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ABSTRACT VOLTERRA EQUATIONS WITH STATE-DEPENDENT DELAY

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ABSTRACT. By using the theory of resolvent families, fixed point theorems and measures of noncompactness, we prove the existence of mild solutions on a compact interval for a semilinear Volterra equation with state-dependent delay. An example is given.

1. Introduction. State-dependent delay equations arise in many applications but fall outside the scope of the rapidly maturing theory of fixed delay equations, and give rise to challenging problems in both the mathematical analysis of the equations and the numerical computation and analysis of solutions. See **[1, 4, 10, 13, 17, 23, 26, 31, 32]** and references therein. Although progress has been made in recent years on some model state-dependent problems, in particular monotone problems with positive or negative feedback, the behavior of more general and realistic systems remains poorly understood.

On the other hand, the theory of abstract Volterra equations has undergone rapid development. To a large extent, this is due to the applications of this theory to problems in mathematical physics, such as viscoelasticity, heat conduction in materials with memory and electrodynamics with memory **[28]**. Many interesting phenomena not found with differential equations but observed in specific examples of Volterra type stimulated research and improved our understanding and knowl-

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edge. This process is still going on, in particular concerning nonlinear problems.

Of concern in this paper is the study of the following class of abstract Volterra equations with state-dependent delay,

$$(1.1)\qquad y(t) - \int_0^t a(t-s)Ay(s)\,ds = f(t, y_{\rho(t,y_t)}), \quad t \in J = [0,b]$$

$$(1.2)\qquad y_0 = \phi \in \mathcal{B},$$

where $a \in L^1_{\text{loc}}(\mathbb{R}_+)$, $A : D(A) \subset E \to E$ is a closed linear operator defined on a Banach space $(E, |\cdot|)$, $f : J \times \mathcal{B} \to E$ and $\rho : J \times \mathcal{B} \to (-\infty, b]$ are suitable functions defined on a phase space $\mathcal{B}$, and $y_t(\theta) = y(t+\theta)$ represents the history of the state from $\theta \in (-\infty, 0]$ up to the present time $t \geq 0$ for any continuous function y defined on $(-\infty, b]$, being $y_t(\cdot) \in \mathcal{B}$.

Abstract evolution equations with state-dependent delay have been studied only recently [**7, 8, 11, 14, 20, 19, 25**]. For example, Benchohra, Litimein and N'Guérékata [**8**] established the existence of solutions for a class of fractional integro-differential inclusions with state-dependent delay in Banach spaces. Hernández [**18**] studied the existence of mild solutions of an initial-value problem for a second-order semilinear functional differential equation with state-dependent delay in a Banach space X. In [**18**], the equation is governed by a linear operator that generates a strongly continuous cosine family in X. Li and Li [**24**] obtained the existence of a periodic solution of an abstract partial functional differential equation which attracts all the solutions exponentially under some assumptions.

Recently, using the technique of measures of noncompactness, Benchohra, Litimein, Trujillo and Velasco [**9**] studied the existence and uniqueness of solutions on a compact interval for (1.1) in the case $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ for $1 < \alpha < 2$. In that paper, the operator A is assumed to be the generator of certain solution operator, defined in [**5**], and closely related to the fractional character of the equation. Notably, the same problem was also studied by Agarwal, De Andrade and Siracusa [**2**]. However, they assumed that A is a sectorial operator defined on a Banach space.

Since abstract one-term fractional differential equations can be modeled as abstract integral equations with the particular kernel $a(t) =$

$g_\alpha(t) := \frac{t^{\alpha-1}}{\Gamma(\alpha)}, \alpha > 0$ [**28**], it is natural to ask if the results of [**9**] remain true when such a kernel is replaced by a more general one.

In this paper, we give an affirmative answer to this question, widely extending the results in [**9**] from the fractional case to the integral one. The main novelty of this work, compared with [**9**], is the flexibility of the hypothesis that we assume on the family of operators that admits the linearized part of (1.1), which is defined by means of the operator A and the kernel $a(t)$. For instance, the operator A can be either the generator of a C_0-semigroup and the kernel $a(t)$ completely positive, or A can be the generator of a bounded analytic semigroup and the kernel $a(t)$ completely monotonic.

Using the technique of measures of noncompactness, a fixed point theorem of Mönch [**27**] and the representation of the solution for the linearization of (1.1) by means of an operator-theoretical approach due to Prüss [**28**], we give first results on existence of mild solutions of (1.1) on a compact interval, under some additional assumptions on the involved data (Theorem 3.3). We finish this paper with an illustrative example.

2. Preliminaries. Let E be a complex Banach space. We denote by $B(E)$ the Banach space of bounded linear operators from E into E. The set of all continuous functions $y : J \to E$ is denoted by $C(J, E)$. This is a Banach space with the sup-norm $\|y\|_\infty = \sup\{|y(t)| : t \in J\}$. We denote by $L^1(J, E)$ the Banach space of all measurable functions $y : J \to E$ with the norm $\|y\|_{L^1} = \int_0^b |y(t)|\, dt$.

We also denote by $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ the complete seminormed linear space of functions $y : (-\infty, 0] \to E$, satisfying the following axiom (see [**16**], [**22**]):

($*$) If $y : (-\infty, b) \to E, b > 0$, is continuous on J and $y_0 \in \mathcal{B}$, then for every $t \in J$:
(i) $y_t \in \mathcal{B}$ and y_t is a $\mathcal{B}-$valued continuous function on J;
(ii) There exists a positive constant H such that $|y(t)| \leq H\|y_t\|_{\mathcal{B}}$;
(iii) There exist two functions $K, M : \mathbb{R}_+ \to \mathbb{R}_+$, K continuous and M locally bounded, such that:

$$\|y_t\|_{\mathcal{B}} \leq K(t) \sup\{\ |y(s)| : 0 \leq s \leq t\} + M(t)\|y_0\|_{\mathcal{B}}.$$

We will denote $K_b = \sup\{K(t) : t \in J\}$ and $M_b = \sup\{M(t) : t \in J\}$.

Definition 2.1 ([**28**])**.** Let $a \in L^1_{\text{loc}}(\mathbb{R}_+)$ be Laplace transformable, that is, $\widehat{a}(\lambda) = \int_0^\infty e^{-\lambda t} a(t)\, dt$ exists and is absolutely convergent for $Re(\lambda) > \omega$. A closed and linear operator A with domain $D(A)$ defined on a Banach space E is called the *generator* of a resolvent family if there exists $\omega > 0$ and a strongly continuous function $S : \mathbb{R}_+ \to B(E)$ such that

$$\frac{1}{\widehat{a}(\lambda)} \in \rho(A) \quad \text{for all } \mathrm{Re}\,(\lambda) > \omega$$

and

$$\frac{1}{\widehat{a}(\lambda)}\left(\frac{1}{\widehat{a}(\lambda)} - A\right)^{-1} x = \int_0^\infty e^{-\lambda t} S(t)x\, dt, \quad \mathrm{Re}\,\lambda > \omega,\ x \in E.$$

In this case, $\{S(t)\}_{t \geq 0}$ is called the resolvent family generated by A.

Remark 2.2. In [**28**], necessary and sufficient conditions are developed for the existence and regularity of $\{S(t)\}_{t\geq 0}$ as well as for its integrability (in several different senses) on the positive half-line. By means of the variation of parameters formulas, the resolvent is then used as an instrument for determining the solutions of the linearized version of (1.1). The role of the resolvent becomes clear here when $a(t) = 1$ or $a(t) = t$ and $S(t)$ is, respectively, the semigroup or cosine family formally generated by A. By means of the subordination principle, we have the following criteria: Let A be the generator of a C_0-semigroup and suppose that the kernel $a(t)$ is completely positive, i.e., the solution of the scalar equation $s(t) - \mu \int_0^t s(t-s)a(s)\, ds = 1$ is positive and nonincreasing for each $\mu > 0$. Then A is the generator of a resolvent family [**28,** Theorem 4.2]. An analogous result holds in case A is the generator of a cosine family [**28,** Theorem 4.3]. Other interesting criteria is the following: Let A be the generator of a bounded analytic semigroup, and suppose that the kernel $a \in C(0,\infty) \cap L^1(0,1)$ is completely monotonic, i.e., can be represented as Laplace transform of a positive measure. Then A is the generator of an (analytic) resolvent family [**28,** Corollary 2.4].

Definition 2.3 ([**6**])**.** Let E be a Banach space and Ω_E the family of bounded subsets of E. The Kuratowski measure of noncompactness is the map $\alpha : \Omega_E \to [0,\infty)$ defined by $\alpha(B) = \inf\{\epsilon > 0 : B \subseteq \cup_{i=1}^n B_i$ and $\operatorname{diam}(B_i) \leq \epsilon\}$.

Proposition 2.4 ([**6**]). *The Kuratowski measure of noncompactness satisfies the following properties.*

(a) $\alpha(B) = 0 \Leftrightarrow \overline{B}$ *is compact* (B *relatively compact*).
(b) $\alpha(B) = \alpha(\overline{B})$.
(c) $A \subset B \Rightarrow \alpha(A) \leq \alpha(B)$.
(d) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$.
(e) $\alpha(cB) = |c|\alpha(B)$; $c \in \mathbb{R}$.
(f) $\alpha(\operatorname{conv} B) = \alpha(B)$.

Theorem 2.5 ([**3, 27**]). *Let D be a bounded, closed and convex subset of a Banach space such that $0 \in D$, and let N be a continuous mapping of D into itself. If ($V = \overline{\operatorname{conv}} N(V)$ or $V = N(V) \cup 0 \Rightarrow \alpha(V) = 0$) for every subset V of D, then N has a fixed point.*

Lemma 2.6 ([**15**]). *If $H \subset C(J, E)$ is a bounded and equicontinuous set and $H(s) = \{x(s) : x \in H\}$, $s \in J$, then*

(i) $\alpha(H(b)) = \sup_{0\leq t\leq b} \alpha(H(t))$.
(ii) $\alpha(\int_0^t x(s)\,ds : x \in H) \leq \int_0^t \alpha(H(s))\,ds$ *for* $t \in J$.

Theorem 2.7 ([**30**]). *Let $\mathcal{B}$ satisfy the axiom $(*)$. Let $H \subset C(J, E)$. Then $\alpha(\{\overline{y}_t : \overline{y} \in H\}) \leq K(t)\alpha(H)$ if H bounded, and also $\alpha(\{\overline{y}_t : \overline{y} \in H\}) \leq K(t) \sup_{0\leq s\leq t} \alpha(H(s))$ if H is equicontinuous.*

3. Main result. The following definition is motivated by the representation of the solution of (1.1) in the linear case.

Definition 3.1. A function $y : (-\infty, b] \to E$ is called a mild solution of (1.1)–(1.2) if $y_0 = \phi$, $y_{\rho(s,y_s)} \in \mathcal{B}$ for every $s \in J$ and

$$(3.1)\quad y(t) = S(t)\phi(0) + \int_0^t S(t-s)\ f(s, y_{\rho(s,y_s)})\,ds, \quad \text{for each } t \in J.$$

Set $\mathcal{R}(\rho^-) = \{\rho(s,\phi) : (s,\phi) \in J \times \mathcal{B}, \rho(s,\phi) \leq 0\}$. Let us consider a continuous function $\rho : J \times \mathcal{B} \to (-\infty, b]$ and the following hypothesis:

(H_ϕ) The function $t \to \phi_t$ is continuous from $\mathcal{R}(\rho^-)$ into $\mathcal{B}$, and there exists a continuous and bounded function $L^\phi : \mathcal{R}(\rho^-) \to (0,\infty)$ such that

$$\|\phi_t\|_{\mathcal{B}} \leq L^\phi(t)\|\phi\|_{\mathcal{B}} \quad \text{for every } t \in \mathcal{R}(\rho^-).$$

We will need the following lemma.

Lemma 3.2 ([**21**]). *Suppose* (H_ϕ), *and let* $y : (-\infty, b] \to E$ *be such that* $y_0 = \phi$ *and* $L^\phi = \sup_{t\in\mathcal{R}(\rho^-)} L^\phi(t)$. *Then*

$$\|y_s\|_{\mathcal{B}} \leq (M_b + L^\phi)\|\phi\|_{\mathcal{B}} + K_b \sup\{|y(\theta)|; \theta \in [0, \max\{0, s\}]\},$$
$$s \in \mathcal{R}(\rho^-) \cup J.$$

Our main result in this paper reads as follows.

Theorem 3.3. *Let us suppose* (H_ϕ) *and the following hypotheses*:

($H1$) *The operator A is the generator of a resolvent family* $\{S(t)\}_{t\geq 0}$.

($H2$) *The function* $f : J \times \mathcal{B} \to E$ *is Carathéodory, that is,* $f(t,\cdot) : \mathcal{B} \to E$ *is continuous and* $f(\cdot, y) : J \to E$ *is measurable for each* $t \in J$ *and* $y \in \mathcal{B}$, *respectively.*

($H3$) *There exist functions* $p \in L^1(J;\mathbb{R}_+)$, $p(0) = 0$, *and* $\psi : \mathbb{R}_+ \to (0,\infty)$ *continuous nondecreasing satisfying*

$$|f(t,u)| \leq p(t)\ \psi(\|u\|_{\mathcal{B}}) \text{ for a.e. } t \in J \text{ and each } u \in \mathcal{B}.$$

($H4$) *For each* $t \in J$ *and bounded* $B \subset \mathcal{B}$, $\alpha(f(t,B)) \leq p(t)\alpha(B)$.

($H5$) *There exists* $r > 0$ *such that* $M\|\phi\|_{\mathcal{B}} + M\psi((M_b + L^\phi)\|\phi\|_{\mathcal{B}} + K_b r)\int_0^b p(s)\,ds \leq r$, *where* $M := \sup_{t\in J:=[0,b]} \|S(t)\|$.

Then the problem (1.1)–(1.2) *has at least one solution on* $(-\infty, b]$.

Proof. Let $Y := \{u \in C(J,E) : u(0) = \phi(0) = 0\}$ be endowed with the uniform convergence topology. The goal is to apply Theorem 2.5 to the operator $N : Y \to Y$ defined by

$$N(y)(t) = S(t)\,\phi(0) + \int_0^t S(t-s)\,f(s, \overline{y}_{\rho(s,\overline{y}_s)})\,ds, \quad t \in J,$$

where $\overline{y} : (-\infty, b] \to E$ verifies $\overline{y}_0 = \phi$ and $\overline{y} = y$ on J. Let $B_r = \{y \in Y : \|y\|_\infty \leq r\}$ be a closed, bounded and convex subset of Y, with r defined in $(H5)$. Then we have:

Step 1. N is continuous on B_r. Let y^n be a sequence such that $y^n \to y$ in Y. Then:

$$\begin{aligned}
&|N(y^n)(t) - N(y)(t)| \\
&= \left| \int_0^t S(t-s)[f(s, \overline{y^n}_{\rho(s,\overline{y^n}_s)}) - f(s, \overline{y}_{\rho(s,\overline{y}_s)})]\, ds \right| \\
&\leq M \int_0^t |f(s, \overline{y^n}_{\rho(s,\overline{y^n}_s)}) - f(s, \overline{y}_{\rho(s,\overline{y}_s)})|\, ds
\end{aligned}$$

that tends to 0 as $n \to +\infty$ by $(H2)$ and then, by the Lebesgue dominated convergence theorem, N is continuous.

Step 2. N maps B_r into itself. To prove this step, we consider hypothesis (H_ϕ) and Lemma 3.2. Thus, for $y \in B_r$, $\|\overline{y}_{\rho(t,\overline{y}_t)}\|_{\mathcal{B}} \leq (M_b + L^\phi)\|\phi\|_{\mathcal{B}} + K_b r$ and, by (H3) and (H5), we have, for each $t \in J$,

$$\begin{aligned}
|N(y)(t)| &\leq M\|\phi\|_{\mathcal{B}} + M \int_0^t |f(s, \overline{y}_{\rho(s,\overline{y}_s)})|\, ds \\
&\leq M\|\phi\|_{\mathcal{B}} + M \int_0^t p(s)\, \psi\left(\|\overline{y}_{\rho(s,\overline{y}_s)}\|_{\mathcal{B}}\right)\, ds \\
&\leq M\|\phi\|_{\mathcal{B}} + M\, \psi\left((M_b + L^\phi)\|\phi\|_{\mathcal{B}} + K_b r\right) \\
&\quad \times \int_0^b p(s)\, ds \leq r.
\end{aligned}$$

Step 3. $N(B_r)$ is bounded and equicontinuous.

By Step 2, $N(B_r) \subset B_r$ is bounded.

To prove the equicontinuity of $N(B_r)$, let $y \in B_r$ and $\tau_1, \tau_2 \in J$, $\tau_2 > \tau_1$. Then, by $(H3)$, Lemma 3.2 and $(H1)$,

$$\begin{aligned}
&|N(y)(\tau_2) - N(y)(\tau_1)| \\
&\leq |S(\tau_2) - S(\tau_1)|\|\phi\|_{\mathcal{B}} \\
&\quad + \int_0^{\tau_1} |S(\tau_2 - s) - S(\tau_1 - s)||f(s, \overline{y}_{\rho(s,\overline{y}_s)})|\, ds
\end{aligned}$$

$$
\begin{aligned}
&+ \int_{\tau_1}^{\tau_2} |S(\tau_2 - s) f(s, \overline{y}_{\rho(s,\overline{y}_s)})| \, ds \\
\leq\; & |S(\tau_2) - S(\tau_1)| \|\phi\|_{\mathcal{B}} \\
&+ \psi\left((M_b + L^{\phi})\|\phi\|_{\mathcal{B}} + K_b r\right) \\
&\times \int_0^{\tau_1} |S(\tau_2 - s) - S(\tau_1 - s)| p(s) \, ds \\
&+ M\psi\left((M_b + L^{\phi})\|\phi\|_{\mathcal{B}} + K_b r\right) \\
&\times \int_{\tau_1}^{\tau_2} p(s) \, ds \quad \xrightarrow[\tau_2 - \tau_1 \to 0]{} 0.
\end{aligned}
$$

Step 4. $\alpha(V) = 0$ for each $V \in B_r$ such that $V \subset \overline{\mathrm{conv}}(N(V) \cup \{0\})$.

Since $V \in B_r$ is bounded and equicontinuous, $v : t \to \alpha(V(t))$ is continuous on J. Then, by Proposition 2.4, for each $t \in J$ we have:

$$
\begin{aligned}
v(t) &\leq \alpha(N(V)(t) \cup \{0\}) \leq \alpha(N(V)(t)) \\
&\leq \alpha\left(S(t)\phi(0)\right) \\
&\quad + \alpha\left(\int_0^t S(t-s) f(s, \overline{y}_{\rho(s,\overline{y}_s)}) \, ds\right),
\end{aligned}
$$

where $\alpha(S(t)\phi(0)) = 0$. Hence, by Lemma 2.6, $(H4)$ and Theorem 2.7, we obtain

$$
\begin{aligned}
v(t) &\leq M \int_0^t p(s) \alpha(\overline{y}_{\rho(s,y_s)} : \overline{y} \in V) \, ds \\
&\leq M \int_0^t p(s) K(s) \sup_{0 \leq \tau \leq s} \alpha(V(\tau)) \, ds \\
&= M \int_0^t p(s) K(s) v(s) \, ds,
\end{aligned}
$$

which implies that $v(t) = 0$ by Gronwall's lemma, and then $V(t)$ is relatively compact in E by Proposition 2.4. Now V is relatively compact in B_r by the Ascoli-Arzelà theorem, and then $\alpha(V) = 0$. Therefore, Theorem 2.5 establishes that N has a fixed point $y \in B_r$, which is solution to the problem (1.1)–(1.2). □

Remark 3.4. If we use Darbo's fixed point theorem instead of Theorem 2.5 and we consider the Kuratowski measure of noncompactness

α_C defined on the family of bounded subsets of the space $C(J,E)$ by

$$\alpha_C(H) = \sup_{t\in J} e^{-\tau L(t)} \alpha(H(t)),$$

where

$$L(t) = \int_0^t \widetilde{l}(s)\, ds, \qquad \widetilde{l}(t) = Ml(t)K(t), \quad \tau > 1.$$

Then $(H4)$ could be replaced by

$(H4)^*$ There exists a nonnegative function $l \in L^1(J, \mathbb{R}^+)$ such that

$$\alpha(f(t,B)) \le l(t)\alpha(B), \quad t \in J.$$

Taking into account Remark 2.2, the following corollaries are immediate. Note that they give simpler hypothesis, based only on the data of the equation.

Corollary 3.5. *Let us suppose* (H_ϕ) *and that* $(H2)$–$(H5)$ *are satisfied. Assume*:

$(H1)^*$ *A is the generator of a* C_0*-semigroup and the kernel* $a(t)$ *is completely positive, or*

$(H1)^{**}$ *A is the generator of a bounded analytic semigroup and the kernel* $a \in C(0,\infty) \cap L^1(0,1)$ *is completely monotonic.*

Then the problem (1.1)–(1.2) *has at least one solution on* $(-\infty, b]$.

4. An example. In this section, we give a simple example to illustrate the feasibility of the assumptions made. For $\sigma \in C(\mathbb{R}, [0,\infty))$,

$$a(t) = \int_0^\infty \frac{t^{\rho-1}}{\Gamma(\rho)}\, d\rho$$

and

$$L_\xi = \frac{\partial^2}{\partial \xi^2} - r, \quad r > 0,$$

we consider the following Volterra equation with state-dependent delay

$$(4.1)\qquad \begin{cases} u(t,\xi) - \int_0^t a(t-s)L_\xi u(s,\xi)\,ds \\ = \frac{te^{-\gamma t+t}|u(t-\sigma(u(t,0)),\xi)|}{3(e^{-t}+e^t)(1+|u(t-\sigma(u(t,0),\xi))|)}, \quad t\in[0,b],\ \xi\in[0,\pi], \\ u(t,0)=u(t,\pi)=0, \quad t\in[0,b], \\ u(\theta,\xi)=u_0(\theta,\xi), \quad \theta\in(-\infty,0],\ \xi\in[0,\pi]. \end{cases}$$

We take

- $E = L^2([0,\pi],\mathbb{R})$.
- $\mathcal{B} := \mathcal{B}_\gamma = \{\phi \in C((-\infty,0],\mathbb{R}) : \lim_{\theta\to-\infty} e^{\gamma\theta}\phi(\theta) \text{ exists}\}$ with $\|\phi\|_{\mathcal{B}_\gamma} = \sup_{\theta\in(-\infty,0]} e^{\gamma\theta}|y(\theta)|$, $\gamma > 0$. Note that $\mathcal{B}$ satisfies the axioms $(*)$ (see **[22]** for details).
- $A = L_\xi : D(A) = \{u \in E : u'' \in E, u(0) = u(\pi) = 0\} \subset E \to E$ that is densely defined in E and is sectorial. By **[28,** Example 2.3] the operator A is a generator of a solution operator on E, and hence $(H1)$ is verified.

Set $y(t)(\xi) := u(t,\xi)$, $t \in [0,b]$, $\xi \in [0,\pi]$, $\phi(\theta)(\xi) := u_0(\theta,\xi)$, $t \in [0,b]$, $\theta \le 0$, $\rho(t,\phi) = t - \sigma(\phi(0,0))$ and

$$f(t,\phi)(\xi) = \frac{te^{-\gamma t+t}\phi(0,\xi)}{3(e^{-t}+e^t)(1+\phi(0,\xi))}, \quad t\in[0,\infty),\ \xi\in[0,\pi].$$

Is not difficult to see that the function f satisfies $(H2)$–$(H5)$ (with $\psi \equiv 1$ and $p(t) = \frac{te^{-\gamma t+t}}{3(e^{-t}+e^t)}$). Let $\phi \in \mathcal{B}_\gamma$ be such that (H_ϕ) holds, and let $t \to \phi_t$ be continuous on $\mathcal{R}(\rho^-)$. Then, by Theorem 3.3, there exists at least one mild solution of (4.1).

Remark 4.1. Completely positive kernels have been introduced in the paper of Clement and Nohel **[12]**. This class of kernels is very natural for Volterra equations with the A generator of a C_0-semigroup. We recall from **[29,** page 326] that completely positive kernels are characterized by the property that the functions

$$f(\lambda) := \frac{1}{\lambda\widehat{a}(\lambda)} \quad \text{and} \quad g(\lambda) := \frac{-\widehat{a}'(\lambda)}{\widehat{a}(\lambda)^2}$$

are completely monotonic on $(0,\infty)$ (the hat indicates Laplace transform). Using this characterization, we easily check that the kernel $a(t) = g_\alpha(t)$ is completely positive for $0 < \alpha < 1$. Hence, the results of the above example remain true for $a(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)}$ in this case, complementing the results of [**9**].

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REFERENCES

1. S. Abbas and M. Benchohra, *Impulsive partial hyperbolic functional differential equations of fractional order with state-dependent delay*, Fract. Calc. Appl. Anal. **13** (2010), 225–244.

2. R.P. Agarwal, B. de Andrade and G. Siracusa, *On fractional integro-differential equations with state-dependent delay*, Comp. Math. Appl. **62** (2011), 1143–1149.

3. R.P. Agarwal, M. Meechan and D. O'Regan. *Fixed point theory and applications*, Cambridge University Press, Cambridge, 2001.

4. E. Ait Dads and K. Ezzinbi, *Boundedness and almost periodicity for some state-dependent delay differential equations*, Electron. J. Differ. Equat. **2002** (2002), 1–13.

5. D. Araya and C. Lizama, *Almost automorphic mild solutions to fractional differential equations*, Nonlin. Anal. **69** (2008), 3692–3705.

6. J. Bana's and K. Goebel, *Measures of noncompactness in Banach spaces*, Marcel Dekker, New York, 1980.

7. M. Belmekki, M. Benchohra and K. Ezzinbi, *Existence results for some partial functional differential equations with state-dependent delay*, Appl. Math. Lett. **24** (2011), 1810–1816.

8. M. Benchohra, S. Litimein and G.M. N'Guérékata, *On fractional integro-differential inclusions with state-dependent delay in Banach spaces*, Appl. Anal. **92** (2013), 335–350.

9. M. Benchohra, S. Litimein, J.J. Trujillo and M.P. Velasco, *Abstract fractional integro-differential equations with state-dependent delay*, Int. J. Evol. Equat. **6** (2012), 25--38.

10. M. Benchohra, I. Medjadj, J.J. Nieto and P. Prakash, *Global existence for functional differential equations with state-dependent delay*, J. Funct. Spaces Appl. **2013**, Art. ID 863561, 7 pages.

11. M. Benchohra and M. Ziane, *Impulsive evolution inclusions with state-dependent delay and multivalued jumps*, Electr. J. Qual. Theor. Diff. Equat. **42** (2013), 21 pp.

12. P. Clement and J.A. Nohel, *Abstract linear and nonlinear Volterra equations preserving positivity*, SIAM J. Math. Anal. **10** (1979), 365–388.

13. J. Dabas and G.R. Gautam, *Impulsive neutral fractional integro-differential equations with state dependent delays and integral condition*, Electr. J. Differ. Equat. **273** (2013), 1–13.

14. J.P. dos Santos, M.M. Arjunan and C. Cuevas, *Existence results for fractional neutral integro-differential equations with state-dependent delay*, Comp. Math. Appl. **62** (2011), 1275–1283.

15. D.J. Guo, V. Lakshmikantham and X. Liu, *Nonlinear integral equations in abstract spaces*, Kluwer Academic Publishers, Dordrecht, 1996.

16. J. Hale and J. Kato, *Phase space for retarded equations with infinite delay*, Funkc. Ekvac. **21** (1978), 11–41.

17. F. Hartung, *On differentiability of solutions with respect to parameters in neutral differential equations with state-dependent delays*, Ann. Mat. Pura Appl. **192** (2013), 17-47.

18. E. Hernández, *Existence of solutions for a second order abstract functional differential equation with state-dependent delay*, Electr. J. Differ. Equat. **2008** (2007), 1–10.

19. E. Hernández, M. Pierri and G. Goncalves, *Existence results for an impulsive abstract partial differential equation with state-dependent delay*, Comp. Math. Appl. **52** (2006), 411–420.

20. E. Hernández, A. Prokopczyk and L. Ladeira, *A note on partial functional differential equations with state-dependent delay*, Nonlin. Anal. Real World Appl. **7** (2006), 510–519.

21. E. Hernández, R. Sakthivel and S.A. Tanaka, *Existence results for impulsive evolution differential equations with state-dependent delay*, Electr. J. Differ. Equat. **2008** (2008), 1–11.

22. Y. Hino, S. Murakami and T. Naito, *Functional differential equations with unbounded delay*, Springer-Verlag, Berlin, 1991.

23. W.S. Li, Y.K. Chang and J.J. Nieto, *Solvability of impulsive neutral evolution differential inclusions with state-dependent delay*, Math. Comp. Model. **49** (2009), 1920–1927.

24. X. Li and Z. Li, *Periodic solutions of an abstract PFDE with state-dependent delay*, Math. Appl. (Wuhan) **24** (2011), 158–163.

25. M. Louihi, M.L. Hbid and O. Arino, *Semigroup properties and the Crandall Liggett approximation for a class of differential equations with state-dependent delays*, J. Differ. Equat. **181** (2002), 1–30.

26. F.M.G. Magpantay, N. Kosovalić and J. Wu, *An age-structured population model with state-dependent delay*: *Derivation and numerical integration*, SIAM J. Numer. Anal. **52** (2014), 735–756.

27. H. Mönch, *Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces*, Nonlin. Anal. **4** (1980), 985--999.

28. J. Prüss, *Evolutionary integral equations and applications*, Mono. Math. **87**, Bikhäuser Verlag, Berlin, 1993.

29. J. Prüss, *Positivity and regularity of hyperbolic Volterra equations in Banach spaces*, Math. Ann. **279** (1987), 317–344.

30. J.S. Shin, *An existence theorem of functional differential equations with infinite delay in a Banach space*, Funkc. Ekvac. **30** (1987), 19–29.

31. B. Slezák, *On the smooth parameter-dependence of the solutions of abstract functional differential equations with state-dependent delay*, Funct. Differ. Equat. **17** (2010), 253–293.

32. Z. Yan and H. Zhang, *Asymptotic stability of fractional impulsive neutral stochastic partial integro-differential equations with state-dependent delay*, Electr. J. Differ. Equat. **206** (2013), 1–29.

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