0^t \left(\int_0^s p(r)\,dr\right) ds & \\ + \int_0^t \Bigg[(\widetilde{N_1}c + \widetilde{N_2}c)K_b & \\ + (N + \widetilde{N})K_b\int_0^s p(r)d\,r\Bigg](\|x\|_s + \|z\|_s)\Bigg]\,ds. & \end{aligned} \tag{3.6}$$

Since $\|x\|_t + \|z\|_t \in C(J_0, X)$, by Gronwall's lemma, there is a constant $G_0 > 0$ such that $\|x\|_t + \|z\|_t \le G_0$, $t \in J$, and $\|x_t\|_{\mathfrak{B}} \le K_b G_0$ and $\|z_t\|_{\mathfrak{B}} \le K_b G_0$, $t \in J_0$. By condition (HI), it is observed that

$$\begin{aligned}\|I_1^j(x_{t_1} + y_{t_1}, z_{t_1} + y'_{t_1})\|_E &\le c_1^j(2K_bG_0 + \overline{M} + M') + d_1^j := \eta_j \\ \|x(t_1^+)\| &= \|x(t_1) + I_1^1(x_{t_1} + y_{t_1}, z_{t_1} + y'_{t_1})\| \\ &\le G_0 + \eta_1 \\ \|z(t_1^+)\| &= \|z(t_1) + I_1^2(x_{t_1} + y_{t_1}, z_{t_1} + y'_{t_1})\| \\ &\le G_0 + \eta_2. \end{aligned} \tag{3.7}$$

When $t \in J_1 = (t_1, t_2]$, let

$$u(t) = \begin{cases} x(t), & t \in (t_1, t_2]; \\ x(t_1^+), & t = t_1. \end{cases}$$

$$v(t) = \begin{cases} z(t), & t \in (t_1, t_2]; \\ z(t_1^+), & t = t_1. \end{cases}$$

Then $u, v \in C([t_1, t_2], X)$.

$$\begin{aligned}
\|u(t)\| \le & \int_0^t \Big(\widetilde{N_1} c K_b + \widetilde{N} K_b \int_0^s p(r)\,dr\Big)(\|x\|_s + \|z\|_s)\,ds \\
& + \int_0^t \Big[\widetilde{N_1} c M + \widetilde{N} \int_0^s p(r)\,dr(\overline{M} + M')\Big]\,ds + \widetilde{N_1} bd \\
& + N\|I_1^1(x_{t_1} + y_{t_1}, z_{t_1} + y'_{t_1})\| \\
& + \widetilde{N}\|I_1^2(x_{t_1} + y_{t_1}, z_{t_1} + y'_{t_1})\|_E\,ds \\
\le & \int_0^{t_1} \left(2\widetilde{N_1} c K_b G_0 + \widetilde{N} \int_0^s 2K_b G_0 p(r)\,dr\right) ds \\
& + \int_0^t \left[N_1 c\overline{M} + \widetilde{N} \int_0^s p(r)\,dr(\overline{M} + M') ds\right] \\
& + \int_{t_1}^t \left(\widetilde{N_1} c K_b + \widetilde{N} K_b \int_0^s p(r)\,dr\right) \\
& \cdot \big(\sup_{t_1 \le \tau \le s} \|u(\tau)\| + \sup_{t_1 \le \tau \le s} \|v(\tau)\|\big)\,ds
\end{aligned} \tag{3.8}$$

$$\begin{aligned}
\|v(t)\| \le & \int_0^{t_1} \Big(2\widetilde{N_2} c K_b G_0 + N \int_0^s 2K_b G_0 p(r)\,dr\Big)(\overline{M} + M')\,ds + \widetilde{N_2} bd \\
& + \int_0^t \left[\widetilde{N}_2 c\overline{M} + N \int_0^s p(r)\,dr(\overline{M} + M')\,ds\right] \\
& + \int_{t_1}^{t_2} \left(\widetilde{N_2} c K_b + N K_b \int_0^s p(r)\,dr\right) \\
& \cdot \big(\sup_{t_1 \le t \le s} \|u(\tau)\| + \sup_{t_1 \le t \le s} \|v(\tau)\|\big)\,ds.
\end{aligned} \tag{3.9}$$

Therefore, from equation (3.8) and (3.9)

$$\begin{aligned}
\sup_{t_1 \le s \le t} \|u(s)\| + \sup_{t_1 \le s \le t} \|v(s)\| \le & \; e_1 + e_2 \\
& + \int_{t_1}^t \Big[\widetilde{N_1} c + \widetilde{N_2} c + (N + \widetilde{N}) \int_0^s p(r)\,dr\Big] K_b
\end{aligned}$$

$$\times\,(\sup_{t_1\le\tau\le s}\|u(\tau)\|+\sup_{t_1\le\tau\le s}\|v(\tau)\|)\,ds, \tag{3.10}$$

where e_1, e_2 are appropriate constants. Using Gronwall's lemma, there exist constants $G_1 > 0$ such that $\|u(t)\| + \|v(t)\| \le G_1$ for $t \in [t_1, t_2]$. So $\|x(t)\| + \|z(t)\| \le G_1$, for $t \in J_1$. Similarly, let $G = \max\{G_0, G_1, \dots, G_n\}$. Then $\|(x, z)\|_b \le G$ and Ω_0 is bounded.

Let $R > G$ and $\Omega_R = \{(x, z) \in S(b) \times S(b) : \|(x, z)\|_b < R\}$. Since $R > G$, so

$$(x,z)\neq\lambda\Gamma(x,z),\quad \text{for all } (x,z)\in\partial\Omega_R. \tag{3.11}$$

Step 2. Suppose $V \subset \overline{\Omega_R}$ is a countable set and $V \subset \overline{\text{co}}(\{0, 0\} \subset \Gamma(V))$. Let

$$\begin{aligned} V_1&=\{x\in S(b): \text{there exists } z\in S(b), (x,z)\in V\},\\ V_2&=\{z\in S(b): \text{there exists } x\in S(b), (x,z)\in V\}. \end{aligned}$$

$$V\subset V_1\times V_2\subset\overline{\text{co}}\,(\{0\}\cup\Gamma_1(V_1\times V_2))\overline{\text{co}}\,(\{0\}\cup\Gamma_2(V_1\times V_2)). \tag{3.12}$$

From equations (3.2), (3.3) and (Hg) (ii), we get that $\Gamma_j((\widetilde{V_1})_i \times (\widetilde{V_2})_i)$, $(j = 1, 2)$ are equicontinuous on $J_i (i = 0, 1, \dots, n)$. From (3.12), it is implied that $(\widetilde{V_k})_i$ $(k = 1, 2)$ are equicontinuous.

Step 3. Now we prove that V_1 and V_2 are relatively compact. We identify $V_k|_{J_i}$ $(k = 1, 2)$ with $\widetilde{V_i}$, where $V_k|_{J_i}$ is the restriction of V_k on $J_i = (t_i, t_{i+1}]$. When $t \in J_0 = [0, t_1]$, from hypotheses (Hf) (iii), (Hg) (v) and Lemma (2.5), we get that:

$$\begin{aligned} \alpha(V_1(t)) &\le \alpha(\Gamma_1(V_1\times V_2)(t))\\ &\le 2\widetilde{N_1}\int_0^t \alpha(g(s,V_{1s}+y_s))\,ds\\ &\quad+2\widetilde{N}\int_0^t\alpha\int_0^s f(r,V_{1\rho(r,x_r)}+y_{\rho(r,x_r)},V_{2r}+y'_r)\,dr\,ds\\ &\le 2\int_0^t\widetilde{N_1}c\alpha(V_{1s}+y_s)\,ds\\ &\quad+2\int_0^t 2\widetilde{N}\int_0^s\mu(r)\,dr(\alpha(V_{1s}+y_s)+\alpha(V_{2s}+y'_s))\,ds \end{aligned}$$

$$
\begin{aligned}
&\leq 2\int_0^t (\widetilde{N_1}c + 2\widetilde{N}\int_0^s \mu(r)\,dr)(\alpha(V_{1s}+y_s)+\alpha(V_{2s}+y'_s))\\
&\leq 2\int_0^t \Big[\widetilde{N_1}cK_b + 2K_b\widetilde{N}\int_0^s \mu(r)\,dr(\sup_{0\leq\tau\leq s}\alpha(V_1(\tau))\\
&\qquad\qquad + \sup_{0\leq\tau\leq s}\alpha(V_1(\tau)))\Big]\,ds \qquad (3.13)
\end{aligned}
$$

$$
\begin{aligned}
\alpha(V_2(t)) &\leq \alpha(\Gamma_2(V_1\times V_2)(t))\\
&\leq 2\widetilde{N_2}\int_0^t \alpha(g(s,V_{1s}+y_s))\,ds\\
&\quad + 2N\int_0^t \alpha\int_0^s f(r,V_{1\rho(r,x_r)}+y_{\rho(r,x_r)},V_{2r}+y'_r)\,dr\,ds\\
&\leq 2\int_0^t \widetilde{N_2}c\alpha(V_{1s}+y_s)\\
&\quad + 2\int_0^t 2N\int_0^s \mu(r)dr(\alpha(V_{1s}+y_s)+\alpha(V_{2s}+y'_s))\,ds\\
&\leq 2\int_0^t \left(\widetilde{N_2}c + 2N\int_0^s \mu(r)\,dr\right)\\
&\cdot(\alpha(V_{1s}+y_s)+\alpha(V_{2s}+y'_s))\,ds\\
&\leq 2\int_0^t \left[\left(\widetilde{N_1}cK_b + 2K_b\widetilde{N}\int_0^s \mu(r)\,dr\right)(\sup_{0\leq\tau\leq s}\alpha(V_1(\tau))\right.\\
&\qquad\qquad \left. + \sup_{0\leq\tau\leq s}\alpha(V_1(\tau)))\right]\,ds. \qquad (3.14)
\end{aligned}
$$

Since $m_j(t) := \sup_{0\leq s\leq t}\alpha(V_j(s))$ $(j=1,2)$ are continuous and non-decreasing functions on J_0. From equations (3.13) and (3.14), we get that

$$
m_1(t)+m_2(t) \leq \int_0^t K\left(c+\int_0^s \mu(r)\,dr\right)(m_1(s)+m_2(s))\,ds, \qquad (3.15)
$$

where K is an appropriate constant. So, by Gronwall's lemma and (3.15), we see that $\alpha(V_k(t)) = 0$, $(k=1,2)$, $t\in J_0$. By Lemma (2.1) (i), we prove that V_k $(k=1,2)$ is relatively compact in $C(J_0,X)$. Since

$$
\alpha(V_{jt_1}+y_{t_1}) \leq \alpha(V_{jt_1}) \leq K_b \sup_{0\leq s\leq t_1}\alpha(V_j(s)) = 0,
$$

also $I_1^j(.,.)$ $(j = 1, 2)$ is continuous, we can show that

$$\alpha(I_1^1(V_{1t_1} + y_{t_1}, V_{2t_1} + y'_{t_1})) = \alpha(I_1^2(V_{1t_1} + y_{t_1}, V_{2t_1} + y'_{t_1})) = 0.$$

Similarly, when $t \in J_1 = [t_1, t_1]$,

$$\begin{aligned}\alpha(V_1(t)) &\leq \alpha(\Gamma_1(V_1 \times V_2)(t)) \\ &\leq 2\int_{t_1}^t \Big[\widetilde{N_1} c K_b + 2K_b \widetilde{N}\int_0^s \mu(r)\,dr(\sup_{0\leq\tau\leq s}\alpha(V_1(\tau)) \\ &\qquad + \sup_{0\leq\tau\leq s}\alpha(V_1(\tau)))\Big]\,ds \end{aligned} \tag{3.16}$$

$$\begin{aligned}\alpha(V_2(t)) \leq 2\int_{t_1}^t \Big[\Big(\widetilde{N_1} c K_b + 2K_b \widetilde{N}\int_0^s \mu(r)\,dr\Big)(\sup_{t_1\leq\tau\leq s}\alpha(V_1(\tau)) \\ + \sup_{t_1\leq\tau\leq s}\alpha(V_1(\tau)))\Big]\,ds.\end{aligned} \tag{3.17}$$

From equations (3.16) and (3.17) we get that

$$\begin{aligned}&\sup_{t_1\leq s\leq t}\alpha(V_1(s)) + \sup_{t_1\leq s\leq t}\alpha(V_2(s)) \\ &\leq \int_{t_1}^t K\Big(\Big\{c + \int_0^s \mu(r)\,dr\Big\}\Big)(\sup_{t_1\leq s\leq t} V_1(s) + \sup_{t_1\leq s\leq t} V_2(s))\,ds,\end{aligned} \tag{3.18}$$

where K is the appropriate constant. So, by Gronwall's lemma and (3.18), we see that $\alpha(V_k(t)) = 0$, $(k = 1, 2)$, $t \in J_1$. By Lemma (2.1) (i), we prove that $V_k, (k = 1, 2)$ is relatively compact in $C(J_1, X)$. Since

$$\alpha(V_{jt_1} + y_{t_1}) \leq \alpha(V_{jt_1}) \leq K_b \sup_{0\leq s\leq t_1}\alpha(V_j(s)) = 0,$$

also $I_2^j(.,.)$ $(j = 1, 2)$ is continuous, we can show that

$$\alpha(I_2^1(V_{1t_1} + y_{t_1}, V_{2t_1} + y'_{t_1})) = \alpha(I_2^2(V_{1t_1} + y_{t_1}, V_{2t_1} + y'_{t_1})) = 0.$$

Similarly, V_k $(k = 1, 2)$ are relatively compact in $C(J_i, X)$, $(i = 2, 3, \ldots, n)$. Thus, V_k $(k = 1, 2)$ are relatively compact in $S(b)$. Now, by Lemma (2.6), we can prove that Γ has a fixed point in $\overline{\Omega_R}$. If (x, z) is a fixed point of Γ on $S(b)$, then $(x + y)$ is a mild solution of problem (1.1). □

3.1. Non-instantaneous impulsive second order neutral differential equation. In this section, we will find the conditions for the existence of mild solution of problem (1.2). Let us define the mild solution as follows.

Definition 3.6. A function $x : (-\infty, a] \to X$ is a mild solution of problem (1.2) if $x_0 = \phi$, $x'(0) = \xi$, $x(.)|_{[0,a]} \in PC^1(X)$, $x(t) = J_i^1(t, x(t-t_1))$, for all $t \in (t_i, s_i]$, $i = 1, \dots, n$, $x'(t) = J_i^2(t, x(t-t_1))$, $t \in (t_i, s_i]$, $i = 1, 2, \dots, n$, and

$$\begin{aligned}x(t) &= C(t)\phi(0) + S(t)\xi - \int_0^t AS(t-s)g(s, x_s)\,ds \\ &\quad + \int_0^t S(t-s)\int_0^s f(r, x_{\rho(s,x_r)}, x'(r))\,dr\,ds, \quad t \in [0, t_1] \\ x(t) &= C(t-s_i)J_i^1(s_i, x(t-t_1)) \\ &\quad + S(t-s_i)J_i^2(s_i, x(t-t_1)) - \int_{s_i}^t AS(t-s)g(s, x_s)\,ds \\ &\quad + \int_{s_i}^t S(t-s)\int_0^s f(s, x_{\rho(r,x_r)}, x'(r))\,dr\,ds, \\ &\qquad \text{for } t \in [s_i, t_{i+1}],\ i = 1, \dots, n. \end{aligned} \tag{3.19}$$

Let $y : (-\infty, a] \to X$ be the function defined by $y_0 = \phi$ and

$$y(t) = C(t)(\phi(0)) + S(t)(\xi) \quad \text{on } [0, t_1].$$

Clearly,

$$\|y_t\|_{\mathfrak{B}} \le K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}}$$

where

$$\|y\|_b = \sup_{0\le t\le b}\|y(t)\|,$$

since

$$S(b) = \{x : (-\infty, b] \longrightarrow X : x_0 = 0, x'(0) = 0,\ x(.)|_J \in PC^1\}.$$

Therefore, $\overline{x} = x + y$ is a mild solution of (1.2).

Theorem 3.7. *If the hypotheses* (Hϕ), (Hf), (Hg), (HJ), (H1) *and* (H2) *hold and the cosine family is equicontinuous, then there exists a mild solution of problem* (1.1).

Proof. Let us define the function $z : (-\infty, 0] \to X$ as

$$z_0 = x_0', \qquad z(t) = x'(t), \quad t \in J.$$

Let

$$\Gamma = (\Gamma_1, \Gamma_2) : S(b) \times S(b) \longrightarrow S(b)$$

be defined as:

(3.20)

$$\Gamma_1(x,z)(t) = \begin{cases} 0, & t \le 0; \\ -\displaystyle\int_0^t AS(t-s)g(s, x_s + y_s)\,ds & \\ +\displaystyle\int_0^t S(t-s)\int_0^s f(r, \overline{x}_{\rho(r,x_r)}, x_r' + y_r')\,dr, & t \in J_1 = [0, t_1]. \end{cases}$$

and $\Gamma_2(x,z)(t) = \Gamma_1(x,z)'(t)$. Therefore,

(3.21)

$$\Gamma_2(x,z)(t) = \begin{cases} 0 & t \le 0 \\ -\displaystyle\int_0^t AC(t-s)g(s, x_s + y_s)\,ds & \\ +\displaystyle\int_0^t C(t-s)\int_0^s f(r, \overline{x}_{\rho(r,x_r)}, x_r' + y_r')\,dr\,ds, & \\ & t \in J_1 = [0, t_1]. \end{cases}$$

(3.22)

$$\Gamma_1(x,z)(t) = \begin{cases} J_i^1(t, x(t-t_1) & t \in (t_i, s_i], \\ C(t-s_i)J_i^1(s_i, x(t-t_1)) & \\ -S(t-s_i)J_i^2(s_i, x(t-t_1)) & \\ -\displaystyle\int_{s_i}^t AS(t-s)g(s, x_s + y_s)\,ds & \\ +\displaystyle\int_{s_i}^t S(t-s)\int_0^s f(r, \overline{x}_{\rho(r,x_r)}, x_r' + y_r')\,dr\,ds, & \\ & t \in J_i = (s_i, t_{i+1}]. \end{cases}$$

and $\Gamma_2(x,z)(t) = \Gamma_1(x,z)'(t)$. Therefore,

$$\Gamma_2(x,z)(t) = \begin{cases} J_i^2(t, x(t-t_1) & t \in (t_i, s_i] \\ AS(t-s_i)J_i^1(s_i, x(t-t_1)) & \\ -C(t-s_i)J_i^2(s_i, x(t-t_1)) & \\ -\displaystyle\int_{s_i}^t AC(t-s)g(s, x_s + y_s)\,ds & \\ +\displaystyle\int_{s_i}^t C(t-s)\int_0^s f(r, \overline{x}_{\rho(r,x_r)}, x'_r + y'_r)\,dr\,ds, & \\ & t \in J_i = (s_i, t_{i+1}]. \end{cases} \tag{3.23}$$

It can be easily proved that Γ is continuous by the Lebesgue dominated convergence theorem, axioms of phase space and the hypotheses (Hϕ), (Hf), (Hg) and (HJ).

Step 1. We show that

$$\Omega_0 = \{(x,z) \in S(b) \times S(b) : (x,z) = \lambda\Gamma(x,z) \quad \text{for some } \lambda \in (0,1)\}$$

is bounded. When $t \in J_0 = [0, t_1]$,

$$\begin{aligned} \|x(t)\| &\le \|\Gamma_1(x,z)(t)\| \\ &\le \widetilde{N_1}\int_0^t \|[c(\|x\|_{\mathfrak{B}} + \overline{M}) + d]\,ds \\ &\quad + \widetilde{N}\int_0^t\int_0^s p(r)(\|x_r\|_{\mathfrak{B}} + \|z_r\|_{\mathfrak{B}} + M' + \overline{M})\,dr\,ds \\ &\le \overline{M}\int_0^t \left(\widetilde{N_1}c + \widetilde{N}\int_0^s p(r)\,dr\right)ds + \widetilde{N_1}bd \\ &\quad + K_b\int_0^t \left(\widetilde{N_1}c + \widetilde{N}\int_0^s p(r)\,dr\right)(\|x\|_s + \|z\|_s)\,ds \\ &\quad + M'\widetilde{N}\int_0^t\int_0^s p(r)\,dr\,ds, \end{aligned} \tag{3.24}$$

$$\begin{aligned} \|z(t)\| &\le \|\Gamma_2(x,z)(t)\| \\ &\le \widetilde{N_2}\int_0^t \|[c(\|x\|_{\mathfrak{B}} + \overline{M}) + d]\,ds \\ &\quad + N\int_0^t\int_0^s p(r)(\|x_r\|_{\mathfrak{B}} + \|z_r\|_{\mathfrak{B}} + M' + \overline{M})\,dr\,ds \end{aligned}$$

$$
\begin{aligned}
&\leq \overline{M}\int_0^t \left(\widetilde{N_2}c + N\int_0^s p(r)\,dr\right) ds + \widetilde{N_2}bd \\
&\quad + K_b\int_0^t \left(\widetilde{N_2}c + N\int_0^s p(r)\,dr\right)(\|x\|_s + \|z\|_s)\,ds \\
&\quad + M'\widetilde{N}\int_0^t\int_0^s p(r)\,dr\,ds. \qquad (3.25)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|x\|_t + \|z\|_t \leq{}& [(\widetilde{N_1} + \widetilde{N_2})bd \\
&+ \overline{M}\int_0^t \left[c(\widetilde{N_1} + \widetilde{N_2}) + (N + \widetilde{N})\int_0^s p(r)\,dr\right] ds \\
&+ M'(\widetilde{N} + N)\int_0^t \left(\int_0^s p(r)\,dr\right) ds \\
&+ \int_0^t \Big[(\widetilde{N_1}c + \widetilde{N_2}c)K_b \\
&+ (N + \widetilde{N})K_b\int_0^s p(r)\,dr\Big](\|x\|_s + \|z\|_s)]\,ds. \qquad (3.26)
\end{aligned}
$$

Since $\|x\|_t + \|z\|_t \in C(J_0, X)$, by Gronwall's lemma, there is a constant $G_0 > 0$ such that

$$\|x\|_t + \|z\|_t \leq G_0, \quad t \in J$$

and

$$\|x_t\|_{\mathfrak{B}} \leq K_b G_0 \quad \text{and} \quad \|z_t\|_{\mathfrak{B}} \leq K_b G_0, \quad t \in J_0.$$

By condition (HJ), it is observed that, for $t \in [t_1, s_1)$,

$$\|J_1^j(t, x(t - t_1))\|_E \leq c_1^j(2K_bG_0 + \overline{M}) + d_1^j := \eta_1^j. \qquad (3.27)$$

When $t \in J_2 = [s_1, t_2]$,

$$
\begin{aligned}
\|x(t)\| &\leq \|\Gamma_1(x,z)(t)\| \\
&\leq N[c_i^1(\|x_{s_i}\|_{\mathfrak{B}}) + d_i^1] + \widetilde{N}[c_i^2\|x_{s_i}\|_{\mathfrak{B}} + d_i^2] \\
&\quad + \widetilde{N_1}\int_{s_i}^t [c(\|x(s)\|_{\mathfrak{B}} + \overline{M}) + d]\,ds \\
&\quad + \widetilde{N}\int_{s_i}^t\int_0^s p(r)(\|x_r\|_{\mathfrak{B}} + \|z_r\|_{\mathfrak{B}} + M' + \overline{M})\,dr\,ds
\end{aligned}
$$

$$
\begin{aligned}
&\leq \overline{M} \int_{s_i}^{t} \Big(\widetilde{N_1} c + \widetilde{N} \int_0^s p(r)\,dr \Big)\,ds \\
&\quad + M' \widetilde{N} \int_{s_i}^{t} \int_0^s p(r)\,dr\,ds \\
&\quad + K_b \int_{s_i}^{t} \Big(\widetilde{N_1} c + \widetilde{N} \int_0^s p(r)\,dr \Big) (\|x\|_s + \|z\|_s)\,ds \\
&\quad + [N(c_i^1 K_b) + \widetilde{N}(c_i^2 K_b)](\|x\| + \|z\|) \\
&\quad + \widetilde{N_1} bd + N(d_i^1) + \widetilde{N}(d_i^2). \qquad (3.28)
\end{aligned}
$$

$$
\begin{aligned}
\|z(t)\| &\leq \|\Gamma_2(x,z)(t)\| \\
&\leq \widetilde{N_1} \|[c_i^1 \|x_{s_i}\|_{\mathfrak{B}} + d_i^1\| \\
&\quad + N[c_i^2 \|x_{s_i}\|_{\mathfrak{B}} + d_i^2 \\
&\quad + \widetilde{N_2} \int_{s_i}^{t} \|[c(\|x(s)\|_{\mathfrak{B}} + \overline{M}) + d]\,ds \\
&\quad + N \int_{s_i}^{t} \int_0^s p(r)(\|x_r\|_{\mathfrak{B}} + \|z_r\|_{\mathfrak{B}} + M' + \overline{M})\,dr\,ds \\
&\leq \overline{M} \int_0^t \Big(\widetilde{N_2} c + N \int_0^s p(r)\,dr \Big)\,ds \\
&\quad + M' \widetilde{N} \int_0^t \int_0^s p(r)\,dr\,ds \\
&\quad + K_b \int_0^t \Big(\widetilde{N_2} c + N \int_0^s p(r)\,dr \Big) (\|x\|_s + \|z\|_s)\,ds \\
&\quad + [\widetilde{N_2}(c_i^1 K_b) + N(c_i^2 K_b)](\|x\| + \|z\|) \\
&\quad + \widetilde{N_2} bd + N(d_i^1) + N(d_i^2) + \widetilde{N_2} bd. \qquad (3.29)
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\|x\|_t + \|z\|_t \leq \Big\{ K &+ \overline{M} \int_0^t \Big[c(\widetilde{N_1} + \widetilde{N_2}) + (N + \widetilde{N}) \int_0^s p(r)\,dr \Big]\,ds \\
&+ M'(\widetilde{N} + N) \int_0^t \Big(\int_0^s p(r)\,dr \Big)\,ds + \int_0^t \Big[(\widetilde{N_1} c + \widetilde{N_2} c) K_b \\
&+ (N + \widetilde{N}) K_b \int_0^s p(r)\,dr \Big] (\|x\|_s + \|z\|_s) \Big]\,ds \Big\}, \qquad (3.30)
\end{aligned}
$$

where K is an appropriate constant. Since $\|x\|_t + \|z\|_t \in C(J_1, X)$, by Gronwall's lemma, there is a constant $G_1 > 0$ such that

$$\|x\|_t + \|z\|_t \leq G_1, \quad t \in J,$$

$$\|x_t\|_{\mathfrak{B}} \leq K_b G_0$$

and

$$\|z_t\|_{\mathfrak{B}} \leq K_b G_1, \quad t \in J_0.$$

By condition (HJ), it is observed that, for $t \in [t_2, s_2)$,

$$\|J_2^j(t, x(t-t_1))\|_E \leq c_2^j(2K_b G_1 + \overline{M}) + d_2^j := \eta_2^j, \quad j = 1, 2. \tag{3.31}$$

Similarly, let

$$G = \max\{G_0, \eta_1, G_1, \eta_2 \cdots G_n\}.$$

Then $\|(x, z)\|_b \leq G$ and Ω_0 is bounded.

Let $R > G$ and

$$\Omega_R = \{(x, z) \in S(b) \times S(b) : \|(x, z)\|_b < R\}.$$

Since $R > G$,

$$(x, z) \neq \lambda\Gamma(x, z) \quad \text{for all } (x, z) \in \partial\Omega_R. \tag{3.32}$$

Step 2. Suppose $V \subset \overline{\Omega_R}$ be a countable set and $V \subset \overline{\text{co}}(\{0, 0\} \subset \Gamma(V))$. Let

$$\begin{aligned} V_1 &= \{x \in S(b) : \text{ there exists } z \in S(b),\ (x, z) \in V\}, \\ V_2 &= \{z \in S(b) : \text{ there exists } x \in S(b), (x, z) \in V\}. \end{aligned}$$

$$\begin{aligned} V \subset V_1 \times V_2 &\subset \overline{\text{co}}(\{0\} \cup \Gamma_1(V_1 \times V_2)) \\ &\times \overline{\text{co}}(\{0\} \cup \Gamma_2(V_1 \times V_2)). \end{aligned} \tag{3.33}$$

From equations (3.22), (3.23) and (Hg) (2), we get that $\Gamma_j((\widetilde{V_1})_i \times (\widetilde{V_2})_i)$, $(j = 1, 2)$ are equicontinuous on $J_i (i = 0, 1, \ldots, n)$. From (3.33), it is seen that $(\widetilde{V_k})_i (k = 1, 2)$ are equicontinuous.

Next, we prove that V_1 and V_2 are relatively compact. We identify $V_k|_{J_i}$ $(k = 1, 2)$ with $\widetilde{V_i}$ where $V_k|_{J_i}$ is the restriction of V_k on

$J_i = (s_i, t_{i+1}]$. When $t \in J_0 = [0, t_1]$, from hypotheses (Hf)(3), (Hg)(5)and Lemma (2.5), we get that:

$$\begin{aligned}
\alpha(V_1(t)) &\leq \alpha(\Gamma_1(V_1 \times V_2)(t)) \\
&\leq 2\widetilde{N_1} \int_0^t \alpha(g(s, V_{1s} + y_s))\, ds \\
&\quad + 2\widetilde{N} \int_0^t \alpha \int_0^s f(r, V_{1\rho(r,x_r)} + y_{\rho(r,x_r)}, V_{2r} + y_r')\, dr\, ds \\
&\leq 2 \int_0^t \widetilde{N_1} c\alpha(V_{1s} + y_s)\, ds \\
&\quad + 2 \int_0^t 2\widetilde{N} \int_0^s \mu(r)\, dr(\alpha(V_{1s} + y_s) + \alpha(V_{2s} + y_s'))\, ds \\
&\leq 2 \int_0^t \Big(\widetilde{N_1} c + 2\widetilde{N} \int_0^s \mu(r)\, dr\Big)(\alpha(V_{1s} + y_s) + \alpha(V_{2s} + y_s')) \\
&\leq 2 \int_0^t \Big[\widetilde{N_1} cK_b + 2K_b\widetilde{N} \int_0^s \mu(r)\, dr(\sup_{0 \leq \tau \leq s} \alpha(V_1(\tau)) \\
&\qquad + \sup_{0 \leq \tau \leq s} \alpha(V_1(\tau)))\Big]\, ds
\end{aligned} \tag{3.34}$$

and

$$\begin{aligned}
\alpha(V_2(t)) &\leq \alpha(\Gamma_2(V_1 \times V_2)(t)) \\
&\leq 2\widetilde{N_2} \int_0^t \alpha(g(s, V_{1s} + y_s))\, ds \\
&\quad + 2N \int_0^t \alpha \int_0^s f(r, V_{1\rho(r,x_r)} + y_{\rho(r,x_r)}, V_{2r} + y_r')\, dr\, ds \\
&\leq 2 \int_0^t \widetilde{N_2} c\alpha(V_{1s} + y_s) \\
&\quad + 2 \int_0^t 2N \int_0^s \mu(r)\, dr(\alpha(V_{1s} + y_s) + \alpha(V_{2s} + y_s'))\, ds \\
&\leq 2 \int_0^t \Big(\widetilde{N_2} c + 2N \int_0^s \mu(r)\, dr\Big)(\alpha(V_{1s} + y_s) + \alpha(V_{2s} + y_s'))\, ds \\
&\leq 2 \int_0^t \Big[\Big(\widetilde{N_1} cK_b + 2K_b\widetilde{N} \int_0^s \mu(r)\, dr\Big)(\sup_{0 \leq \tau \leq s} \alpha(V_1(\tau))
\end{aligned}$$

$$+ \sup_{0\le\tau\le s} \alpha(V_1(\tau)))\Big]\, ds, \tag{3.35}$$

since $m_j(t) := \sup_{0\le s\le t} \alpha(V_j(s))$ $(j = 1, 2)$ are continuous and non-decreasing functions on J_0. From equations (3.34) and (3.35) we get that

$$m_1(t) + m_2(t) \le \int_0^t K(c + \int_0^s \mu(r)dr)(m_1(s) + m_2(s)0)\, ds, \tag{3.36}$$

where K is an appropriate constant. So, by Gronwall's lemma and (3.36), we see that $\alpha(V_k(t)) = 0$ $(k = 1, 2)$, $t \in J_0$. By Lemma (2.1) (i), we prove that $V_k, (k = 1, 2)$ is relatively compact in $C(J_0, X)$. Since

$$\alpha(V_{jt_1} + y_{t_1}) \le \alpha(V_{jt_1}) \le K_b \sup_{0\le s\le t_1} \alpha(V_j(s)) = 0,$$

also $J_1^j(.,.)$ $(j = 1, 2)$ is continuous, we can show that

$$\alpha(J_1^1(V_{1t_1} + y_{t_1})) = \alpha(J_1^2(V_{1t_1} + y_{t_1})) = 0.$$

Similarly, when $t \in J_1 = [t_1, s_1]$,

$$\begin{aligned}
\alpha(V_1(t)) &\le \alpha(\Gamma_1(V_1 \times V_2)(t)) \\
&\le 2\int_{t_1}^t \Big[\widetilde{N_1} c K_b + 2K_b \widetilde{N} \int_0^s \mu(r)\, dr(\sup_{0\le\tau\le s} \alpha(V_1(\tau)) \\
&\qquad\qquad + \sup_{0\le\tau\le s} \alpha(V_1(\tau)))\Big]\, ds \\
&\quad + \int_0^t cK_b \sup_{t_1\le s\le t} \alpha(V_1(s))\, ds,
\end{aligned} \tag{3.37}$$

$$\begin{aligned}
\alpha(V_2(t)) \le 2\int_{t_1}^t \Big[\Big(\widetilde{N_1} c K_b + 2K_b \widetilde{N} \int_0^s \mu(r)\, dr\Big)(\sup_{t_1\le\tau\le s} \alpha(V_1(\tau)) \\
+ \sup_{t_1\le\tau\le s} \alpha(V_1(\tau)))\Big]\, ds + cK_b \sup_{t_1\le s\le t} \alpha(V_2(s))\, ds.
\end{aligned} \tag{3.38}$$

From equations (3.37) and (3.38) we get that

$$\sup_{t_1\le s\le t} \alpha(V_1(s)) + \sup_{t_1\le s\le t} \alpha(V_2(s))$$

(3.39)
$$\leq \int_{t_1}^{t} \Big(K \Big\{ c + \int_0^s \mu(r)\, dr \Big\} + cK_b \Big) (\sup_{t_1 \leq s \leq t} V_1(s) + \sup_{t_1 \leq s \leq t} V_2(s))\, ds,$$

where K is the appropriate constant. So, by Gronwall's lemma and (3.39) we see that $\alpha(V_k(t)) = 0$, $(k = 1,2)$, $t \in J_1$. By Lemma (2.1) (i), we prove that V_k $(k = 1,2)$ is relatively compact in $C(J_1, X)$. Since

$$\alpha(V_{jt_1} + y_{t_1}) \leq \alpha(V_{jt_1}) \leq K_b \sup_{0 \leq s \leq t_1} \alpha(V_j(s)) = 0,$$

also $J_2^j(.,.)$ $(j = 1,2)$ is continuous and we can show that

$$\alpha(J_2^1(V_{1t_1} + y_{t_1})) = \alpha(J_2^2(V_{1t_1} + y_{t_1})) = 0.$$

Similarly V_k $(k = 1,2)$ are relatively compact in $C(J_i, X)$ $(i = 2,3,\ldots,n)$. Thus, V_k $(k = 1,2)$ are relatively compact in $S(b)$. Now by Lemma (2.6), we can prove that Γ has fixed point in $\overline{\Omega_R}$. If (x,z) is a fixed point of Γ on $S(b)$ then $(x+y)$ is a mild solution of problem (1.2). $\square$

Remark 3.8. We can also apply the above methodology to the following:

$$\begin{aligned} \frac{d^2}{dt^2} x(t) &= A\Big(x(t) - \int_0^t g(\tau, x_\tau)\, d\tau \Big) \\ &\qquad + \int_0^t f(t, x_{\rho(t,x_t)})\, dt, \quad t \in [0,b],\ t \neq t_i, \\ &\qquad\qquad i = 1, \ldots, n \\ x_0 &= \phi \in \mathfrak{B}, \\ x'(0) &= \xi \in X, \\ \Delta x(t_i) &= I_i^1(x_{t_i}), \quad i = 1,2,\ldots,n \\ \Delta x'(t) &= I_i^2(x_{t_i}), \quad i = 1,2,\ldots,n. \end{aligned} \tag{3.40}$$

Here $0 = t_0 < t_1 \leq t_2 \cdots < t_n \leq t_{n+1} = b$ are prefixed numbers.

$$\frac{d^2}{dt^2} x(t) = A\Big(x(t) - \int_0^t g(\tau, x_\tau)\, d\tau \Big) + \int_0^t f(t, x_{\rho(t,x_t)})\, dt,$$
$$t \in (s_i, t_{i+1}], \quad i = 0, \ldots, n$$

$$
\begin{aligned}
x_0 &= \phi \in \mathfrak{B}, \\
x'(0) &= \xi \in X, \\
x(t) &= J_i^1(t, x(t-t_1)), \quad t \in (t_i, s_i],\ i = 1, 2, \dots, n \\
x'(t) &= J_i^2(t, x(t-t_1)), \quad t \in (t_i, s_i],\ i = 1, 2, \dots, n.
\end{aligned} \tag{3.41}
$$

Here $0 = t_0 = s_0 < t_1 \le s_1 \le t_2 \cdots < t_n \le s_n \le t_{n+1} = b$ are prefixed numbers. The mild solution of (3.40) is defined as

Definition 3.9. A function $x : (-\infty, a] \to X$ is a mild solution of the problem (3.40) if $x_0 = \phi$, $x'(0) = \xi$, $x(.)|_{[0,b]} \in PC^1(X)$, and

$$
\begin{aligned}
x(t) = C(t)\phi(0) &+ S(t)\xi \\
&+ \int_0^t g(s, x_s)\, ds - \int_0^t C(t-s) g(s, x_s)\, ds \\
&+ \int_0^t S(t-s) \int_0^s f(r, x_{\rho(r, x_r)})\, dr\, ds \\
&+ \sum_{0<t_i<t} C(t-t_i) I_i^1(x_{t_i}) + \sum_{0<t_i<t} S(t-t_i) I_i^2(x_{t_i}).
\end{aligned} \tag{3.42}
$$

We define $S(b) = \{x : (-\infty, b] \to X : x_0 = 0, x'(0) = 0,\ x(.)|_J \in PC^1\}$. We define

$$\Gamma =: S(b) \times S(b) \to S(b)$$

as

$$
\Gamma(x)(t) = \begin{cases}
0, & t \le 0; \\
+ \int_0^t C(t-s) g(s, x_s + y_s)\, ds & \\
+ \int_0^t g(s, x_s + y_s)\, ds & \\
+ \int_0^t S(t-s) \int_0^s f(r, \overline{x}_{rho(r, x_r)})\, dr & \\
+ \sum_{0<t_i<t} C(t-t_i) I_i^1(x_{t_i} + y_{t_i}) & \\
+ \sum_{0<t_i<t} S(t-t_i) I_i^2(x_{t_i} + y_{t_i}), & t \in J,
\end{cases} \tag{3.43}
$$

and proceed as in the first case of Theorem 3.5.

Definition 3.10. A function $x : (-\infty, a] \to X$ is a mild solution of problem (3.41) if $x_0 = \phi$, $x'(0) = \xi$, $x(.)|_{[0,a]} \in PC^1(X)$, $x(t) = J_i^1(t, x(t-t_1))$ for all $t \in (t_i, s_i]$, $i = 1, \dots, n$, $x'(t) = J_i^2(t, x(t-t_1))$, $t \in (t_i, s_i]$, $i = 1, 2, \dots, n$, and

(3.44)
$$\begin{aligned} x(t) &= C(t)\phi(0) + S(t)\xi - \int_0^t C(t-s) g(s, x_s)\, ds + \int_0^t g(s, x_s)\, ds \\ &\quad + \int_0^t S(t-s) \int_0^s f(r, x_{\rho(s,x_r)})\, dr\, ds, \quad t \in [0, t_1], \\ x(t) &= C(t-s_i) J_i^1(s_i, x(t-t_1)) + S(t-s_i) J_i^2(s_i, x(t-t_1)) \\ &\quad - \int_{s_i}^t C(t-s) g(s, x_s)\, ds + \int_0^t g(s, x_s)\, ds \\ &\quad + \int_{s_i}^t S(t-s) \int_0^s f(s, x_{\rho(r,x_r)})\, dr\, ds, \end{aligned}$$

(3.45)
$$\text{for } t \in [s_i, t_{i+1}],\ i = 1, \dots, n.$$

We define $\Gamma : S(b) \times S(b) \to S(b)$ as

(3.46)
$$\Gamma(x)(t) = \begin{cases} 0 & t \le 0; \\ -\int_0^t C(t-s) g(s, x_s + y_s)\, ds + \int_0^t g(s, x_s)\, ds & \\ + \int_0^t S(t-s) \int_0^s f(r, \overline{x}_{\rho(r,x_r)})\, dr & t \in J_1 = [0, t_1]. \end{cases}$$

(3.47)
$$\Gamma(x)(t) = \begin{cases} J_i^1(t, x(t-t_1) & t \in (t_i, s_i], \\ C(t-s_i) J_i^1(s_i, x(t-t_1)) & \\ -S(t-s_i) J_i^2(s_i, x(t-t_1)) & \\ -\int_{s_i}^{t_i} C(t-s) g(s, x_s + y_s)\, ds & \\ +\int_{s_i}^{t_i} g(s, x_s)\, ds & \\ +\int_0^t C(t-s) \int_0^s f(r, \overline{x}_{\rho(r,x_r)}, x_r' + y_r')\, dr\, ds, & \\ & t \in J_1 = [0, t_1], \end{cases}$$

and proceed as in Theorem (3.5).

4. Examples. In this section, we discuss a partial differential equation applying the abstract results of this paper. We discuss the partial differential equation in two examples. In Example 4.1, the instantaneous impulsive differential system is studied, while in Example 4.2, the non-instantaneous impulsive differential system is studied. As a result, the dynamics and solutions of these two examples will be different as we can perceive from equations (3.9) and (3.19). In this application, $\mathfrak{B}$ is the phase space $C_0 \times L^2(h, X)$, see **[31]**.

Example 4.1. Consider the second order neutral differential equation with instantaneous impulses

$$\begin{aligned}
\frac{\partial^2}{\partial t^2}x(t,\sigma) &= (i\triangle - iV(\sigma))\Big(x(t,\sigma) - \int_{-\infty}^{t}\int_0^{\pi} x(s,\sigma-\upsilon)\,ds\Big)\\
&\quad + \int_{-\infty}^{t}\Bigg(a(x) + B(x(s,\sigma - h(x(s,\sigma))))\\
&\qquad \cdot \sin\left(\frac{t}{\epsilon}\right)\Bigg)\,ds, \quad t \in [0,a],\ \sigma \in [0,\pi],\\
x(t,0) &= x(t,\pi) = 0, \quad t \in [0,a],\\
x(s,\sigma) &= \phi(s,\sigma), \quad -\infty \le s \le 0,\ 0 \le \sigma \le \pi,\\
\frac{\partial}{\partial t}x(0,\sigma) &= \xi(\sigma), \quad 0 \le \sigma \le \pi,\\
\Delta x(t_i)(\sigma) &= \int_{\infty}^{t_i} n_i^1(t_i - s)x(s,\sigma)\,ds, \quad i = 1,\ldots,n\\
\Delta x'(t_i)(\sigma) &= \int_{\infty}^{t_i} n_i^1(t_i - s)x(s,\sigma)\,ds, \quad i = 1,\ldots,n, \qquad (4.1)
\end{aligned}$$

where $\phi \in H^1([0,\pi])$, $\xi \in X$ and $0 = t_0 = s_0 < t_1 \le s_1 \le t_2, \ldots, t_n \le s_n \le t_{n+1} = a$. Here,

$$\begin{aligned}
X &= L^2([0,\pi]),\\
\mathfrak{B} &= PC_0 \times L^2(\rho, X),\\
A &\subset D(A) \subset X \longrightarrow X
\end{aligned}$$

is the map defined by $A = (i\triangle - iV))$ with domain $D(A) = H^2 \cap H_0^1$. It is well known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in R}$ on X. Also, A has a discrete spectrum, and the following properties hold:

(C1) $A\phi = -\sum_{n=1}^{\infty} \lambda_n^2 \langle \phi, z_n \rangle z_n$, where $\phi \in D(A)$, λ_n, z_n and $n \in \mathfrak{N}$ are eigenvalues and eigenvectors of A.

(C2) $C(t)\phi = \sum_{n=1}^{\infty} \cos(\lambda_n t)\langle \phi, z_n \rangle z_n$ and

$$S(t)\phi = \sum_{n=1}^{\infty} \frac{\sin(\lambda_n t)}{n} < \phi, \quad z_n > z_n,$$

for $\phi \in X$.

By defining the maps $\rho, g, f : [0,a] \times \mathfrak{B} \times X \to X$ by

$$\begin{aligned}
\rho(t,\sigma) &:= \sigma - h(x(s,\sigma)) \\
g(\psi)(\sigma) &:= \int_{-\infty}^{t} \int_0^{\pi} x(s, \sigma - \upsilon)\, ds), \\
f(\psi)(\sigma) &:= \int_{-\infty}^{t} \left(a(x) + B(x(s, \sigma - h(x(s,\sigma)))) \sin\left(\frac{t}{\epsilon}\right)\right),
\end{aligned}$$

the system (4.2) can be transformed into system (1.1). Assume that the functions

$$\rho_i : \mathbb{R} \longrightarrow [0,\infty), \qquad m : \mathbb{R} \longrightarrow \mathbb{R}$$

are piecewise continuous. $g(t,.)$, I_i $(i = 1, \ldots, n)$ f are bounded linear operators. Also, we can prove that g is $D(A)$-valued. Thus, we take $Y = D(A)$. Therefore, if $\iota : Y \to X$ is the inclusion, then $t \to AS(t)$ is uniformly continuous into $L(Y,X)$ and

$$\|AS(t)\|_{L(Y,X)} \leq 1 \quad \text{for } t \in [0,a].$$

Hence, by assumptions (Hϕ), (Hf), (Hg), (HI), (H1), (H2) and Theorem (3.4), it is ensured that a mild solution to problem (4.2) exists.

Example 4.2. Consider the second order neutral differential equation with non-instantaneous impulses

$$\begin{aligned}
&\frac{\partial^2}{\partial t^2}\left(x(t,\sigma) - \int_{-\infty}^{t} \int_0^{\pi} x(s, \sigma - \upsilon)\, ds\right) \\
&= (i\triangle - iV(\sigma))x(t,\sigma) \\
&\quad + \int_{-\infty}^{t} \left(a(x) + B(x(s, \sigma - h(x(s,\sigma)))) \sin\left(\frac{t}{\epsilon}\right)\right) ds, \\
&\qquad t \in [0,a], \ \sigma \in [0,\pi],
\end{aligned}$$

$$
\begin{aligned}
x(t,0) &= x(t,\pi) = 0, \quad t \in [0,a], \\
x(s,\sigma) &= \phi(s,\sigma), \quad -\infty \le s \le 0,\ 0 \le \sigma \le \pi, \\
\frac{\partial}{\partial t} x(0,\sigma) &= \xi(\sigma), \quad 0 \le \sigma \le \pi, \\
x(t)(\sigma) &= \int_{t_i}^{s_i} n_i^1(t-t_1)x(s,\sigma)\,ds, \quad t \in [s_i,t_i],\ i = 1,\dots,n \\
x'(t)(\sigma) &= \int_{t_i}^{s_i} n_i^1(t-t_1)x(s,\sigma)\,ds, \quad t \in [s_i,t_i],\ i = 1,\dots,n,
\end{aligned}
\tag{4.2}
$$

where $\phi \in H^1([0,\pi])$, $\xi \in X$,

$$0 = t_0 = s_0 < t_1 \le s_1 \le t_2, \dots, t_n \le s_n \le t_{n+1} = a.$$

Here, $X = L^2([0,\pi])$, $\mathfrak{B} = PC_0 \times L^2(\rho, X)$, $A \subset D(A) \subset X \to X$ is the map defined by $A = (i\triangle - iV))$ with domain $D(A) = H^2 \cap H_0^1$. It is well known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in R}$ on X. Also, A has a discrete spectrum, and the following properties hold:

(C1)
$$A\phi = -\sum_{n=1}^{\infty} \lambda_n^2 \langle \phi, z_n\rangle z_n,$$

where $\phi \in D(A)$, λ_n, z_n, $n \in \mathfrak{N}$ are eigenvalues and eigenvectors of A.

(C2)
$$C(t)\phi = \sum_{n=1}^{\infty} \cos(\lambda_n t)\langle \phi, z_n\rangle z_n$$

and

$$S(t)\phi = \sum_{n=1}^{\infty} \frac{\sin(\lambda_n t)}{n} < \phi, z_n > z_n, \quad \text{for } \phi \in X.$$

By defining maps $\rho,\ g,\ f : [0,a] \times \mathfrak{B} \times X \to X$ as in Example 4.1, system (4.2) can be transformed into system (1.2). Assume that the functions

$$\rho_i : \mathbb{R} \longrightarrow [0,\infty), \qquad m : \mathbb{R} \longrightarrow \mathbb{R}$$

are piecewise continuous. Hence, by assumptions (Hϕ), (Hf), (Hg), (HJ), (H1), (H2) and Theorem (3.5), it is ensured that mild solution to problem (4.2) exists.

5. Conclusion. In this paper, we establish the existence of the mild solution of the non-instantaneous impulsive partial neutral second order functional differential equation (1.1) using the Kuratowski measure of noncompactness and the Mónch fixed point theorem. The compactness Lipschitz condition and other restrictive conditions have been removed.

Acknowledgments. The authors would like to express sincere gratitude to the reviewer for his valuable suggestions.

REFERENCES

1. W.G. Aiello, H.I. Freedman and J. Wu, *Analysis of a model representing stage-structured population growth with state-dependent time delay*, SIAM J. Appl. Math. **52** (1992), 855–869.

2. D. Alexander, D. Michael and L. Elena, *On equations with delay depending on solution*, Nonlinear Anal. TMA **49** (2002), 689–701.

3. O. Arino, K. Boushaba and A. Boussouar, *A mathematical model of the dynamics of the phytoplankton-nutrient system*, in *Spatial heterogeneity in ecological models* (Alcalá de Henares, 1998), Nonlin. Anal. RWA **1** (2000), 69–87.

4. K. Balachandran and S. Anthoni, *Marshal existence of solutions of second order neutral functionaldifferential equations*, Tamkang J. Math. **30** (1999), 299–309.

5. K. Balachandran, D.G. Park and S. Anthoni, *Marshal existence of solutions of abstract-nonlinear second-order neutral functional integrodifferential equations*, Comp. Math. Appl. **46** (2003), 1313–1324.

6. J. Banas and K. Goebel, *Measure of noncompactness in Banach space*, Lect. Notes Pure Appl. Math. **60**, M. Dekker, New York, 1980.

7. M. Benchohra, J. Henderson and S.K. Ntouyas, *Existence results for impulsive multivalued semilinear neutral functional differential inclusions in Banach spaces*, J. Math. Anal. Appl. **263** (2001), 763–780.

8. ______, *Impulsive differential equations and inclusions*, Hindawi Publishing Corporation, New York, 2006.

9. Y. Cao, J. Fan and T.C. Gard, *The effects of state-dependent time delay on a stage-structured population growth model*, Nonlinear Anal. TMA **19** (1992), 95–105.

10. Y.K. Chang, A. Anguraj and M. Mallika Arjunan, *Existence results for impulsive neutral functional differential equations with infinite delay*, Nonlin. Anal. **2** (2008), 209–218.

11. Y.K. Chang, W.T. Li, *Existence results for impulsive dynamic equations on time scales with nonlocal initial conditions*, Math. Comp. Model. **43** (2006), 377–384.

12. Y.K. Chang and J.J. Nieto, *Existence of solutions for impulsive neutral integro-differential inclusions with nonlocal initial conditions via fractional operators*, Numer. Funct. Anal. Optim. **30** (2009), 227–244.

13. G. Chen and D.L. Russell, *A mathematical model for linear elastic systems with structural damping*, Quart. Appl. Math. **39** (1982), 433-454.

14. S. Chen and R. Triggiani, *Proof of extensions of two conjectures on structural damping for elastic systems*, Pac. J. Math. **136** (1989), 15-55.

15. Sanjukta Das, D.N. Pandey and N. Sukavanam, *Approximate controllability of a functional differential equation with deviated argument*, Nonlin. Dyn. Syst. Theor. **14** (2014), 265–277.

16. ______, *Exact controllability of an impulsive semilinear system with deviated argument in a Banach space*, J. Diff. Equat., Article ID 461086, 6 pages (2014), `http://dx.doi.org/10.1155/2014/461086`.

17. ______, *Approximate controllability of a second-order neutral differential equation with state dependent delay*, Diff. Equat. Dyn. Syst. Article 218 (2014), `http://dx.doi.org/10.1007/s12591-014-0218-6`.

18. G. Di Blasio, K. Kunisch and E. Sinestrari, *Mathematical models for the elastic beam with structural damping*, Appl. Anal. **48** (1993), 133-156.

19. H.O. Fattorini, *Second order linear differential equations in Banach spaces*, North-Holland Math. Stud. **108**, North-Holland, Amsterdam, 1985.

20. C. Fengde, S. Dexian and S. Jinlin, *Periodicity in a food-limited population model with toxicants and state dependent delays*, J. Math. Anal. Appl. **288** (2003), 136–146.

21. X. Fu and K. Ezzinbi, *Existence of solutions for neutral functional differential evolution equations with nonlocal conditions*, Nonlin. Anal. **54** (2003), 215–227.

22. J.A. Goldstein, *Semigroups of linear operators and applications*, Oxford University Press, New York, 1985.

23. Jerome A. Goldstein, *Semigroups of linear operators and applications*, Oxford University Press, Oxford, 1983.

24. M.E. Gurtin and A.C. Pipkin, *A general theory of heat conduction with finite wave speeds*, Arch. Rat. Mech. Anal. **31** (1968), 113–126.

25. W.M. Haddad, V. Chellabonia, S.G. Nersesov and G. Sergey, *Impulsive and hybrid dynamical systems*: *Stability, dissipativity, and control*, Princeton University Press, Princeton, NJ, 2006.

26. J.K. Hale and J. Kato, *Phase space for retarded equations with infinite delay*, Funk. Ekvac. **21** (1978), 11–41.

27. E. Hernández, *Existence results for partial neutral integrodifferential equations with unbounded delay*, J. Math. Anal. Appl. **292** (2004), 194–210.

28. E. Hernandez and H. Henríquez, *Existence results for partial neutral functional differential equation with unbounded delay*, J. Math. Anal. Appl. **222** (1998), 452–475.

29. E.M. Hernandez, Hernan R. Henriquez and Mark A. McKibben, *Existence results for abstract impulsive second-order neutral functional differential equations*, Nonlin. Anal. **70** (2009), 2736–2751.

30. E.M. Hernández, M. Rabello and Herná R. Henríuez, *Existence of solutions for impulsive partial neutral functional differential equations*, J. Math. Anal. Appl. **331** (2007), 1135–1158.

31. Eduardo Hernández and Donal O'Regan, *On a new class of abstract impulsive differential equations*, Proc. Amer. Math. Soc. **S0002-9939** (2012), 11613–11612.

32. F. Huang, *Some problems for linear elastic systems with damping*, Acta Math. Sci. **10** (1990), 319-326 (in English).

33. V. Lakshmikanthan, D.D. Bainov and P.S. Simeonov, *Theory of impulsive differential equations*, World Scientific, Singapore, 1989.

34. I. Lasiecka, D. Lukes and L. Pandolfi, *A feedback synthesis of boundary control problem for a plate equation with structural damping*, Appl. Math. Comp. Sci. **4** (1994), 5-18.

35. Haeng Joo Lee, Jeongyo Park and Jong Yeoul Park, *Existence results for second-order neutral functional differential and integrodifferential inclusions in Banach spaces*, Electr. J. Differ. Equat. **96** (2002), 13 pages (electronic).

36. Eduardo Hernández Morales, Mark A. McKibben and Hernán R. Henrquez, *Existence results for partial neutral functional differential equations with state-dependent delay*, Math. Comp. Model. **49** (2009), 1260–1267.

37. J.J. Nieto and D. O'Regan, *Variational approach to impulsive differential equations*, Nonlin. Anal. RWA **10** (2009), 680–690.

Department of Mathematics, Indian Institute of Technology Roorkee, Uttarakhand, India
Email address: **dassanjukta44@gmail.com**

Department of Mathematics, Indian Institute of Technology Roorkee, Uttarakhand, India
Email address: **dwij.iitk@gmail.com**

Department of Mathematics, Indian Institute of Technology Roorkee, Uttarakhand, India
Email address: **nsukavanam@gmail.com**