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NON-AUTONOMOUS IMPULSIVE CAUCHY PROBLEMS OF PARABOLIC TYPE INVOLVING NONLOCAL INITIAL CONDITIONS

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ABSTRACT. We consider a nonautonomous impulsive Cauchy problem of parabolic type involving a nonlocal initial condition in a Banach space X, where the operators in linear part (possibly unbounded) depend on time t and generate an evolution family. New existence theorems of mild solutions to such a problem, in the absence of compactness and Lipschitz continuity of the impulsive item and nonlocal item, are established. The non-autonomous impulsive Cauchy problem of neutral type with nonlocal initial condition is also considered. Comparisons with available literature are also given. Finally, as a sample of application, these results are applied to a system of partial differential equations with impulsive condition and nonlocal initial condition. Our results essentially extend some existing results in this area.

1. Introduction. The dynamics of evolving processes is often subjected to abrupt changes at certain moments such as shocks, harvesting and natural disasters. Often these short-term perturbations are treated as having acted instantaneously or in the form of "impulses" (see [**4**]). One of the emerging branches of the study associated with "impulses" is the theory of impulsive differential equations (inclusions). It is significant to study this class of equations (inclusions), because, in this way, a large class of physical processes in population biology, the diffusion of chemicals, the spread of heat, the radiation of electromagnetic waves, and so forth, can be analyzed. These processes usually have short-time

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perturbations during their evolution. The perturbations are performed discretely, and their duration is negligible in comparison with the total duration of the processes. Some advances concerning this topic can be found, for instance, in the monographs of Bainov and Simeonov [**4**], Benchohra et al. [**7**], Lakshmikantham et al. [**20**], Samoilenko and Perestyuk [**29**] and the papers of Ahmed [**1, 2**], Cardinali and Rubbioni [**9**], Hernández and Aki [**19**], Liu [**22**], Liu and Willms [**24**], Nieto and O'Regan [**26**] and Rogovchenko [**28**].

In particular, stimulated by the observation that nonlocal initial conditions are more realistic than usual ones in treating physical problems, the study of impulsive differential equations (inclusions) with nonlocal initial conditions has been investigated to a large extent; for significant works along this line, we refer to, e.g., [**3, 7, 13, 14, 17**] and the references therein. Please see [**8, 10, 11, 30**] for more detailed information related to the importance of nonlocal initial conditions in applications.

Let X be a Banach space with norm $\|\cdot\|$. We consider the nonautonomous impulsive Cauchy problem of parabolic type involving nonlocal initial condition

$$\begin{cases} u'(t) = A(t)u(t) + F(t, u(t)), \quad t \in [0, T]\backslash\{t_1, \dots, t_n\}, \\ \Delta u(t_i) = I_i(u(t_i)), \quad i = 1, \dots, n, \\ u(0) = H(u), \end{cases} \tag{1.1}$$

in X, where $T > 0$, $(A(t))_{t\in[0,T]}$ is a family of (possibly unbounded) linear operators depending on time and having the domains $(D(A(t)))_{t\in[0,T]}$, $0 < t_1 < t_2 < \cdots < t_n < T$ are pre-fixed numbers, $\Delta u(t_i)$ represents the jump of the function u at t_i, which is defined by $u(t_i^+) - u(t_i^-)$, where $u(t_i^+) = \lim_{h\to 0^+} u(t_i + h)$ and $u(t_i^-) = \lim_{h\to 0^-} u(t_i + h)$ denote respectively the right and left limits of $u(t)$ at $t = t_i$, and F, H, I_i $(i = 1, \dots, n)$ are appropriate functions to be specified later. As can be seen, H constitutes a nonlocal condition and $\Delta u(t_i) = u(t_i^+) - u(t_i^-)$ constitutes an impulsive condition.

As usual, the solution $t \to u(t)$ with the points of discontinuity at the moments t_i $(i = 1, \dots, n)$ follows that $u(t_i) = u(t_i^-)$, that is, at which it is continuous from the left.

We mention that in [**21**], Liang et al. studied the existence and uniqueness of mild and classical solutions to the Cauchy problem (1.1) in the autonomous case (i.e., $A(t) \equiv A$), with nonlocal item H being

Lipschitz continuous, compact, or not Lipschitz continuous and not compact, and impulsive item I_i being Lipschitz continuous or compact.

In the present paper, we will combine this earlier work and extend the study to the Cauchy problem (1.1), which is more general than those in many previous publications. New results concerning the existence of mild solutions to the Cauchy problem (1.1), in the absence of compactness and Lipschitz continuity of the impulsive and nonlocal items, are established by using evolution family and the approximating technique. Then, we treat the generalization to the Cauchy problem (1.1) in the form

(1.2)
$$\begin{cases} \dfrac{d}{dt}[u(t)-g(t,u(t))] = A(t)u(t) + F(t,u(t)), \quad t \in [0,T]\backslash\{t_1,\dots,t_n\}, \\ \Delta u(t_i) = I_i(u(t_i)), \quad i = 1,\dots,n, \\ u(0) = H(u), \end{cases}$$

where g is an appropriate function to be specified later.

Let us point out that the work of this paper has two wedges; on the one hand, we will extend the study of autonomous impulsive Cauchy problems with nonlocal initial conditions to non-autonomous ones. On the other hand, we will obtain the existence theorems of mild solutions to the Cauchy problems (1.1) and (1.2) under weaker conditions on the impulsive and nonlocal items. The results obtained in this paper are generalizations of related results (see Remarks 1.1 and 1.2 below). Moreover, even for corresponding abstract impulsive Cauchy problems without nonlocal initial conditions, the results here are new.

***Remark* 1.1.** As the reader can see, the hypotheses on the impulsive and nonlocal items in our theorems are reasonably weak and different from those in many previous papers such as **[3, 7]**, and the proofs provided are concise.

***Remark* 1.2.** One will see that Theorem 3.8 below essentially extends the main results of the previous research in several ways; as far as the mild solution of Cauchy problems (1.1) is concerned, by dropping the compactness and Lipschitz continuity of the impulsive item from the hypotheses. This distinguishes the present paper from earlier works on impulsive Cauchy problems.

***Remark* 1.3.** Note that the techniques in the proofs of our theorems are essentially different from those used in **[15]**.

2. Preliminaries. In this section, we introduce some notation, establish some conventions and describe some results which are essential tools in later sections.

Let $C([c,d];X)$ for $-\infty < c < d < +\infty$ be the Banach space of all continuous functions from $[c,d]$ into X with uniform norm topology. $\mathscr{L}(X)$ stands for the Banach space of all bounded linear operators from X to X equipped with its natural topology.

Write

$$J_0 = [0,t_1], \quad J_i = (t_i,t_{i+1}], \quad i = 1,\dots,n,$$

with $t_0 = 0, t_{n+1} = T$, and let u_i be the restriction of a function u to J_i $(i = 0,1,\dots,n)$. Consider the set of functions

$$\begin{aligned} PC([0,T];X) = \{u : [0,T] \to X; u_i \in C(J_i;X), i = 0,1,\dots,n, \\ \text{and } u(t_i^+) \text{ and } u(t_i^-) \text{ exist and satisfy} \\ u(t_i) = u(t_i^-) \text{ for } i = 1,\dots,n\}. \end{aligned}$$

Endowed with the norm

$$\|u\|_{PC} = \max\{\sup_{t\in J_i} \|u_i(t)\|; \quad i = 0,1,\dots,n\},$$

it is easy to show that $PC([0,T];X)$ is a Banach space (see **[18]**).

Let r be a finite positive constant, and put

$$\Omega_r = \{u \in PC([0,T];X); \|u(t)\| \le r, \quad \text{for all } t \in [0,T]\},$$

which is a convex closed subset of $PC([0,T];X)$.

For a set $B \subset PC([0,T];X)$, we denote by

$$B|_{\overline{J}_i} = \{u \in C(\overline{J}_i;X) : u(t_i) = v(t_i^+), u(t) = v(t), t \in J_i, v \in B\}, \\ i = 0,1,\dots,n.$$

We present the following lemma, which is useful for our further study in this work.

Lemma 2.1. *A set $B \subset PC([0,T];X)$ is precompact in $PC([0,T];X)$ if and only if the set $B|_{\overline{J}_i}$ is precompact in $C(\overline{J}_i;X)$ for each $i = 0, 1, \ldots, n$.*

Definition 2.2. An operator family $\{U(t,s)\}_{0\le s\le t\le T} \subset \mathscr{L}(X)$ on X is called a (strongly continuous) evolution family if

(1) $U(t,r)U(r,s) = U(t,s)$, $U(t,t) = I$ for $0 \le s \le r \le t \le T$.
(2) The map $(t,s) \mapsto U(t,s)$ is strongly continuous for $0 \le s \le t \le T$.

In this paper $\{A(t)\}_{t\in[0,T]}$ is assumed to be a family of linear operators defined in X. Frow now on, Hypotheses (a)–(b) (parabolicity conditions) below will be assumed throughout.

(a) The domain $D(A(t)) = D$ of $A(t)$, $0 \le t \le T$, is dense in X and independent of t.
(b) For every $t \in [0,T]$ and $\lambda \in \mathbf{C}$ with $\operatorname{Re}\lambda \le 0$, the resolvent $(\lambda + A(t))^{-1}$ exists in $\mathscr{L}(X)$ and satisfies

$$\|(\lambda + A(t))^{-1}\|_{\mathscr{L}(X)} \le \frac{M_0}{1+|\lambda|}, \quad t \in [0,T],\ \operatorname{Re}\lambda \le 0,$$

for a constant M_0.
(c) There are constants $\alpha \in (0,1]$ and M_1 such that

$$\|(A(t) - A(s))A(r)^{-1}\|_{\mathscr{L}(X)} \le M_1|t-s|^\alpha, \quad t,s,r \in [0,T].$$

Under the Hypotheses (a)–(c), there is a unique evolution family $\{U(t,s)\}_{0\le s\le t\le T}$ satisfying:

(i) $\|U(t,s)\|_{\mathscr{L}(X)} \le M$ for $0 \le s \le t \le T$.
(ii) For $0 \le s < t \le T$, $U(t,s) : X \to D$ and $t \to U(t,s)$ is strongly differentiable in X. The derivative $(\partial/\partial_t)U(t,s) \in \mathscr{L}(X)$, and it

is strongly continuous on $0 \le s < t \le T$. Moreover,

$$\frac{\partial U(t,s)}{\partial t} - A(t)U(t,s) = 0 \quad \text{for } 0 \le s < t \le T,$$
$$\left\| \frac{\partial U(t,s)}{\partial t} \right\|_{\mathscr{L}(X)} = \|A(t)U(t,s)\|_{\mathscr{L}(X)} \le \frac{M'}{t-s} \quad \text{for } 0 \le s < t \le T,$$
$$\|A(t)U(t,s)A(s)^{-1}\|_{\mathscr{L}(X)} \le M'' \quad \text{for } 0 \le s \le t \le T.$$

Here M, M' and M'' are positive constants.

(iii) For every $u \in D$ and $t \in (0,T]$, $U(t,s)u$ is differentiable with respect to s on $0 \le s < t \le T$ and

$$\frac{\partial^+ U(t,s)u}{\partial s} = U(t,s)A(s)u.$$

In this case, $\{U(t,s)\}_{0 \le s \le t \le T}$ is called the evolution family associated with $\{A(t)\}_{t \in [0,T]}$. "Evolution family" is also called evolution system, evolution operator, evolution process, propagator or fundamental solution. More details can be found in, e.g., **[12, 23, 27, 31]**.

The considerations of this paper also need the following result.

Lemma 2.3. (Krasnoselskii's fixed point theorem). *Let E be a Banach space, B a bounded closed and convex subset of E, and let F_1, F_2 be maps of B into E such that $F_1 x + F_2 y \in B$ for every pair $x, y \in B$. If F_1 is a contraction and F_2 is completely continuous, then the equation $F_1 x + F_2 x = x$ has a solution on B.*

Lemma 2.3 is classical and can be found in many books.

3. Main results. The present section is devoted to the study of mild solutions to Cauchy problem (1.1). We start with the definition of mild solutions.

Definition 3.1. A solution $u \in PC([0,T];X)$ of the impulsive integral equation

$$\begin{aligned} u(t) &= U(t,0)H(u) + \int_0^t U(t,s)F(s,u(s))\,\mathrm{d}s \\ &\quad + \sum_{0<t_i<t} U(t,t_i)I_i(u(t_i^-)), \quad t \in [0,T], \end{aligned}$$

is called a mild solution of Cauchy problem (1.1).

Let us first introduce our basic assumptions:

(H_1) The evolution family $\{U(t,s)\}_{0\le s\le t\le T}$ is compact, specifically, $U(t,s)$ maps bounded subsets of X into precompact subsets of X for all $0\le s<t\le T$.

(H_2) $F:[0,T]\times X\mapsto X$ is a Carathéodory function; there exists a function $f_r(\cdot)\in L^1(0,T;\mathbf{R}^+)$ such that for almost every $t\in[0,T]$ and all $u\in X$ satisfying $\|u\|\le r$,

$$\|F(t,u)\|\le f_r(t),\quad \text{and}\quad \liminf_{r\to+\infty}\frac{\|f_r\|_{L^1(0,T)}}{r}=\sigma<\infty.$$

(H_3) (i) $H: PC([0,T];X)\mapsto X$ is continuous; there exists a nondecreasing function $\Phi:\mathbf{R}^+\to\mathbf{R}^+$ such that, for all $u\in\Omega_r$,

$$\|H(u)\|\le\Phi(r),\quad\text{and}\quad \liminf_{r\to+\infty}\frac{\Phi(r)}{r}=\mu<\infty.$$

(ii) There is an $\eta\in(0,t_1)$ such that, for any $u,w\in PC([0,T];X)$ satisfying $u(t)=w(t)$ $(t\in[\eta,T])$, $H(u)=H(w)$.

(H_4) For every $i=1,\dots,n$, $I_i:X\mapsto X$ is continuous, there exists a nondecreasing function $\Psi_i:\mathbf{R}^+\to\mathbf{R}^+$ such that, for all $u\in X$ satisfying $\|u\|\le r$,

$$\|I_i(u)\|\le\Psi_i(r),\quad\text{and}\quad \liminf_{r\to+\infty}\frac{\Psi_i(r)}{r}=\gamma_i<\infty.$$

***Remark* 3.2.** Let us note that if, for each $t\in[0,T]$ and some $\lambda\in\rho(A(t))$ (the resolvent set of $A(t)$), the resolvent $R(\lambda,A(t))$ is a compact operator, then $U(t,s)$ is a compact operator whenever $t>s$ (see **[16,** Proposition 2.1]). Moreover, the compactness of $U(t,s)$ for $t>s$ implies the continuity in uniform operator topology.

***Remark* 3.3.** Assumption (H_3) (ii) is the case when the values of the solution $u(t)$ for t near zero do not affect $H(u)$. A case in point was presented by Deng **[11]**, where the operator H is given as follows: $H(u)=\sum_{i=1}^p C_iu(s_i)$, where C_i $(i=1,\dots,p)$ are given constants and $0<s_1<\cdots<s_{p-1}<s_p<+\infty$ $(p\in\mathbf{N})$, which is used to describe the diffusion phenomenon of a small amount of gas in a transparent tube.

Before proving the main theorems, we first present some lemmas.

Lemma 3.4. *Under hypotheses* (H_1) *and* (H_2), *the operator* $\Gamma^1 : \Omega_r \mapsto PC([0,T];X)$, *defined by*

$$(\Gamma^1 u)(t) = \int_0^t U(t,s)F(s,u(s))\,\mathrm{d}s, \quad u \in \Omega_r,\ t \in [0,T],$$

is compact.

Proof. A standard argument, taking into account hypotheses (H_1) and (H_2), shows that the assertion of the lemma remains true (see also the proof of Lemma 3.6 below). Here, we omit the details. □

Write $\Theta := \{m; m \in \mathbf{N}^+ \text{ and } mT \geq 1\}$. It is clear that

$$\|U(t,s)\|_{\mathscr{L}(X)} \leq M \quad \text{for all } 0 \leq s \leq t \leq \frac{1}{m}\ (m \in \Theta).$$

Now, consider, for each $m \in \Theta$, an operator Γ_m on $PC(0,T;X)$ defined by

$$\begin{aligned}(\Gamma_m u)(t) \quad &= \quad U(t,0)U\left(\frac{1}{m},0\right)H(u) + \int_0^t U(t,s)F(s,u(s))\,\mathrm{d}s \\ &\quad + \sum_{0<t_i<t} U(t,t_i)U\left(\frac{1}{m},0\right)I_i(u(t_i^-)), \quad t \in [0,T].\end{aligned}$$

Lemma 3.5. *Let hypotheses* (H_1)–(H_4) *hold, except for* (H_3) (ii). *Then* Γ_m *has at least one fixed point* $u_m \in PC(0,T;X)$ *for each* $m \in \Theta$, *provided that*

$$M\sigma + M^2\left(\mu + \sum_{i=1}^n \gamma_i\right) < 1. \tag{3.1}$$

Proof. Fix $m \in \Theta$. From our hypotheses on F, H and I_i ($i = 1,\ldots,n$) and (3.1), it is easy to see that Γ_m, mapping $PC(0,T;X)$ into itself, is well defined and there exists a $k_0 > 0$ such that

$$M\|f_{k_0}\|_{L^1(0,T)} + M^2\left(\Phi(k_0) + \sum_{i=1}^n \Psi_i(k_0)\right) \leq k_0,$$

from which we see that, for $t \in [0,T]$ and $u \in \Omega_{k_0}$,

$$\begin{aligned}\|(\Gamma_m u)(t)\| \le \left\| U(t,0)U\left(\frac{1}{m},0\right)\right\|_{\mathscr{L}(X)} \|H(u)\| \\ + \int_0^t \|U(t,s)\|_{\mathscr{L}(X)} \|F(s,u(s))\|\,\mathrm{d}s \\ + \sum_{0<t_i<t} \left\| U(t,t_i)U\left(\frac{1}{m},0\right)\right\|_{\mathscr{L}(X)} \|I_i(u(t_i^-))\| \\ \le M^2\Phi(k_0) + M\int_0^t f_{k_0}(s)\,\mathrm{d}s + M^2\sum_{i=1}^n \Psi_i(k_0) \\ \le k_0.\end{aligned}$$

This proves that Γ_m maps Ω_{k_0} into itself.

Let $\{u_q\}_{q=1}^\infty \subset \Omega_{k_0}$ be a sequence such that $u_q \to u$ as $q \to \infty$ in $PC(0,T;X)$. Observe, by the continuity of H, I_i ($i = 1,\dots,n$), and F with respect to the second argument, that for each $t \in [0,T]$,

$$\begin{aligned}\|(\Gamma_m u_q)(t) - (\Gamma_m u)(t)\| \le M^2\|H(u_q) - H(u)\| \\ + M\int_0^t \|F(s,u_q(s)) - F(s,u(s))\|\,\mathrm{d}s \\ + M^2\sum_{i=1}^n \|I_i(u_q(t_i^-)) - I_i(u(t_i^-))\| \\ \longrightarrow 0, \quad \text{as } q \to \infty,\end{aligned}$$

due to the Lebesgue dominated convergence theorem. That is,

$$\|\Gamma_m u_q - \Gamma_m u\|_{PC} \longrightarrow 0, \quad \text{as } q \to \infty.$$

Accordingly, Γ_m is continuous on Ω_{k_0}.

Next, to be able to apply Schauder's second fixed point theorem to obtain a fixed point of Γ_m, we need to prove that Γ_m is compact on Ω_{k_0}. Since $H(\Omega_{k_0})$ is bounded in X in view of (H_3) (i) and $U((1/m),0)$ is compact in X in view of (H_1), we justify that, for each $t \in [0,T]$,

$$\left\{U(t,0)U\left(\frac{1}{m},0\right)H(u); u \in \Omega_{k_0}\right\} \text{ is precompact in } X,$$

by the boundedness of $U(t,0)$ ($t \in [0,T]$) and, for $0 \le s \le t \le T$,

$$\begin{aligned}\left\|U(t,0)U\left(\frac{1}{m},0\right)H(u)-U(s,0)U\left(\frac{1}{m},0\right)H(u)\right\| \\ = \left\|[U(t,0)-U(s,0)]U\left(\frac{1}{m},0\right)H(u)\right\| \\ \longrightarrow 0, \quad \text{as } t\to s,\end{aligned}$$

by the strong continuity of $U(t,s)$ and the compactness of $U((1/m),0)$ $H(u)$ in X. Thus, we verify, with the aid of Arzela-Ascoli's theorem, that

$$U(t,0)U\left(\frac{1}{m},0\right)H(\cdot),$$
$$\text{mapping } \Omega_{k_0} \text{ into } PC([0,T];X), \text{ is compact.}$$

The same idea can be used to prove that

$$\sum_{0<t_i<t} U(t,t_i)U\left(\frac{1}{m},0\right)I_i(\cdot(t_i^-)),$$
$$\text{mapping } \Omega_{k_0} \text{ into } PC([0,T];X), \text{ is compact.}$$

In fact, this can be seen from Lemma 2.1 and the observations that, for every $t\in \overline{J}_i$ $(i=1,\dots,n)$,

$$\left\{U(t,t_i)U\left(\frac{1}{m},0\right)I_i(u(t_i^-)); u\in\Omega_{k_0}\right\} \text{ is precompact in } X$$

due to the compactness of $U((1/m),0)$, and for $t_i\le s\le t\le t_{i+1}$ $(i=1,\dots,n)$,

$$\begin{aligned}\left\|U(t,t_i)U\left(\frac{1}{m},0\right)I_i(u(t_i^-))-U(s,t_i)U\left(\frac{1}{m},0\right)I_i(u(t_i^-))\right\| \\ = \left\|[U(t,t_i)-U(s,t_i)]U\left(\frac{1}{m},0\right)I_i(u(t_i^-))\right\| \\ \longrightarrow 0, \quad \text{as } t\to s\end{aligned}$$

by the strong continuity of $U(t,s)$ and the compactness of

$$U((1/m),0)I_i(u(t_i^-)) \quad (i=1,\dots,n) \quad \text{in } X.$$

Consequently, we have proved, noticing Lemma 3.4, that Γ_m is compact on Ω_{k_0}. This enables us to deduce that Γ_m has at least one fixed point $u_m\in\Omega_{k_0}$ for each $m\in\Theta$. The proof is then completed. $\square$

Suppose that u_m, coming from Lemma 3.5, is a fixed point of Γ_m corresponding to $m \in \Theta$. Then, from Lemma 3.5, it follows that there exists a $k_0 > 0$ such that $u_m \in \Omega_{k_0}$ for all $m \in \Theta$ and u_m satisfies the integral equation

(3.2)

$$u_m(t) = U(t,0)U\left(\frac{1}{m},0\right)H(u_m) + \int_0^t U(t,s)F(s,u_m(s))\,\mathrm{d}s \\ + \sum_{0<t_i<t} U(t,t_i)U\left(\frac{1}{m},0\right)I_i(u_m(t_i^-)), \quad t \in [0,T],\ m \in \Theta.$$

Now, write

$$(\Gamma^2 u_m)(t) = U(t,0)U\left(\frac{1}{m},0\right)H(u_m),$$
$$(\Gamma^3 u_m)(t) = \sum_{0<t_i<t} U(t,t_i)U\left(\frac{1}{m},0\right)I_i(u_m(t_i^-)),$$

and let $\mu \in (0,\eta)$ be fixed with η being the constant in (H_3) (ii).

Lemma 3.6. *Under the hypotheses of Lemma* 3.5, $\{\Gamma^2 u_m; m \in \Theta\}_{[\mu,t_1]}$ *is precompact in* $C([\mu,t_1];X)$, $\{\Gamma^2 u_m; m \in \Theta\}_{\overline{J}_i}$ *for each* $i = 1,\dots,n$ *is precompact in* $C(\overline{J}_i;X)$, *and* $\{\Gamma^i u_m; m \in \Theta\}$ *for* $i = 1,3$ *is precompact in* $PC([0,T];X)$.

Proof. From the compactness of $U(t,0)$ ($t \in [\mu,T]$) in X, the boundedness of $U((1/m),0)$, Remark 3.2, and (H_3) (i), it is not difficult to see that $\{U(t,0)U((1/m),0)H(u_m); m \in \Theta\}$ for each $t \in [\mu,t_1]$ is precompact in X, and for $s_1,s_2 \in [\mu,t_1]$ with $s_1 \le s_2$,

$$\left\|U(s_2,0)U\left(\frac{1}{m},0\right)H(u_m) - U(s_1,0)U\left(\frac{1}{m},0\right)H(u_m)\right\| \\ = \left\|(U(s_2,0) - U(s_1,0))U\left(\frac{1}{m},0\right)H(u_m)\right\| \\ \longrightarrow 0, \quad \text{as } s_2 \to s_1,$$

uniformly for $m \in \Theta$. Hence, an application of Arzela-Ascoli's theorem justifies that $\{\Gamma^2 u_m; m \in \Theta\}_{[\mu,t_1]}$ is precompact in $C([\mu,t_1];X)$. The same idea can be used to prove that, for each $i = 1, \ldots, n$, $\{\Gamma^2 u_m; m \in \Theta\}_{\overline{J}_i}$ is precompact in $C(\overline{J}_i; X)$.

Next, we treat $\{\Gamma^1 u_m; m \in \Theta\}$. Let $t \in (0, t_1]$ be fixed. For any $\varepsilon \in (0, t)$, note that $U(t, t-\varepsilon/2) \in \mathscr{L}(X)$ and $U(t-\varepsilon/2, t-\varepsilon)$ is compact in X by (H_1). Therefore, as

$$\begin{aligned}(\Gamma^1 u_m)(t) &= \int_0^t U(t,s)F(s,u_m(s))\,\mathrm{d}s \\ &= \int_{t-\varepsilon}^t U(t,s)F(s,u_m(s))\,\mathrm{d}s \\ &\quad + U\Big(t, t-\frac{\varepsilon}{2}\Big)U\Big(t-\frac{\varepsilon}{2}, t-\varepsilon\Big) \\ &\quad \times \int_0^{t-\,\varepsilon} U(t-\varepsilon, s)F(s,u_m(s))\,\mathrm{d}s,\end{aligned}$$

and

$$\bigg\| \int_{t-\varepsilon}^t U(t,s)F(s,u_m(s))\,\mathrm{d}s \bigg\| \le M \int_{t-\varepsilon}^t f_{k_0}(s)\,\mathrm{d}s \longrightarrow 0 \quad \text{as } \varepsilon \to 0,$$

uniformly for $m \in \Theta$ due to (H_2), we conclude, using total boundedness, that, for each $t \in (0, t_1]$, $\{(\Gamma^1 u_m)(t); m \in \Theta\}$ is relatively compact in X. Let $\delta > 0$ be small enough. Furthermore, for the case when $0 < s_1 < s_2 \le t_1$, in view of (H_2) and Remark 3.2, we have

$$\begin{aligned}&\|(\Gamma^1 u_m)(s_2) - (\Gamma^1 u_m)(s_1)\| \\ &\quad \le \int_{s_1}^{s_2} \|U(s_2,s)F(s,u_m(s))\|\,\mathrm{d}s \\ &\qquad + \int_0^{s_1-\delta} \|(U(s_2,s) - U(s_1,s))F(s,u_m(s))\|\,\mathrm{d}s \\ &\qquad + \int_{s_1-\delta}^{s_1} \|(U(s_2,s) - U(s_1,s))F(s,u_m(s))\|\,\mathrm{d}s \\ &\quad \le M \int_{s_1}^{s_2} f_{k_0}(s)\,\mathrm{d}s \\ &\qquad + \sup_{s\in[0,s_1-\delta]} \|(U(s_2,s) - U(s_1,s))\|_{\mathscr{L}(X)} \int_0^{s_1-\delta} f_{k_0}(s)\,\mathrm{d}s\end{aligned}$$

$$+\, 2M \int_{s_1-\delta}^{s_1} f_{k_0}(s)\,\mathrm{d}s \longrightarrow 0 \quad \text{as } s_2 - s_1 \to 0,\ \delta \to 0,$$

uniformly for $m \in \Theta$. For the case when $0 = s_1 < s_2 \le t_1$, since

$$\left\| \int_0^{s_2} U(s_2, s) F(s, u_m(s))\,\mathrm{d}s \right\| \le M \int_0^{s_2} f_{k_0}(s)\,\mathrm{d}s,$$

in view of (H_2), $\|(\Gamma^1 u_m)(t_2)\|$ can be made small when s_2 is small independently of u_m ($m \in \Theta$). Thus, we verify that $\{\Gamma^1 u_m; m \in \Theta\}|_{J_0}$ is precompact in $C(J_0; X)$, with the aid of Arzela-Ascoli's theorem. The same idea can be used to prove that, for each $i = 1, \dots, n$, $\{\Gamma^1 u_m; m \in \Theta\}_{\overline{J}_i}$ is precompact in $C(\overline{J}_i; X)$. Accordingly, we obtain, thanks to Lemma 2.1, that

$$\{\Gamma^1 u_m; m \in \Theta\} \quad \text{is precompact in } PC([0,T]; X).$$

As shown in the above arguments, $\{u_m; m \in \Theta\}_{[\mu, t_1]}$ is precompact in $C([\mu, t_1]; X)$. Accordingly, one can assume, without loss of generality, that $u_m \to u_0$ in $C([\mu, t_1]; X)$ as $m \to \infty$, which implies in particular that $u_m(t_1) \to u(t_1)$ in X as $m \to \infty$. So, in view of the continuity of I_1, we obtain $I_1(u_m(t_1)) \to I_1(u(t_1))$ in X as $m \to \infty$. That is, $\{I_1(u_m(t_1)); m \in \Theta\}$ is relatively compact in X. This, together with the strong continuity of $U(t,s)$ gives that

$$\begin{aligned}
&\left\| I_1(u(t_1)) - U\left(\frac{1}{m}, 0\right) I_1(u_m(t_1)) \right\| \\
&\le \left\| U\left(\frac{1}{m}, 0\right) (I_1(u(t_1)) - I_1(u_m(t_1))) \right\| \\
&\quad + \left\| I_1(u(t_1)) - U\left(\frac{1}{m}, 0\right) I_1(u(t_1)) \right\| \\
&\le M \| I_1(u(t_1)) - I_1(u_m(t_1)) \| \\
&\quad + \left\| I_1(u(t_1)) - U\left(\frac{1}{m}, 0\right) I_1(u(t_1)) \right\| \longrightarrow 0 \quad \text{as } m \to \infty.
\end{aligned}$$

Hence, we deduce that

$$\left\{ U\left(\frac{1}{m}, 0\right) I_1(u_m(t_1)); m \in \Theta \right\} \quad \text{is relatively compact in } X. \tag{3.3}$$

So $\{U(t, t_1) U((1/m), 0) I_1(u_m(t_1^-)); m \in \Theta\}|_{(t_1, t_2]}$ is also relatively compact due to the compactness of $U(t, t_1)$ for $t > t_1$. At the same time, from (3.3), it follows that for $s_1, s_2 \in \overline{J}_1$ with $s_1 \leq s_2$,

$$\begin{aligned}&\left\|U(s_2,t_1)U\left(\frac{1}{m},0\right)I_1(u_m(t_1^-)) - U(s_1,t_1)U\left(\frac{1}{m},0\right)I_1(u_m(t_1^-))\right\|\\&= \left\|[U(s_2,t_1) - U(s_1,t_1)]U\left(\frac{1}{m},0\right)I_1(u(t_1^-))\right\| \longrightarrow 0, \quad \text{as } s_2 \to s_1,\end{aligned}$$

uniformly for $m \in \Theta$. Therefore, we find that

$$\{U(t,t_1)U((1/m,0)I_1(u_m(t_1^-));\ m \in \Theta\}|_{\overline{J}_1}$$

is precompact in $C(\overline{J}_1; X)$ in view of the Arzela-Ascoli theorem. A similar argument enables us to conclude that, for each $i = 2, \dots, n$, $\{U(t,t_i)U((1/m),0)I_i(u_m(t_i^-)); m \in \Theta\}|_{\overline{J}_i}$ is precompact in $C(\overline{J}_i; X)$. Accordingly, we deduce, thanks to Lemma 2.1, that

$$\{\Gamma^3 u_m; m \in \Theta\} \quad \text{is precompact in } PC([0,T]; X).$$

The proof is completed. $\square$

Lemma 3.7. *Let the hypotheses* (H_1)–(H_4) *hold. Then* $\{\Gamma^2 u_m; m \in \Theta\}|_{[0,\eta]}$ *is precompact in* $C([0,\eta]; X)$.

Proof. From Lemma 3.6 it follows readily that $\{u_m; m \in \Theta\}_{[\mu,t_1]}$ is precompact in $C([\mu, t_1]; X)$ and $\{u_m; m \in \Theta\}_{\overline{J}_i}$ for each $i = 1, \dots, n$ is precompact in $C(\overline{J}_i; X)$. Write

$$\overline{u}_m(t) = \begin{cases} u_m(t) & \text{if } t \in [\eta, T], \\ u_m(\eta) & \text{if } t \in [0, \eta]. \end{cases}$$

It is clear that $H(u_m) = H(\overline{u}_m)$ in view of (H_3) (ii). Also, without loss of generality, we may assume that $\overline{u}_m \to u$ in $PC(0,T;X)$ as $m \to \infty$. This, together with the continuity of H and the strong continuity of $U(t,s)$, gives that, for any $t \in [0,\eta]$,

$$\begin{aligned}&\left\|U(t,0)H(u) - U(t,0)U\left(\frac{1}{m},0\right)H(u_m)\right\|\\&= \left\|U(t,0)H(u) - U(t,0)U\left(\frac{1}{m},0\right)H(\overline{u}_m)\right\|\end{aligned}$$

$$\le \left\| U(t,0)H(u) - U(t,0)U\left(\frac{1}{m},0\right)H(u) \right\|$$
$$+ \left\| U(t,0)U\left(\frac{1}{m},0\right)(H(u) - H(\overline{u}_m)) \right\|$$
$$\le M \left\| H(u) - U\left(\frac{1}{m},0\right)H(u) \right\| + M^2\|H(u) - H(\overline{u}_m)\|$$
$$\longrightarrow 0 \quad \text{as } m \to \infty,$$

from which we see that the set $\{U(t,0)U((1/m),0)H(u_m); m \in \Theta\}$ for any $t \in [0,\eta]$ is relatively compact in X. This means, in particular, that $\{U((1/m),0)H(u_m); m \in \Theta\}$ is relatively compact in X, which, together with the strong continuity of $U(t,s)$, yields that for $s_1, s_2 \in [0,\eta]$ with $s_1 \le s_2$,

$$\left\| U(s_2,0)U\left(\frac{1}{m},0\right)H(u_m) - U(s_1,0)U\left(\frac{1}{m},0\right)H(u_m) \right\|$$
$$\le \left\| (U(s_2,0) - U(s_1,0))U\left(\frac{1}{m},0\right)H(u_m) \right\| \longrightarrow 0 \quad \text{as } s_2 \to s_1,$$

uniformly for $m \in \Theta$. Consequently, we conclude that the assertion of the lemma holds due to Arzela-Ascoli's theorem. This completes the proof. □

Now, we can state our main result of this section.

Theorem 3.8. *Let the hypotheses* (H_1)–(H_4) *hold. Then the Cauchy problem* (1.1) *has at least one mild solution provided that*

$$M\sigma + M^2\left(\mu + \sum_{i=1}^{n} \gamma_i\right) < 1.$$

***Remark* 3.9.** Note that, in Theorem 3.8, the impulsive item and nonlocal item only verify the continuity and the growth conditions.

Proof of Theorem 3.8. Let u_m, coming from Lemma 3.5, be a fixed point of Γ_m corresponding to $m \in \Theta$. Combining Lemma 3.6 and

Lemma 3.7, one can see, thanks to Lemma 2.1, that

$$\{u_m; m \in \Theta\}\text{is precompact in } PC([0,T];X),$$

which implies that there is a subsequence of $\{u_m; m \in \Theta\}$, again denoted by $\{u_m\}$, and a $u \in PC(0,T;X)$ such that $u_m \to u$ in $PC(0,T;X)$ as $m \to \infty$. Note that $u_m \in PC(0,T;X)$ satisfies the integral equation (3.2). Letting $m \to \infty$ on both sides of (3.2), one finds, noticing the continuity of H, I_i $(i = 1, \ldots, n)$, and F with respect to the second argument, that u is a mild solution of Cauchy problem (1.1). This completes the proof. □

The following corollaries are generalizations of Theorem 3.8.

Corollary 3.10. *Under the hypotheses* (H_1), (H_2) *and* (H_4), *for every* $u_0 \in X$, *the non-autonomous impulsive Cauchy problem*

$$\begin{cases} u'(t) = A(t)u(t) + F(t, u(t)), & t \in [0,T]\backslash\{t_1, \ldots, t_n\}, \\ \Delta u(t_i) = I_i(u(t_i)), & i = 1, \ldots, n, \\ u(0) = u_0, \end{cases}$$

has at least one mild solution, provided that $M\sigma + M^2 \sum_{i=1}^{n} \gamma_i < 1$.

Corollary 3.11. *Assume that hypotheses* (H_1), (H_2) *and* (H_4) *are satisfied. Then the non-autonomous impulsive Cauchy problem with nonlocal initial condition*

$$\begin{cases} u'(t) = A(t)u(t) + F(t, u(t)), & t \in [0,T]\backslash\{t_1, \ldots, t_n\}, \\ \Delta u(t_i) = I_i(u(t_i)), & i = 1, \ldots, n, \\ u(0) = \sum_{i=1}^{p} C_i u(s_i), & 0 < s_1 < \cdots < s_{p-1} < s_p < T, \end{cases}$$

where C_i $(i = 1, \ldots, p)$ *are given constants, has at least one mild solution provided that* $M\sigma + M^2(\sum_{i=1}^{p} |C_i| + \sum_{i=1}^{n} \gamma_i) < 1$.

Proof. Define

$$H(u) = \sum_{i=1}^{p} C_i u(s_i), \quad u \in PC([0,T];X).$$

Then it follows readily that function H satisfies the hypothesis (H_2) with $\Phi(r) = r \sum_{i=1}^{p} |C_i|$ and $\mu = \sum_{i=1}^{p} |C_i|$. Hence, the conclusion holds due to Theorem 3.8. The proof is completed. □

4. Cauchy problems of neutral type. In this section, we extend the results obtained in Section 3 to the Cauchy problem of neutral type (1.2).

It follows from (c) that there exist constants $M_1 > 0$, $0 < \alpha \le 1$, such that

$$\|A(t)A^{-1}(0)\|_{\mathscr{L}(X)} \le M_1 T^{\alpha} \tag{4.1}$$

for all $t \in [0, T]$. Let X^1 denote the Banach space D endowed with the graph norm $\|u\|_1 = \|A(0)u\|$ for $u \in X^1$.

Definition 4.1. A mild solution to Cauchy problem (1.2) is a function $u \in PC([0, T]; X)$ satisfying the integral equation

$$\begin{aligned} u(t) = {} & U(t,0)[H(u) - g(0, u(0))] + g(t, u(t)) \\ & + \int_0^t U(t,\tau)A(\tau)g(\tau, u(\tau))\,\mathrm{d}\tau \\ & + \int_0^t U(t,\tau)F(\tau, u(\tau))\,\mathrm{d}\tau \\ & + \sum_{0<t_i<t} U(t, t_i)I_i(u(t_i^-)), \quad t \in [0, T]. \end{aligned} \tag{4.2}$$

***Remark* 4.2.** It will be seen later that the integrals on right side in (4.2), being ones in sense of Bochner (see **[25]**), are reasonable.

Assume that

(H_5) (i) The function $g : [0, T] \times X \to X^1$ is continuous and $g(\cdot, u(\cdot)) = g(\cdot, w(\cdot))$ for any $u, w \in PC([0, T]; X)$ with $u(t) = w(t)$ ($t \in [\eta, T]$).

(ii) There exist a constant L_g and a nondecreasing function $\Upsilon : \mathbf{R}^+ \to \mathbf{R}^+$ such that

$$\|g(t, u) - g(t, v)\|_1 \le L_g \|u - v\|,$$
$$\|g(t, u)\|_1 \le \Upsilon(\|u\|)$$

for all $t \in [0,T]$, $u, v \in X$ and

$$\liminf_{r\to+\infty} \frac{\Upsilon(r)}{r} = \sigma_1.$$

Our main result in this section is the following theorem.

Theorem 1. *If assumptions* (H_1)–(H_5) *are satisfied together with*

$$M\sigma + M^2\left(\mu + \sum_{i=1}^{n} \gamma_i\right) + \left((M+1)\|A^{-1}(0)\|_{\mathscr{L}(X)} + MM_1T^{\alpha+1}\right)\max\{\sigma_1, L_g\} < 1,$$

then the Cauchy problem (1.2) *admits at least one mild solution.*

Proof. Fix $m \in \Theta$. Let us assume that the mapping Γ_m is defined the same as in Section 3. Write, for $u \in PC([0,T];X)$,

$$\begin{aligned}(\Gamma^4 u)(t) &= -U(t,0)g(0,u(0)) + g(t,u(t)) \\ &\quad + \int_0^t U(t,\tau)A(\tau)g(\tau,u(\tau))\,\mathrm{d}\tau \\ &:= -U(t,0)g(0,u(0)) + (\Gamma_g^4 u)(t), \quad t \in [0,T].\end{aligned}$$

It is clear that Γ^4, mapping $PC(0,T;X)$ into itself, is well defined. Also, from the proof of Lemma 3.5, (H_5) (ii) and (4.1), we observe that, for $t \in [0,T]$ and $u, v \in \Omega_r$,

$$\begin{aligned}\|(\Gamma_m u + \Gamma^4 v)(t)\| \le M^2\Phi(r) + M\int_0^t f_r(s)\,\mathrm{d}s + M^2\sum_{i=1}^{n}\Psi_i(r) \\ + ((M+1)\|A^{-1}(0)\|_{\mathscr{L}(X)} + MM_1T^{\alpha+1})\Upsilon(r),\end{aligned}$$

which, together with (4), yields that there exists a $k_0 > 0$ such that $\Gamma_m u + \Gamma^4 v$ maps Ω_{k_0} into itself for every pair $u, v \in \Omega_{k_0}$. Moreover, one finds that Γ^4 is a contraction on Ω_{k_0}. In fact, this can be seen from

(4) and

$$
\begin{aligned}
\|(\Gamma^4 u - \Gamma^4 v)(t)\| \le\; & M L_g \|A^{-1}(0)\|_{\mathscr{L}(X)} \|u(0)-v(0)\| \\
& + L_g \|A^{-1}(0)\|_{\mathscr{L}(X)} \|u(t)-v(t)\| \\
& + M M_1 T^\alpha L_g \int_0^t \|u(\tau)-v(\tau)\| \, \mathrm{d}\tau \\
\le\; & ((M+1) L_g \|A^{-1}(0)\|_{\mathscr{L}(X)} + M M_1 L_g T^{\alpha+1}) \|u-v\|_{PC}
\end{aligned}
$$

for $t \in [0,T]$ and $u, v \in \Omega_{k_0}$. At the same time, as proved in Lemma 3.5, Γ_m is continuous and compact on Ω_{k_0}.

We see, from the arguments above, that $\Gamma_m + \Gamma^4$ verifies all conditions of Lemma 2.3, which enables us to deduce that $\Gamma_m + \Gamma^4$ has at least one fixed point $v_m \in \Omega_{k_0}$ for every $m \in \Theta$, that is, v_m satisfies the integral equation

$$
\begin{aligned}
v_m(t) = {} & U(t,0)\left[U\left(\frac{1}{m}, 0\right) H(v_m) - g(0, u_m(0))\right] + g(t, v_m(t)) \\
& + \int_0^t U(t,\tau) A(\tau) g(\tau, v_m(\tau)) \, \mathrm{d}\tau \\
& + \int_0^t U(t,\tau) F(\tau, v_m(\tau)) \, \mathrm{d}\tau \\
& + \sum_{0<t_i<t} U(t,t_i) U\left(\frac{1}{m}, 0\right) I_i(v_m(t_i^-)), \quad t \in [0,T].
\end{aligned}
$$

Now, consider the set $\{v_m; m \in \Theta\}$. Let $\mu' \in (0,\eta)$ be fixed with η being the constant in (H_3) (ii) and (H_5) (i), and assume that the mappings Γ^i $(i = 1,2,3)$ are defined the same as in Section 3. Note that

$$
\begin{gathered}
v_m(t) = (\Gamma^1 v_m)(t) + (\Gamma^2 v_m)(t) + (\Gamma^3 v_m)(t) + (\Gamma^4 v_m)(t), \\
t \in [0,T], \quad m \in \Theta.
\end{gathered}
$$

From the compactness of $U(t,0)$ for $t \in (0,T]$ and (H_5) (ii), it follows readily that $\{U(t,0)g(0,v_m(0)); m \in \Theta\}|_{[\mu',t_1]}$ is precompact in $C([\mu',t_1];X)$ and $\{U(t,0)g(0,v_m(0)); m \in \Theta\}_{\overline{J}_i}$ for each $i = 1,\dots,n$ is precompact in $C(\overline{J}_i;X)$. Also, by Lemma 3.6 we note that $\{\Gamma^2 v_m; m \in \Theta\}_{[\mu',t_1]}$ is precompact in $C([\mu',t_1];X)$, $\{\Gamma^2 v_m; m \in \Theta\}_{\overline{J}_i}$ for each $i =$

$1, \ldots, n$ is precompact in $C(\overline{J}_i; X)$ and $\{\Gamma^1 v_m; m \in \Theta\}$ is precompact in $PC([0,T]; X)$. Denote

$$\widetilde{u}(t) = \begin{cases} u(t) & \text{if } t \in [\mu', T], \\ u(\mu') & \text{if } t \in [0, \mu'] \end{cases}$$

for $u \in C([\mu', t_1]; X)$. Therefore, for $u, v \in C([\mu', t_1]; X)$ and $t \in [\mu', t_1]$, we have

$$\begin{aligned} \big\|(\Gamma_g^4 u)(t) - (\Gamma_g^4 v)(t)\big\| &\le \|A^{-1}(0)\|_{\mathscr{L}(X)} \|g(t, u(t)) - g(t, v(t))\|_1 \\ &\quad + M M_1 T^\alpha \int_0^t \|g(\tau, \widetilde{u}(\tau)) - g(\tau, \widetilde{v}(\tau))\|_1 \mathrm{d}\tau \\ &\le L_g \|A^{-1}(0)\|_{\mathscr{L}(X)} \|u(t) - v(t)\| \\ &\quad + M M_1 L_g T^{\alpha+1} \sup_{0 \le t \le t_1} \|\widetilde{u}(t) - \widetilde{v}(t)\| \\ &\le \big(L_g \|A^{-1}(0)\|_{\mathscr{L}(X)} + M M_1 L_g T^{\alpha+1}\big) \\ &\quad \times \sup_{\mu' \le t \le t_1} \|u(t) - v(t)\|, \end{aligned}$$

which gives that Γ_g^4, mapping $C([\mu', t_1]; X)$ into itself, is Lipschitz continuous. Hence, one has

$$\begin{aligned} \beta(\{v_m; m \in \Theta\}|_{[\mu', t_1]}) &\le \beta(\{\Gamma^1 v_m; m \in \Theta\}_{[\mu', t_1]}) \\ &\quad + \beta(\{\Gamma^2 v_m; m \in \Theta\}_{[\mu', t_1]}) \\ &\quad + \beta(\{U(t,0) g(0, v_m(0)); m \in \Theta\}|_{[\mu', t_1]}) \\ &\quad + \beta(\{\Gamma^4 v_m; m \in \Theta\}_{[\mu', t_1]}) \\ &\le \beta(\{\Gamma_g^4 v_m; m \in \Theta\}_{[\mu', t_1]}) \\ &\le (L_g \|A^{-1}(0)\|_{\mathscr{L}(X)} + M M_1 L_g T^{\alpha+1}) \\ &\quad \times \beta(\{v_m; m \in \Theta\}|_{[\mu', t_1]}), \end{aligned}$$

where $\beta(\cdot)$ stands for the Hausdorff measure of noncompactness (see **[5]**), from which together with (4), we see that $\beta(\{v_m; m \in \Theta\}|_{[\mu', t_1]}) = 0$. This proves that $\{v_m; m \in \Theta\}|_{[\mu', t_1]}$ is precompact in $C([\mu', t_1]; X)$.

The same idea as the last part of the proof in Lemma 3.6 can be used to prove that $\{v_m; m \in \Theta\}_{\overline{J}_i}$ is precompact in $C(\overline{J}_i; X)$. Furthermore, by applying a similar argument as that in Lemma 3.7, we have that $\{\Gamma^2 v_m; m \in \Theta\}|_{[0,\eta]}$ is precompact in $C([0, \eta]; X)$.

As proven in the above arguments, we obtain

$$\beta(\{v_m; m \in \Theta\}|) \\ \leq ((M+1)L_g \|A^{-1}(0)\|_{\mathscr{L}(X)} + MM_1 L_g T^{\alpha+1})\beta(\{v_m; m \in \Theta\}|),$$

which, together with (4), implies that

$$\{v_m; m \in \Theta\} \quad \text{is precompact in } PC([0,T]; X).$$

Moreover, following from the same idea as the proof in Theorem 3.8, we obtain that the theorem remains true. This completes the proof. □

5. An example. To illustrate our abstract results, in this section let us consider a system of partial differential equations with impulsive and nonlocal initial conditions, which does not aim at generality but indicates how our theorem can be applied to a concrete problem. Such an example is inspired directly from the work of Fan and Li [**15,** Example 6.2] and Wang and Yang [**30,** Example 4.1].

Consider the following system

(5.1)

$$\begin{cases} \dfrac{\partial u(t,x)}{\partial t} = \dfrac{\partial^2 u(t,x)}{\partial x^2} - a(t)u(t,x) + b(t)u(t,x)\sin u^2(t,x), \\ \qquad\qquad\qquad\qquad t \in [0,T]\backslash\{t_1,\dots,t_n\},\ x \in [0,\pi], \\ u(t_i^+,x) - u(t_i^-,x) = \dfrac{u^{1/3}(t_i^-,x)}{1+|u(t_i^-,x)|}, \quad x \in [0,\pi],\ i = 1,\dots,n, \\ u(0,x) = u_0(x) + \displaystyle\sum_{i=1}^{p} C_i u^{1/3}(s_i,x), \quad x \in [0,\pi],\ 0<s_1<\cdots<s_p<T, \end{cases}$$

supplemented with homogeneous Dirichlet boundary condition $u(t,0) = u(t,\pi) = 0$ ($t \in [0,T]$), where $0 < t_1 < t_2 < \cdots < t_n < T$ are pre-fixed numbers. We assume that

(H$_a$) $a : [0,T] \to \mathbf{R}$ is a continuously differentiable function and

$$a_{\min} := \min_{t\in[0,T]} a(t) > -1.$$

(H$_b$) $b \in L^1(0,T;\mathbf{R}^+)$ and $\|b\|_{L^1(0,T)} < 1$.

Here, our objective is to show the existence of mild solutions to system (5.1).

Take $X = L^2[0,\pi]$ with the norm $\|\cdot\|_{L^2[0,\pi]}$ and inner product $(\cdot,\cdot)_2$. Define an operator $B : D(B) \subset X \to X$ by

$$\begin{aligned} Bu &= \frac{\partial^2}{\partial x^2}u, \quad u \in D(B),\\ D(B) &= \{u \in X; u, u' \quad \text{are absolutely continuous},\\ &\qquad\qquad u'' \in X, \text{ and } u(0) = u(\pi) = 0\}. \end{aligned}$$

It is well known that B has a discrete spectrum, and its eigenvalues are $-n^2$, $n \in \mathbf{N}^+$ with the corresponding normalized eigenvectors $y_n(x) = \sqrt{2/\pi}\sin(nx)$. More details about these facts can be seen from the monograph **[27]** of Pazy.

Put

$$\begin{aligned} D(A(t)) &= D(B), \quad t \in [0,T],\\ A(t)u &= Bu - a(t)u, \quad u \in D(A(t)). \end{aligned}$$

Then, from our hypotheses, it is clear that $(A(t))_{t\in[0,T]}$ satisfies conditions (a)–(c), which ensures that it generates an evolution family $\{U(t,s)\}_{0\le s\le t\le T}$:

$$U(t,s)u = \sum_{n=1}^{\infty} e^{-(\int_s^t a(\tau)\,d\tau + n^2(t-s))}(u, y_n)_2 y_n$$
$$\text{for } 0 \le s \le t \le T, \ u \in X.$$

A direct calculation gives

$$\|U(t,s)\|_{\mathscr{L}(X)} \le e^{-(1+a_{min})(t-s)} \quad \text{for } 0 \le s \le t \le T.$$

Note also that, for each $t, s \in [0,T]$ with $t > s$, the operator $U(t,s)$ is a nuclear operator, which implies the compactness of $U(t,s)$ for $t > s$.

Define

$$\begin{aligned} u(t)(x) &= u(t,x),\\ F(t,u(t))(x) &= b(t)u(t,x)\sin u^2(t,x),\\ I_i(u(t_i))(x) &= \frac{u^{1/3}(t_i^-,x)}{1+|u(t_i^-,x)|},\\ H(u)(x) &= u_0(x) + \sum_{i=1}^{p} C_i u^{1/3}(s_i,x). \end{aligned}$$

Note that system (5.1) can be reformulated as the abstract Cauchy problem (1.1) and the hypotheses (H_1)–(H_4) hold with

$$\begin{aligned} f_r(t) &= b(t)r, \\ \Psi_i(r) &= \pi^{1/3}r^{1/3}(i=1,\dots,n), \\ \Phi(r) &= \|u_0\|_{L^2[0,\pi]} + \pi^{1/3}r^{1/3}\sum_{i=1}^{p}|C_i|, \\ \sigma &= \|b\|_{L^1(0,T)}, \\ \gamma_i &= 0(i=1,\dots,n), \\ \mu &= 0, \quad M = 1. \end{aligned}$$

Hence, we deduce, under Hypotheses (H_a) and (H_b), that system (5.1) has at least one mild solution due to Theorem 3.8.

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