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VOLTERRA-TYPE OPERATORS FROM ANALYTIC MORREY SPACES TO BLOCH SPACE

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ABSTRACT. In this note, we study the boundedness and compactness of integral operators I_g and T_g from analytic Morrey spaces to Bloch space. Furthermore, the norm and essential norm of those operators are given.

1. Introduction. Let $\mathbb{D} = \{z : |z| < 1\}$ and $\partial\mathbb{D} = \{z : |z| = 1\}$ denote, respectively, the open unit disc and the unit circle in the complex plane $\mathbb{C}$. Let $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$ and $dm(z) = \frac{1}{\pi}\, dx\, dy$ the normalized area Lebesgue measure.

The aim of this paper is to characterize the boundedness and compactness of two Volterra type operators I_g and T_g from the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ to the classical Bloch space B, and from the little analytic Morrey spaces $\mathcal{L}_0^{2,\lambda}$ to the little Bloch space B_0. Also, we estimate the essential norm of I_g and T_g.

The *Morrey* space was initially introduced in 1938 by Morrey **[21]** to show that certain systems of partial differential equations (PDEs) had Hölder continuous solutions. In the past, the Morrey space has been heavily studied in different areas. For example, Adams and Xiao studied Morrey spaces which is defined on Euclidean spaces $\mathbb{R}^n$ by potential theory and Hausdorff capacity in **[1, 2]**. Cascante, Fàbrega, Ortega **[11]** (partially) and Wang and Xiao **[28]** studied holomorphic Campanato spaces on the open unit ball $\mathbb{B}^n$ of $\mathbb{C}^n$. But here we will be

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mostly interested in the analytic Morrey spaces $\mathcal{L}^{2,\lambda}$ in the unit disk. It was introduced and studied by Wu and Xie in [**31**].

For an arc $I \subset \partial\mathbb{D}$, let $|I| = \frac{1}{2\pi}\int_I |d\zeta|$ be the normalized arc length of I,

$$f_I = \frac{1}{|I|}\int_I f(\zeta)\frac{|d\zeta|}{2\pi}, \quad f \in H(\mathbb{D}),$$

and let $S(I)$ be the Carleson box based on I with

$$S(I) = \left\{z \in \mathbb{D} : 1 - |I| \leq |z| < 1, \frac{z}{|z|} \in I\right\}.$$

Clearly, if $I = \partial\mathbb{D}$, then $S(I) = \mathbb{D}$.

Let $\mathcal{L}^{2,\lambda}(\mathbb{D})$ represent the analytic Morrey spaces of all analytic functions $f \in H^2$ on $\mathbb{D}$ such that

$$\sup_{I\subset\partial\mathbb{D}}\left(\frac{1}{|I|^\lambda}\int_I |f(\zeta) - f_I|^2\frac{|d\zeta|}{2\pi}\right)^{1/2} < \infty,$$

where $0 < \lambda \leq 1$ and the Hardy space H^2 consist of analytic functions f in $\mathbb{D}$ satisfying

$$\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta < \infty.$$

Similarly to the relation between BMOA space and VMOA space, we have that $f \in \mathcal{L}_0^{2,\lambda}(\mathbb{D})$, the little Morrey spaces, if $f \in \mathcal{L}^{2,\lambda}(\mathbb{D})$ and

$$\lim_{|I|\to 0}\left(\frac{1}{|I|^\lambda}\int_I |f(\zeta) - f_I|^2\frac{|d\zeta|}{2\pi}\right)^{1/2} = 0.$$

Xiao and Xu [**36**] studied the composition operators of $\mathcal{L}^{2,\lambda}$ spaces. Cascante, Fàbrega and Ortega [**11**] studied the Corona theorem of $\mathcal{L}^{2,\lambda}$. Wang and Xiao [**29**] characterized the first and second preduals of the analytic Morrey spaces. Xiao and Yuan [**37**] studied analytic Campanato spaces (including the analytic Morrey spaces) in terms of the Möbius mappings and the Littlewood-Paley forms. It is a useful tool for the study of harmonic analysis and partial differential equations. We refer the interested reader to [**21, 22, 42**].

The following lemma gives some equivalent conditions of $\mathcal{L}^{2,\lambda}(\mathbb{D})$ (see [**32,** Theorem 3.1] or [**35,** Theorem 3.21]).

Lemma 1.1. *Suppose that* $0 < \lambda < 1$ *and* $f \in H(\mathbb{D})$. *Let* $a \in \mathbb{D}$, $\varphi_a(z) = (a-z)/(1-\overline{a}z)$. *Then the following statements are equivalent*:

(i) $f \in \mathcal{L}^{2,\lambda}(\mathbb{D})$;

(ii) $\displaystyle\sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^\lambda}\int_{S(I)} |f'(z)|^2(1-|z|^2)\,dm(z) < \infty$;

(iii) $\displaystyle\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\int_{\mathbb{D}} |f'(z)|^2(1-|\varphi_a(z)|^2)\,dm(z) < \infty$.

From the lemma above, we can define the norm of the function $f \in \mathcal{L}^{2,\lambda}(\mathbb{D})$ and its equivalent formula as follows:

$$\begin{aligned}\|f\|_{\mathcal{L}^{2,\lambda}} &= |f(0)| + \sup_{I\subset\partial\mathbb{D}}\left(\frac{1}{|I|^\lambda}\int_{S(I)} |f'(z)|^2(1-|z|^2)\,dm(z)\right)^{1/2}\\ &\approx |f(0)| + \sup_{a\in\mathbb{D}}\left((1-|a|^2)^{1-\lambda}\int_{\mathbb{D}} |f'(z)|^2(1-|\varphi_a(z)|^2)\,dm(z)\right)^{1/2}.\end{aligned}$$

It is known that $\mathcal{L}^{2,1}(\mathbb{D}) = \mathrm{BMOA}$ and if $0<\lambda<1$, $\mathrm{BMOA} \subsetneq \mathcal{L}^{2,\lambda}(\mathbb{D})$. For more information on BMOA and VMOA, see [**15**].

A function f analytic on the unit disk is said to belong to the Bloch space B if

$$\|f\|_B = \sup_{z\in\mathbb{D}}\{(1-|z|^2)|f'(z)|\} < \infty,$$

and to the little Bloch space B_0 if $f \in B$ and

$$\lim_{|z|\to 1^-}(1-|z|^2)|f'(z)| = 0.$$

It is well known that B is a Banach space under the norm $|||f|||_B = |f(0)| + \|f\|_B$ and B_0 is a closed subspace of B. See [**5**]. By [**7, 34**], together with [**8,** Lemma 2.1], we have the following equivalent statements about the norm of $f \in B$.

Proposition 1.2. *For all $p \in (1,\infty)$,*

$$\|\|f\|\|_B \approx |f(0)| + \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^2 (1-|\varphi_a(z)|^2)^p dm(z) \right)^{1/2}$$
$$\approx |f(0)| + \sup_{I\subset\partial\mathbb{D}} \left(\frac{1}{|I|^p} \int_{S(I)} |f'(z)|^2 (1-|z|^2)^p dm(z) \right)^{1/2}.$$

Suppose that $g : \mathbb{D} \to \mathbb{C}$ is a holomorphic map. The integral operator T_g, called the *Volterra-type operator*, is defined as

$$T_g f(z) = \int_0^z f(w) g'(w)\, dw, \quad z \in \mathbb{D},\ f \in H(\mathbb{D}).$$

In [**23**], Pommerenke introduced the operator T_g and showed that T_g is a bounded operator on the Hardy space H^2 if and only if $g \in$ BMOA.

The companion integral operator I_g is analogously defined as

$$I_g f(z) = \int_0^z f'(w) g(w)\, dw, \quad z \in \mathbb{D},\ f \in H(\mathbb{D}).$$

The boundedness, compactness or essential norm of T_g and I_g between spaces of analytic functions were investigated by many authors. Aleman and Siskakis in [**4**] studied the integral operator T_g on the Bergman space, and then Aleman considered with Cima T_g acting on the Hardy space in [**3**]. Siskakis and Zhao [**27**] also investigated T_g on the BMOA space. T_g on the Q_p space was studied by Xiao in [**33**]. Li and Stević in [**19**] studied the boundedness and compactness of T_g and I_g on the Zygmund spaces and the little Zygmund spaces. Constantin in [**12**] considered the boundedness and compactness of T_g on *Fock* spaces. Ye in [**39**] studied products of Volterra-type operators and composition operators on logarithmic Bloch space. Ye and Gao in [**40**] gave the boundedness and compactness of T_g between different weighted Bloch spaces.

There are some articles about the integral operator acting on the Morrey space. For example, Wu in [**30**] considered T_g from Hardy to analytic Morrey spaces. Li, Liu and Lou [**17**] characterized the boundedness and essential norms of T_g and I_g on analytic Morrey spaces (see also the related references therein).

Now, we need two spaces. Let $\alpha > -1$. Recall that $f \in H(\mathbb{D})$ belongs to the weighted space H_α^∞ if it satisfies

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f(z)| < \infty.$$

When $\alpha > -1$, H_α^∞ endowed with the norm $\|f\|_\alpha = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f(z)|$ is a Banach space. This space is connected with the study of growth conditions of analytic functions and was also studied in detail, see **[9, 10, 25, 26, 38]**. The space H_α^∞ is used in the characterizations of the boundedness and essential norm of I_g. Then we conclude the boundedness and essential norm of T_g by introducing the following Bloch-Morrey type spaces.

Definition 1.3. Let $0 < \lambda \le 1$ and $p > 1$. The *Bloch-Morrey* type space $B\mathcal{L}^{p,\lambda}$ is the set of all $g \in H(\mathbb{D})$ such that

$$M(g) = \sup_{I\subset\partial\mathbb{D}}\left(\frac{1}{|I|^{p-\lambda+1}}\int_{S(I)}|g'(z)|^2(1-|z|^2)^p dm(z)\right)^{1/2} < \infty.$$

The corresponding subspace $B\mathcal{L}_0^{p,\lambda}$, the little *Bloch-Morrey* type space, can be defined as

$$B\mathcal{L}_0^{p,\lambda}=\left\{g \in B\mathcal{L}^{p,\lambda}, \lim_{|I|\to 0}\left(\frac{1}{|I|^{p-\lambda+1}}\int_{S(I)}|g'(z)|^2(1-|z|^2)^p dm(z)\right)^{1/2} = 0\right\}.$$

It is easy to prove that $B\mathcal{L}^{p,\lambda}$ is a Banach space under the norm

$$\|g\|_{B\mathcal{L}^{p,\lambda}} = |g(0)| + M(g).$$

Clearly, $B\mathcal{L}^{p,1} = B$. From **[8]**, we know that $\|g\|_{B\mathcal{L}^{p,\lambda}}$ is comparable with the norm

$$|g(0)| + \sup_{a\in\mathbb{D}}\left(\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}z|^2}\right)^{p+1-\lambda}|g'(z)|^2(1-|z|^2)^p dm(z)\right)^{1/2}.$$

Notation. For two functions F and G, if there is a constant $C > 0$ dependent only on indexes $p, \lambda, \ldots$, such that $F \le CG$, then we say that $F \lesssim G$. Furthermore, denote that $F \approx G$ (F is comparable with G) whenever $F \lesssim G \lesssim F$.

2. $\mathcal{L}^{2,\lambda}$ vs B. Evidently, when $0<\lambda<1$, BMOA $\subsetneq \mathcal{L}^{2,\lambda}(\mathbb{D})$. On the other hand BMOA $\subsetneq B$. Does $\mathcal{L}^{2,\lambda}$ and B have the inclusion relation? We claim that the answer is negative by the following two proposition.

Proposition 2.1. *$\mathcal{L}^{2,\lambda} \nsubseteq B$.*

Proof. Consider $g(z)=(\log\frac{1}{1-z})^2$, which is obviously not a Bloch function. We get that $g(z)\in\mathcal{L}^{2,\lambda}$. Indeed,

$$\begin{aligned}\sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\int_{\mathbb{D}}|g'(z)|^2(1-|\varphi_a(z)|^2)\,dm(z)&\\ &\lesssim \sup_{a\in\mathbb{D}}(1-|a|^2)^{1-\lambda}\\ &\times\int_{\mathbb{D}}|\frac{1}{1-z}|^2\log^2\frac{1}{1-|z|}(1-|\varphi_a(z)|^2)\,dm(z)\\ &\lesssim \int_{\mathbb{D}}|\frac{1}{1-z}|^2(1-|z|^2)^{1-\lambda}\log^2\frac{1}{1-|z|}dm(z)\\ &=\int_0^1\int_0^{2\pi}|\frac{1}{1-re^{i\theta}}|^2d\theta(1-r^2)^{1-\lambda}\log^2\frac{1}{1-r}\,dr\\ &=\int_0^1(1-r^2)^{-\lambda}\log^2\frac{1}{1-r}\,dr<\infty.\end{aligned}$$

This finishes the proof. □

Conversely, the function

$$f(z)=\sum_{n=0}^{\infty}z^{2^n}$$

is a Bloch function (see **[6]**), and it is well known that it has a radial limit almost nowhere. Consequently, f does not belong to any of the Hardy spaces, and so $f\notin\mathcal{L}^{2,\lambda}$. Now we have the following proposition.

Proposition 2.2. *$B \nsubseteq \mathcal{L}^{2,\lambda}$.*

3. Boundedness of I_g and T_g from $\mathcal{L}^{2,\lambda}$ to B. In this section, we prove the boundedness and estimate the norms of I_g and T_g. The following lemmas will be used through this paper.

Lemma 3.1. *Let $0 < \lambda < 1$ and $b \in \mathbb{D}$. We set functions $f_b(z)$ and $F_b(z)$ as*

$$f_b(z) = (1-|b|^2)^{(1-\lambda)/2}(\varphi_b(z) - b),\ F_b(z) = (1-|b|^2)(1-\overline{b}z)^{(\lambda-3)/2}.$$

Then $f_b(z) \in \mathcal{L}^{2,\lambda}(\mathbb{D})$ and $F_b(z) \in \mathcal{L}^{2,\lambda}(\mathbb{D})$. Particularly, we have $f_b(z) \in \mathcal{L}_0^{2,\lambda}(\mathbb{D})$ and $F_b(z) \in \mathcal{L}_0^{2,\lambda}(\mathbb{D})$. Moreover, $\|f_b\|_{\mathcal{L}^{2,\lambda}} \lesssim 1$, $\|F_b\|_{\mathcal{L}^{2,\lambda}} \lesssim 1$.

Proof. See [**17,** Lemma 4]. From its proof, we further deduce that $f_b(z) \in \mathcal{L}_0^{2,\lambda}(\mathbb{D})$ and $F_b(z) \in \mathcal{L}_0^{2,\lambda}(\mathbb{D})$. □

We get a result about the growth rate of functions in $\mathcal{L}^{2,\lambda}(\mathbb{D})$ from [**17**].

Lemma 3.2. *Let $0 < \lambda < 1$. If $f \in \mathcal{L}^{2,\lambda}(\mathbb{D})$, then*

$$|f(z)| \lesssim \frac{\|f\|_{\mathcal{L}^{2,\lambda}}}{(1-|z|^2)^{(1-\lambda)/2}}, \quad z \in \mathbb{D}.$$

We first consider the boundedness of $I_g : \mathcal{L}^{2,\lambda} \to B$.

Theorem 3.3. *Let $0 < \lambda < 1$ and $g \in H(\mathbb{D})$. Then $I_g : \mathcal{L}^{2,\lambda} \to B$ is bounded if and only if $g \in H^\infty_{(\lambda-1)/2}$. Moreover, the operator norm satisfies*

$$\|I_g\| \approx \|g\|_{(\lambda-1)/2}.$$

Proof. For $0 < \lambda < 1$ and $1 < 2 - \lambda$, we set $B = Q_{2-\lambda}$.

Let $g \in H^\infty_{(\lambda-1)/2}$. For any $f \in \mathcal{L}^{2,\lambda}(\mathbb{D})$, we have

$$\begin{aligned}\|I_g f\|_B &\approx \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} |f'(z)|^2 |g(z)|^2 (1-|\varphi_a(z)|^2)^{2-\lambda} dm(z) \right)^{1/2} \\ &= \sup_{a\in\mathbb{D}} \Bigg((1-|a|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(z)|^2 \\ &\quad \times (1-|\varphi_a(z)|^2) |g(z)|^2 \left(\frac{1-|z|^2}{|1-\overline{a}z|^2} \right)^{1-\lambda} dm(z) \Bigg)^{1/2} \\ &\lesssim \|g\|_{(\lambda-1)/2} \|f\|_{\mathcal{L}^{2,\lambda}},\end{aligned}$$

which implies that I_g is bounded and $\|I_g\| \lesssim \|g\|_{(\lambda-1)/2}$.

On the other hand, let I_g be bounded. For any $b \in \mathbb{D}$, considering functions $f_b(z)$ in Lemma 3.1, we have $\|f_b\|_{\mathcal{L}^{2,\lambda}} \lesssim 1$. Then

$$\begin{aligned}\|I_g\| &\gtrsim \|I_g f_b\|_B \\ &\approx \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} |f_b'(z)|^2 |g(z)|^2 (1-|\varphi_a(z)|^2)^{2-\lambda} dm(z) \right)^{1/2} \\ &= \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} \frac{(1-|b|^2)^{1+\lambda}}{|1-\overline{b}z|^4} |g(z)|^2 (1-|\varphi_a(z)|^2)^{1+1-\lambda} dm(z) \right)^{1/2} \\ &\geqslant \left(\int_{\mathbb{D}} \frac{(1-|b|^2)^2}{|1-\overline{b}z|^4} |g(z)|^2 \left(\frac{1-|z|^2}{|1-\overline{b}z|^2} \right)^{1-\lambda} (1-|\varphi_b(z)|^2)\, dm(z) \right)^{1/2} \\ &= \left(\int_{\mathbb{D}} |\varphi_b'(z)|^2 |g(z)|^2 \left(\frac{1-|z|^2}{|1-\overline{b}z|^2} \right)^{1-\lambda} (1-|\varphi_b(z)|^2)\, dm(z) \right)^{1/2} \\ &= \left(\int_{\mathbb{D}} |g(\varphi_b(w))|^2 \left(\frac{1-|\varphi_b(w)|^2}{|1-\overline{b}\varphi_b(w)|^2} \right)^{1-\lambda} (1-|w|^2)\, dm(w) \right)^{1/2} \\ &\gtrsim \left| \frac{g(b)}{(1-|b|^2)^{(1-\lambda)/2}} \right|,\end{aligned} \tag{3.1}$$

where we used **[41,** Lemma 4.12] in the last inequality. Since b is arbitrary, we have $\|I_g\| \gtrsim \|g\|_{(\lambda-1)/2}$. The proof is finished. □

With the space $B\mathcal{L}^{p,\lambda}$, we can establish the boundedness of $T_g : \mathcal{L}^{2,\lambda} \to B$ as in the following theorem.

Theorem 3.4. *Suppose that* $0 < \lambda < 1$ *and* $g \in H(\mathbb{D})$. *Then the following conditions are equivalent*:

(i) $T_g : \mathcal{L}^{2,\lambda} \to B$ *is bounded*;
(ii) $g \in B\mathcal{L}^{p,\lambda}$ *for all* $p \in (1, \infty)$;
(iii) $g \in B\mathcal{L}^{p,\lambda}$ *for some* $p \in (1, \infty)$.

Moreover,

$$\|T_g\| \approx M(g).$$

Proof. (i) $\Rightarrow$ (ii). Suppose that $T_g : \mathcal{L}^{2,\lambda} \to B$ is bounded. For any $I \subset \partial\mathbb{D}$, let $b = (1 - |I|)\zeta \in \mathbb{D}$, where ζ is the center of I. Then

$$(1 - |b|^2) \approx |1 - \overline{b}z| \approx |I|, \quad z \in S(I).$$

Considering the functions $F_b(z)$ in Lemma 3.1, we have $\|F_b\|_{\mathcal{L}^{2,\lambda}} \lesssim 1$. This, together with Proposition 2.2, we obtain that for any $p \in (1, \infty)$,

$$\begin{aligned}\frac{1}{|I|^{p-\lambda+1}} &\int_{S(I)} |g'(z)|^2 (1 - |z|^2)^p dm(z) \\ &\approx \frac{1}{|I|^p} \int_{S(I)} |F_b(z)|^2 |g'(z)|^2 (1 - |z|^2)^p dm(z) \\ &\lesssim \|T_g F_b\|_B^2 \leq \|T_g\|^2 \|F_b\|^2_{\mathcal{L}^{2,\lambda}} \lesssim \|T_g\|^2.\end{aligned}$$

Since I is arbitrary, we have $M(g) \lesssim \|T_g\|$.

(ii) $\Rightarrow$ (iii). It is obvious.

(iii) $\Rightarrow$ (i). Suppose that fixed $p \in (1, \infty)$ and $M(g) < \infty$. For $f \in \mathcal{L}^{2,\lambda}(\mathbb{D})$ and any $I \subset \partial\mathbb{D}$, by Lemma 3.2, it follows that

(3.2)
$$\begin{aligned}\|T_g f\|_B &\approx \sup_{I \subset \partial\mathbb{D}} \left(\frac{1}{|I|^p} \int_{S(I)} |f(z)|^2 |g'(z)|^2 (1 - |z|^2)^p dm(z) \right)^{1/2} \\ &\lesssim \|f\|_{\mathcal{L}^{2,\lambda}} \cdot \sup_{I \subset \partial\mathbb{D}} \left(\frac{1}{|I|^p} \int_{S(I)} |g'(z)|^2 (1 - |z|^2)^{p+\lambda-1} dm(z) \right)^{1/2}.\end{aligned}$$

To the end, for a given subarc I of $\partial\mathbb{D}$, let $\mathcal{D}_n(I)$ represent the set of 2^n subarcs of length $2^{-n}|I|$ obtained by n successive bipartition of I. For each $J \in \mathcal{D}_n(I)$, write $T(J)$ for the top half Carleson box of $S(J)$, i.e.,

$$T(J) = \left\{ z \in S(J) : \frac{z}{|z|} \in J, 1 - |J| < |z| < 1 - \frac{|J|}{2} \right\}.$$

Then

$$S(I) = \bigcup_{n=0}^{\infty} \bigcup_{J \in \mathcal{D}_n(I)} T(J).$$

Noting that $z \in T(J)$, $1 - |z| \approx |J|$, one has

$$\begin{aligned}
&\int_{S(I)} |g'(z)|^2 (1-|z|^2)^{p+\lambda-1} dm(z) \\
&= \sum_{n=0}^{\infty} \sum_{J \in \mathcal{D}_n(I)} \int_{T(J)} |g'(z)|^2 (1-|z|^2)^{p+\lambda-1} dm(z) \\
&\approx \sum_{n=0}^{\infty} \sum_{J \in \mathcal{D}_n(I)} \int_{T(J)} |J|^{\lambda-1} |g'(z)|^2 (1-|z|^2)^{p} dm(z) \\
&\leq \sum_{n=0}^{\infty} \sum_{J \in \mathcal{D}_n(I)} \int_{S(J)} |J|^{\lambda-1} |g'(z)|^2 (1-|z|^2)^{p} dm(z) \\
&\leq \sum_{n=0}^{\infty} \sum_{J \in \mathcal{D}_n(I)} M(g)^2 |J|^{\lambda-1} |J|^{p-\lambda+1} \qquad (3.3) \\
&= \sum_{n=0}^{\infty} 2^n M(g)^2 |J|^p = \sum_{n=0}^{\infty} (2^n)^{1-p} M(g)^2 |I|^p \\
&\lesssim M(g)^2 |I|^p.
\end{aligned}$$

Now invoking (3.2),

$$\|T_g f\|_B \lesssim M(g) \cdot \|f\|_{\mathcal{L}^{2,\lambda}}.$$

As a result, $\|T_g\| \lesssim M(g)$. □

Theorem 3.4 has an interesting consequence.

Corollary 3.5. *Let $0 < \lambda < 1$ and $1 < p < q < \infty$. Then $B\mathcal{L}^{p,\lambda} = B\mathcal{L}^{q,\lambda}$.*

4. Essential norm of I_g and T_g from $\mathcal{L}^{2,\lambda}$ to B. Let X and Y be Banach spaces. The essential norm of a bounded operator $T : X \to Y$, $\|T\|_{e,X\to Y}$, is defined as the distance from T to the space of compact operators,

$$\|T\|_{e,X\to Y} = \inf\{\|T - K\|_{X\to Y} : K \text{ is any compact operator}\},$$

where the norm of T is denoted by $\|\cdot\|_{X\to Y}$.

Since that T is compact if and only if $\|T\|_{e,X\to Y} = 0$, then the estimation of $\|T\|_{e,X\to Y}$ indicates the condition for T to be compact. For some recent results related to the essential norm, see [**16, 17, 18, 24**] and the references therein.

In this section, we estimate the essential norm of I_g and T_g from $\mathcal{L}^{2,\lambda}$ to B. We need some auxiliary results.

Lemma 4.1. *Let $0 < \lambda < 1$. For $0 < t < 1$, $z \in \mathbb{D}$, $f_t(z) = f(tz)$. If $f \in \mathcal{L}^{2,\lambda}(\mathbb{D})$, then $f_t \in \mathcal{L}_0^{2,\lambda}(\mathbb{D})$ and $\|f_t\|_{\mathcal{L}^{2,\lambda}} \le \|f\|_{\mathcal{L}^{2,\lambda}}$.*

Proof. If $f \in \mathcal{L}^{2,\lambda}(\mathbb{D})$ and $0 < t < 1$, then f_t is analytic on the closed unit disk $\overline{\mathbb{D}}$. A simple computation shows that $f_t \in \mathcal{L}_0^{2,\lambda}(\mathbb{D})$. In addition, by the Poisson formula, we have

$$f_t(z) = \int_0^{2\pi} f(ze^{i\theta}) \frac{1-t^2}{|e^{i\theta}-t|^2} \frac{d\theta}{2\pi}, \quad z \in \mathbb{D}.$$

Then,

$$\begin{aligned}
&\sup_{a\in\mathbb{D}} (1-|a|^2)^{1-\lambda} \int_{\mathbb{D}} |f_t'(z)|^2 (1-|\varphi_a(z)|^2)\, dm(z) \\
&\quad \le \sup_{a\in\mathbb{D}} (1-|a|^2)^{1-\lambda} \\
&\qquad \times \int_{\mathbb{D}} \int_0^{2\pi} |f'(ze^{i\theta})|^2 \frac{1-t^2}{|e^{i\theta}-t|^2} \frac{d\theta}{2\pi} (1-|\varphi_a(z)|^2)\, dm(z) \\
&\quad = \int_0^{2\pi} \sup_{a\in\mathbb{D}} (1-|a|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(ze^{i\theta})|^2 \\
&\qquad\quad (1-|\varphi_a(z)|^2)\, dm(z) \frac{1-t^2}{|e^{i\theta}-t|^2} \frac{d\theta}{2\pi} \\
&\quad \le \sup_{a\in\mathbb{D}} (1-|a|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(z)|^2 (1-|\varphi_a(z)|^2)\, dm(z)
\end{aligned}$$

$$\cdot \int_0^{2\pi} \frac{1-t^2}{|e^{i\theta}-t|^2} \frac{d\theta}{2\pi}$$
$$= \sup_{a\in\mathbb{D}} (1-|a|^2)^{1-\lambda} \int_{\mathbb{D}} |f'(z)|^2 (1-|\varphi_a(z)|^2)\, dm(z).$$

Thus, $\|f_t\|_{\mathcal{L}^{2,\lambda}} \le \|f\|_{\mathcal{L}^{2,\lambda}}$. □

By Lemma 3.2 and standard arguments (see, e.g., [**13,** Proposition 3.11]), the following lemma follows.

Lemma 4.2. *Assume that g is an analytic function on $\mathbb{D}$. Then T_g (or I_g) $: \mathcal{L}^{2,\lambda} \to B$ is compact if and only if T_g (or I_g) $: \mathcal{L}^{2,\lambda} \to B$ is bounded, and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{L}^{2,\lambda}$ which converges to zero uniformly on $\mathbb{D}$ as $k\to\infty$, $\|T_g f_k\|_B \to 0$ (or $\|I_g f_k\|_B \to 0$) as $k\to\infty$.*

Lemma 4.3. *Suppose that $0<\lambda<1$ and $p>1$. For $g \in B\mathcal{L}^{p,\lambda}$, define the following operators $T_{g,r} : \mathcal{L}^{2,\lambda} \to B$:*

$$T_{g,r} f(z) = \int_0^z f(rw) g'(w)\, dw,$$

where $r\in(0,1)$. Then $T_{g,r}$ is compact.

Proof. Let $\{f_n\}$ be such that $\|f_n\|_{\mathcal{L}^{2,\lambda}} \le 1$ and $f_n \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $n\to\infty$. We are required to show that $\lim_{n\to\infty} \|T_{g,r} f_n\|_B = 0$. In fact, since $\|g\|_B \lesssim M(g)$, we have

$$|g'(z)| \lesssim \frac{M(g)}{1-|z|^2}.$$

From $\|f_n\|_{\mathcal{L}^{2,\lambda}} \le 1$ and Lemma 3.2, it yields that $(1-|r|^2)^{(1-\lambda)/2}|f_n(rz)| \lesssim 1$. Thus,

$$\|T_{g,r} f_n\|_B = \sup_{z\in\mathbb{D}} (1-|z|^2)|f_n(rz)||g'(z)|$$
$$\lesssim M(g) \sup_{z\in\mathbb{D}} \frac{1}{(1-r^2)^{(1-\lambda)/2}}.$$

Accordingly, by the dominated convergence theorem, one reaches

$$\lim_{n\to\infty} \|T_{g_r} f_n\|_B = 0.$$ □

Now, we present the main result of this section.

Theorem 4.4. *Suppose* $0 < \lambda < 1$ *and* $g \in H(\mathbb{D})$. *If* $I_g : \mathcal{L}^{2,\lambda} \to B$ *is bounded, then*

$$\|I_g\|_{e,\mathcal{L}^{2,\lambda}\to B} \approx \|g\|_{(\lambda-1)/2}.$$

Proof. Choose the zero operator $O : \mathcal{L}^{2,\lambda} \to B : f \mapsto 0$. Since O is compact and $\|O\| = 0$, we get

$$\|I_g\|_{e,\mathcal{L}^{2,\lambda}\to B} = \inf_K \|I_g - K\| \leq \|I_g\| \lesssim \|g\|_{(\lambda-1)/2}.$$

Conversely, choose the sequence $\{b_n\} \subset \mathbb{D}$ such that $|b_n| \to 1$ as $n \to \infty$. Considering the sequence of functions $f_n(z) = (1 - |b_n|^2)^{(1-\lambda)/2}(\varphi_{b_n}(z) - b_n)$, we obtain $\|f_n\|_{\mathcal{L}^{2,\lambda}} \lesssim 1$ by Lemma 3.1. By an easy calculation,

$$f_n(z) = -(1 - |b_n|^2)^{(1+\lambda)/2} \int_0^z \frac{dw}{(1 - \overline{b_n} z)^2},$$

and thus f_n converges to zero uniformly on compact subsets of $\mathbb{D}$. Then $\|Kf_n\|_B \to 0$ as $n \to \infty$ for any compact operator K. So

$$\begin{aligned}\|I_g - K\| &\gtrsim \limsup_{n\to\infty} \|(I_g - K)f_n\|_B \\ &\geq \limsup_{n\to\infty} (\|I_g f_n\|_B - \|Kf_n\|_B) \\ &\geq \limsup_{n\to\infty} \|I_g f_n\|_B.\end{aligned}$$

By (3.1), we have

$$\|I_g - K\| \gtrsim \limsup_{n\to\infty} \left| \frac{g(b_n)}{(1 - |b_n|^2)^{(1-\lambda)/2}} \right|.$$

Then the arbitrary choice of the sequence $\{b_n\}$ implies

$$\|I_g\|_{e,\mathcal{L}^{2,\lambda}\to B} \gtrsim \|g\|_{(\lambda-1)/2}. \qquad \square$$

Theorem 4.5. *Suppose* $0 < \lambda < 1$ *and* $g \in H(\mathbb{D})$. *If* $T_g : \mathcal{L}^{2,\lambda} \to B$ *is bounded, then*

$$\|T_g\|_{e,\mathcal{L}^{2,\lambda}\to B} \approx \limsup_{|a|\to 1} \left(\int_{\mathbb{D}} \left(\frac{1 - |a|^2}{|1 - \overline{a} z|^2} \right)^{p+1-\lambda} |g'(z)|^2 (1-|z|^2)^p \, dm(z) \right)^{1/2}.$$

Proof. For any $r_n \in (0,1)$ such that $r_n \to 1$ as $n \to \infty$, we introduce $T_{g,r_n} : \mathcal{L}^{2,\lambda} \to B$ which is compact. Letting $s \in (0,1)$, we have

$$\begin{aligned}
&\|T_g\|_{e,\mathcal{L}^{2,\lambda}\to B} \leq \|T_g - T_{g,r_n}\| \approx \sup_{\|f\|_{\mathcal{L}^{2,\lambda}}=1} \|T_g - T_{g,r_n}\|_B \\
&= \sup_{\|f\|_{\mathcal{L}^{2,\lambda}}=1} \\
&\times \sup_{a\in\mathbb{D}} \left(\int_{\mathbb{D}} |f(z) - f(r_n z)|^2 |g'(z)|^2 (1-|\varphi_a(z)|^2)^{p+1-\lambda} dm(z) \right)^{1/2} \\
&\leq \sup_{\|f\|_{\mathcal{L}^{2,\lambda}}=1} \\
&\times \sup_{|a|\leq s} \left(\int_{\mathbb{D}} |f(z) - f(r_n z)|^2 |g'(z)|^2 (1-|\varphi_a(z)|^2)^{p+1-\lambda} dm(z) \right)^{1/2} \\
&\quad + \sup_{\|f\|_{\mathcal{L}^{2,\lambda}}=1} \\
&\times \sup_{|a|> s} \left(\int_{\mathbb{D}} |f(z) - f(r_n z)|^2 |g'(z)|^2 (1-|\varphi_a(z)|^2)^{p+1-\lambda} dm(z) \right)^{1/2} \\
&\leq \sup_{\|f\|_{\mathcal{L}^{2,\lambda}}=1} \\
&\times \sup_{|a|\leq s} \left(\int_{\mathbb{D}} |f(z) - f(r_n z)|^2 |g'(z)|^2 (1-|\varphi_a(z)|^2)^{p+1-\lambda} dm(z) \right)^{1/2} \\
&\quad + 2 \sup_{|a|>s} \left(\int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\overline{a}z|^2} \right)^{p+1-\lambda} |g'(z)|^2 (1-|z|^2)^p dm(z) \right)^{1/2} \\
&\triangleq K_1 + K_2.
\end{aligned}$$

Since $|a| \leq s$ is a closed set of $\mathbb{D}$ and $g \in B\mathcal{L}^{p,\lambda}$, the dominated convergence theorem yields $K_1 \to 0$ as $n \to \infty$.

Now, letting $n \to \infty$ and then letting $s \to 1$, we get

$$\begin{aligned}
&\|T_g\|_{e,\mathcal{L}^{2,\lambda}\to B} \\
&\qquad \lesssim \limsup_{|a|\to 1} \left(\int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^{p+1-\lambda} |g'(z)|^2 (1-|z|^2)^p dm(z) \right)^{1/2}.
\end{aligned}$$

Conversely, let $I_n \subset \partial\mathbb{D}$ be such that $|I_n| \to 0$ as $n \to \infty$. ζ_n is the center of arc I and $b_n = (1-|I_n|)\zeta$, so

$$(1-|b_n|^2) \approx |1-\overline{b_n}z| \approx |I_n|, \quad z \in S(I_n).$$

Consider the function $F_n(z) = (1-|b_n|^2)(1-\overline{b_n}z)^{(\lambda-3)/2}$. Then $\|F_n\|_{\mathcal{L}^{2,\lambda}} \lesssim 1$ and $F_n \to 0$ uniformly on the compact subsets of $\mathbb{D}$ as $n \to \infty$ by Lemma 3.1. Thus, $\|KF_n\|_B \to 0$ for any compact operator K. Therefore,

$$\begin{aligned}
\|T_g - K\| &\gtrsim \limsup_{n\to\infty} \|(T_g-K)F_n\|_B \\
&\geq \limsup_{n\to\infty}(\|T_gF_n\|_B - \|KF_n\|_B) \\
&\geq \limsup_{n\to\infty} \|T_g f_n\|_B \\
&\gtrsim \limsup_{n\to\infty} \left(\frac{1}{|I_n|^p}\int_{S(I_n)} |F_n(z)|^2|g'(z)|^2(1-|z|^2)^p dm(z)\right)^{1/2} \\
&\approx \limsup_{n\to\infty} \Big(\frac{1}{|I_n|^{p+1-\lambda}}\int_{S(I_n)} |g'(z)|^2(1-|z|^2)^p dm(z)\Big)^{1/2}.
\end{aligned}$$

Since the sequence $\{I_n\}$ is arbitrary, we conclude

$$\begin{aligned}
&\|T_g\|_{e,\mathcal{L}^{2,\lambda}\to B} \\
&\gtrsim \limsup_{|I|\to 0} \left(\frac{1}{|I|^{p+1-\lambda}}\int_{S(I)} |g'(z)|^2(1-|z|^2)^p dm(z)\right)^{1/2} \\
&\approx \limsup_{|a|\to 1} \left(\int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^{p+1-\lambda} |g'(z)|^2(1-|z|^2)^p dm(z)\right)^{1/2}.
\end{aligned}$$

This completes the proof. □

We have the following corollary about their compactness.

Corollary 4.6. *Suppose that* $0<\lambda<1$ *and* $p>1$*. Then*:

(i) $I_g : \mathcal{L}^{2,\lambda} \to B$ *is compact if and only if* $g = 0$.
(ii) $T_g : \mathcal{L}^{2,\lambda} \to B$ *is compact if and only if* $g \in B\mathcal{L}_0^{p,\lambda}$.

5. Boundedness and essential norm of I_g and T_g from $\mathcal{L}_0^{2,\lambda}$ to B_0.

Theorem 5.1. *Let* $0<\lambda<1$ *and* $g\in H(\mathbb{D})$. *Then* $I_g:\mathcal{L}_0^{2,\lambda}\to B_0$ *is bounded if and only if* $g\in H^\infty_{(\lambda-1)/2}$. *Moreover,*

$$\|I_g\|\approx\|g\|_{(\lambda-1)/2}.$$

Proof. Necessity. Assume that $I_g:\mathcal{L}_0^{2,\lambda}\to B_0$ is bounded. Then it is clear that $I_g:\mathcal{L}_0^{2,\lambda}\to B$ is bounded. The necessity of Theorem 3.3, together with $f_b(z)\in\mathcal{L}_0^{2,\lambda}(\mathbb{D})$, proves $g\in H^\infty_{(\lambda-1)/2}$.

Sufficiency. Let $g\in H^\infty_{(\lambda-1)/2}$. Then, from Theorem 3.3, $I_g:\mathcal{L}^{2,\lambda}\to B$ is bounded, and hence $I_g:\mathcal{L}_0^{2,\lambda}\to B$ is bounded. It suffices to prove that, for any $f\in\mathcal{L}_0^{2,\lambda}$, $I_gf\in B_0$. In fact, for any $f\in\mathcal{L}_0^{2,\lambda}$, we have

$$\begin{aligned}
&\lim_{|a|\to1}\int_{\mathbb{D}}|f'(z)|^2|g(z)|^2(1-|\varphi_a(z)|^2)^{2-\lambda}dm(z)\\
&=\lim_{|a|\to1}(1-|a|^2)^{1-\lambda}\\
&\quad\times\int_{\mathbb{D}}|f'(z)|^2(1-|\varphi_a(z)|^2)|g(z)|^2\left(\frac{1-|z|^2}{|1-\overline{a}z|^2}\right)^{1-\lambda}dm(z)\\
&\le\|g\|_{(\lambda-1)/2}\cdot\lim_{|a|\to1}(1-|a|^2)^{1-\lambda}\\
&\quad\times\int_{\mathbb{D}}|f'(z)|^2(1-|\varphi_a(z)|^2)\,dm(z)=0.
\end{aligned}$$

Consequently, $I_g:\mathcal{L}_0^{2,\lambda}\to B_0$ is bounded. □

Theorem 5.2. *Suppose that* $0<\lambda<1$ *and* $g\in H(\mathbb{D})$. *If* $I_g:\mathcal{L}_0^{2,\lambda}\to B_0$ *is bounded, then*

$$\|I_g\|_{e,\mathcal{L}_0^{2,\lambda}\to B_0}\approx\|g\|_{(\lambda-1)/2}.$$

Proof. As a matter of fact, if $g\in H^\infty_{(\lambda-1)/2}$, then for any $f\in\mathcal{L}_0^{2,\lambda}$, $I_gf\in B_0$. Since $f_n(z)\in\mathcal{L}_0^{2,\lambda}$ (see Theorem 4.4), we complete the proof as the same as in the proof of Theorem 4.4. □

Theorem 5.3. *Suppose that* $0 < \lambda < 1$ *and* $g \in H(\mathbb{D})$. *Then the following conditions are equivalent*:

(i) $T_g : \mathcal{L}_0^{2,\lambda} \to B_0$ *is bounded*;
(ii) $g \in B\mathcal{L}_0^{p,\lambda}$ *for all* $p \in (1,\infty)$;
(iii) $g \in B\mathcal{L}_0^{p,\lambda}$ *for some* $p \in (1,\infty)$.

Moreover,

$$\|T_g\| \approx M(g).$$

Proof. (i) $\Rightarrow$ (ii). Suppose that T_g is bounded. For any $I \subset \partial\mathbb{D}$, let $b = (1-|I|)\zeta$, where ζ is the center of I. Then

$$(1-|b|^2) \approx |1-\bar{b}z| \approx |I|, \ z \in S(I).$$

Concerning the functions $F_b(z)$ in Lemma 3.1, we have $F_b(z) \in \mathcal{L}_0^{2,\lambda}(\mathbb{D})$; thus, $T_g(F_b(z)) \in B_0$. Furthermore, from Proposition 2.2, it yields $g \in B\mathcal{L}_0^{p,\lambda}$.

(ii) $\Rightarrow$ (iii). It is obvious.

(iii) $\Rightarrow$ (i). Fix $p \in (1,\infty)$ and $g \in B\mathcal{L}_0^{p,\lambda}$. Then from Theorem 3.4, $T_g : \mathcal{L}^{2,\lambda} \to B$ is bounded, and hence $T_g : \mathcal{L}_0^{2,\lambda} \to B$ is bounded. It suffices to prove that, for any $f \in \mathcal{L}_0^{2,\lambda}$, $T_g f \in B_0$. Indeed, $g \in B\mathcal{L}_0^{p,\lambda}$, for every $\varepsilon > 0$ there is a constant $\delta > 0$ such that, as $|J| < \delta$,

$$\frac{1}{|J|^{p-\lambda+1}} \int_{S(J)} |g'(z)|^2 (1-|z|^2)^p \, dm(z) < \varepsilon.$$

With the above δ, for any $|I| < \delta$, we break up $S(I)$ in the same way as in Theorem 3.4. Then, by (3.2),

$$\begin{aligned}
&\int_{S(I)} |g'(z)|^2 (1-|z|^2)^{p+\lambda-1} \, dm(z) \\
&\lesssim \sum_{n=0}^{\infty} \sum_{J \in \mathcal{D}_n(I)} \int_{S(J)} |J|^{\lambda-1} |g'(z)|^2 (1-|z|^2)^p \, dm(z) \\
&\leq \sum_{n=0}^{\infty} \sum_{J \in \mathcal{D}_n(I)} \varepsilon |J|^{\lambda-1} |J|^{p-\lambda+1} \\
&\lesssim \varepsilon |I|^p;
\end{aligned}$$

namely,

$$\lim_{|I|\to 0}\frac{1}{|I|^p}\int_{S(I)}|g'(z)|^2(1-|z|^2)^{p+\lambda-1}dm(z)=0.$$

Now, it is easy to see that

$$\begin{aligned}&\lim_{|I|\to 0}\left(\frac{1}{|I|^p}\int_{S(I)}|f(z)|^2|g'(z)|^2(1-|z|^2)^p dm(z)\right)^{1/2}\\ &\lesssim \|f\|_{\mathcal{L}^{2,\lambda}}\cdot\lim_{|I|\to 0}\left(\frac{1}{|I|^p}\int_{S(I)}|g'(z)|^2(1-|z|^2)^{p+\lambda-1}dm(z)\right)^{1/2}=0.\end{aligned}$$

In conclusion, $T_g:\mathcal{L}_0^{2,\lambda}\to B_0$ is bounded. □

Lemma 5.4. *Suppose that* $0<\lambda<1$, $1<p$ *and* $g\in B\mathcal{L}_0^{p,\lambda}$ *and the operator* $T_{g,r}:\mathcal{L}_0^{2,\lambda}\to B_0$ *satisfies*

$$T_{g,r}f(z)=\int_0^z f(rw)g'(w)\,dw,$$

where $r\in(0,1)$. *Then* $T_{g,r}:\mathcal{L}_0^{2,\lambda}\to B_0$ *is compact.*

Proof. Since $g\in B\mathcal{L}_0^{p,\lambda}$, it follows from Lemma 4.3 that $T_{g,r}:\mathcal{L}^{2,\lambda}\to B$ is compact, and hence $T_{g,r}:\mathcal{L}_0^{2,\lambda}\to B$ is compact. As a matter of fact, in Theorem 5.3, if $g\in B\mathcal{L}_0^{p,\lambda}$, then for any $f\in\mathcal{L}_0^{2,\lambda}$, $T_gf\in B_0$. Together with Lemma 4.1, we conclude $f\in\mathcal{L}_0^{2,\lambda}$, $T_{g,r}f\in B_0$, so that $T_{g,r}:\mathcal{L}_0^{2,\lambda}\to B_0$ is compact. □

Theorem 5.5. *Let* $0<\lambda<1$ *and* $g\in H(\mathbb{D})$. *If* $T_g:\mathcal{L}_0^{2,\lambda}\to B_0$ *is bounded, then*

$$\|T_g\|_{e,\mathcal{L}_0^{2,\lambda}\to B_0}\approx\limsup_{|a|\to 1}\left(\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\overline{a}z|^2}\right)^{p+1-\lambda}|g'(z)|^2(1-|z|^2)^p dm(z)\right)^{1/2}.$$

Proof. Noting that $F_n(z)\in\mathcal{L}_0^{2,\lambda}$ and the compact operator $T_{g,r}$ in Lemma 5.4, we can obviously complete the proof as the same as in proof of Theorem 4.2. □

The following corollary is an immediate consequence of the above theorem.

Corollary 5.6. *Let* $0 < \lambda < 1$ *and* $p > 1$. *Then*

(i) $I_g : \mathcal{L}_0^{2,\lambda} \to B_0$ *is compact if and only if* $g = 0$.

(ii) $T_g : \mathcal{L}_0^{2,\lambda} \to B_0$ *is compact if and only if* $T_g : \mathcal{L}_0^{2,\lambda} \to B_0$ *is bounded if and only if* $g \in B\mathcal{L}_0^{p,\lambda}$.

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