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EXISTENCE OF CONTINUOUS SOLUTIONS OF QUADRATIC VOLTERRA INTEGRAL INCLUSIONS

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ABSTRACT. Using the theory of measures of noncompactness and applying a new method, we prove the existence of solutions of quadratic Volterra integral inclusions.

1. Introduction. In this paper, we discuss the following quadratic Volterra integral inclusions of the form

$$x(t) \in h(t) + g(t, x(t)) \int_0^t k(t,s)F(s, x(s))\, ds, \quad t \in I = [0,1], \tag{1.1}$$

where $g : I \times \Re \to \Re$ is a given function, and $F : I \times \Re \to \Re$ is multivalued with nonempty compact values. Throughout this paper, the map $x \to F(t,x)$ is either upper semi-continuous or lower semi-continuous for almost every $t \in I$.

The study of quadratic integral equations has received much attention over the last 30 years or so. For instance, Cahlon and Eskin [**9**] prove the existence of positive solutions in the space $C[0,1]$ and $C^\alpha[0,1]$ of an integral equation of the Chandrasekhar H-equation with perturbation. Argyros [**1**] investigates a class of quadratic equations with a nonlinear perturbation. Banaś et al. [**3**] prove a few existence theorems for some quadratic integral equations. Banaś and Rzepka [**4**] study the Volterra quadratic integral equation on unbounded interval. Banaś and Sadarangani [**5**] study the solvability of Volterra-Stieltjes integral equation. In [**7, 8, 15**], the authors prove the existence of nondecreasing solutions of a quadratic integral equation. Dhage [**12, 13**] proves an existence theorem for a certain differential inclusions in Banach algebras.

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Dhage [**14**] proves the existence results of some nonlinear functional integral equations.

The purpose of this paper is to continue the study of those authors. Using the theory of measures of noncompactness and applying a new method, we prove the existence results of quadratic Volterra integral inclusions.

The organization of this work is as follows. In Section 2, we recall some definitions and theorems about the measure of noncompactness and fixed point theorems. In Section 3, we give theorems on the existence of continuous solutions of quadratic Volterra integral inclusions (1.1). Finally, in Section 4, examples are given to show the applications of our results.

2. Preliminaries. This section is devoted to collecting some definitions and results which will be needed throughout this paper.

Let $(X, \|\cdot\|)$ be a Banach space, and let $\wp(X) = \{Y \subset X : Y \neq \varnothing\}$, $\wp_{cl}(X) = \{Y \subset X : Y \text{ closed}\}$, $\wp_{bd}(X) = \{Y \subset X : Y \text{ bounded}\}$, $\wp_{cp}(X) = \{Y \subset X : Y \text{ compact}\}$, $\wp_{cv,cp}(X) = \{Y \subset X : Y \text{ convex and compact}\}$. A multi-valued map $T : X \to \wp(X)$ has convex (closed) values if $T(x)$ is convex (closed) for all $x \in X$. We say that T is bounded on bounded sets if $T(S)$ is bounded in X for each bounded set S of X, i.e., $\sup_{x\in S}\{\sup \|y\| : y \in T(x)\} < \infty$.

The multi-valued map T is called *lower semi-continuous* (lsc), respectively, *upper semi-continuous* (usc)) if the set $\{x \in X : T(x) \cap S \neq \varnothing\}$ is open (respectively, closed) for any open (respectively, closed) set S in X. If T is lsc and usc, then T is continuous. T is compact if $\overline{T(S)} = \overline{\bigcup_{x\in S}(T(x))}$ is a compact subset of X, for any bounded subset $S \subset X$, and it is completely continuous if $\overline{T(S)}$ is compact for all bounded sets $S \subset X$. If a multi-valued map T is completely continuous with nonempty compact values, then T is usc. if and only if T has a closed graph (i.e., $x_n \to x$, $y_n \to y$, $y_n \in T(x_n)$ imply $y \in T(x)$). A point $x \in X$ is called a fixed point of T if $x \in T(x)$.

For any $A, B \subset X$ (X is a Banach algebra), let us denote

$$A \cdot B = \{a \cdot b : a \in A, b \in B\},$$

and

$$\|A\| = \sup\{\|x\| : x \in A\}.$$

A Hausdorff measure of noncompactness χ of a bounded set $S \subset X$ is a nonnegative real number $\chi(S)$ defined by

$$\chi(S) = \inf\left\{ r > 0, S \subset \bigcup_{i=1}^{n} B(x_i, r), x_i \in S \right\}.$$

The details of measures of noncompactness and their properties appear in Deimling [**11**] and Zeidler [**18**].

Next, let us suppose that M is a nonempty subset of a Banach space X and the operator $T : M \to X$ is continuous and transforms bounded sets onto bounded ones. We say that T satisfies the Darbo condition (with constant $k \geq 0$) with respect to a measure of noncompactness χ if, for any bounded subset S of M, we have

$$\chi(TS) \leq k\chi(S).$$

If T satisfies the Darbo condition with $k < 1$, then it is called a contraction with respect to χ.

Remark 2.1. If T is Lipschitz with a Lipshitz constant k, then $\chi T(S) \leq k\chi(S)$ for any bounded subset S of X.

For our purposes, we will need the following results.

Theorem 2.2 [**2**]. *If $S_1, S_2 \in \wp_{bd}(X)$, then*

$$\chi(S_1 \cdot S_2) \leq \chi(S_1)\|S_2\| + \chi(S_2)\|S_1\|.$$

Theorem 2.3 [**10**]. *Let Q be a nonempty, bounded, closed and convex subset of the Banach space E and χ a measure of noncompactness in E. Let $T : Q \to Q$ be a contraction with respect to χ. Then T has a fixed point in the set Q.*

Theorem 2.4 [**16**]. *If $W \subseteq X$ is nonempty, bounded, closed and convex, the continuous map $F : W \to 2^W$ is a closed χ contraction map with $F(x)$ a nonempty, convex and compact subset of W for each $x \in W$. Then F has at least one fixed point in W.*

3. Main results. In this section, by using the measure of noncompactness defined in Section 2, we study the existence results of the quadratic Volterra integral inclusions (1.1).

We first give the existence results for (1.1) when F is upper semicontinuous. Here we list the hypotheses which will be required further on.

(1) $h : I \to \Re$ is a continuous function.

(2) $g : I \times \Re \to \Re$ is a continuous function, and there exists a constant $k \geq 0$ such that

$$|g(t,x) - g(t,y)| \leq k|x-y|,$$

for all $t \in I$ and $x, y \in \Re$.

(3) For each $t \in I$, $k(t,s)$ is measurable on $[0,t]$ and $k(t) = \operatorname{ess\,sup} |k(t,s)|, 0 \leq s \leq t$, is bounded on $[0,1]$, let $K = \sup_{0 \leq t \leq 1} |k(t)|$.

(4) The map $t \to k_t$ is continuous from $[0,1]$ to $L^\infty[0,1]$; here $k_t(s) = k(t,s)$.

(5) $F : I \times \Re \to \Re$ is a multi-valued function with nonempty, convex and compact values. $t \to F(t,x)$ is measurable for every $x \in \Re$, $x \to F(t,x)$ is usc for almost every $t \in I$.

(6) There exist a function $L \in L^1(0,1;\Re^+)$ and a nondecreasing continuous function $\Omega : \Re^+ \to \Re^+$ such that

$$|F(t,x)| \leq L(t)\Omega(|x|),$$

for all $x \in \Re$ and almost every $t \in I$.

Now we give the existence results under the above hypotheses.

Theorem 3.1. *Under assumptions* (1)–(6), *equation* (1.1) *has at least one solution* $x \in C(I)$ *provided that there exists a constant* R *such that*

$$\int_0^1 L(s)\,ds < \frac{1}{K(kR+b)} \int_a^R \frac{1}{\Omega(s)}\,ds, \tag{3.1}$$

where $a = \max\{|h(t)| : t \in I\}$ *and* $b = \max\{|g(t,0)| : t \in I\}$.

Proof. Let us consider the operator T defined on the space $C(I)$ by the formula

$$(Tx)(t) = h(t) + (Ax)(t) \cdot (Bx)(t),$$

where $(Ax)(t) = g(t, x(t))$, $(Bx)(t) = \{w(t) : w(t) = \int_0^t k(t,s)v(s)\,ds$, $v(s) \in F(s, x(s))$, almost everywhere $s \in I\}$. From [**17**], we know that B is usc, completely continuous and has nonempty, convex, compact values.

We shall show that the multi-valued function T has at least one fixed point; the fixed point is then a solution of problem (1.1).

In order to apply Theorem 2.4, we will give the proof in several steps.

Step 1. T has nonempty, compact and convex values. Indeed, if $y_1, y_2 \in T(x)$, then there exist $w_1, w_2 \in B(x)$, such that for each $t \in I$, we have

$$y_i(t) = h(t) + (Ax)(t) \cdot w_i(t), \quad i = 1, 2.$$

Let $0 \le \lambda \le 1$. Then for each $t \in I$, we have

$$\lambda y_1(t) + (1-\lambda)y_2(t) = h(t) + (Ax)(t)(\lambda w_1(t) + (1-\lambda)w_2(t)).$$

Since B has convex values, we have

$$\lambda y_1 + (1-\lambda)y_2 \in T(x).$$

Next, from Theorem 2.2 and the fact that B has compact values, it follows that

$$\begin{aligned}\chi(Tx) &= \chi(Ax \cdot Bx)\\ &\le \|Bx\|\chi(Ax) + \|Ax\|\chi(Bx)\\ &= 0,\end{aligned}$$

for every $x \in C(I)$.

Step 2. T maps W into itself. In view of Assumption (3.1), we infer that there exists a constant $\varepsilon > 0$ such that

$$\int_0^1 L(s)\,ds = A \int_{a+\varepsilon}^{R} \frac{1}{\Omega(s)}\,ds,$$

where $A = 1/(K(kR + b))$.

Then there exists a positive integer n such that

$$A \int_{a+\varepsilon}^{a+n\varepsilon} \frac{1}{\Omega(s)}\,ds < \int_0^1 L(s)\,ds \le A \int_{a+\varepsilon}^{a+(n+1)\varepsilon} \frac{1}{\Omega(s)}\,ds.$$

Therefore, there exists a sequence $\{t_0, t_1, t_2, \dots, t_{n-1}, t_n\}$ such that

$$0 = t_0 < t_1 < t_2 < \cdots < t_{n-1} < t_n = 1,$$

for which we have

$$\begin{aligned}
\int_0^{t_1} L(s)\,ds &= A\int_{a+\varepsilon}^{a+2\varepsilon} \frac{1}{\Omega(s)}\,ds,\\
\int_{t_1}^{t_2} L(s)\,ds &= A\int_{a+2\varepsilon}^{a+3\varepsilon} \frac{1}{\Omega(s)}\,ds,\\
&\cdots\cdots\cdots,\\
\int_{t_{n-2}}^{t_{n-1}} L(s)\,ds &= A\int_{a+(n-1)\varepsilon}^{a+n\varepsilon} \frac{1}{\Omega(s)}\,ds,\\
\int_{t_{n-1}}^{1} L(s)\,ds &\le A\int_{a+n\varepsilon}^{a+(n+1)\varepsilon} \frac{1}{\Omega(s)}\,ds.
\end{aligned}$$

We denote $W = \{x \in C(I), \|x_i\| = \sup\{|x(t)| : t \in [t_{i-1}, t_i]\} \le a + i\varepsilon,\ i = 1, 2, \dots, n\}$. Then $W \subseteq C(I)$ is nonempty, bounded, closed and convex.

For any $x \in W$, there exists $v(t) \in F(t, x(t))$ such that

$$\begin{aligned}
(Tx)(t) &= h(t) + (Ax)(t)\cdot(Bx)(t)\\
&= \Big\{y(t) : y(t) = h(t) + g(t, x(t))\int_0^t k(t,s)v(s)\,ds\Big\}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
|y(t)| &= \left|h(t) + g(t, x(t))\int_0^t k(t,s)v(s)\,ds\right|\\
&\le a + (|g(t, x(t)) - g(t, 0)| + |g(t, 0)|)\left|\int_0^t k(t,s)v(s)\,ds\right|\\
&\le a + K(k|x(t)| + b)\int_0^t L(s)\Omega(|x(s)|)\,ds\\
&\le a + K(k(a + n\varepsilon) + b)\int_0^t L(s)\Omega(|x(s)|)\,ds
\end{aligned}$$

$$\leq a + K(kR+b)\int_0^t L(s)\Omega(|x(s)|)\,ds,$$

and

$$\begin{aligned}
\|y\|_i &= \sup\{|(Tx)(t)| : t \in [t_{i-1}, t_i]\} \\
&\leq \sup\left\{a + K(kR+b)\int_0^t L(s)\Omega(|x(s)|)\,ds : t \in [t_{i-1}, t_i]\right\} \\
&\leq a + K(kR+b)\int_0^{t_i} L(s)\Omega(|x(s)|)\,ds \\
&\leq a + K(kR+b)\Bigg[\int_0^{t_1} L(s)\Omega(|x(s)|)\,ds \\
&\qquad + \int_{t_1}^{t_2} L(s)\Omega(|x(s)|)\,ds \\
&\qquad\qquad + \cdots + \int_{t_{i-1}}^{t_i} L(s)\Omega(|x(s)|)\,ds\Bigg] \\
&\leq a + K(kR+b)\Bigg[\int_0^{t_1} L(s)ds\Omega(a+\varepsilon) + \int_{t_1}^{t_2} L(s)\,ds\Omega(a+2\varepsilon) \\
&\qquad\qquad + \cdots + \int_{t_{i-1}}^{t_i} L(s)\,ds\Omega(a+i\varepsilon)\Bigg] \\
&\leq a + K(kR+b)A\Bigg[\int_{a+\varepsilon}^{a+2\varepsilon} \frac{1}{\Omega(s)}\,ds\Omega(a+\varepsilon) \\
&\qquad + \int_{a+2\varepsilon}^{a+3\varepsilon} \frac{1}{\Omega(s)}\,ds\Omega(a+2\varepsilon) \\
&\qquad\qquad + \cdots + \int_{a+i\varepsilon}^{a+(i+1)\varepsilon} \frac{1}{\Omega(s)}\,ds\Omega(a+i\varepsilon)\Bigg] \\
&\leq a + K(kR+b)Ai\varepsilon \\
&\leq a + i\varepsilon,
\end{aligned}$$

which implies that $T : W \to W$ is a bounded operator.

Step 3. T is contraction with respect to χ. Let S be a bounded subset of W. Then

$$\begin{aligned}
\chi(T(S)) &= \chi(A(S)\cdot B(S)) \\
&\leq \|B(S)\|\chi(A(S)) + \|A(S)\|\chi(B(S)) \\
&\leq k\|B(S)\|\chi(S),
\end{aligned}$$

and

$$
\begin{aligned}
\|B(S)\| &= \sup\{\|w\| : w \in B(S)\} \\
&= \sup\left\{\|w\| : w(t) = \int_0^t k(t,s)v(s)\,ds, v(s) \in F(s,x(s));\ x \in S\right\} \\
&\le \sup\left\{\int_0^1 KL(s)\Omega(|x(s)|)ds; x \in S\right\} \\
&\le \sup\left\{\int_0^{t_1} KL(s)\Omega(|x(s)|)\,ds + \int_{t_1}^{t_2} KL(s)\Omega(|x(s)|)\,ds \right. \\
&\qquad\qquad \left. + \cdots + \int_{t_{n-1}}^1 KL(s)\Omega(|x(s)|)\,ds;\ x \in S\right\} \\
&\le K\Omega(a+\varepsilon)\int_0^{t_1} L(s)\,ds + K\Omega(a+2\varepsilon)\int_{t_1}^{t_2} L(s)\,ds \\
&\qquad\qquad + \cdots + K\Omega(a+n\varepsilon)\int_{t_{n-1}}^1 L(s)\,ds \\
&\le K\Omega(a+\varepsilon)A\int_{a+\varepsilon}^{a+2\varepsilon} \frac{1}{\Omega(s)}\,ds + K\Omega(a+2\varepsilon)A\int_{a+2\varepsilon}^{a+3\varepsilon} \frac{1}{\Omega(s)}\,ds \\
&\qquad\qquad + \cdots + K\Omega(a+n\varepsilon)A\int_{a+n\varepsilon}^{a+(n+1)\varepsilon} \frac{1}{\Omega(s)}\,ds \\
&\le KAn\varepsilon.
\end{aligned}
$$

Consequently, we get

$$
\begin{aligned}
\chi(T(S)) &= kKAn\varepsilon\chi(S) \\
&< kK\frac{1}{K(kR+b)}(R-a)\chi(S) \\
&\le \frac{k(R-a)}{kR+b}\chi(S).
\end{aligned}
$$

Step 4. T has a closed graph. Let $y_n \to y$, $y_n \in T(x_n)$ and $x_n \to x$. We shall prove that $y \in T(x)$.

Now, $y_n \in T(x_n)$ means that there exists $w_n \in B(x_n)$ such that

$$y_n(t) = h(t) + g(t, x_n(t))w_n(t).$$

We must prove that there exists $w \in B(x)$ such that

$$y(t) = h(t) + g(t, x(t))w(t).$$

Since B is usc and completely continuous, we have $w_n \to w$, and $w \in B(x)$. Then

$$y_n(t) \longrightarrow h(t) + g(t, x(t))w(t),$$

and hence

$$y(t) = h(t) + g(t, x(t))w(t).$$

Therefore, T has closed values on W.

Finally, due to the fixed point Theorem 2.4, T has at least one fixed point $u \in T(u)$, and u is a solution to equation (1.1). Thus, we have completed the proof. □

Remark 3.2. By using a new method, we prove that the operator T maps bounded set W into itself and is a contraction map. If $g(t, x) = 1$, we can get the main results in [**19**].

Theorem 3.3. *Under assumptions* (1)–(6), *equation* (1.1) *has at least one solution $x \in C(I)$, provided that there exists a constant R such that*

$$a + K(kR + b)\Omega(R) \int_0^1 L(s)\, ds \le R. \tag{3.2}$$

Proof. In view of (3.2), we have

$$\int_0^1 L(s))\, ds \le \frac{R - a}{K(kR + b)\Omega(R)} < \frac{1}{K(kR + b)} \int_a^R \frac{1}{\Omega(s)}\, ds.$$

Then, applying Theorem 3.1, we obtain the desired assertion. □

Next, we study the case where F need not have convex values. Our approach is based on the fixed point Theorem 2.3 combined with a selection theorem due to Bressan and Colombo [**6**] for lower semi-continuous multivalued operators with decomposable values.

We suppose that

($5'$) $F : I \times \Re \to \Re$ is a multi-valued function with nonempty and compact values. $t \to F(t,x)$ is measurable for every $x \in \Re$, $x \to F(t,x)$ is lsc, for almost every $t \in I$.

Theorem 3.4. *Under assumptions* (1)–(4), ($5'$) *and* (6), *equation* (1.1) *has at least one solution* $x \in C(I)$ *provided that there exists a constant* R *satisfying* (3.1).

Proof. From [**6, 17**], there exists a continuous function $f : C(I) \to L^1(I,\Re)$ such that $f(x) \in \mathrm{F}(x)$ for all $x \in C(I)$, where F is defined by

$$\mathrm{F}(x) = \{v \in L^1(I,\Re), v(t) \in F(t,x(t)),\ \text{almost everywhere } t \in I\}.$$

Let us consider the operator T defined on space $C(I)$ by the formula

$$(Tx)(t) = h(t) + (Ax)(t) \cdot (Bx)(t),$$

where $(Ax)(t) = g(t,x(t))$ and $(Bx)(t) = \int_0^t k(t,s)(f(x))(s)\,ds$.

Taking into account assumptions (1)–(4), ($5'$) and (6), we infer that the function Tx is continuous on I for any $x \in C(I)$, i.e., the operator T transforms space $C(I)$ into itself.

As in Theorem 3.1, T maps bounded set W into itself.

By the Arzelá-Ascoli theorem, we can conclude that B is completely continuous. Let S be a bounded subset of W. We have

$$\begin{aligned}\chi(TS) &= \chi(AS \cdot BS)\\ &\le \|BS\|\chi(AS) + \|AS\|\chi(BS)\\ &\le k\|BS\|\chi(S)\\ &\le \frac{k(R-a)}{kR+b}\chi(S);\end{aligned}$$

therefore, T is a contraction with respect to χ.

Due to the fixed point Theorem 2.3, T has at least one fixed point $u = T(u)$, and u is a solution of equation (1.1). Thus, we have completed the proof. □

4. Examples.

Example 4.1. Consider the following quadratic integral equation

$$x(t) \in 1 + \frac{1}{3} \arctan x(t) \int_0^t F(s, x(s))\, ds, \quad t \in I, \tag{4.1}$$

where $F(t,x) = tI_x, I_x = [-x, x]$, $x \in \Re$.

Obviously, this equation is a particular case of equation (1.1), where $h(t) = 1$, $g(t, x(t)) = (1/3) \arctan x(t)$, $k(t,s) = 1$, F satisfies assumptions (5)–(6) and $\int_0^1 L(s)\, ds = 1$, $\Omega(x) = x$.

We know there exists a constant $R = e$ such that

$$1 < \frac{1}{(1/3)e} \int_1^e \frac{1}{s}\, ds = \frac{3}{e}.$$

So, by Theorem 3.1, we conclude that equation (4.1) has at least one solution.

Remark 4.2. For the above equation, we cannot obtain a constant R such that

$$1 + \frac{1}{3} R^2 \leq R.$$

By using Theorem 3.3, we do not know whether or not equation (4.1) has a solution. Thus, Theorem 3.1 is more general than Theorem 3.3.

Example 4.3. Consider the following differential equation

$$\begin{cases} (x(t)/g(t,x(t)))' \in F(t, x(t)) & \text{almost everywhere } t \in I, \\ x(0) = 0, \end{cases} \tag{4.2}$$

where g satisfies assumption (2) and $g(t,x) \neq 0$ for all $t \in I$ and $x \in \Re$, f satisfies assumptions (5) and (6).

Then equation (4.2) can be regarded as the following quadratic integral equation

$$x(t) \in g(t, x(t)) \int_0^t F(s, x(s))\, ds, \quad t \in I = [0,1], \tag{4.3}$$

if there exists a constant R such that

$$\int_0^1 L(s)\,ds < \frac{1}{kR+b}\int_0^R \frac{1}{\Omega(s)}\,ds.$$

Then, applying Theorem 3.1, we can prove that equation (4.2) has at least one solution in $C(I)$.

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