JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 27, Number 1, Spring 2015

ASYMPTOTIC ERROR ANALYSIS OF PROJECTION AND MODIFIED PROJECTION METHODS FOR NONLINEAR INTEGRAL EQUATIONS

REKHA P. KULKARNI AND T.J. NIDHIN

Communicated by Kendall Atkinson

ABSTRACT. Consider a nonlinear operator equation $x - K(x) = f$, where K is a Urysohn integral operator with a smooth kernel. Using the orthogonal projection onto a space of discontinuous piecewise polynomials of degree $\leq r$, previous authors have established an order $r+1$ convergence for the Galerkin solution, $2r+2$ for the iterated Galerkin solution, $3r+3$ for the modified projection solution and $4r+4$ for the iterated modified projection solution. Equivalent results have also been established for the interpolatory projection at Gauss points. In this paper, the iterated Galerkin/iterated collocation solution and the iterated modified projection solution are shown to have asymptotic series expansions. The Richardson extrapolation can then be used to improve the order of convergence to $2r+4$ in the case of the iterated Galerkin/iterated collocation method and to $4r+6$ in the case of the iterated modified projection method. Numerical results are given to illustrate this improvement in the orders of convergence.

1. Introduction. Let $X = L^\infty[0,1]$, and consider a Urysohn integral operator

$$K(x)(s) = \int_0^1 \kappa(s,t,x(t))\,dt, \quad s \in [0,1],\ x \in X,$$

where the kernel $\kappa(s,t,u)$ is a real valued continuous function. For $f \in X$, we are interested in a solution of

$$x - K(x) = f. \tag{1.1}$$

2010 AMS *Mathematics subject classification.* Primary 45L10, 65J15, 65R20.
Keywords and phrases. Urysohn integral operator, Galerkin method, collocation at Gauss points, extrapolation.
Received by the editors on June 25, 2014, and in revised form on October 2, 2014.
DOI:10.1216/JIE-2015-27-1-67

We assume that the above equation has a unique solution φ. Let a and b be real numbers such that

$$\left[\min_{s\in[0,1]} \varphi(s), \max_{s\in[0,1]} \varphi(s)\right] \subset (a,b).$$

Define

$$\Omega = [0,1] \times [0,1] \times [a,b].$$

For $r \geq 0$, let X_n be the space of piecewise polynomials of degree $\leq r$ with respect to a uniform partition of $[0,1]$ with n subintervals. Let $h = 1/n$. We consider two types of projections from $L^\infty[0,1]$ to X_n. Let π_n be either the restriction to $L^\infty[0,1]$ of the orthogonal projection from $L^2[0,1]$ to X_n or the interpolatory projection at $r+1$ Gauss points in each subinterval of the partition. If π_n is the orthogonal projection, then in the classical Galerkin method, (1.1) is approximated by

$$\varphi_n^G - \pi_n K(\varphi_n^G) = \pi_n f.$$

If π_n is an interpolatory projection, then the collocation solution φ_n^C is obtained by solving

$$\varphi_n^C - \pi_n K(\varphi_n^C) = \pi_n f.$$

The above projection methods have been studied in research literature. See Krasnoselskii **[12]**, Krasnoselskii, Vainikko et al. **[13]** and Krasnoselskii and Zabreiko **[14]**.

The iterated Galerkin solution is defined by

$$\varphi_n^S = K(\varphi_n^G) + f$$

and the iterated collocation solution is defined in a similar fashion.

The iterated projection methods for a Urysohn integral operator with Green's function type kernel are analyzed in **[4]**. Assume that

$$\frac{\partial \kappa}{\partial u} \in C^{r+1}(\Omega), \ \frac{\partial^2 \kappa}{\partial u^2} \in C(\Omega) \quad \text{and} \quad f \in C^{r+1}[0,1].$$

Then the following orders of convergence can be deduced from the error estimates of **[4]**:

$$\|\varphi_n^G - \varphi\|_\infty = O(h^{r+1}), \quad \|\varphi_n^C - \varphi\|_\infty = O(h^{r+1}), \tag{1.2}$$

$$\|\varphi_n^S - \varphi\|_\infty = O(h^{2r+2}). \tag{1.3}$$

In [**8, 9**], the following modified projection method is proposed:

$$\varphi_n^M - \mathcal{K}_n^M(\varphi_n^M) = f, \tag{1.4}$$

where

$$\mathcal{K}_n^M(x) = \pi_n \mathcal{K}(x) + \mathcal{K}(\pi_n x) - \pi_n \mathcal{K}(\pi_n x), \quad x \in X. \tag{1.5}$$

It is a generalization of the modified projection method in the linear case, which was proposed in [**15**].

As in the case of the iterated Galerkin method, we perform one step of iteration and define the iterated modified projection solution as:

$$\widetilde{\varphi}_n^M = \mathcal{K}(\varphi_n^M) + f. \tag{1.6}$$

Let

$$\kappa, \frac{\partial \kappa}{\partial u} \in C^{2r+2}(\Omega) \quad \text{and} \quad f \in C^{2r+2}[0,1].$$

Then, the following orders of convergence for the modified projection solution and its iterated version are proved in [**10**]:

$$\|\varphi_n^M - \varphi\|_\infty = O(h^{3r+3}), \tag{1.7}$$

$$\|\tilde{\varphi}_n^M - \varphi\|_\infty = O(h^{4r+4}). \tag{1.8}$$

In [**7**], a non-linear integral operator with a kernel of the form $\kappa(s,t)\Psi(x(t))$ is considered. The kernel $\kappa(s,t)$ is assumed to be of the type of Green's function, and the approximating operator is considered to be the Nyström operator obtained by replacing the integral by the composite trapezoidal rule. Asymptotic expansions for approximate solutions at the node points are obtained and Richardson extrapolation is employed to improve orders of convergence.

In the case of a linear integral equation of the second kind, asymptotic series expansions for the iterated Galerkin/iterated collocation solutions are proved by McLean [**18**] and, for the iterated modified projection solution, are proved by Kulkarni and Grammont [**16**]. Asymptotic series expansions for the iterated collocation method are also obtained in Lin et al. [**17**]. The aim of this paper is to extend these results to the present case of nonlinear integral equations and obtain asymptotic series expansions for φ_n^S and $\widetilde{\varphi}_n^M$. Richardson extrapolation can then be used to obtain approximate solutions of higher order.

The paper is organized as follows. In Section 2, notation is set and some preliminary results are proved for later use. In Section 3, asymptotic series expansions for the iterated Galerkin/iterated collocation solutions as well as for the iterated modified projection solution associated with both the orthogonal projection and the interpolatory projection at Gauss points are obtained. Numerical solutions are given in Section 4.

2. Preliminaries. Let α be a positive integer. For $x \in C^\alpha[0,1]$, we define

$$\|x\|_{\alpha,\infty} = \sum_{i=0}^{\alpha} \|x^{(i)}\|_\infty,$$

where $x^{(i)}$ denotes the ith derivative of x.

Consider the following uniform partition of $[0,1]$:

$$0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1, \tag{2.1}$$

and let

$$h = \frac{1}{n}.$$

We consider two types of projections from $L^\infty[0,1]$ to X_n.

(1) The map π_n is the restriction to $L^\infty[0,1]$ of the orthogonal projection from $L^2[0,1]$ to X_n.
(2) Let $\tau_0, \tau_1, \ldots, \tau_r$ be the $r+1$ Gauss points in $[0,1]$, and let

$$t_{ij} = (i - 1 + \tau_j)h, \quad i = 1, \ldots, n, \ j = 0, 1, \ldots r,$$

be the collocation points. Define $\pi_n : C[0,1] \to X_n$ as

$$(\pi_n x)(t_{i,j}) = x(t_{i,j}), \quad i = 1, \ldots, n, \ j = 0, 1, \cdots, r.$$

This map is extended to $L^\infty[0,1]$ as in **[3]** and then $\pi_n : L^\infty[0,1] \to X_n$ is a projection.

In both cases,

$$\pi_n x \longrightarrow x, \ x \in C[0,1],$$

and, if $x \in C^{r+1}[0,1]$,

$$\|x - \pi_n x\|_\infty \le C_1 \|x^{(r+1)}\|_\infty h^{r+1}, \tag{2.2}$$

where C_1 is a constant independent of n (see Chatelin **[6]**). Also,

$$\sup_n \|\pi_n\|_{L^\infty[0,1]\to L^\infty[0,1]} < \infty. \tag{2.3}$$

Let $x, y \in C^{r+1}[0,1]$. If π_n is the restriction of the orthogonal projection to $L^\infty[0,1]$, then we deduce the following estimate from (2.2):

$$\begin{aligned} \left| \int_0^1 x(t)(I-\pi_n)y(t)\,dt \right| &= |\langle x,\, (I-\pi_n)\overline{y}\rangle| \\ &= |\langle (I-\pi_n)x,\, (I-\pi_n)\overline{y}\rangle| \\ &\leq (C_1)^2 \|x^{(r+1)}\|_\infty \|y^{(r+1)}\|_\infty h^{2r+2}, \end{aligned} \tag{2.4}$$

where $\overline{y}(t)$ denotes the complex conjugate of $y(t)$.

If π_n is the interpolatory projection at $r+1$ Gauss points, then for $x \in C^{r+1}[0,1]$ and $y \in C^{2r+2}[0,1]$, the following error estimate is proved in **[5]**:

$$\left| \int_0^1 x(t)(I-\pi_n)y(t)\,dt \right| \leq C_2 \|x\|_{r+1,\infty} \|y\|_{2r+2,\infty} h^{2r+2}, \tag{2.5}$$

where C_2 is a constant independent of n.

Assume that

$$\kappa \in C^{2r+2}(\Omega) \quad \text{and} \quad \frac{\partial \kappa}{\partial u} \in C^{2r+4}(\Omega).$$

Then K is a compact operator from $L^\infty[0,1]$ to $C^{2r+2}[0,1]$. As in Section 1, we assume that (1.1) has a unique solution φ. We also assume that $f \in C^{2r+2}[0,1]$. Then, since,

$$\varphi - K(\varphi) = f, \tag{2.6}$$

it follows that $\varphi \in C^{2r+2}[0,1]$.

The operator K is then Fréchet differentiable, and the Fréchet derivative is given by

$$(K'(\psi)g)(s) = \int_0^1 \frac{\partial \kappa}{\partial u}(s,t,\psi(t))\, g(t)\,dt.$$

For $\delta > 0$, let

$$\mathcal{B}(\varphi, \delta) = \{\psi \in X : \|\varphi - \psi\|_\infty \leq \delta\}$$

denote the closed δ neighborhood of φ.

Since $\partial\kappa/\partial u \in C^{2r+4}(\Omega)$, it follows that K' is Lipschitz continuous in a neighborhood in $\mathcal{B}(\varphi,\delta)$ of φ, that is, there exists a constant γ such that

$$\|K'(\varphi) - K'(\psi)\| \leq \gamma \|\varphi - \psi\|_\infty, \quad \psi \in \mathcal{B}(\varphi,\delta). \tag{2.7}$$

Also, the second derivative $K''(\psi)$ is a bi-linear function and is given by

$$(K''(\psi)(g_1,g_2))(s) = \int_0^1 \frac{\partial^2\kappa}{\partial u^2}(s,t,\psi(t))\, g_1(t) g_2(t)\, dt.$$

In general, the ith derivative $K^{(i)}(\psi)$ is i-linear and is given by

$$\left(K^{(i)}(\psi)(g_1,\ldots,g_i)\right)(s) = \int_0^1 \frac{\partial^i\kappa}{\partial u^i}(s,t,\psi(t))\, g_1(t)\ldots g_i(t)\, dt. \tag{2.8}$$

We define

$$\|K^{(i)}(\psi)\| = \sup_{\substack{\|g_j\|_\infty \leq 1 \\ j=1,\ldots,i}} \left\|K^{(i)}(\psi)(g_1,\ldots,g_i)\right\|_\infty, \quad i = 1,\ldots,5.$$

It follows from (2.8) that

$$\|K^{(i)}(\psi)\| \leq \max_{s,t\in[0,1]} \left|\frac{\partial^i\kappa}{\partial u^i}(s,t,\psi(t))\right|, \quad i = 1,\ldots,5. \tag{2.9}$$

Consider

$$(K'(\varphi)g)(s) = \int_0^1 \frac{\partial\kappa}{\partial u}(s,t,\varphi(t))\, g(t)\, dt.$$

As φ is fixed, we write

$$\ell(s,t) = \frac{\partial\kappa}{\partial u}(s,t,\varphi(t)), \quad s,t \in [0,1].$$

Since $\partial\kappa/\partial u \in C^{2r+4}(\Omega)$, it follows that

$$\ell(\cdot,\cdot) \in C^{2r+4}([0,1]\times[0,1]).$$

The operator $K'(\varphi)$ is compact. Assume that 1 is not an eigenvalue of $K'(\varphi)$. Then it can be shown that

$$M = (I - K'(\varphi))^{-1} K'(\varphi)$$

is a compact linear integral operator (see **[17]**) given by

$$(Mg)(s) = \int_0^1 m(s,t)g(t)\,dt,$$

where the smoothness of kernel m is the same as that of kernel ℓ of the integral operator $K'(\varphi)$, that is,

$$m(\cdot,\cdot) \in C^{2r+4}([0,1]\times[0,1]).$$

Let π_n be either the restriction of the orthogonal projection to $L^\infty[0,1]$ or the interpolatory projection at $r+1$ Gauss points.

We prove some preliminary results.

Lemma 2.1. *Let*

$$\frac{\partial\kappa}{\partial u} \in C^{2r+6}(\Omega).$$

Then

$$\|M(I-\pi_n)K^{(i)}(\varphi)\| = O(h^{2r+2}), \quad i=1,\ldots,5. \tag{2.10}$$

Proof. For $i=1,\ldots,5$,

$$K^{(i)}(\varphi)(g_1,\ldots,g_i)(s) = \int_0^1 \frac{\partial^i\kappa}{\partial u^i}(s,t,\varphi(t))\,g_1(t)\ldots g_i(t)\,dt$$

and, for $j=1,\ldots,2r+2$,

$$\Big[K^{(i)}(\varphi)(g_1,\ldots,g_i)\Big]^{(j)}(s) = \int_0^1 \frac{\partial^{i+j}\kappa}{\partial s^j\partial u^i}(s,t,\varphi(t))\,g_1(t)\ldots g_i(t)\,dt.$$

Let

$$C_{2+i} = \sum_{j=0}^{2r+2}\max_{s,t\in[0,1]}\left|\frac{\partial^{i+j}\kappa}{\partial s^j\partial u^i}(s,t,\varphi(t))\right|, \quad i=1,\ldots,5.$$

Then

$$\Big\|K^{(i)}(\varphi)(g_1,\ldots,g_i)\Big\|_{2r+2,\infty} \le C_{2+i}\|g_1\|_\infty\cdots\|g_i\|_\infty, \quad i=1,\ldots,5. \tag{2.11}$$

For a fixed $s \in [0,1]$ and for $i = 1, \ldots, 5$, we have

$$M(I - \pi_n)K^{(i)}(\varphi)(g_1, \ldots, g_i)(s) = \int_0^1 m(s,t)(I - \pi_n)K^{(i)}(\varphi)(g_1, \ldots, g_i)(t)\, dt.$$

Let

$$m_s(t) = m(s,t), \quad t \in [0,1].$$

Using the estimate (2.4) in the case of orthogonal projection and the estimate (2.5) in the case of interpolatory projection, we obtain

$$\left| M(I - \pi_n)K^{(i)}(\varphi)(g_1, \ldots, g_i)(s) \right| \leq C_2 \|m_s\|_{r+1,\infty} \left\| K^{(i)}(\varphi)(g_1, \ldots, g_i) \right\|_{2r+2,\infty} h^{2r+2}.$$

Let

$$\|m\|_{r+1,\infty} = \sum_{j=0}^{r+1} \max_{s,t \in [0,1]} \left| \frac{\partial^j m}{\partial t^j}(s,t) \right|.$$

Then, from (2.11),

$$\left\| M(I - \pi_n)K^{(i)}(\varphi)(g_1, \ldots, g_i) \right\|_\infty \leq C_2 C_{2+i} \|m\|_{r+1,\infty} \|g_1\|_\infty \ldots \|g_i\|_\infty h^{2r+2}$$

and hence, for $i = 1, \ldots, 5$,

$$\left\| M(I - \pi_n)K^{(i)}(\varphi) \right\| = \sup_{\substack{\|g_j\|_\infty \leq 1 \\ j=1,\ldots,i}} \left\| M(I - \pi_n)K^{(i)}(\varphi)(g_1, \ldots, g_i) \right\|_\infty \leq C_2 C_{2+i} \|m\|_{r+1,\infty} h^{2r+2},$$

which completes the proof. □

As the proof of the following result is similar to that of the above lemma, we skip its proof.

Lemma 2.2. *Let*

$$\kappa \in C^{2r+2}(\Omega), \qquad \frac{\partial \kappa}{\partial u} \in C^{3r+4}(\Omega) \quad \textit{and} \quad f \in C^{2r+2}[0,1].$$

Then:

$$\|M(I-\pi_n)K'(\varphi)(I-\pi_n)\varphi\|_\infty = O(h^{4r+4}),$$
$$\|M(I-\pi_n)K'(\varphi)(I-\pi_n)K'(\varphi)\| = O(h^{4r+4}),$$
$$\|M(I-\pi_n)K'(\varphi)(I-\pi_n)K''(\varphi)\| = O(h^{4r+4}),$$
$$\|M(I-\pi_n)K'(\varphi)(I-\pi_n)K^{(3)}(\varphi)\| = O(h^{4r+4}),$$
$$\|M(I-\pi_n)K'(\varphi)(I-\pi_n)K'(\varphi)(I-\pi_n)\varphi\|_\infty = O(h^{6r+6}).$$

Lemma 2.3. *Let*

$$\kappa \in C^{2r+2}(\Omega),\ \frac{\partial \kappa}{\partial u} \in C^{3r+6}(\Omega) \quad \textit{and} \quad f \in C^{2r+2}[0,1].$$

Then

$$\left\| M\left(\left(K_n^M\right)'(\varphi) - K'(\varphi)\right)\left(\varphi_n^M - \varphi\right)\right\|_\infty = O(h^{6r+6}). \tag{2.12}$$

Proof. Recall from (1.5) that

$$K_n^M(\varphi) = \pi_n K(\varphi) + K(\pi_n\varphi) - \pi_n K(\pi_n\varphi).$$

Hence,

$$\left(K_n^M\right)'(\varphi) = \pi_n K'(\varphi) + (I-\pi_n)K'(\pi_n\varphi)\pi_n$$

and

$$\begin{aligned}\left(K_n^M\right)'(\varphi) - K'(\varphi) &= (\pi_n - I)K'(\varphi) + (I-\pi_n)K'(\pi_n\varphi)\pi_n \\ &= -(I-\pi_n)K'(\varphi)(I-\pi_n) \\ &\quad + (I-\pi_n)(K'(\pi_n\varphi) - K'(\varphi))\pi_n.\end{aligned}$$

Thus,

$$\begin{aligned}&M\left(\left(K_n^M\right)'(\varphi) - K'(\varphi)\right)\left(\varphi_n^M - \varphi\right) \\ &\quad = -M(I-\pi_n)K'(\varphi)(I-\pi_n)\left(\varphi_n^M - \varphi\right) \\ &\qquad + M(I-\pi_n)(K'(\pi_n\varphi) - K'(\varphi))\pi_n\left(\varphi_n^M - \varphi\right).\end{aligned} \tag{2.13}$$

Using (1.4) and (2.6), we write the first term in the above expression as:

$$
\begin{aligned}
&-M(I-\pi_n)K'(\varphi)(I-\pi_n)\left[\varphi_n^M-\varphi\right] \\
&\qquad = -M(I-\pi_n)K'(\varphi)(I-\pi_n)\left[K_n^M\left(\varphi_n^M\right)-K(\varphi)\right] \\
&\qquad = -M(I-\pi_n)K'(\varphi)(I-\pi_n) \\
&\qquad\quad \left[K_n^M\left(\varphi_n^M\right)-K_n^M(\varphi)-\left(K_n^M\right)'(\varphi)\left(\varphi_n^M-\varphi\right)\right] \\
&\qquad\quad -M(I-\pi_n)K'(\varphi)(I-\pi_n)\left[K_n^M(\varphi)-K(\varphi)\right] \\
&\qquad\quad -M(I-\pi_n)K'(\varphi)(I-\pi_n)\left[\left(K_n^M\right)'(\varphi)\left(\varphi_n^M-\varphi\right)\right].
\end{aligned} \tag{2.14}
$$

By [**10,** Lemma 3.3],

$$\|K_n^M\left(\varphi_n^M\right)-K_n^M(\varphi)+\left(K_n^M\right)'(\varphi)\left(\varphi_n^M-\varphi\right)\|_\infty = O\left(h^{6r+6}\right)$$

and hence, by Lemma 2.1,

$$
\begin{aligned}
&\|M(I-\pi_n)K'(\varphi)(I-\pi_n) \\
&\left[K_n^M\left(\varphi_n^M\right)-K_n^M(\varphi)-\left(K_n^M\right)'(\varphi)\left(\varphi_n^M-\varphi\right)\right]\|_\infty = O\left(h^{8r+8}\right).
\end{aligned} \tag{2.15}
$$

Next,

$$K_n^M(\varphi)-K(\varphi) = (I-\pi_n)\left[K(\pi_n\varphi)-K(\varphi)\right].$$

Choose n big enough so that $\pi_n\varphi \in \mathcal{B}(\varphi,\delta)$. Then, by Taylor's theorem,

$$
\begin{aligned}
K(\pi_n\varphi)-K(\varphi) &= K'(\varphi)(\pi_n\varphi-\varphi)+\frac{1}{2}K''(\varphi)(\pi_n\varphi-\varphi)^2 \\
&\quad + \frac{1}{6}K^{(3)}(\varphi)(\pi_n\varphi-\varphi)^3+\frac{1}{24}K^{(4)}(\xi)(\pi_n\varphi-\varphi)^4,
\end{aligned}
$$

for some $\xi \in \mathcal{B}(\varphi,\delta)$. Then, by Lemma 2.1 and Lemma 2.2, we obtain

$$\|M(I-\pi_n)K'(\varphi)(I-\pi_n)\left[K_n^M(\varphi)-K(\varphi)\right]\|_\infty = O\left(h^{6r+6}\right). \tag{2.16}$$

Consider the third term:

$$
\begin{aligned}
&-M(I-\pi_n)K'(\varphi)(I-\pi_n)\left(K_n^M\right)'(\varphi)\left(\varphi_n^M-\varphi\right) \\
&\qquad = -M(I-\pi_n)K'(\varphi)(I-\pi_n)K'(\pi_n\varphi)\pi_n\left(\varphi_n^M-\varphi\right) \\
&\qquad = -M(I-\pi_n)K'(\varphi)(I-\pi_n) \\
&\qquad\qquad (K'(\pi_n\varphi)-K'(\varphi))\pi_n\left(\varphi_n^M-\varphi\right) \\
&\qquad\qquad -M(I-\pi_n)K'(\varphi)(I-\pi_n)K'(\varphi)\pi_n\left(\varphi_n^M-\varphi\right).
\end{aligned}
$$

Then, by the estimates (1.7), (2.3), (2.7), and by Lemma 2.1 and Lemma 2.2, we obtain

$$\begin{aligned}
&\|M(I-\pi_n)K'(\varphi)(I-\pi_n)\left(K_n^M\right)'(\varphi)\left(\varphi_n^M-\varphi\right)\|_\infty \\
&\quad\leq \|M(I-\pi_n)K'(\varphi)\|\|(I-\pi_n)\|\|K'(\pi_n\varphi)-K'(\varphi)\|\|\pi_n\| \\
&\quad \|\varphi_n^M-\varphi\|_\infty \\
&\qquad +\|M(I-\pi_n)K'(\varphi)(I-\pi_n)K'(\varphi)\|\|\pi_n\|\|\varphi_n^M-\varphi\|_\infty \\
&\quad = O\left(h^{6r+6}\right).
\end{aligned} \tag{2.17}$$

From (2.14)–(2.17), we deduce

$$\|M(I-\pi_n)K'(\varphi)(I-\pi_n)\left(\varphi_n^M-\varphi\right)\|_\infty = O\left(h^{6r+6}\right). \tag{2.18}$$

Note that, for n large enough,

$$\begin{aligned}
&(K'(\pi_n\varphi)-K'(\varphi))\pi_n\left(\varphi_n^M-\varphi\right) \\
&\quad = K''(\varphi)(\pi_n\varphi-\varphi,\pi_n\left(\varphi_n^M-\varphi\right)) \\
&\qquad + \frac{1}{2}K^{(3)}(\varphi)\left((\pi_n\varphi-\varphi)^2,\pi_n\left(\varphi_n^M-\varphi\right)\right) \\
&\qquad + \frac{1}{6}K^{(4)}(\xi)\left((\pi_n\varphi-\varphi)^3,\pi_n\left(\varphi_n^M-\varphi\right)\right),
\end{aligned}$$

for some $\xi \in \mathcal{B}(\varphi,\delta)$. Hence, using the estimates (1.7), (2.2), (2.3) and Lemma 2.1, we obtain

$$\|M(I-\pi_n)(K'(\pi_n\varphi)-K'(\varphi))\pi_n\left(\varphi_n^M-\varphi\right)\|_\infty = O\left(h^{6r+6}\right). \tag{2.19}$$

The required result follows from (2.13), (2.18) and (2.19). □

For $i = 1, 2, \ldots$, let $B_i(\tau)$ denote the Bernoulli polynomial of degree i. Then

$$B_i(\tau) = (-1)^i B_i(1-\tau), \quad \tau \in [0,1]. \tag{2.20}$$

Let

$$B_0(\tau) = 1.$$

We quote the Euler-MacLaurin formula for future reference:

If $\psi \in C^{p+1}[0,1]$, then, for $0 \leq \tau \leq 1$,

$$h \sum_{i=1}^{n} \psi[(i-1+\tau)h] = \int_0^1 \psi(t)\, dt + \sum_{j=1}^{p} \frac{B_j(\tau)}{j!}[\psi^{(j-1)}(1) - \psi^{(j-1)}(0)]h^j + O(h^{p+1}). \tag{2.21}$$

Let

$$q(s,t) = \frac{\partial^2 \kappa}{\partial u^2}(s,t,\varphi(t)), \quad s,t \in [0,1].$$

We are interested in an asymptotic series expansion for $K''(\varphi)(\pi_n\psi - \psi)^2$. We need to consider two different cases.

2.1. Orthogonal projection. Let $\eta_0, \eta_1, \ldots$ be the sequence of orthonormal polynomials in $L^2[0,1]$. Then η_p is a polynomial of degree p and

$$\langle \eta_p,\, \eta_q \rangle = \delta_{pq} \quad \text{for all } p, q \geq 0.$$

For $i = 1, \ldots, n$, define η_{ip} on $[(i-1)h, ih]$ by

$$\eta_{ip}[(i-1+\tau)h] = h^{-1/2}\eta_p(\tau), \quad 0 \leq \tau \leq 1,$$

and then extend by zero to $[0,1]$. The functions η_{ip}, $i = 1, \ldots, n$,$p = 0, \ldots, r$, form an orthonormal basis for X_n, and the orthogonal projection π_n from $L^2[0,1]$ to X_n is given by

$$\pi_n \psi = \sum_{i=1}^{n} \sum_{p=0}^{r} \langle \psi,\, \eta_{ip} \rangle \eta_{ip}.$$

Let

$$\Lambda_{r+1}(\sigma,\tau) = \sum_{p=0}^{r} \eta_p(\sigma)\eta_p(\tau)$$

and

$$\chi_j(\tau) = \int_0^1 \Lambda_{r+1}(\sigma,\tau) \frac{(\sigma-\tau)^j}{j!}\, d\sigma, \quad j = 1, \ldots, r+2.$$

The following asymptotic series expansion can be deduced using results from **[18]**: If $\partial\kappa/\partial u \in C^{2r+4}(\Omega)$ and $\psi \in C^{2r+4}[0,1]$, then

$$K'(\varphi)(\pi_n\psi - \psi) = T(\psi)h^{2r+2} + O(h^{2r+4}), \tag{2.22}$$

$$M(\pi_n\psi - \psi) = U(\psi)h^{2r+2} + O(h^{2r+4}), \tag{2.23}$$

where

$$\begin{aligned} T(\psi)(s) &= b_{2r+2,2r+2}\, K'(\varphi)\psi^{(2r+2)}(s) \\ &\quad + \sum_{i=1}^{2r+1} b_{2r+2,i}\left[\left(\frac{\partial}{\partial t}\right)^{2r+1-i} \ell(s,t)\psi^{(i)}(t)\right]_{t=0}^{1}, \end{aligned}$$

$$\begin{aligned} U(\psi)(s) &= b_{2r+2,2r+2}\, M\psi^{(2r+2)}(s) \\ &\quad + \sum_{i=1}^{2r+1} b_{2r+2,i}\left[\left(\frac{\partial}{\partial t}\right)^{2r+1-i} m(s,t)\psi^{(i)}(t)\right]_{t=0}^{1}, \end{aligned}$$

with

$$b_{2r+2,i} = \int_0^1 \int_0^1 \Lambda_{r+1}(\sigma,\tau)\frac{B_{2r+2-i}(\tau)}{(2r+2-i)!}\,\frac{(\sigma-\tau)^i}{i!}\,d\sigma\,d\tau, \quad i = 1,\dots,2r+2.$$

Note that the coefficients $b_{2r+2,i}$ are independent of h.

We now obtain asymptotic series expansions for $K''(\varphi)(\pi_n\psi - \psi)^2$ and for $K^{(3)}(\varphi)(\pi_n\psi - \psi)^3$. The proof is similar to that of **[18,** Theorem 5.1**]**. Note that, in the proof of (2.22) and of (2.23), we need $\ell(\cdot,\cdot) \in C^{2r+4}[0,1]$ and $m(\cdot,\cdot) \in C^{2r+4}[0,1]$. The smoothness conditions on the kernel $q(\cdot,\cdot)$ of $K''(\varphi)$ and on ψ in the following lemma are less stringent.

Lemma 2.4. *Let* $q(\cdot,\cdot) \in C([0,1]\times[0,1])$ *and* $\psi \in C^{r+3}[0,1]$. *Then*

$$K''(\varphi)(\pi_n\psi - \psi)^2 = V_1(\psi)h^{2r+2} + O(h^{2r+4}), \tag{2.24}$$

where

$$V_1(\psi) = \left(\int_0^1 \chi_{r+1}(\tau)^2 d\tau\right)K''(\varphi)\left(\psi^{(r+1)}\right)^2$$

is independent of h. Also,

$$K^{(3)}(\varphi)(\pi_n\psi - \psi)^3 = V_2(\psi)h^{3r+3} + O(h^{3r+4}), \tag{2.25}$$

where

$$V_2(\psi) = \left(\int_0^1 \chi_{r+1}(\tau)^3 d\tau\right) K^{(3)}(\varphi)\left(\psi^{(r+1)}\right)^3$$

is independent of h.

Proof. Note that

$$\begin{aligned} K''(\varphi)(\pi_n\psi - \psi)^2(s) &= \int_0^1 \frac{\partial^2 k}{\partial u^2}(s,t,\varphi(t))\,(\pi_n\psi - \psi)^2(t)\,dt \\ &= \sum_{i=1}^n \int_{(i-1)h}^{ih} q(s,t)\,(\pi_n\psi - \psi)^2(t)\,dt \\ &= \int_0^1 h\sum_{i=1}^n q(s,(i-1+\tau)h)\,(\pi_n\psi - \psi)^2 \\ &\qquad\qquad [(i-1+\tau)h]\,d\tau. \end{aligned} \tag{2.26}$$

We write

$$\begin{aligned} &(\pi_n\psi - \psi)[(i-1+\tau)h] \\ &= \sum_{p=0}^r \left\{\int_{(i-1)h}^{ih} [\psi(u) - \psi[(i-1+\tau)h]]\,\eta_{ip}(u)\,du\right\}\eta_{ip}[(i-1+\tau)h]. \end{aligned}$$

Let $u = (i-1+\sigma)h$ in the above expression to obtain

$$\begin{aligned} &(\pi_n\psi - \psi)[(i-1+\tau)h] \\ &\quad = \int_0^1 \left(\sum_{p=0}^r \eta_p(\sigma)\eta_p(\tau)\right)(\psi[(i-1+\sigma)h] - \psi[(i-1+\tau)h])\,d\sigma. \end{aligned}$$

By Taylor's theorem,

$$
\begin{aligned}
&(\pi_n\psi - \psi)[(i-1+\tau)h] \\
&\qquad = \sum_{j=1}^{r+2} \left\{ \int_0^1 \Lambda_{r+1}(\sigma,\tau)\frac{(\sigma-\tau)^j}{j!}\, d\sigma \right\} \psi^{(j)}[(i-1+\tau)h]h^j \\
&\qquad\quad + O(h^{r+3}) \\
&\qquad = \sum_{j=1}^{r+2} \chi_j(\tau)\psi^{(j)}[(i-1+\tau)h]h^j + O(h^{r+3}).
\end{aligned}
$$

We recall the Christoffel-Darboux identity ([**11,** page 342]):

$$
\Lambda_{r+1}(\sigma,\tau) = \frac{a_r}{a_{r+1}} \frac{\eta_{r+1}(\sigma)\eta_r(\tau) - \eta_r(\sigma)\eta_{r+1}(\tau)}{\sigma - \tau},
$$

where a_r is the leading coefficient of the polynomial η_r.

Since η_r is orthogonal to polynomials of degree $\le r-1$, we obtain

$$
\chi_j(\tau) = \int_0^1 \Lambda_{r+1}(\sigma,\tau)\frac{(\sigma-\tau)^j}{j!}\, d\sigma = 0, \quad j = 1,\dots,r.
$$

Hence,

$$
\begin{aligned}
&(\pi_n\psi - \psi)[(i-1+\tau)h] \\
&\qquad = \chi_{r+1}(\tau)\psi^{(r+1)}[(i-1+\tau)h]h^{r+1} \\
&\qquad\quad + \chi_{r+2}(\tau)\psi^{(r+2)}[(i-1+\tau)h]h^{r+2} \\
&\qquad\quad + O(h^{r+3}).
\end{aligned}
$$

Substituting the above expression in (2.26), we obtain

$$
\begin{aligned}
&K''(\varphi)(\pi_n\psi - \psi)^2(s) \\
&\quad = h^{2r+2} \int_0^1 \chi_{r+1}(\tau)^2\, h \sum_{i=1}^{n} q(s,(i-1+\tau)h) \\
&\qquad \Big(\psi^{(r+1)}[(i-1+\tau)h]\Big)^2\, d\tau \\
&\qquad + 2h^{2r+3} \int_0^1 \chi_{r+1}(\tau)\chi_{r+2}(\tau)\, h \sum_{i=1}^{n} q(s,(i-1+\tau)h) \qquad (2.27)\\
&\qquad \psi^{(r+1)}[(i-1+\tau)h]\psi^{(r+2)}[(i-1+\tau)h]\, d\tau
\end{aligned}
$$

$$+ O(h^{2r+4}).$$

By the Euler-MacLaurin series expansion (2.21),

$$\begin{aligned} & h \sum_{i=1}^{n} \Big[q(s,(i-1+\tau)h) \psi^{(r+1)}((i-1+\tau)h)^2 \Big] \\ & \quad = \int_0^1 q(s,t) \psi^{(r+1)}(t)^2 dt + B_1(\tau) \left[q(s,t) \Big(\psi^{(r+1)} \Big)^2 (t) \right]_{t=0}^{1} h \\ & \quad + O(h^2) \\ & = K''(\varphi) \Big(\psi^{(r+1)} \Big)^2 (s) + B_1(\tau) \left[q(s,t) \Big(\psi^{(r+1)} \Big)^2 (t) \right]_{t=0}^{1} h \\ & \quad + O(h^2) \end{aligned} \tag{2.28}$$

and

$$\begin{aligned} & h \sum_{i=1}^{n} \Big[q(s,(i-1+\tau)h) \psi^{(r+1)}((i-1+\tau)h) \psi^{(r+2)}((i-1+\tau)h) \Big] \\ & = \int_0^1 q(s,t) \psi^{(r+1)}(t) \psi^{(r+2)}(t) \, dt + O(h) \\ & = K''(\varphi) \left(\psi^{(r+1)}, \psi^{(r+2)} \right) (s) + O(h). \end{aligned} \tag{2.29}$$

Hence,

$$\begin{aligned} & K''(\varphi)(\pi_n \psi - \psi)^2(s) \\ & = \left(\int_0^1 \chi_{r+1}(\tau)^2 d\tau \right) K''(\varphi) \Big(\psi^{(r+1)} \Big)^2 (s) \, h^{2r+2} \\ & \quad + \left(\int_0^1 \chi_{r+1}(\tau)^2 B_1(\tau) \, d\tau \right) \left[q(s,t) \Big(\psi^{(r+1)} \Big)^2 (t) \right]_{t=0}^{1} h^{2r+3} \\ & \quad + 2 \left(\int_0^1 \chi_{r+1}(\tau) \chi_{r+2}(\tau) \, d\tau \right) K''(\varphi) \left(\psi^{(r+1)}, \psi^{(r+2)} \right) (s) \, h^{2r+3} \\ & \quad + O(h^{2r+4}). \end{aligned}$$

Since, for $\tau \in [0,1]$,

$$\eta_p(1-\tau) = (-1)^p \eta_p(\tau),$$

it follows that, for $\sigma, \tau \in [0,1]$,

$$\Lambda_{r+1}(\sigma,\tau) = \sum_{p=0}^{r} \eta_p(\sigma)\eta_p(\tau) = \Lambda_{r+1}(1-\sigma, 1-\tau).$$

Hence, for $\tau \in [0,1]$ and for $j = 1, \dots, r+2$,

$$\chi_j(\tau) = \int_0^1 \Lambda_{r+1}(\sigma,\tau)\frac{(\sigma-\tau)^j}{j!}\,d\sigma = (-1)^j\chi_j(1-\tau).$$

Also, for $\tau \in [0,1]$,

$$B_1(\tau) = -B_1(1-\tau).$$

As a consequence,

$$\int_0^1 \chi_{r+1}(\tau)^2 B_1(\tau)\,d\tau = 0 \quad \text{and} \quad \int_0^1 \chi_{r+1}(\tau)\chi_{r+2}(\tau)\,d\tau = 0.$$

Hence for $s \in [0,1]$,

$$K''(\varphi)(\pi_n\psi-\psi)^2(s) = \left(\int_0^1 \chi_{r+1}(\tau)^2 d\tau\right) K''(\varphi)\left(\psi^{(r+1)}\right)^2(s)\, h^{2r+2} + O(h^{2r+4}),$$

which completes the proof of (2.24).

The proof of (2.25) is similar. □

Remark 2.5. If $r = 0$, then

$$V_2(\psi) = \left(\int_0^1 \chi_1(\tau)^3 d\tau\right) K^{(3)}(\varphi)\left(\psi^{(1)}\right)^3.$$

Since

$$\chi_1(\tau) = -\chi_1(1-\tau), \quad \tau \in [0,1],$$

it follows that

$$V_2(\psi) = 0$$

and

$$K^{(3)}(\varphi)(\pi_n\psi - \psi)^3(s) = O(h^4) = O(h^{2r+4}). \tag{2.30}$$

If $r \geq 1$, then $3r+3 \geq 2r+4$, and hence from (2.25),

$$K^{(3)}(\varphi)(\pi_n\psi - \psi)^3(s) = O(h^{2r+4}). \tag{2.31}$$

2.2. Interpolatory projection. Recall that $\tau_0, \tau_1, \ldots, \tau_r$ are the $r+1$ Gauss points in $[0,1]$. Let

$$\omega_{r+1}(\tau) = \prod_{p=0}^{r} (\tau - \tau_p)$$

and, for $i = r+1, \ldots, 2r+2$,

$$\Phi_i(\tau) = \int_0^1 \frac{(\sigma - \tau)^{i-r-1}}{(i-r-1)!} \, \frac{[\tau_0, \tau_1, \ldots, \tau_r, \tau](\cdot - \sigma)_+^r}{r!} \, d\sigma. \tag{2.32}$$

Since the Gauss points are symmetric in $[0,1]$, it follows that, for $\tau \in [0,1]$,

$$\begin{gathered} \omega_{r+1}(\tau) = (-1)^{r+1} \omega_{r+1}(1-\tau), \\ \Phi_i(\tau) = (-1)^{i-r-1} \Phi_i(1-\tau), \\ i = r+1, \ldots, 2r+2. \end{gathered}$$

The following asymptotic series expansions can be deduced using results from **[18]**: If $\partial\kappa/\partial u \in C^{2r+4}(\Omega)$ and $\psi \in C^{2r+4}[0,1]$, then

$$K'(\varphi)(\pi_n \psi - \psi) = T(\psi) h^{2r+2} + O(h^{2r+4}), \tag{2.33}$$

$$M(\pi_n \psi - \psi) = U(\psi) h^{2r+2} + O(h^{2r+4}), \tag{2.34}$$

where, for $s \in [0,1]$,

$$\begin{aligned} T(\psi)(s) &= d_{2r+2,2r+2} \, K'(\varphi) \psi^{(2r+2)}(s) \\ &\quad + \sum_{i=r+1}^{2r+1} d_{2r+2,i} \left[\left(\frac{\partial}{\partial t} \right)^{2r+1-i} \ell(s,t) \psi^{(i)}(t) \right]_{t=0}^{1}, \end{aligned}$$

$$\begin{aligned} U(\psi)(s) &= d_{2r+2,2r+2} \, M \psi^{(2r+2)}(s) \\ &\quad + \sum_{i=r+1}^{2r+1} d_{2r+2,i} \left[\left(\frac{\partial}{\partial t} \right)^{2r+1-i} m(s,t) \psi^{(i)}(t) \right]_{t=0}^{1}, \end{aligned}$$

with

$$\begin{gathered} d_{2r+2,i} = -\int_0^1 \Phi_i(\tau) \frac{B_{2r+2-i}(\tau)}{(2r+2-i)!} \, \omega_{r+1}(\tau) \, d\tau, \\ i = r+1, \ldots, 2r+2. \end{gathered}$$

Note that the coefficients $d_{2r+2,i}$ are independent of h.

We now obtain an asymptotic series expansion for $K''(\varphi)(\pi_n\psi-\psi)^2$ and for $K^{(3)}(\varphi)(\pi_n\psi - \psi)^3$. As in the proof of [**18,** Theorem 4.1], the crucial step is the error formula in the interpolating polynomial in terms of the divided differences.

Lemma 2.6. *Let* $q(\cdot,\cdot) \in C([0,1]\times[0,1])$ *and* $\psi \in C^{r+3}[0,1]$. *Then*

$$K''(\varphi)(\pi_n\psi - \psi)^2 = V_1(\psi)h^{2r+2} + O(h^{2r+4}), \tag{2.35}$$

where

$$V_1(\psi) = \left(\int_0^1 \omega_{r+1}(\tau)^2\Phi_{r+1}(\tau)^2 d\tau\right) K''(\varphi)\left(\psi^{(r+1)}\right)^2$$

and

$$K^{(3)}(\varphi)(\pi_n\psi - \psi)^3 = h^{3r+3}V_2(\psi) + O(h^{3r+4}), \tag{2.36}$$

where

$$V_2(\psi) = \left(\int_0^1 \omega_{r+1}(\tau)^3\Phi_{r+1}(\tau)^3 d\tau\right) K^{(3)}(\varphi)\left(\psi^{(r+1)}\right)^3.$$

Proof. Note that

$$\begin{aligned} K''(\varphi)(\pi_n\psi - \psi)^2(s) &= \int_0^1 \frac{\partial^2 k}{\partial u^2}(s,t,\varphi(t))\,(\pi_n\psi-\psi)^2(t)\,dt \\ &= \sum_{i=1}^n \int_{(i-1)h}^{ih} q(s,t)\,(\pi_n\psi-\psi)^2(t)\,dt. \end{aligned}$$

Since

$$(\pi_n\psi)(t_{ij}) = \psi(t_{ij}), \quad j=0,1,\ldots,r,\ i=1,\ldots,n,$$

we obtain

$$(\pi_n\psi)(t)-\psi(t) = [t_{i0},t_{i1},\ldots,t_{ir},t]\psi\left(\prod_{p=0}^r (t-t_{ip})\right),\ t\in[(i-1)h, ih].$$

Thus,

$$K''(\varphi)(\pi_n\psi-\psi)^2(s) = \sum_{i=1}^{n}\int_{(i-1)h}^{ih} q(s,t)\left(([t_{i0},t_{i1},\ldots,t_{ir},t]\psi)\prod_{p=0}^{r}(t-t_{ip})\right)^2 dt.$$

Let $t=(i-1+\tau)h$. Then

$$\prod_{p=0}^{r}(t-t_{ip}) = h^{r+1}\prod_{p=0}^{r}(\tau-\tau_p) = h^{r+1}\omega_{r+1}(\tau)$$

and

$$K''(\varphi)(\pi_n\psi-\psi)^2(s) = h^{2r+3}\sum_{i=1}^{n}\int_0^1 q(s,(i-1+\tau)h) \{[t_{i0},t_{i1},\ldots,t_{ir},(i-1+\tau)h]\psi\}^2\ \omega_{r+1}(\tau)^2\,d\tau. \tag{2.37}$$

By Peano representation for divided differences ([**19**]), we get

$$[t_{i0},t_{i1},\ldots,t_{ir},t]\psi = \int_{(i-1)h}^{ih}\frac{[t_{i0},t_{i1},\ldots,t_{ir},t](\cdot-z)_+^r}{r!}\psi^{(r+1)}(z)\,dz.$$

By putting

$$t=(i-1+\tau)h,\quad z=(i-1+\sigma)h$$

in the above expression, we obtain

$$[t_{i0},t_{i1},\ldots,t_{ir},t]\psi = \int_0^1\frac{[\tau_0,\tau_1,\ldots,\tau_r,\tau](\cdot-\sigma)_+^r}{r!}\psi^{(r+1)}((i-1+\sigma)h)\,d\sigma.$$

By Taylor's theorem,

$$\psi^{(r+1)}((i-1+\sigma)h) = \psi^{(r+1)}((i-1+\tau)h)+\psi^{(r+2)}((i-1+\tau)h)(\sigma-\tau)h+O(h^2).$$

Then

$$[t_{i0}, t_{i1}, \ldots, t_{ir}, t]\psi = \psi^{(r+1)}((i-1+\tau)h)\Phi_{r+1}(\tau) + h\,\psi^{(r+2)}((i-1+\tau)h)\Phi_{r+2}(\tau) + O(h^2).$$

Substituting the above expression into (2.37), we obtain

$$\begin{aligned}
&K''(\varphi)(\pi_n\psi - \psi)^2(s) \\
&\quad = h^{2r+3}\int_0^1 \omega_{r+1}(\tau)^2 \sum_{i=1}^n q(s,(i-1+\tau)h) \\
&\quad \Big\{\psi^{(r+1)}((i-1+\tau)h)\Phi_{r+1}(\tau) \\
&\quad\quad + h\,\psi^{(r+2)}((i-1+\tau)h)\Phi_{r+2}(\tau)\Big\}^2\, d\tau + O(h^{2r+4}) \\
&= h^{2r+2}\int_0^1 h\sum_{i=1}^n \Big[q(s,(i-1+\tau)h)\psi^{(r+1)}((i-1+\tau)h)^2\Big] \\
&\quad \omega_{r+1}(\tau)^2\Phi_{r+1}(\tau)^2 d\tau \\
&\quad + 2h^{2r+3}\int_0^1 h\sum_{i=1}^n \Big[q(s,(i-1+\tau)h)\psi^{(r+1)} \\
&\quad ((i-1+\tau)h)\psi^{(r+2)}((i-1+\tau)h)\Big] \\
&\quad \omega_{r+1}(\tau)^2\Phi_{r+1}(\tau)\Phi_{r+2}(\tau)d\tau + O(h^{2r+4}).
\end{aligned}$$

Using (2.28) and (2.29), we obtain

$$\begin{aligned}
K''(\varphi)(\pi_n\psi - \psi)^2(s) = &\left(\int_0^1 \omega_{r+1}(\tau)^2\Phi_{r+1}(\tau)^2 d\tau\right) \\
&K''(\varphi)\Big(\psi^{(r+1)}\Big)^2(s)\ h^{2r+2} \\
&+ \left(\int_0^1 B_1(\tau)\omega_{r+1}(\tau)^2\Phi_{r+1}(\tau)^2 d\tau\right) \\
&\left[q(s,t)\Big(\psi^{(r+1)}\Big)^2(t)\right]_{t=0}^1 h^{2r+3} \\
&+ 2\left(\int_0^1 \omega_{r+1}(\tau)^2\Phi_{r+1}(\tau)\Phi_{r+2}(\tau)\, d\tau\right)
\end{aligned}$$

$$K''(\varphi)\left(\psi^{(r+1)}, \psi^{(r+2)}\right)(s)\, h^{2r+3}$$
$$+ O(h^{2r+4}).$$

Since, for $\tau \in [0,1]$,

$$\Phi_{r+1}(\tau) = \Phi_{r+1}(1-\tau),$$
$$\Phi_{r+2}(\tau) = -\Phi_{r+2}(1-\tau),$$
$$\omega_{r+1}(\tau) = (-1)^{r+1}\omega_{r+1}(1-\tau)$$

and

$$B_1(\tau) = -B_1(1-\tau),$$

the terms containing h^{2r+3} vanish and, hence, for $s \in [0,1]$,

$$K''(\varphi)(\pi_n\psi - \psi)^2(s)$$
$$= \left(\int_0^1 \omega_{r+1}(\tau)^2\Phi_{r+1}(\tau)^2 d\tau\right) K''(\varphi)\left(\psi^{(r+1)}\right)^2(s)\, h^{2r+2} + O(h^{2r+4}),$$

which completes the proof of (2.35).

The proof of (2.36) is similar. □

Remark 2.7. If $r = 0$, then

$$V_2(\psi) = \left(\int_0^1 \omega_1(\tau)^3\Phi_1(\tau)^3 d\tau\right) K^{(3)}(\varphi)\left(\psi^{(1)}\right)^3.$$

Since

$$\Phi_1(\tau) = \Phi_1(1-\tau), \quad \text{and} \quad \omega_1(\tau) = -\omega_1(1-\tau), \quad \tau \in [0,1],$$

it follows that

$$V_2(\psi) = 0$$

and

$$K^{(3)}(\varphi)(\pi_n\psi - \psi)^3 = O(h^4) = O(h^{2r+4}). \tag{2.38}$$

If $r \geq 1$, then $3r + 3 \geq 2r + 4$, and hence, from (2.36),

$$K^{(3)}(\varphi)(\pi_n\psi - \psi)^3 = O(h^{2r+4}). \tag{2.39}$$

3. Main results.

3.1. Iterated collocation/iterated Galerkin method. Let φ_n denote either the Galerkin solution, φ_n^G, or the collocation solution, φ_n^C, and let φ_n^S denote the iterated Galerkin or the iterated collocation solution. Note that

$$\pi_n \varphi_n^S = \varphi_n.$$

Theorem 3.1. *Let $r \geq 0$ and X_n be the space of piecewise polynomials of degree $\leq r$ with respect to the uniform partition* (2.1). *Let π_n be either the restriction to $L^\infty[0,1]$ of the orthogonal projection from $L^2[0,1]$ to X_n or the interpolatory projection at $r+1$ Gauss points in each subinterval of the partition. Assume that*

$$\kappa \in C^{2r+4}(\Omega), \ \frac{\partial \kappa}{\partial u} \in C^{3r+6}(\Omega) \quad \textit{and} \quad f \in C^{2r+4}[0,1].$$

Let φ be the unique solution of (1.1) *and φ_n^S the iterated Galerkin or the iterated collocation solution. Assume that* 1 *is not an eigenvalue of $K'(\varphi)$. Then*

$$\varphi_n^S = \varphi + \eta\, h^{2r+2} + O(h^{2r+4}), \tag{3.1}$$

where the function η is independent of h.

Proof. Throughout this proof, we use the following relation:

$$\varphi_n - \varphi = \pi_n(\varphi_n^S - \varphi) - (I - \pi_n)\varphi.$$

We quote the following identity from [**3,** equation (2.28)].

$$\begin{aligned}\varphi_n^S - \varphi = (I - K'(\varphi))^{-1} \{&[K(\varphi_n) - K(\varphi) - K'(\varphi)(\varphi_n - \varphi)] \\ &- M(I-\pi_n)[K(\varphi_n) - K(\varphi) - K'(\varphi)(\varphi_n - \varphi)] \\ &-M(I-\pi_n)K'(\varphi)(\varphi_n - \varphi) - M(I-\pi_n)\varphi\}\,.\end{aligned} \tag{3.2}$$

Choose n big enough so that $\varphi_n \in \mathcal{B}(\varphi, \delta)$. By Taylor's theorem and estimate (1.2),

$$\begin{aligned}&K(\varphi_n) - K(\varphi) - K'(\varphi)(\varphi_n - \varphi) \\ =\ &\frac{K''(\varphi)}{2}(\varphi_n - \varphi)^2 + \frac{K^{(3)}(\varphi)}{6}(\varphi_n - \varphi)^3 + \frac{K^{(4)}(\xi)}{24}(\varphi_n - \varphi)^4\end{aligned}$$

$$= \frac{K^{''}(\varphi)}{2}(\varphi_n - \varphi)^2 + \frac{K^{(3)}(\varphi)}{6}(\varphi_n - \varphi)^3 + O(h^{4r+4}).$$

We write

$$\begin{aligned} K^{''}(\varphi)(\varphi_n - \varphi)^2 &= K^{''}(\varphi)[\pi_n(\varphi_n^S - \varphi) - (I - \pi_n)\varphi]^2 \\ &= K^{''}(\varphi)\left(\pi_n(\varphi_n^S - \varphi)\right)^2 \\ &\quad -2K^{''}\left(\varphi)(\pi_n(\varphi_n^S - \varphi), (I - \pi_n)\varphi\right) \\ &\quad +K^{''}(\varphi)((I - \pi_n)\varphi)^2 . \end{aligned} \tag{3.3}$$

Using the estimates (1.3), (2.2) and (2.3), we obtain

$$\|K^{''}(\varphi)(\pi_n(\varphi_n^S - \varphi))^2\|_\infty \leq \|K''(\varphi)\|\|\pi_n\|^2\|\varphi_n^S - \varphi\|_\infty^2 = O(h^{4r+4}), \tag{3.4}$$

whereas

$$\begin{aligned} &\|K^{''}\left(\varphi)(\pi_n(\varphi_n^S - \varphi), (I - \pi_n)\varphi\right)\|_\infty \\ &\quad \leq \|K''(\varphi)\|\|\pi_n\|\|\varphi_n^S - \varphi\|_\infty\|\varphi - \pi_n\varphi\|_\infty \\ &\quad = O(h^{3r+3}). \end{aligned}$$

If $r \geq 1$, then $3r + 3 \geq 2r + 4$, and we can write

$$\|K^{''}\left(\varphi)(\pi_n(\varphi_n^S - \varphi), (I - \pi_n)\varphi\right)\|_\infty = O(h^{2r+4}). \tag{3.5}$$

Let $r = 0$. Then, for a fixed $s \in [0, 1]$,

$$\begin{aligned} &K^{''}\left(\varphi)(\pi_n(\varphi_n^S - \varphi), (I - \pi_n)\varphi\right)(s) \\ &\quad = \sum_{i=1}^{n} \int_{(i-1)h}^{ih} q(s,t)(\pi_n(\varphi_n^S - \varphi))(t)(I - \pi_n)\varphi(t)\, dt \\ &\quad = \sum_{i=1}^{n} \beta_i \int_{(i-1)h}^{ih} q(s,t)(I - \pi_n)\varphi(t)\, dt, \end{aligned}$$

where

$$\beta_i = \pi_n(\varphi_n^S - \varphi)(t), \quad t \in [(i-1)h, ih].$$

Since

$$|\beta_i| \leq \|\pi_n\|\|\varphi_n^S - \varphi\|_\infty, \quad i = 1, \ldots, n,$$

we get

$$\left|K^{''}\left(\varphi)(\pi_n(\varphi_n^S - \varphi), (I - \pi_n)\varphi\right)(s)\right|$$

$$\leq \|\pi_n\| \|\varphi_n^S - \varphi\|_\infty \sum_{i=1}^{n} \left| \int_{(i-1)h}^{ih} q(s,t)(I - \pi_n)\varphi(t)\, dt \right| \leq C h^{2r+4}.$$

Hence,

$$\left\| K^{''}(\varphi)(\pi_n(\varphi_n^S - \varphi), (I - \pi_n)\varphi) \right\|_\infty = O(h^{2r+4}). \tag{3.6}$$

By Lemma 2.4, in the case of the orthogonal projection and, by Lemma 2.6 in the case of the interpolatory projection, we have

$$K^{''}(\varphi)(\pi_n\varphi - \varphi)^2 = V_1(\varphi)\, h^{2r+2} + O(h^{2r+4}). \tag{3.7}$$

Combining the results from (3.3)–(3.7), we obtain

$$K^{''}(\varphi)(\varphi_n - \varphi)^2 = V_1(\varphi)\, h^{2r+2} + O(h^{2r+4}). \tag{3.8}$$

Note that

$$\begin{aligned} K^{(3)}(\varphi)(\varphi_n - \varphi)^3 &= K^{(3)}(\varphi) \left(\pi_n(\varphi_n^S - \varphi) - (I - \pi_n)\varphi\right)^3 \\ &= K^{(3)}(\varphi) \left(\pi_n(\varphi_n^S - \varphi)\right)^3 \\ &\quad - 3K^{(3)}(\varphi) \left((\pi_n(\varphi_n^S - \varphi))^2, (I - \pi_n)\varphi\right) \\ &\quad + 3K^{(3)}(\varphi) \left(\pi_n(\varphi_n^S - \varphi), ((I - \pi_n)\varphi)^2\right) \\ &\quad - K^{(3)}(\varphi)((I - \pi_n)\varphi)^3. \end{aligned}$$

By Remark 2.5 in the case of the orthogonal projection and Remark 2.7 in the case of the interpolatory projection at Gauss points, and using the estimates (1.3), (2.2) and (2.3), we obtain

$$\|K^{(3)}(\varphi)(\varphi_n - \varphi)^3\|_\infty = O(h^{2r+4}).$$

Hence, from (3.3), (3.8) and the above result,

$$\begin{aligned} &(I - K'(\varphi))^{-1} \left[K(\varphi_n) - K(\varphi) - K'(\varphi)(\varphi_n - \varphi)\right] \\ &\qquad = \frac{1}{2}(I - K'(\varphi))^{-1} V_1(\varphi) h^{2r+2} + O(h^{2r+4}). \end{aligned} \tag{3.9}$$

By Lemma 2.1 and the estimate (1.2),

$$\begin{aligned} &\|M(I - \pi_n)[K(\varphi_n) - K(\varphi) - K'(\varphi)(\varphi_n - \varphi)]\|_\infty \\ &\leq \frac{1}{2} \|M(I - \pi_n) K''(\varphi)\| \|\varphi_n - \varphi\|_\infty^2 \end{aligned}$$

$$+ \frac{1}{6}\|M(I-\pi_n)K^{(3)}(\varphi)\|\|\varphi_n-\varphi\|_\infty^3$$
$$+ \frac{1}{24}\|M\|\|(I-\pi_n)\|\|K^{(4)}(\xi)\|\|\varphi_n-\varphi\|_\infty^4$$
$$= O(h^{4r+4}). \tag{3.10}$$

We write

$$M(I-\pi_n)K'(\varphi)(\varphi_n-\varphi) = M(I-\pi_n)K'(\varphi)\pi_n(\varphi_n^S-\varphi) - M(I-\pi_n)K'(\varphi)(I-\pi_n)\varphi.$$

Using estimate (1.3) and Lemmas 2.1 and 2.2, we then obtain

$$\|M(I-\pi_n)K'(\varphi)(\varphi_n-\varphi)\|_\infty = O(h^{4r+4}). \tag{3.11}$$

Lastly, by the asymptotic expansion (2.23) in the case of orthogonal projection and by the asymptotic expansion (2.34) in the case of interpolatory projection, we obtain

$$M(\pi_n\varphi-\varphi) = U(\varphi)h^{2r+2} + O(h^{2r+4}). \tag{3.12}$$

The asymptotic expansion (3.1) then follows from (3.2), (3.9)–(3.12) with

$$\eta = \frac{1}{2}(I-K'(\varphi))^{-1}V_1(\varphi) + U(\varphi). \qquad \square$$

We can now apply one step of Richardson extrapolation and obtain approximations of φ of higher order. Define

$$\varphi_n^{E_1} = \frac{2^{2r+2}\varphi_{2n}^S - \varphi_n^S}{2^{2r+2}-1}.$$

Then we have the following result.

Corollary 3.2. *Under the assumptions of Theorem* 3.1*,*

$$\|\varphi-\varphi_n^{E_1}\|_\infty = O(h^{2r+4}). \tag{3.13}$$

Proof. Note that

$$\varphi_n^S = \varphi + \eta\, h^{2r+2} + O(h^{2r+4})$$

and

$$\varphi_{2n}^S = \varphi + \eta\left(\frac{h}{2}\right)^{2r+2} + O(h^{2r+4}).$$

Hence,

$$\varphi_n^{E_1} = \frac{2^{2r+2}\varphi_{2n}^S - \varphi_n^S}{2^{2r+2}-1} = \varphi + O(h^{2r+4}),$$

which completes the proof. □

3.2. Iterated modified projection method.

Theorem 3.3. *Let $r \geq 0$ and X_n be the space of piecewise polynomials of degree $\leq r$ with respect to the uniform partition* (2.1)*. Let π_n be either the restriction to $L^\infty[0,1]$ of the orthogonal projection from $L^2[0,1]$ to X_n or the interpolatory projection at $r+1$ Gauss points in each subinterval of the partition. Assume that*

$$\kappa \in C^{2r+4}(\Omega), \quad \frac{\partial \kappa}{\partial u} \in C^{3r+6}(\Omega) \quad \textit{and} \quad f \in C^{2r+4}[0,1].$$

Let φ be the unique solution of (1.1) *and φ_n^M the unique solution of* (1.4) *in $B(\varphi, \delta)$. Assume that* 1 *is not an eigenvalue of $K'(\varphi)$. Then*

$$\widetilde{\varphi}_n^M = \varphi + \zeta\, h^{4r+4} + O\left(h^{4r+6}\right), \tag{3.14}$$

where the function ζ is independent of h.

Proof. From (1.6) and (2.6), we obtain

$$\widetilde{\varphi}_n^M - \varphi = K(\varphi_n^M) - K(\varphi).$$

We quote the following result from Grammont et al. **[10]**:

For n large enough,

$$K(\varphi_n^M) - K(\varphi) = K'(\varphi)(\varphi_n^M - \varphi) + R(\varphi_n^M - \varphi),$$

with

$$(R(\varphi_n^M - \varphi))(s) = \int_0^1 (1-\theta)\left(K''(\varphi + \theta(\varphi_n^M - \varphi))\,(\varphi_n^M - \varphi)^2\right)(s)\, d\theta, \quad s \in [0,1].$$

Let

$$C_8 = \max_{\substack{s,t \in [0,1] \\ |u| \leq \|\varphi\|_\infty + \delta}} \left| \frac{\partial^2 k}{\partial u^2}(s,t,u) \right|.$$

Then, by (1.7),

$$\|R(\varphi_n^M - \varphi)\|_\infty \leq \frac{C_8}{2}(\|\varphi_n^M - \varphi\|_\infty)^2 = O(h^{6r+6}).$$

Thus,

$$\widetilde{\varphi}_n^M - \varphi = K'(\varphi)(\varphi_n^M - \varphi) + O(h^{6r+6}). \tag{3.15}$$

Note that

$$\begin{aligned} K'(\varphi)(\varphi_n^M - \varphi) &= -(I - K'(\varphi))^{-1} K'(\varphi) \\ &\quad \left[K(\varphi) - K'(\varphi)\varphi - K_n^M(\varphi_n^M) + K'(\varphi)\varphi_n^M\right] \\ &= -M\left[K(\varphi) - K_n^M(\varphi_n^M) + K'(\varphi)(\varphi_n^M - \varphi)\right]. \end{aligned}$$

We write

$$\begin{aligned} K'(\varphi)(\varphi_n^M - \varphi) &= -M\left[K(\varphi) - K_n^M(\varphi)\right] \\ &\quad + M\left[K_n^M(\varphi_n^M) - K_n^M(\varphi)\right. \\ &\qquad \left. -(K_n^M)'(\varphi)(\varphi_n^M - \varphi)\right] \\ &\quad + M((K_n^M)'(\varphi) - K'(\varphi))(\varphi_n^M - \varphi). \end{aligned} \tag{3.16}$$

Since, by definition,

$$K_n^M(\varphi) = \pi_n K(\varphi) + K(\pi_n\varphi) - \pi_n K(\pi_n\varphi),$$

we have

$$\begin{aligned} M(K(\varphi) - K_n^M(\varphi)) &= M(I - \pi_n)(K(\varphi) - K(\pi_n\varphi)) \\ &= M(\pi_n - I)\left[K(\pi_n\varphi) - K(\varphi)\right. \\ &\qquad \left. -K'(\varphi)(\pi_n\varphi - \varphi)\right] \\ &\quad +M(\pi_n - I)K'(\varphi)(\pi_n\varphi - \varphi). \end{aligned} \tag{3.17}$$

By Taylor's theorem, for n large enough,

$$\begin{aligned} &K(\pi_n\varphi) - K(\varphi) - K'(\varphi)(\pi_n\varphi - \varphi) \\ &= \frac{K''(\varphi)}{2}(\pi_n\varphi - \varphi)^2 + \frac{K^{(3)}(\varphi)}{6}(\pi_n\varphi - \varphi)^3 \\ &\quad + \frac{K^{(4)}(\varphi)}{24}(\pi_n\varphi - \varphi)^4 + \frac{K^{(5)}(\varphi)}{120}(\pi_n\varphi - \varphi)^5 \\ &\quad + \frac{K^{(6)}(\xi)}{720}(\pi_n\varphi - \varphi)^6, \end{aligned}$$

for some $\xi \in \mathcal{B}(\varphi, \delta)$.

Using estimate (2.2) and Lemma 2.1, we obtain

$$\begin{aligned} &M(\pi_n - I)\left[K(\pi_n\varphi) - K(\varphi) - K'(\varphi)(\pi_n\varphi - \varphi)\right] \\ &\quad = \frac{1}{2}M(\pi_n - I)K''(\varphi)(\pi_n\varphi - \varphi)^2 \\ &\qquad + \frac{1}{6}M(\pi_n - I)K^{(3)}(\varphi)(\pi_n\varphi - \varphi)^3 + O(h^{6r+6}). \end{aligned}$$

Using the asymptotic expansions (2.23) and (2.24) in the case of orthogonal projection and the asymptotic expansions (2.34) and (2.35) in the case of interpolatory projection, we obtain

$$M(\pi_n - I)K''(\varphi)(\pi_n\varphi - \varphi)^2 = U(V_1(\varphi))\, h^{4r+4} + O\left(h^{4r+6}\right).$$

In a similar fashion, using the asymptotic expansions (2.23) and (2.25) in the case of orthogonal projection and asymptotic expansions (2.34) and (2.36) in the case of interpolatory projection, we obtain

$$M(\pi_n - I)K^{(3)}(\varphi)(\pi_n\varphi - \varphi)^3 = U(V_2(\varphi))\, h^{5r+5} + O\left(h^{5r+6}\right).$$

By Remark 2.5 in the case of orthogonal projection and by Remark 2.7 in the case of interpolatory projection, we have the following result:

If $r = 0$, then $V_2(\varphi) = 0$. On the other hand, if $r \geq 1$, then $5r + 5 \geq 4r + 6$. Hence,

$$\|M(\pi_n - I)K^{(3)}(\varphi)(\pi_n\varphi - \varphi)^3\|_\infty = O\left(h^{4r+6}\right).$$

As a consequence,

$$\begin{aligned} \text{(3.18)}\quad &M(\pi_n - I)\left[K(\pi_n\varphi) - K(\varphi) - K'(\varphi)(\pi_n\varphi - \varphi)\right] \\ &\qquad = \frac{U(V_1(\varphi))}{2}h^{4r+4} + O\left(h^{4r+6}\right). \end{aligned}$$

Using the asymptotic expansions (2.22) and (2.23) in the case of orthogonal projection and the asymptotic expansions (2.33) and (2.34) in the case of interpolatory projection, we obtain

$$\text{(3.19)}\qquad M(\pi_n - I)K'(\varphi)(\pi_n\varphi - \varphi) = h^{4r+4}U(T(\varphi)) + O\left(h^{4r+6}\right).$$

Hence, using (3.17), (3.18) and (3.19), we obtain

(3.20)
$$M(K(\varphi) - K_n^M(\varphi)) = \left(U(T(\varphi)) + \frac{U(V_1(\varphi))}{2}\right) h^{4r+4} + O\left(h^{4r+6}\right).$$

By [**10,** Lemma 3.3],

(3.21)
$$\left\|M\left[K_n^M(\varphi_n^M) - K_n^M(\varphi) - (K_n^M)'(\varphi)(\varphi_n^M - \varphi)\right]\right\|_\infty = O(h^{6r+6})$$

and, by Lemma 2.3,

$$\left\|M((K_n^M)'(\varphi) - K'(\varphi))(\varphi_n^M - \varphi)\right\|_\infty = O(h^{6r+6}). \tag{3.22}$$

From (3.16), (3.20), (3.21) and (3.22), it follows that

$$K'(\varphi)(\varphi_n^M - \varphi) = -\left(U(T(\varphi)) + \frac{U(V_1(\varphi))}{2}\right) h^{4r+4} + O\left(h^{4r+6}\right).$$

The asymptotic expansion (3.14) follows from (3.15) using the above result with

$$\zeta = -\left(U(T(\varphi)) + \frac{U(V_1(\varphi))}{2}\right). \qquad \square$$

We can now apply one step of the Richardson extrapolation and obtain approximations of φ of higher order. Define

$$\varphi_n^{E_2} = \frac{2^{4r+4}\widetilde{\varphi}_{2n}^M - \widetilde{\varphi}_n^M}{2^{4r+4} - 1}.$$

Then we have the following result.

Corollary 3.4. *Under the assumptions of Theorem* 3.3*,*

$$\|\varphi - \varphi_n^{E_2}\|_\infty = O(h^{4r+6}). \tag{3.23}$$

Remark 3.5. We refer to [**10**] for the implementation details and the discussion of the complexity of various methods discussed in this paper.

4. Numerical results.

In this section, we illustrate the improvement of orders of convergence by Richardson extrapolation obtained in Corollaries 3.2 and 3.4.

Consider

$$\varphi(s) - \int_0^1 \frac{ds}{s+t+\varphi(t)} = f(s), \quad 0 \le s \le 1, \tag{4.1}$$

where f is so chosen that

$$\varphi(t) = \frac{1}{t+1}$$

is a solution of (4.1).

We consider the following uniform partition of $[0,1]$:

$$0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n}{n} = 1, \tag{4.2}$$

and choose X_n to be the space of the piecewise constant and piecewise linear polynomials with respect to the above partition. The computations are done for $n = 2, 4, 8, 16$ and 32.

If X_n is the space of piecewise constant functions, then in both the cases of orthogonal and interpolatory projection at the midpoints, the order of convergence in the Sloan method is $1/n^2$, and in iterated version of the modified projection method is $1/n^4$. We need to evaluate certain integrals numerically, and it is necessary to choose the numerical quadrature rule which has order of convergence at least $1/n^4$. We choose composite 2 point Gaussian quadrature with respect to the uniform partition of $[0,1]$ with 256 intervals.

If X_n is the space of piecewise linear functions, then in the cases of interpolatory projection at Gauss 2 points, the order of convergence in the Sloan method is $1/n^4$ and an iterated version of the modified projection method is $1/n^8$. In this case, we choose composite 2 point Gaussian quadrature with respect to the uniform partition of $[0,1]$ with n^2 intervals.

4.1. Orthogonal projection. Let X_n be the space of piecewise constant functions ($r = 0$) with respect to the partition (4.2) and $\pi_n : L^\infty[0,1] \to X_n$ the restriction of the orthogonal projection from $L^2[0,1]$ to X_n.

The expected orders of convergence from (1.2), (1.3) and (3.13) are as follows:

- Galerkin solution: $\delta^G = 1$,
  Sloan solution: $\delta^S = 2$,
  Extrapolated solution: $\delta^{E_1} = 4$,

whereas the expected orders of convergence from (1.7), (1.8) and (3.23) are as follows:

- Modified projection solution: $\delta^M = 3$,
  Iterated modified projection solution: $\delta^{IM} = 4$,
  Extrapolated solution: $\delta^{E_2} = 6$.

TABLE 4.1. Orthogonal projection: $r = 0$**.**

	Galerkin		Sloan		Extrapolation	
n	$\|\varphi - \varphi_n^G\|_\infty$	δ^G	$\|\varphi - \varphi_n^S\|_\infty$	δ^S	$\|\varphi - \varphi_n^{E_1}\|_\infty$	δ^{E_1}
2	1.85×10^{-1}		8.05×10^{-3}			
4	1.05×10^{-1}	0.81	1.99×10^{-3}	2.01	2.32×10^{-5}	
8	5.65×10^{-2}	0.90	4.97×10^{-4}	2.00	2.13×10^{-6}	3.45
16	2.91×10^{-2}	0.96	1.24×10^{-5}	2.00	1.42×10^{-7}	3.91
32	1.45×10^{-2}	1.01	3.10×10^{-5}	2.00	8.99×10^{-9}	3.98

TABLE 4.2. Orthogonal projection: $r = 0$**.**

	Modified Projection		Iterated Modified Projection		Extrapolation	
n	$\|\varphi - \varphi_n^M\|_\infty$	δ^M	$\|\varphi - \tilde{\varphi}_n^M\|_\infty$	δ^{IM}	$\|\varphi - \varphi_n^{E_2}\|_\infty$	δ^{E_2}
2	4.40×10^{-3}		7.61×10^{-5}			
4	6.71×10^{-4}	2.71	4.40×10^{-6}	4.11	2.32×10^{-7}	
8	9.44×10^{-5}	2.83	2.66×10^{-7}	4.05	2.13×10^{-9}	5.31
16	1.25×10^{-5}	2.92	1.65×10^{-8}	4.01	1.42×10^{-10}	5.85
32	1.57×10^{-6}	2.99	1.03×10^{-9}	4.00	8.99×10^{-12}	5.84

It is seen from Tables 4.1 and 4.2 that the computed orders of convergence match well with the theoretical orders of convergence.

4.2. Interpolatory projection. For $r = 0, 1$, let X_n be the space of piecewise polynomials of degree $\leq r$ with respect to the partition (4.2). The collocation points are chosen to be $r + 1$ Gauss points in each subinterval.

The expected orders of convergence from (1.2), (1.3) and (3.13) are as follows:

- Collocation solution: $\delta^C = r + 1$,
  Sloan solution: $\delta^S = 2r + 2$,
  Extrapolated Solution: $\delta^{E_1} = 2r + 4$,

whereas the expected orders of convergence from (1.7), (1.8) and (3.23) are as follows:

- Modified projection solution: $\delta^M = 3r + 3$, Iterated modified projection solution: $\delta^{IM} = 4r + 4$, Extrapolated solution: $\delta^{E_2} = 4r + 6$.

TABLE 4.3. Interpolation at midpoints: $r = 0$.

	Collocation		Sloan		Extrapolation	
n	$\|\varphi - \varphi_n^C\|_\infty$	δ^C	$\|\varphi - \varphi_n^S\|_\infty$	δ^S	$\|\varphi - \varphi_n^{E_1}\|_\infty$	δ^{E_1}
2	1.93×10^{-1}		1.27×10^{-2}			
4	1.08×10^{-1}	0.84	3.15×10^{-3}	2.01	1.82×10^{-5}	
8	5.73×10^{-2}	0.91	7.86×10^{-4}	2.00	2.00×10^{-6}	3.18
16	2.93×10^{-2}	0.97	1.96×10^{-4}	2.00	1.37×10^{-7}	3.87
32	1.45×10^{-2}	1.01	4.91×10^{-5}	2.00	8.77×10^{-9}	3.97

TABLE 4.4. Interpolation at midpoints: $r = 0$.

	Modified Projection		Iterated Modified Projection		Extrapolation	
n	$\|\varphi - \varphi_n^M\|_\infty$	δ^M	$\|\varphi - \tilde{\varphi}_n^M\|_\infty$	δ^{IM}	$\|\varphi - \varphi_n^{E_2}\|_\infty$	δ^{E_2}
2	6.92×10^{-3}		3.30×10^{-4}			
4	1.02×10^{-3}	2.76	2.13×10^{-5}	3.95	7.31×10^{-7}	
8	1.40×10^{-4}	2.87	1.34×10^{-6}	3.99	6.81×10^{-9}	6.75
16	1.82×10^{-5}	2.94	8.37×10^{-8}	4.00	8.46×10^{-11}	6.33
32	2.27×10^{-6}	3.00	5.23×10^{-9}	4.00	1.01×10^{-12}	6.39

TABLE 4.5. Interpolation at Gauss 2 points: $r = 1$.

	Collocation		Sloan		Extrapolation	
n	$\|\varphi - \varphi_n^C\|_\infty$	δ^C	$\|\varphi - \varphi_n^S\|_\infty$	δ^S	$\|\varphi - \varphi_n^{E_1}\|_\infty$	δ^{E_1}
2	7.60×10^{-2}		1.36×10^{-3}			
4	2.64×10^{-2}	1.53	8.18×10^{-5}	4.05	3.38×10^{-6}	
8	7.92×10^{-3}	1.74	4.68×10^{-6}	4.13	4.68×10^{-7}	2.85
16	2.13×10^{-3}	1.90	2.84×10^{-7}	4.04	9.09×10^{-9}	5.68
32	5.17×10^{-4}	2.04	1.76×10^{-8}	4.01	1.44×10^{-10}	5.98

TABLE 4.6. Interpolation at Gauss 2 points: $r = 1$.

	Modified Projection		Iterated Modified Projection		Extrapolation	
n	$\|\varphi - \varphi_n^M\|_\infty$	δ^M	$\|\varphi - \tilde{\varphi}_n^M\|_\infty$	δ^{IM}	$\|\varphi - \varphi_n^{E_2}\|_\infty$	δ^{E_2}
2	5.06×10^{-4}		6.47×10^{-5}			
4	1.07×10^{-5}	5.56	2.09×10^{-7}	8.27	4.37×10^{-8}	
8	1.85×10^{-7}	5.86	8.45×10^{-10}	7.95	2.67×10^{-11}	10.68
16	3.07×10^{-9}	5.90	3.35×10^{-12}	7.98	4.73×10^{-14}	9.14
32	4.74×10^{-11}	6.02	1.34×10^{-14}	7.96	2.11×10^{-15}	4.49

It is seen from Tables 4.3–4.6 that the computed orders of convergence match well with the theoretical orders of convergence.

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Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai 400 076, India
Email address: **rpk@math.iitb.ac.in**

Department of Mathematics, Indian Institute of Technology Bombay, Powai, Mumbai 400 076, India
Email address: **nidhin@math.iitb.ac.in**