JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 26, Number 4, Winter 2014

ON A NEW CLASS OF INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT. We consider various initial-value problems for ordinary integro-differential equations of first order that are characterized by convolution-terms, where all factors depend on the solutions of the equations. Applications of such problems are descriptions of certain glass-transition phenomena based on mode-coupling theory, for instance. We will prove results concerning well-posedness of such problems and the asymptotic behaviour of their solutions.

1. Introduction. Mode-coupling theory of glass-transition lead to initial-value problems for ordinary integro-differential equation (**[11]**), i.e., problems of the kind

$$\phi(t) + \dot{\phi}(t) + \int_0^t F(\phi(t-s))\dot{\phi}(s)\,ds = 0 \quad (t \in (0,\infty)),\ \phi(0) = 1, \tag{1}$$

where $F : \mathbb{R} \to \mathbb{R}$ is a so-called kernel-function and $\phi : [0,\infty) \to \mathbb{R}$ is a correlation-function. Especially the long-time limits of solutions (if they exist) are of physical interest, i.e., in the case of $\lim_{t\to\infty} \phi(t) = 0$, the considered undercooled liquid stays viscous and, in case of $\lim_{t\to\infty} \phi(t) \neq 0$, the liquid transitions into a glass. Physically relevant kernel-functions are of polynomial type, e.g., $F(x) = v_1 x + v_2 x^2$ $(v_1, v_2 \geq 0)$. Problem (1) is equivalent to the following integral equation

$$\phi(t) = f(t) + \int_0^t g(\phi(t-s))h(\phi(s))\,ds, \tag{2}$$

Keywords and phrases. Well-posedness, integro-differential equations, asymptotic behaviour, glass-transition.

Received by the editors on February 25, 2014, and in revised form on September 21, 2014.

DOI:10.1216/JIE-2014-26-4-497

with $f(t) = 1 - t$, $g(x) = 1 - x$ and $h(x) = 1 + F(x)$. Further glass-forming models work with more-parametric kernel-functions, i.e.,

$$\phi(t) + \dot{\phi}(t) + \int_0^t F(\phi(t-s), t-s, s)\dot{\phi}(s)\, ds = 0 \tag{3}$$
$$(t \in (0, \infty)), \quad \phi(0) = 1,$$

where $F : \mathbb{R} \times [0, \infty) \times [0, \infty) \to \mathbb{R}$ (see **[6]**, **[9]**, **[15]**), or with complex-valued equations (**[10]**).

The kernels of the convolution-terms of all three equations (1)–(3) are dependent on the solutions of the equations, i.e., they are given by functions $k = F(\phi)$, respectively $k = F(\phi, \cdot)$. This is the main difference from integral equations as studied extensively in the literature (e.g., equations of Volterra-type, see **[7, 8, 13]** or **[19]**) and to mainly considered integro-differential equations from **[1, 2, 3, 4]**.

Until now, only two works are known to us that deal with integro-differential equations whose convolution terms are of a similar type as in problems (1)–(3), namely, **[12, 20]**. In **[12]**, well-posedness and asymptotic behavior results have been proved for problem (1) under the restriction, that F is an absolutely monotone function.[1] In **[20]**, integro-differential equations of second order were studied, i.e., equations with semilinear structure which are essentially different from the equations in (1)–(3).

In this work, we aim to prove results for the problems (1)–(3) for a wider class of kernel-functions than introduced in **[12]**. In Section 2, we will extend the class of kernel-functions from absolutely monotone functions to monotonically increasing ones. In Section 3, we will present a class of kernel-functions that lead to ill-posed problems, i.e., we will prove under certain assumptions the existence of so called blow-up solutions, that are unbounded on a bounded interval of time. In Section 4, we will follow an ansatz from **[20]** to obtain results under smallness-conditions on the data. Problem (3) will be discussed in Section 5. In Section 6, we will present some comments on systems with real- and complex-valued equations. Examples and applications that use the results of Sections 2–6 are the subject of Section 7.

2. Monotone kernel-functions. In this chapter we consider the following problem for an ordinary integro-differential equation:

$$(4) \quad \phi(t) + \dot{\phi}(t) + \int_0^t F(\phi(t-s))\dot{\phi}(s)ds = 0 \ (t \in (0,\infty)), \quad \phi(0) = 1,$$

with a kernel-function $F : \mathbb{R} \to \mathbb{R}$. Problem (4) is equivalent to the following fixed-point problem

$$(5) \quad \phi(t) = 1 + \int_0^t F(\phi(s)) - \phi(s) - \phi(t-s)F(\phi(s))ds \ (t \in [0,\infty)).$$

Theorem 2.1. *Let $F : \mathbb{R} \to \mathbb{R}$ satisfy*:

(i) *there exists $x_0 < 1 : F(x_0) = x_0/(1-x_0)$,*
(ii) *$F|_{[x_0,1]}$ is differentiable, monotonically increasing and locally Lipschitz-continuous.*

Then problem (4) has a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$, and ϕ is monotonically decreasing with $x_0 \leq \phi(t) \leq 1$ for all $t \in [0,\infty)$.

Proof. We define $\widetilde{F} : \mathbb{R} \to \mathbb{R}$ by

$$\widetilde{F}(x) := \left\{ \begin{array}{ll} F(1), & x > 1 \\ F(x), & x_0 \leq x \leq 1 \\ F(x_0), & x < x_0 \end{array} \right\}.$$

Let $(F_n)_{n\in\mathbb{N}} \subseteq C^1(\mathbb{R},\mathbb{R})$ be a sequence of monotonically increasing functions such that

$$\sup_{x\in\mathbb{R}} |\tilde{F}(x) - F_n(x)| \overset{n\to\infty}{\longrightarrow} 0.$$

Due to the boundedness of F_n for all $n \in \mathbb{N}$, one can easily prove by using the Banach fixed-point theorem that problem (5) with kernel-function F_n has a unique solution $\phi_n \in C^2([0,\infty),\mathbb{R})$ for all $n \in \mathbb{N}$. Analogously, we see that problem (5) with kernel-function $\widetilde{F}$ has a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$. Considering problem (4) with F_n instead of F, differentiation of the equation with respect to t leads to

$$\ddot{\phi}_n(t) \leq -(1 + F(1))\dot{\phi}_n(t),$$

for $t \in [0, t')$, where

$$t' := \left\{ \begin{array}{ll} \inf\{t > 0 : \dot{\phi}_n(t) = 0\}, & \{t > 0 : \dot{\phi}_n(t) = 0\} \neq \emptyset \\ \infty, & \{t > 0 : \dot{\phi}_n(t) = 0\} = \emptyset \end{array} \right\}.$$

Using Gronwall's inequality, we obtain, for $t \in [0, t')$

$$\dot{\phi}_n(t) \leq -e^{-(1+F_n(1))t}.$$

If $\{t > 0 : \dot{\phi}_n(t) = 0\}$ was not empty, this inequality would lead to $\dot{\phi}_n(t') < 0$, which would contradict the assumption. It follows that ϕ_n is monotonically decreasing for all $n \in \mathbb{N}$. By using Gronwall's inequality, one can easily prove that, for all $N, \varepsilon > 0$, there exists a $k > 0$ only dependent on N, ε and $\widetilde{F}$ such that

$$\sup_{0 \leq t \leq N} |\phi(t) - \phi_n(t)| \leq k \sup_{x \in \mathbb{R}} |\widetilde{F} - F_n|.$$

It follows that ϕ is monotonically decreasing, and we obtain with (4)

$$\phi(t) = -\dot{\phi}(t) - \int_0^t \widetilde{F}(\phi(t-s))\dot{\phi}(s)\, ds \geq -F(x_0)(\phi(t) - 1),$$

i.e., $x_0 \leq \phi(t) \leq 1$ for all $t \in [0, \infty)$. We conclude $\widetilde{F}(\phi(t)) = F(\phi(t))$ for all $t \in [0, \infty)$, so ϕ is a solution of (4) with kernel-function F. Uniqueness follows from the Banach fixed-point theorem applied to (5). $\square$

Corollary 2.2. *Let $x_0 < 1$ be the maximal point of intersection of F and G given by $G(x) = x/(1-x)$. Then*

$$\phi(t) \stackrel{t \to \infty}{\longrightarrow} x_0.$$

Proof. From Theorem 2.1, we know that there exists a $g \geq x_0$ such that $\phi(t) \to g$ if $t \to \infty$. This implies the existence of a sequence $(t_n)_{n \in \mathbb{N}} \subseteq [0, \infty)$ with $t_n \to \infty$, $\phi(t_n) \to g$ and $\dot{\phi}(t_n) \to 0$ if $n \to \infty$.

We have, for every $0 \le t_1 < t$,

$$\begin{aligned}
&\Big| \int_0^t F(\phi(t-s))\dot{\phi}(s)\, ds - F(g)(g-1) \Big| \\
&\le \Big| \int_0^{t_1} F(\phi(t-s))\dot{\phi}(s) ds - F(g)(g-1) \Big| \\
&\quad + \Big| \int_{t_1}^t F(\phi(t-s))\dot{\phi}(s) ds \Big| \\
&=: I_1 + I_2.
\end{aligned}$$

We have, for fixed t_1,

$$I_1 \stackrel{t\to\infty}{\longrightarrow} |F(g)||\phi(t_1) - g|$$

and

$$I_2 \le C\left(|\phi(t) - g| + |\phi(t_1) - g|\right) \stackrel{t\to\infty}{\longrightarrow} C|\phi(t_1) - g|,$$

where $C := \sup_{t\in[0,\infty)} |F(\phi(t))|$. For $\varepsilon > 0$ arbitrary, we choose t_1 large enough such that

$$|\phi(t_1) - g| < \min\left\{ \frac{\varepsilon}{2|F(g)|}, \frac{\varepsilon}{4C} \right\},$$

and it follows

$$\limsup_{t\to\infty} I_1 + I_2 < \varepsilon.$$

Using (4), we obtain

$$F(g) = \frac{g}{1-g} = G(g),$$

and from Theorem 2.1: $g = x_0$ because x_0 is the maximal point of intersection of F and G and $g \ge x_0$. □

We will now formulate a result concerning the rate of convergence in the special case of $x_0 = 0$. The case $x_0 \ne 0$ will be discussed later.

Theorem 2.3. *Assume $F \in C^1([0,1],\mathbb{R})$ is monotonically increasing, and suppose*

(i) $F(x) < x/(1-x)$ *for* $x \in (0,1)$,
(ii) $F(0) = 0$,
(iii) $F'(0) < 1$.

Then there exists a constant $s_0 > 0$ *such that*

$$\lim_{t\to\infty} e^{s_0 t}\phi(t) = 0.$$

Proof. One easily proves that there is a constant $\varepsilon_0 \in (0,1)$ such that

$$F(x) \le \frac{x}{1+\varepsilon_0 - x} =: G_{\varepsilon_0}(x) \quad (x \in [0,1]).$$

G_{ε_0} is an absolute monotone function and fulfils $G'_{\varepsilon_0}(0) < 1$. It has been shown in **[12]** that there exist $x_0 > 0$ and $\varepsilon \in (0,1)$ such that, for all $n \in \mathbb{N}$ and $x > x_0$,

$$\int_{x_0}^{x} t^n F(\phi(t))\,dt \le \int_{x_0}^{x} t^n G_{\varepsilon_0}(\phi(t))\,dt \le (1-\varepsilon)\int_{x_0}^{x} t^n \phi(t)\,dt. \tag{6}$$

Applying estimate (6) to the techniques of Section 7 from **[12]**, one proves

$$\int_0^\infty t^n \phi(t)\,dt < \infty$$

for all $n \in \mathbb{N}$ and finally the requested result. □

The restriction $F'(0) < 1$ in Theorem 2.3 implies $G'_{\varepsilon_0}(0) < 1$, which was needed for proving estimate (6). The question concerning rates of convergence in case of $F'(0) = 1$ is not answered yet. In the following theorem we will approach a certain class of functions that fulfil this property.

Theorem 2.4. *Let* $F \in C^0([0,1],\mathbb{R})$ *be differentiable and monotonically increasing with the following condition*

there exists $c \in (0,1]$ *such that for all* $x \in [0,1] : 0 \le F(x) \le cx$.

Then the solution ϕ *of* (4) *with kernel-function* F *fulfils, for all* $t \in [0,\infty)$,

$$\phi(t) \le c^{-1/2} t^{-1/2}.$$

Proof. Applying the variation of constants formula to (4) leads to

$$\begin{aligned}
\phi(t) &= e^{-t} - e^{-t}\int_0^t e^s \int_0^s F(\phi(s-r))\dot\phi(r)\,dr\,ds \\
&\le e^{-t} - e^{-t}\int_0^t e^s \int_0^s c\phi(s-r)\dot\phi(r)\,dr\,ds \\
&= e^{-t} - e^{-t}\int_0^t e^s \left(\frac{d}{ds} c \int_0^s \phi(s-r)\phi(r)dr - c\phi(s)\right) ds \\
&\le e^{-t} + e^{-t}\int_0^t e^s c \int_0^s \phi(s-r)\phi(r)\,dr\,ds \\
&\quad - c\int_0^t \phi(t-s)\phi(s)\,ds + e^{-t}\int_0^t e^s \phi(s)\,ds.
\end{aligned}$$

By using Gronwall's inequality, we obtain

$$e^t c \int_0^t \phi(t-s)\phi(s)\,ds \le e^t,$$

and due to the monotonicity of ϕ it follows

$$\phi(t) \le c^{-1/2} t^{-1/2}$$

for all $t \in [0,\infty)$. □

Remark 2.5.

(i) We consider problem (4) with $F(x) = x$ ($x \in \mathbb{R}$), and we assume that there are $k, \delta > 0$ such that, for all $t \in [0,\infty)$,

$$\phi(t) \le k \frac{1}{(1+t)^{1/2+\delta}} =: h(t).$$

It follows from (5) that

$$\phi(t) \geq 1 - \int_0^t h(t-s)h(s)\,ds.$$

It has been shown in **[14]** that there exists a constant $k_1 > 0$ such that

$$\int_0^t h(t-s)h(s)\,ds \leq k_1 \frac{1}{(1+t)^{2\delta}} \stackrel{t\to\infty}{\longrightarrow} 0,$$

so we have $\phi(t) \to c$ $(t \to \infty)$ for a $c \geq 1$, that contradicts the conclusion of Theorem 2.4. Due to this example, the rate of convergence in Theorem 2.4 is optimal.

(ii) The results of Theorems 2.3 and 2.4 can be generalized to the case of a maximal $g \neq 0$ that fulfils $F(g) = g/(1-g)$. This can be done in a similar way as presented in **[12]** by defining

$$\widetilde{F}(x) := [F((1-g)x + g) - F(g)]\,(1-g)$$

and

$$\widetilde{\phi}(t) := \frac{\phi((1-g)t) - g}{1-g}.$$

Then one has

$$\begin{aligned} \widetilde{\phi}(t) + \dot{\widetilde{\phi}}(t) + \int_0^t \widetilde{F}(\widetilde{\phi}(t-s))\dot{\widetilde{\phi}}(s)ds = 0 \quad (t \in (0,\infty)), \\ \widetilde{\phi}(0) = 1. \end{aligned} \tag{7}$$

Applying Theorems 2.3 and 2.4 to problem (7), one obtains similar results for the general case.

(iii) The results of this chapter can easily be extended to more general (not necessary physically relevant) cases with initial conditions $\phi(0) \neq 1$ and inhomogeneous right-hand sides $f : [0,\infty) \to \mathbb{R}$ that fulfil $\overline{f} := \lim_{t\to\infty} f(t) < \infty$ and $f(0) < \phi(0)$.[2] The limit-equation from Theorem 2.1, respectively Corollary 2.2, then proceeds to

$$F(x) = \frac{x - \overline{f}}{\phi(0) - x}.$$

To prove results concerning rates of convergence of the solutions as seen in Theorems 2.3 and 2.4, it will be necessary to call for additional decay rates of f.

3. Blow-up solutions. In the previous section, we discussed the existence of global solutions of problem (4) under certain restrictions on the kernel-function F. It is a natural question whether one can always expect global solutions or whether there are kernel-functions such that related solutions are unbounded on a bounded interval $[0, T)$ for a $T > 0$, i.e., they only exist on $[0, T)$ and produce a so called blow-up at time T. In this chapter, we will prove the existence of such blow-up solutions under certain conditions on the kernel-function. We start by quoting a version of a lemma from **[8]** for Volterra-integral equations.

Lemma 3.1. *Let $g \in C^1(\mathbb{R}, \mathbb{R})$ be monotonically increasing with $g(x) > 0$ if $x > 0$, $k \in C^1((0, \infty), \mathbb{R})$ nonnegative and monotonically increasing with*

$$K(x) = \int_0^x k(s)\, ds > 0 \quad \text{if } x > 0,$$

and assume $f \in C^0([0, \infty), \mathbb{R})$ is nonnegative and monotonically increasing. Furthermore, let g satisfy

$$\limsup_{x \to \infty} \frac{x}{g(x)} < \infty$$

and

$$\int_0^\infty \frac{g'(s)}{g(s)} K^{-1}\left(\frac{s}{g(s)}\right) ds < \infty.$$

If $u : [0, T) \to \mathbb{R}$ is a solution of the following Volterra-integral equation

$$u(t) = f(t) + \int_0^t k(t - s) g(u(s))\, ds,$$

with maximal interval of existence $[0, T)$ such that $u(t) > 0$ for all $t \in (0, T)$, then $T < \infty$ and $u(t) \to \infty$ if $t \to T$.

Theorem 3.2. *Let $F \in C^1((-\infty,1],\mathbb{R})$ be monotonically increasing with $F(x) < -1$ for $x \in (-\infty,1]$, and let there exist $x_0 \in (-\infty,1]$ and $\varepsilon > 0$ such that for $x \in (-\infty,x_0)$*

$$F(x) \leq \varepsilon x - (\varepsilon + 1). \tag{8}$$

Furthermore, let there exist $\delta > 0$ such that

$$\int_{-\infty}^{-\delta} \frac{F'(x)\sqrt{-x}}{(-F(x))^{3/2}}\,dx < \infty. \tag{9}$$

Then there is a $T > 0$, so that problem (4) *has a unique solution $\phi : [0,T) \to \mathbb{R}$ that satisfies $\phi(t) \to -\infty$ if $t \to T$, i.e., there is no global solution for* (4)*.*

Proof. We assume that problem (4) has a global solution $\phi : [0,\infty) \to \mathbb{R}$, and we aim to produce a contradiction with the help of Lemma 3.1. We define, for $t, x \in [0,\infty)$,

$$\begin{aligned} &k(t) := 1 - \phi(t), \qquad f(t) := t,\\ &u(t) := 1 - \phi(t) \text{ and } g(x) := -1 - F(1-x). \end{aligned}$$

With this, u is a global solution of the following Volterra-integral equation

$$u(t) = f(t) + \int_0^t k(t-s)g(u(s))\,ds.$$

It is easy to see that (8) implies $x/g(x) \leq 1/\varepsilon$ for $x \geq 1 - x_0$, i.e.,

$$\limsup_{x\to\infty} \frac{x}{g(x)} < \infty.$$

Due to $F(x) < -1$ ($x \in (-\infty,1]$), one has by using (4): $\dot{\phi}(t) \leq -1$, i.e., $\phi(t) \leq 1 - t$ ($t \in [0,\infty)$). It follows that $k(t) \geq t$ and from that $K(x) \geq x/2$, so we obtain for $x \in [0,\infty)$

$$K^{-1}(x) \leq \sqrt{2x}.$$

It follows that

$$\int_0^\infty \frac{g'(s)}{g(s)} K^{-1}\left(\frac{s}{g(s)}\right) ds \le \sqrt{2} \int_{-\infty}^{1} \frac{F'(x)\sqrt{1-x}}{(-1-F(x))^{3/2}}\, dx.$$

The integral on the right-hand side is bounded because of (9) and

$$\int_{-\infty}^{-\delta_1} \frac{F'(x)\sqrt{1-x}}{(-1-F(x))^{3/2}}\, dx \le \frac{1}{2} \int_{-\infty}^{-\delta_1} \frac{F'(x)\sqrt{-x}}{(-F(x))^{3/2}}\, dx,$$

where $\delta_1 \ge \delta$ was chosen suitably. Using Lemma 3.1, we obtain a contradiction to the assumption from the beginning. This finishes the proof. □

4. Kernels under smallness-conditions. In this chapter, we aim at results for well-posedness and asymptotic behavior of solutions of (4) without using monotonicity conditions on the kernel-functions. This will be done by regarding the convolution-integral term in (4) as a small perturbation of the linear equation, so that the exponential decaying solution of the linear part will dominate. First of all, we consider the following related linear problem

$$\begin{aligned} \phi(t) + \dot{\phi}(t) + \int_0^t m(t-s)\dot{\phi}(s)\, ds &= 0 \quad (t \in (0,\infty)), \\ \phi(0) &= 1, \end{aligned} \tag{10}$$

where $m : [0,\infty) \to \mathbb{R}$.

Theorem 4.1. *Let $m \in C^1([0,\infty),\mathbb{R})$ satisfy $m(0) > -1$, $\lim_{t\to\infty} m(t) = 0$ and*

$$|m'(t)| \le k e^{-c_1 t}$$

for all $t \in [0,\infty)$, where $k, c_1 > 0$ such that $c(c_1 - c) > k$ with $c := 1 + m(0)$. Then problem (10) has a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$ that satisfies

$$\left|\dot{\phi}(t)\right| \le e^{t[k-c(c_1-c)]/(c_1-c)} \quad \text{and} \quad |\phi(t)| \le \frac{-(c_1-c)}{k-c(c_1-c)} e^{t[k-c(c_1-c)]/(c_1-c)}.$$

Proof. Equation (10) is equivalent to the following fixed-point equation:

$$\phi(t) = 1 + \int_0^t m(s) - \phi(s) - m(s)\phi(t-s)\,ds. \tag{11}$$

By using the Banach fixed-point theorem, it is easy to prove that (11) has a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$. Differentiation of (10) with respect to t and variation of constants lead to

$$e^{ct}\dot{\phi}(t) = -1 - \int_0^t \int_r^t e^{cs} m'(s-r)\,ds\dot{\phi}(r)\,dr.$$

Using the conditions on m, we obtain

$$e^{ct}|\dot{\phi}(t)| \leq 1 + \frac{k}{c_1 - c}\int_0^t e^{cr}|\dot{\phi}(r)|\,dr.$$

By using Gronwall's inequality, one has

$$|\dot{\phi}(t)| \leq e^{t[k-c(c_1-c)]/(c_1-c)}$$

and it follows the existence of a $g \in \mathbb{R}$ such that

$$|\phi(t) - g| \leq \frac{-(c_1-c)}{k - c(c_1-c)} e^{t[k-c(c_1-c)]/(c_1-c)}.$$

By using similar techniques as presented in the proof of Corollary 2.2, it is easy to see that

$$\lim_{t\to\infty} \left| \int_0^t m(t-s)\dot{\phi}(s)\,ds \right| = 0,$$

and it follows $g = 0$. $\square$

We will now discuss the nonlinear problem (4). Assume $F \in C^1(\mathbb{R},\mathbb{R})$ with $F(1) > -1$, and let $c := 1 + F(1)$ and $c_1 > c$ constants be chosen arbitrarily.

(i) Let $k > 0$ be such that $c(c_1 - c) > k \geq (c-1)(c_1 - c)$,
(ii) let $\alpha > 0$ satisfy $(\alpha+1)[k - c(c_1-c)]/(c_1-c) \leq -c_1$ and
(iii) let $v_1 > 0$ fulfil $v_1(-(c_1-c)/[k-c(c_1-c)])^\alpha \leq k$.

In addition to that, suppose that F satisfies the smallness conditions

$$F(0) = 0 \text{ and } |F'(x)| \leq v_1 |x|^\alpha \tag{12}$$

for $x \in \mathbb{R}$. We define $X := \{f \in C^1([0,\infty),\mathbb{R}) | f, f' \text{ are bounded}\}$ together with the norm $\|f\|_X := \max\{\|f\|_\infty, \|f'\|_\infty\}$ and the following subset of X:

$$C := \left\{ f \in X \,\middle|\, f(0) = 1, \forall t \in [0,\infty): \begin{array}{l} |f(t)| \leq \frac{-(c_1-c)}{k-c(c_1-c)} e^{t[k-c(c_1-c)]/(c_1-c)} \\ |f'(t)| \leq e^{t[k-c(c_1-c)]/(c_1-c)} \end{array} \right\}.$$

$C \subseteq X$ is bounded, closed, convex and due to (i) not being empty. We define

$$T : C \longrightarrow C, \qquad v \longmapsto Tv := u_v,$$

where u_v is the solution of the linear problem (11) with kernel-function $m := F \circ v$. Due to conditions (i)–(iii) and Theorem 4.1, we easily see that T is well-defined. By using the Schauder fixed-point theorem, we obtain a fixed-point $\phi \in C$ of T that is a solution of (4) with kernel-function F. Due to the equivalence of (4) and (5), Banach fixed-point arguments on (5) lead to the uniqueness of the solution ϕ of (4) in X. Altogether, we have proved the following:

Theorem 4.2. *Assume $F \in C^1(\mathbb{R},\mathbb{R})$ with $F(1) > -1$ and $F(0) = 0$. Furthermore, let $c := 1 + F(1)$ and $c_1 > c$.*

- (i) *Let $k > 0$ be such that $c(c_1 - c) > k \geq (c-1)(c_1 - c)$,*
- (ii) *let $\alpha > 0$ satisfy $(\alpha+1)[k - c(c_1 - c)]/(c_1 - c) \leq -c_1$ and*
- (iii) *let $v_1 > 0$ fulfil $v_1(-(c_1 - c)/[k - c(c_1 - c)])^\alpha \leq k$.*

In addition to that, suppose

$$|F'(x)| \leq v_1 |x|^\alpha \text{ for } x \in \left[\frac{c_1 - c}{k - c(c_1 - c)} - \delta, \frac{-(c_1 - c)}{k - c(c_1 - c)} + \delta \right],$$

for $\delta > 0$. Then problem (4) has a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$ that satisfies

$$\left|\dot\phi(t)\right| \leq e^{t[k-c(c_1-c)]/(c_1-c)} \text{ and } |\phi(t)| \leq \frac{-(c_1 - c)}{k - c(c_1 - c)} e^{t[k-c(c_1-c)]/(c_1-c)}.$$

Corollary 4.3. *Let $\varepsilon \in (0,1)$ and $f \in C^1([-4/(3\varepsilon), 4/(3\varepsilon)],\mathbb{R})$ be twice differentiable in $x = 0$, and suppose $f(0) = f'(0) = 0$ and $f(1) > -1$.*

Then there exists a constant $\kappa_0 \in (0,1]$ *such that the problem* (4) *with kernel-function* $F := \kappa \cdot f$ *has a unique solution* $\phi \in C^1([0,\infty),\mathbb{R})$ *for all* $\kappa \in (0,\kappa_0]$, *with*

$$|\phi(t)| \leq \frac{4}{3+3\kappa f(1)} e^{-t(3+3\kappa f(1))/4} \quad \textit{and} \quad |\dot{\phi}(t)| \leq e^{-t(3+3\kappa f(1))/4}.$$

Proof. We define for a $\kappa > 0$ to be determined later

$$c_\kappa := 1 + \kappa f(1), \qquad \alpha_\kappa := 1, \qquad k_\kappa := \frac{1}{8} c_\kappa^2,$$
$$c_{1\kappa} := \frac{3}{2} c_\kappa \quad \text{and} \quad v_{1\kappa} := \frac{3}{32} c_\kappa^3.$$

Let $\kappa_1 > 0$ be such that, for all $\kappa \in (0,\kappa_1]$,

$$\frac{4}{3} \geq 1 + \kappa f(1) > \varepsilon.$$

Due to $\kappa \leq \kappa_1$ the constants defined above fulfil the conditions of Theorem 4.2. By consequence of the conditions on f it is easy to show that there exists $M > 0$ such that $|f'(x)| \leq M|x|$ for all $x \in [-4/(3\varepsilon), 4/(3\varepsilon)]$. Defining $\kappa_2 := (3/32)\varepsilon^3(1/M)$ and $\kappa_0 := \min\{\kappa_1,\kappa_2\}$, we obtain for all $\kappa \in (0,\kappa_0]$ and $x \in [-4/(3\varepsilon), 4/(3\varepsilon)]$

$$\kappa |f'(x)| \leq v_{1\kappa} |x|^{\alpha_\kappa}.$$

Application of Theorem 4.2 to the kernel-function $\kappa \cdot f$ finishes the proof. □

As a consequence of Corollary 4.3, it is easy to prove the following

Corollary 4.4. *Let* $\varepsilon \in (0,1)$ *and* $F \in C^1([-4/(3\varepsilon), 4/(3\varepsilon)],\mathbb{R})$ *with* $F(0) = F'(0) = 0$, $-1 < F(1) \leq 1/3$ *and*

$$|F'(x)| \leq \frac{3}{32}(1+F(1))^3|x|, \quad x \in \left[-\frac{4}{3\varepsilon}, \frac{4}{3\varepsilon}\right],$$

then problem (4) *with kernel-function* F *has a unique continuously differentiable solution that decays exponentially.*

Remark 4.5. The results of this chapter can easily be extended to the more general case of inhomogeneous right-hand sides f and arbitrary initial conditions. Under the additional assumptions that the derivative of f decays exponentially and that the long-time limit of f is zero, one can construct a similar self-mapping as above. The smallness-parameters on the kernel-function F will additionally depend on the decay-parameters of f' and on $\phi(0)$.

The condition $F'(0) = 0$ from Theorem 4.2 is too restrictive for some applications in physics. This restriction was necessary due to the fact that the convolution of an exponentially decaying solution with itself decays with a worse rate than the function. We will see that, under the weaker expectation of polynomially decaying solutions, one can work without this restriction. We start formulating a special case of Theorem 2.2 from **[18]**.

Lemma 4.6. *Let $d > 0$, $n > 1$ and $f(x) := 1/(d+x)^n$ for $x \in [0,\infty)$. Then one has*

$$\left| \int_0^x f(x-y)f(y)\,dy \right| \leq \frac{2^{n+2}}{(n-1)d^{n-1}} \frac{1}{(d+x)^n}, \quad x \in [0,\infty).$$

Theorem 4.7. *Assume $F \in C^1(\mathbb{R},\mathbb{R})$ with $F(0) = 0$ and $F(1) > -1$. Furthermore, let $n > 1$, $K := n^n$, $k > K$ and $a > 0$ with $a \leq [(k-K)(n-1)^2 n^{2n-2}]/32Kk^2 4^n$. In addition to that, suppose $|F'(x)| \leq a$ for $x \in [-k/((n-1)n^{n-1}), k/((n-1)n^{n-1})]$. Then there exists a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$ of (4) with kernel-function F that satisfies*

$$\left|\dot{\phi}(t)\right| \leq \frac{k}{(n+t)^n} \quad \textit{and} \quad |\phi(t)| \leq \frac{k}{n-1}\frac{1}{(n+t)^{n-1}}.$$

Remark 4.8. We easily see that $a \leq (1/128)(n-1)^2/(4^n n^2) \overset{n\to\infty}{\longrightarrow} 0$, i.e., better rates of decay for ϕ need stronger restrictions on F.

Proof of Theorem 4.7. We define similarly as in the case of exponentially decaying solutions $X := \{f \in C^1([0,\infty),\mathbb{R}) | f, f' \text{ are bounded}\}$ with the norm $\|f\|_X := \max\{\|f\|_\infty, \|f'\|_\infty\}$ and

$$C := \left\{ f \in X \,\middle|\, f(0) = 1, \forall t \in [0,\infty) : \begin{array}{r} |f(t)| \leq (k/(n-1))(1/(n+t)^{n-1}) \\ \text{and} \\ |f'(t)| \leq k/(n+t)^n \hfill \end{array} \right\}.$$

We consider the following mapping

$$T : C \longrightarrow C, \qquad v \longmapsto Tv := u_v,$$

where u_v is the unique solution of the linear problem

$$u_v(t) + \dot{u}_v(t) + \int_0^t m(t-s,s)\,ds = 0, \quad \phi(0) = 1,$$

with $m(t,s) := F(v(t))\dot{v}(s)$ for $t,s \in [0,\infty)$. To show that T is well-defined, we consider the following equation using variation of constants formula

$$\begin{aligned}\dot{u}_v(t) = -e^{-t} - \int_0^t e^{-(t-s)} \int_0^s F'(v(s-r))\dot{v}(s-r)\dot{v}(r)\,dr\,ds\\ - \int_0^t e^{-(t-s)} F(1)\dot{v}(s)\,ds.\end{aligned}$$

Due to $e^{-t} \leq K(1/(n+t)^n)$ for $t \geq 0$, Lemma 4.6 and the conditions on F, it follows

$$|\dot{u}_v(t)| \leq \left(K + \frac{16Kak^2 4^n}{(n-1)^2 n^{2n-2}} + \frac{4K|F(1)|2^n k}{(n-1)n^{n-1}}\right)\frac{1}{(n+t)^n}.$$

Considering the conditions on the constants, we obtain $u_v \in C$.[3] Using the Schauder fixed-point theorem, one can easily prove the existence of a fixed-point $\phi \in C$ of T, which is a solution of (4). Uniqueness follows with the same argument as in the case of exponentially decaying solutions by working with the Banach fixed-point theorem on problem (5). $\square$

5. More parametric kernel-functions. In this chapter we aim to apply the techniques from Sections 2 and 4 to more parametric problems of the following kind

$$\text{(13)} \qquad \begin{aligned}\phi(t) + \dot{\phi}(t) + \int_0^t F(\phi(t-s), t-s, t)\dot{\phi}(s)\,ds = 0, \quad t \in [0,\infty),\\ \phi(0) = 1,\end{aligned}$$

where $F : \mathbb{R} \times [0,\infty) \times [0,\infty) \to \mathbb{R}$. Physically relevant kernel-functions are of separate type, like $F(x,s,t) = f(x)g(s,t) + c$ with functions $f : \mathbb{R} \to \mathbb{R}$, $g : [0,\infty) \times [0,\infty) \to \mathbb{R}$ and constants $c \in \mathbb{R}$. We start formulating a result based on monotonicity-methods from Section 2.

Theorem 5.1. *Assume $f : \mathbb{R} \to \mathbb{R}$, $g : [0,\infty) \times [0,\infty) \to \mathbb{R}$, $(s,t) \mapsto g(s,t)$ and $c \in \mathbb{R}$, and suppose the following conditions:*

(i) *there exists $\overline{g} := \lim_{t\to\infty} g(t,t)$,*
(ii) *there exists $x_0 < 1 : f(x_0)\overline{g} + c = x_0/(1-x_0)$,*
(iii) *f is differentiable and locally Lipschitz-continuous on $[x_0, 1]$,*
(iv) *g is partially differentiable with partial derivatives $g_1 := \partial g/\partial s$ and $g_2 := \partial g/\partial t$,*
(v) *g is locally bounded,*
(vi) *one of the two following conditions is fulfilled on $[x_0, 1] \times [0,\infty) \times [0,\infty)$:*

$$\text{a) } f' \geq 0,\ g \geq 0 \quad \textit{and} \quad f \geq 0,\ g_1 \leq 0,\ g_1 + g_2 \leq 0,$$
$$\text{b) } f' \leq 0,\ g \leq 0 \quad \textit{and} \quad f \leq 0,\ g_1 \geq 0,\ g_1 + g_2 \geq 0.$$

Then problem (13) with kernel-function $F := f \cdot g + c$ has a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$ that is monotonically decreasing with $x_0 \leq \phi(t) \leq 1$ for all $t \in [0,\infty)$.

Proof. We define

$$\widetilde{f} := \left\{ \begin{array}{ll} f(1), & x > 1 \\ f(x), & x_0 \leq x \leq 1 \\ f(x_0), & x < x_0 \end{array} \right\}.$$

Let $(f_n)_{n\in\mathbb{N}} \subseteq C^0([0,\infty),\mathbb{R})$ be a sequence of differentiable locally Lipschitz-continuous functions that satisfies

$$\|f_n - \widetilde{f}\|_\infty \overset{t\to\infty}{\longrightarrow} 0, \quad f_n(x)\widetilde{f}(x) \geq 0 \quad (x \in \mathbb{R})$$

and

$$f_n'(x) \left\{ \begin{array}{ll} \geq 0, & \text{if condition (vi) a) is satisfied} \\ \leq 0, & \text{else} \end{array} \right\}.$$

Due to the boundedness of $\widetilde{f}$ and f_n and to conditions (iii)–(v), one can easily prove, by using Banach's fixed-point theorem, that problem (13) with kernel-function $F_n := f_n \cdot g + c$ has a unique solution $\phi_n \in$

$C^1([0,\infty),\mathbb{R})$ for all $n \in \mathbb{N}$ and that problem (13) with kernel-function $\widetilde{F} := \widetilde{f} \cdot g + c$ has a unique solution $\widetilde{\phi} \in C^1([0,\infty),\mathbb{R})$. Furthermore, this proves the uniqueness of any solution $\phi \in C^1([0,\infty),\mathbb{R})$ of (13). Differentiating the equation from (13) with kernel-function F_n with respect to t, one obtains, due to $\dot{\phi}_n(0) < 0$ and condition (vi),

$$\ddot{\phi}_n \leq -(1 + f_n(1)g(0,t) + c)\dot{\phi}_n(t),$$

for $t \in [0,t_0)$, where $t_0 > 0$ is minimal such that $\dot{\phi}_n(t) < 0$ for all $t \in [0,t_0)$.[4] Gronwall's inequality leads to $t_0 = \infty$, i.e., ϕ_n is monotonically decreasing for all $n \in \mathbb{N}$. Using Gronwall's inequality once again, one can easily show by considering the conditions (iii)–(v) that $\sup_{0\leq t\leq N} |\widetilde{\phi}(t) - \phi_n(t)| \overset{n\to\infty}{\longrightarrow} 0$ for all $N > 0$, i.e., $\widetilde{\phi}$ is monotonically decreasing. With this, one has for all $s_1, s_2, s_3 \in [0,\infty)$ with $s_2 \leq s_3$,

$$\widetilde{f}(\widetilde{\phi}(s_1))g(s_2,s_3) \overset{(vi)}{\geq} f(x_0)g(s_2,s_3) \overset{(i),(vi)}{\geq} f(x_0)\lim_{t\to\infty} g(t,t) \overset{(ii)}{=} \frac{x_0}{1-x_0} - c.$$

Using this, (13) and Gronwall's inequality, we obtain

$$\widetilde{\phi}(t) \geq e^{-1/(1-x_0)t} + \int_0^t e^{-1/(1-x_0)(t-s)} \frac{x_0}{1-x_0}\, ds \overset{t\to\infty}{\longrightarrow} x_0,$$

i.e., one has $x_0 \leq \widetilde{\phi}(t) \leq 1$ for all $t \in [0,\infty)$. Due to $\widetilde{f}(\widetilde{\phi}(s_1))g(s_2,s_3) = f(\widetilde{\phi}(s_1))g(s_2,s_3)$ for all $s_1, s_2, s_3 \in [0,\infty)$, $\widetilde{\phi}$ is a solution of (13) with kernel-function $F = f \cdot g + c$. □

If the limit $\overline{g}$ satisfies $\lim_{t\to\infty} g(t_n^1, t_n^2) = \overline{g}$ for all sequences $(t_n^i)_{n\in\mathbb{N}} \subseteq [0,\infty)$ with $t_n^i \overset{n\to\infty}{\longrightarrow} \infty$, $i = 1,2$, one has the convergence of ϕ to the maximal $\xi \in [x_0, 1]$ that fulfils

$$f(\xi)\overline{g} + c = \frac{\xi}{1-\xi}. \tag{14}$$

This can be proved analogously to Corollary 2.2, by using

$$\lim_{t\to\infty} f(\phi(t-s))g(t-s,t) = f(\overline{\phi})\overline{g},$$

where $\overline{\phi}$ is the limit of ϕ that exists due to Theorem 5.1.

Theorem 5.2. *Assume additionally to the conditions of Theorem* 5.1,

(vii) $f(x)\overline{g} + c \left\{ \begin{array}{ll} < \frac{x}{1-x}, & x > 0 \\ = 0, & x = 0 \end{array} \right\}, \quad x \in [0,1],$

(viii) $f'(0)\overline{g} < 1,$

(ix) $g_2(s,t) \left\{ \begin{array}{ll} \leq 0, & f(x) \geq 0 \textit{ for all } x \in [0,1], \\ \geq 0, & f(x) \leq 0 \textit{ for all } x \in [0,1] \end{array} \right\}, \quad s,t \in [0,\infty).$

Then one has, for all $n \in \mathbb{N}$*,*

$$\lim_{t\to\infty} t^n \phi(t) = 0. \tag{15}$$

If, additionally,

(x) $\overline{g} = 0 \Rightarrow f(0)g(0,0) = 0,$

then one has the existence of a constant $s_0 > 0$ *such that*

$$\lim_{t\to\infty} e^{s_0 t}\phi(t) = 0. \tag{16}$$

Proof. Due to Theorem 5.1 one has $\lim_{t\to\infty}\phi(t) = 0$. We define $H(x) := f(x)\overline{g} + c$ ($x \in [0,1]$). Similarly to the proof of Theorem 2.3, we obtain, using conditions (vii) and (viii), the existence of an $\varepsilon_0 > 0$ such that $H(x) \leq G_{\varepsilon_0}(x) < \frac{x}{1-x}$ for all $x \in [0,1)$, and this leads to

$$\exists \delta \in (0,1), t_0 \in [0,\infty) \quad \forall t \geq t_0 : f(\phi(t))g(t,t) + c \leq (1-\delta)\phi(t),$$

which proves an analogue to estimate (6). Following the same steps as in the proof of Theorem 2.3, respectively of the proof of Theorem 5 from **[12]**, one can prove (15). Doing this, the following equation comes up

$$\begin{aligned} &\int_0^t (f(\phi(t-s))g(t-s,t) + c)\,\dot{\phi}(s)\,ds \\ &= \frac{d}{dt}\int_0^t (f(\phi(s))g(s,t) + c)\,\phi(t-s) - (f(\phi(s))g(s,s) + c)\,ds \\ &\qquad - \int_0^t f(\phi(s))g_2(s,t)\phi(t-s)\,ds. \end{aligned} \tag{17}$$

Condition (ix) is needed to estimate the last integral-term of (17). To prove (16), we distinguish between two cases. In the case of $\overline{g} \neq 0$, we obtain for $x \in [0,1]$, $s,t \in [0,\infty)$

$$f(x)g(s,t) + c \le \kappa H(x) \le \kappa G_{\varepsilon_0}(1)x,$$

where $\kappa := \max\{(1/\overline{g}) \sup_{s,t\in[0,\infty)} g(s,t), 1\}$. In the case of $\overline{g} = 0$, one has $c = 0$ and

$$f(x)g(s,t) \le f(x)g(0,0) \overset{(x)}{\le} \sup_{x\in[0,1]} |f'(x)|g(0,0)x.$$

Using this, one can use the techniques in **[12,** Section 7] to prove (16). □

Remark 5.3. Theorem 5.2 only considers the case of $\xi = 0$ as the maximal $\xi \le 1$ that fulfils (14), which leads to the limit of the solution $\overline{\phi} = \xi = 0$. In the case of $\overline{\phi} \neq 0$, we define $\widetilde{\phi}(t) := [\phi((1-\overline{\phi})t) - \overline{\phi}]/(1-\overline{\phi})$, $\widetilde{f}(x) := f((1-\overline{\phi})x + \overline{\phi})(1-\overline{\phi})$, $\widetilde{g}(s,t) := g((1-\overline{\phi})s, (1-\overline{\phi})t)$ and $\widetilde{c} := -f(\overline{\phi})\overline{g}(1-\overline{\phi})$. Using (14), one has

$$\widetilde{\phi}(t) + \dot{\widetilde{\phi}}(t) + \int_0^t \left(\tilde{f}\left(\widetilde{\phi}(t-s)\right)\widetilde{g}(t-s,s) + \widetilde{c}\right)\dot{\widetilde{\phi}}(s)\,ds = 0, \tag{18}$$
$$\widetilde{\phi}(0) = 1.$$

Applying Theorem 5.2 to (18), one obtains asymptotic results for this case.

We will now formulate a result for problem (13) using smallness-conditions based on Section 4. We start by considering the related linear problem

$$\phi(t) + \dot{\phi}(t) + \int_0^t m(t-s,t)\dot{\phi}(s)\,ds = 0, \quad \phi(0) = 1, \tag{19}$$

where $m \in C^1([0,\infty)\times[0,\infty),\mathbb{R})$ is a fixed kernel. Problem (19) is equivalent to the following problem of an integral-equation

$$\phi(t) = 1 + \int_0^t m(s,s) - \phi(s) - m(s,t)\phi(t-s)\,ds + \int_0^t \int_0^s m_2(r,s)\phi(s-r)\,dr\,ds, \tag{20}$$

where $m_2(s,t) := (d/dt)m(s,t)$ $(m_1(s,t) := (d/ds)m(s,t))$. Banach's fixed-point theorem leads to a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$ of (20), respectively (19).

Lemma 5.4. *Assume the following conditions*:

(i) $m(0,t) \geq -1+\varepsilon$ *for an* $\varepsilon > 0$ *and for all* $t \in [0,\infty)$.
(ii) $|m(0,t)| \leq c$ *for a* $c > 0$ *and for all* $t \in [0,\infty)$.
(iii) $|m_1(s,t) + m_2(s,t)| \leq ke^{-c_1 s}$ *for all* $s,t \in [0,\infty)$, *where* $c_1 > 1 + c$ *and* $k > 0$ *such that* $k/(c_1 - c - 1) < \varepsilon$.
(iv) $\lim_{s,t\to\infty} m(s,t) = 0$.

Then the solution ϕ *of* (19) *satisfies, for all* $t \in [0,\infty)$,

$$|\phi(t)| \leq \frac{1}{\kappa}e^{-\kappa t} \quad \text{and} \quad \left|\dot{\phi}(t)\right| \leq e^{-\kappa t},$$

with $\kappa := \varepsilon - k/(c_1 - c - 1) > 0$.

Proof. Differentiation of (19) with respect to t and the variation of constants formula lead to

$$e^{c(t)}\dot{\phi}(t) = -1 + \int_0^t \int_r^t e^{c(s)}(m_1+m_2)(s-r,s)\dot{\phi}(s)\,ds\,dr,$$

with $c(t) := \int_0^t 1 + m(0,s)\,ds$. One has with (i) and (ii)

$$|c(t) - c(s)| \leq (1+c)|t-s| \quad \text{and} \quad c(t) \geq \varepsilon t.$$

Using (iii), we obtain

$$e^{c(t)}\left|\dot{\phi}(t)\right| \leq 1 + \frac{k}{c_1 - c - 1}\int_0^t e^{c(r)}\left|\dot{\phi}(r)\right| dr.$$

Gronwall's inequality and condition (iv) finish the proof. □

Using Lemma 5.4 we will extend the result to the nonlinear problem (13). Assume $F \in C^1(\mathbb{R} \times [0,\infty) \times [0,\infty), \mathbb{R})$ with derivatives $F_1(x,s,t) := \frac{\partial}{\partial x}F(x,s,t)$, $F_2(x,s,t) := \frac{\partial}{\partial s}F(x,s,t)$ and $F_3(x,s,t) := \frac{\partial}{\partial t}F(x,s,t)$, and suppose $F(1,0,t) \geq -1+\varepsilon$ for a $\varepsilon > 0$ and for all $t \in [0,\infty)$.

(i) Let $k > 0$ be such that $\kappa := \varepsilon - k/(c_1 - c - 1) > 0$ and $\kappa \leq 1$ with $c > 0$ and $c_1 > c+1$.
(ii) Let $v_1, v_2, \alpha, \beta, a_1, a_2 \geq 0$ be such that $v_1\frac{1}{\kappa^\alpha} + v_2\frac{1}{\kappa^\beta} \leq k$, $(\alpha+1)\kappa - a_1 \geq c_1$ and $\beta\kappa - a_2 \geq c_1$.

Furthermore, let F satisfy the following smallness-conditions for $x, s, t \in \mathbb{R} \times [0,\infty) \times [0,\infty)$:

(iii) $F(0,s,t) = 0$ and $|F(1,0,t)| \leq c$,
(iv) $|F_1(x,s,t)| \leq v_1|x|^\alpha e^{a_1 s}$,
(v) $|F_2(x,s,t) + F_3(x,s,t)| \leq v_2|x|^\beta e^{a_2 s}$.
(vi) $\forall N, M > 0\ \exists L > 0\ \forall x, y \in [-M, M]$ for all $s, t \in [0, N]$: $|F_3(x,s,t) - F_3(y,s,t)| \leq L|x-y|$.

We define $X := \{f \in C^1([0,\infty),\mathbb{R}) | f, f' \text{ are bounded}\}$, with the norm $\|f\|_X := \max\{\|f\|_\infty, \|f'\|_\infty\}$ and

$$C := \left\{ f \in X \,\middle|\, f(0) = 1, \forall t \in [0,\infty) : |f(t)| \leq \frac{1}{\kappa}e^{-\kappa t}, |f'(t)| \leq e^{-\kappa t} \right\}.$$

We consider the following self-mapping

$$T : C \longrightarrow C, \qquad v \longmapsto Tv := u_v,$$

where u_v is the solution of the linear problem (19) with kernel-function $m(s,t) := F(u(s),s,t)$. Due to conditions (i)–(v), T is well-defined. Since $C \subseteq X$ is bounded, closed and convex, Schauder's fixed-point theorem leads to a fixed-point $\phi \in C$ of T, i.e., to an exponentially decaying solution of (13). Uniqueness follows from condition (vi) by applying Banach's fixed-point theorem to (20). With this, we have proved the following

Theorem 5.5. *Assume* $F \in C^1(\mathbb{R} \times [0,\infty) \times [0,\infty), \mathbb{R})$ *and suppose* $F(1,0,t) \geq -1 + \varepsilon$ *for a* $\varepsilon > 0$ *(*$t \in [0,\infty)$*) and the conditions* (i)–(vi). *Then there exists a unique solution* $\phi \in C^1([0,\infty), \mathbb{R})$ *such that*

$$|\phi(t)| \leq \frac{1}{\kappa} e^{-\kappa t} \quad \textit{and} \quad \left|\dot{\phi}(t)\right| \leq e^{-\kappa t}$$

for all $t \in [0,\infty)$.

6. Comments on systems with real- and complex-valued equations. In this section we consider the following problem for a system of a real- with a complex-valued equation[5]

(21)

$$\begin{aligned}
&\text{(i)} \quad \dot{\phi}_1(t) + \omega_1 \phi_1(t) + \omega_1 \int_0^t \tfrac{f_1(\overline{\phi_1(t-s)}, \phi_2(t-s), t-s)}{1 - ip_1} \dot{\phi}_1(s)\, ds = 0, \\
&\qquad\qquad \phi_1(0) = \phi_1^0, \\
&\text{(ii)} \quad \dot{\phi}_2(t) + \omega_2 \phi_2(t) + \omega_2 \int_0^t \tfrac{f_2(\phi_2(t-s), \Re\phi_1(t-s), t-s)}{1 + p_2} \dot{\phi}_2(s)\, ds = 0, \\
&\qquad\qquad \phi_2(0) = \phi_2^0,
\end{aligned}$$

where $\phi_1^0 \in \mathbb{C}$, $\phi_2^0 \in \mathbb{R}$, $\omega_1 \in \mathbb{C}$ with $\Re(\omega_1) > 0$, $\omega_2, p_1, p_2 \in \mathbb{R}$ with $\omega_2 > 0$, $f_1 : \mathbb{C} \times \mathbb{R} \times [0,\infty) \to \mathbb{C}$ and $f_2 : \mathbb{R} \times \mathbb{R} \times [0,\infty) \to \mathbb{R}$. The functions f_1 and f_2 are of linear type $f_1(x_1, x_2, s) = \alpha_1 x_1 \phi(s) + \alpha_2 x_2 \phi(s)$ and $f_2(x_1, x_2, s) = \beta_1 x_1 \phi(s) + \beta_2 x_2 \phi(s)$, with $\alpha_{1,2}, \beta_{1,2} \in (0,\infty)$ and $\phi : [0,\infty) \to \mathbb{R}$ is the solution of (4) with kernel-function $F(x) = v_1 x + v_2 x^2$ ($v_{1,2} > 0$) that satisfies[6]

$$|\phi(t)| \leq \frac{k}{n-1} \frac{1}{(d+t)^{n-1}} \qquad \text{and} \qquad |\dot{\phi}(t)| \leq \frac{k}{(d+t)^n}$$

where $k, d > 0$ and $n > 1$. We will sketch techniques which will lead to well-posedness and asymptotic behaviour results for (21). As compared to Section 4, we will need to work with the related linear problems

(22)

$$\begin{aligned}
&\text{(i)} \quad \dot{\phi}_1(t) + \omega_1 \phi_1(t) + \omega_1 \int_0^t m_1(t-s, s)\, ds = 0, \quad \phi_1(0) = \phi_1^0 \in \mathbb{C}, \\
&\text{(ii)} \quad \dot{\phi}_2(t) + \omega_2 \phi_2(t) + \omega_2 \int_0^t m_2(t-s, s)\, ds = 0, \quad \phi_2(0) = \phi_2^0 \in \mathbb{R},
\end{aligned}$$

with $m_1 : [0,\infty) \times [0,\infty) \to \mathbb{C}$ and $m_2 : [0,\infty) \times [0,\infty) \to \mathbb{R}$ both differentiable.

(i) Let $M_1, M_2 > 0$ be such that, for all $t \in [0,\infty)$,

$$\frac{1}{(n/\Re(\omega_1)+t)^n} \leq \frac{M_1}{(d+t)^n} \qquad \text{and} \qquad \frac{1}{(n/\omega_2+t)^n} \leq \frac{M_2}{(d+t)^n}.$$

(ii) Let $k_1 > 0$ be such that $k_1 > (n/\Re(\omega_1))^n M_1|\phi_1^0|$ and $k_1 \geq (n-1)d^{n-1}|\phi_1^0|$.

(iii) Let $k_2 > 0$ be such that $k_2 > (n/\omega_2)^n M_2|\phi_2^0|$ and $k_2 \geq (n-1)d^{n-1}|\phi_2^0|$.

We look for solutions $(\phi_1, \phi_2) \in \mathscr{C}_1 \times \mathscr{C}_2$, where

$$\mathscr{C}_1 := \left\{ f \in X_1 : f(0) = \phi_1^0, \begin{array}{l} |f'(t)| \leq \frac{k_1}{(d+t)^n} \\ |f(t)| \leq \frac{k_1}{n-1}\frac{1}{(d+t)^{n-1}} \end{array} \right\},$$

$$\mathscr{C}_2 := \left\{ f \in X_2 : f(0) = \phi_2^0, \begin{array}{l} |f'(t)| \leq \frac{k_2}{(d+t)^n} \\ |f(t)| \leq \frac{k_2}{n-1}\frac{1}{(d+t)^{n-1}} \end{array} \right\},$$

$$X_1 := \left\{ f \in C^1([0,\infty),\mathbb{C}) | f, f' \text{ are bounded} \right\},$$

$$X_2 := \left\{ f \in C^1([0,\infty),\mathbb{R}) | f, f' \text{ are bounded} \right\},$$

with norms

$$\|f\|_{X_1} = \max\left\{\|f\|_\infty, \|f'\|_\infty\right\}$$

and

$$\|f\|_{X_2} = \max\left\{\|f\|_\infty, \|f'\|_\infty\right\}.$$

Let $u \in \mathscr{C}_2$ be arbitrary but fixed. We consider $T_1 : \mathscr{C}_1 \to \mathscr{C}_1$, $w \mapsto T_1 w$ a solution of (22) (i) with kernel-function

$$m_1(t,s) := \frac{f_1(\overline{w(t)}, u(t), t)\dot{w}(s)}{1 - ip_1}.$$

Due to Lemma 4.6 and some smallness-conditions on α_1 and α_2, T_1 is well-defined. Applying the Schauder fixed-point theorem, we obtain a fixed-point $F_1(u) \in \mathscr{C}_1$ for T_1. With this, we define $T_2 u$ as the solution of (22) (ii) with kernel-function

$$m_2(t,s) = \frac{f_2(u(t), \Re(F_1(u)(t)), t)}{1 + p_2}\dot{u}(s).$$

By using smallness-conditions on β_1 and β_2, we obtain $T_2 u \in \mathscr{C}_2$. This defines a self-mapping $T_2 : \mathscr{C}_2 \to \mathscr{C}_2$, that has a fixed-point $\phi_2 \in \mathscr{C}_2$ as one can prove similarly as for T_1. By construction, the pair $(\phi_1, \phi_2) \in \mathscr{C}_1 \times \mathscr{C}_2$ with $\phi_1 := F_1(\phi_2)$ is a solution of (21). For more details, we refer the reader to [**16,** Chapter 6].

7. Examples and applications.

Example 7.1 (Results of Section 2)**.**

(i) We consider problem (4) with kernel-function $F(x) = (1/2)\sin(x)$. Applying Theorem 2.3, we obtain a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$ that decays exponentially. In case of $F(x) = \sin(x)$, condition $F'(0) < 1$ is not fulfilled. Using Theorem 2.4, we obtain $\phi(t) \leq t^{-1/2}$.

(ii) The rate of convergence for problem (4) with kernel-function $F(x) = x + x^2$ is not answered, yet. With Corollary 2.2, we obtain $\phi(t) \to 0$ if $t \to \infty$.

Example 7.2 (Results of Section 3)**.**

(i) Considering $F(x) = -x^2 + 2x - 2 - \tau$ for $\tau > 0$, Theorem 3.2 proves the existence of a unique solution $\phi \in C^1([0,T),\mathbb{R})$, with $T \in (0,\infty)$ such that $\phi(t) \to -\infty$ if $t \to T$, i.e., there is no global solution for problem (4).

(ii) Condition (9) is not fulfilled for $F(x) = x - 2$, i.e., this condition can be interpreted such that F has to decrease faster than any linear function for $x \to -\infty$.

Example 7.3 (Results of Section 4)**.**

(i) We consider the kernel-function

$$F(x) = \frac{27}{1472}(x^2 - x^4).$$

Applying Corollary 4.3 to F, one obtains the existence of a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$ for problem (4) that fulfils

$$|\phi(t)| \leq \frac{4}{3} e^{-(3/4)t} \quad \text{and} \quad |\dot{\phi}(t)| \leq e^{-(3/4)t}.$$

(ii) Let $w_0 > 0$ be the unique real root of the polynomial $P(x) = 3 - 73x + 9x^2 - 3x^3$. In the case of $F(x) = \pm w_0 x^2$ we obtain a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$ that satisfies

$$|\phi(t)| \leq \frac{4}{3-3w_0} e^{-t(3-3w_0)/4} \quad \text{and} \quad |\dot{\phi}(t)| \leq e^{-t(3-3w_0)/4}.$$

(iii) Let $n = 2$, $K = 4$, $k = 8$ and $a = 1/8192$. From Theorem 4.7, every function $F : [-4,4] \to \mathbb{R}$ that satisfies $|F'(x)| \leq a$ leads to a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$ with

$$|\phi(t)| \leq \frac{8}{2+t} \quad \text{and} \quad |\dot{\phi}(t)| \leq \frac{8}{(2+t)^2},$$

e.g., $F(x) = \pm(1/8192)x$ or $F(x) = \pm(1/73728)(x + x^2)$.

Remark 7.4. Some of the functions from Examples 7.1 and 7.3 are not absolutely monotone on $[0,1]$. By this, we see that the results in this work extend the class of kernel-functions introduced in **[12]**.

Example 7.5 (Results of Section 5)**.**

(i) Let $f(x) := x + x^2 + \tau$ $(\tau > 0)$, $g(s,t) := 1/(1+s^2)$, $c := 0$. From Theorem 5.2 the solution ϕ of problem (13) with kernel-function $F := f \cdot g + c$ satisfies

$$\forall n \in \mathbb{N} : \lim_{t\to\infty} t^n \phi(t) = 0.$$

Condition (x) is not fulfilled.

(ii) In the case of $f(x) = x+x^2$, $g(s,t) = 1/(1+s^2)+\tau$ $(\tau \in [0,1))$, $c = 0$ one has

$$\exists s_0 > 0 : \lim_{t\to\infty} e^{s_0 t}\phi(t) = 0.$$

(iii) Let $f(x) := x + x^2$, $g(s,t) := 1 + 1/(1+s^2)$, $c := 0$, then Theorem 5.2 is not applicable. We have, from Theorem 5.1,

$$\lim_{t\to\infty} \phi(t) = 0.$$

(iv) We consider the following physically relevant problem introduced in **[9]**[7]:

$$[\phi(t) + \dot{\phi}(t)] + \int_0^t \frac{f(\phi(t-s))}{1+\gamma^2(t-s)^2}\dot{\phi}(s)\,ds = 0, \quad \phi(0) = 1, \tag{23}$$

where $\gamma \in \mathbb{R}$ and $f : [0,1] \to \mathbb{R}$. If f is differentiable and locally Lipschitz-continuous such that $f(x) \geq 0$ and $f'(x) \geq 0$ for all $x \in [0,1]$, then application of Theorem 5.2 to (23) proves the existence of a unique solution $\phi \in C^1([0,\infty),\mathbb{R})$ that is monotonically decreasing and satisfies

$$\forall n \in \mathbb{N} : \lim_{t\to\infty} t^n \phi(t) = 0.$$

If, additionally, $f(0) = 0$, one has

$$\exists s_0 > 0 : \lim_{t\to\infty} e^{s_0 t}\phi(t) = 0.$$

(v) We consider the following problem from **[5]**[8]:

$$\phi(t) + \dot{\phi}(t) + \int_0^t \frac{f(\phi(t-s))}{1+\gamma^2\sin^2(\omega(t-s))}\dot{\phi}(s)\,ds = 0, \quad \phi(0) = 1, \tag{24}$$

with $\gamma, \omega \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$. If one defines $F(x,s,t) := f(x)/(1+\gamma^2\sin^2(\omega s))$, one can apply Theorem 5.5 to problem (24). In the case of $\omega = \gamma = 1$, the kernel-functions $f(x) = \pm(75/1324)x^2$ and $f(x) = \pm(16875/3018752)(x^2 + x^4)$ lead to unique solutions $\phi \in C^1([0,\infty),\mathbb{R})$ that satisfy

$$|\phi(t)| \leq \frac{16}{15}e^{-15/16t} \quad \text{and} \quad |\dot{\phi}(t)| \leq e^{-15/16t}.$$

(vi) If $\widetilde{f} \in C^1(\mathbb{R},\mathbb{R})$ satisfies $\widetilde{f}(0) = \widetilde{f}'(0) = 0$ twice differentiable in $x = 0$, one can prove the existence of a $\tau_0 \in (0,1]$ such that problem (24) with kernel-function $f := \tau_0\widetilde{f}$ has a unique solution that decays exponentially (see **[16,** Corollary 5.21]).

Example 7.6 (Results of Section 6)**.** [9]
Let $\phi \in C^1([0,\infty),\mathbb{R})$ with $\phi(0) = 1$ and

$$|\phi(t)| \leq \frac{8}{2+t} \quad \text{and} \quad |\dot{\phi}(t)| \leq \frac{8}{(2+t)^2}$$

(compare Example 7.3 (iii)). Furthermore, let $\phi_1^0 = \phi_2^0 = 1$, $\omega_1 = \omega_2 = 2$ and $p_1 = p_2 = 1$. If one sets $n = 2$, $k = 8$, $d = 2$, $M_1 = M_2 = 4$, $k_1 = k_2 = 5$ and $\varepsilon_1 = \varepsilon_2 = 1$, then application of **[16,** Theorem 6.2] to problem (21) proves under the restrictions $\alpha_1 + \alpha_2 \leq \sqrt{2}/204800$ and $\beta_1 + \beta_2 \leq 1/102400$ the existence of a unique solution $(\phi_1, \phi_2) \in C^1([0,\infty), \mathbb{R}) \times C^1([0,\infty), \mathbb{R})$ that satisfies

$$|\phi_1(t)| \leq \tfrac{5}{2+t},\ |\dot{\phi_1}(t)| \leq \tfrac{5}{(2+t)^2},\ |\phi_2(t)| \leq \tfrac{5}{2+t} \text{ and } |\dot{\phi_2}(t)| \leq \tfrac{5}{(2+t)^2}.$$

Acknowledgments. This work is based on the Ph.D. thesis **[16]**. The techniques of Section 4 can be extended to treat comparable problems of partial integro-differential equations of first order (see **[17]**).

ENDNOTES

1. $F \in C^\infty$, $F \geq 0$, $F^{(n)} \geq 0$ $(n \in \mathbb{N})$.
2. This condition is necessary to obtain monotonically decreasing solutions.
3. To obtain an estimate for $|u_v(t)|$, one can use similar techniques as used in the proof of Theorem 4.1.
4. $t_0 = \infty$ is possible.
5. See **[10]**, $\Re(z)$ denotes the real-part of a complex number $z \in \mathbb{C}$. Problems of that kind describe models with coefficients that depend additionally on an external force. The distinction of the wave-vector into a part that is parallel and into another one that is perpendicular to the force-vector leads to a system with a real- and a complex-valued equation.
6. See Theorem 4.7.
7. Such problems describe effects of shearing like upcoming loss of memory that is mathematically described by a decaying term which is added to the kernel-function.
8. Problems of that kind arise from externally applied shear strains.
9. In **[10]**, no explicit values for the parameters are given. In order to give an idea about what is meant by smallness-conditions, the parameters are chosen arbitrarily.

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