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SOLVABILITY AND EXISTENCE OF ASYMPTOTICALLY STABLE SOLUTIONS FOR A VOLTERRA-HAMMERSTEIN INTEGRAL EQUATION ON AN INFINITE INTERVAL

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ABSTRACT. By applying a fixed point theorem of Krasnosel'skii type, we study the solvability and existence of asymptotically stable solutions for a nonlinear Volterra-Hammerstein integral equation on an infinite interval. In order to illustrate the results obtained, two examples are also given.

1. Introduction. In this paper, we consider the following nonlinear functional integral equation

$$
\begin{aligned}
x(t) &= Q(t) + f(t, x(t), x(\pi(t))) \\
&\quad + \int_0^{\mu(t)} V(t, s, x(\sigma_1(s)), \dots, x(\sigma_p(s)))\, ds \\
&\quad + \int_0^{\infty} G(t, s, x(\chi_1(s)), \dots, x(\chi_q(s)))\, ds, \quad t \in \mathbf{R}_+,
\end{aligned} \tag{1.1}
$$

where $Q : \mathbf{R}_+ \to E$; $f : \mathbf{R}_+ \times E^2 \to E$; $V : \Delta_\mu \times E^p \to E$; $G : \mathbf{R}_+^2 \times E^q \to E$ are supposed to be continuous and $\Delta_\mu = \{(t,s) \in \mathbf{R}_+ \times \mathbf{R}_+ : s \le \mu(t)\}$, the functions μ, π, $\sigma_1, \dots, \sigma_p$, $\chi_1, \dots, \chi_q \in C(\mathbf{R}_+; \mathbf{R}_+)$ are continuous and E is a Banach space.

We call the integral equation (1.1) a Volterra-Hammerstein integral equation because it includes the well-known Volterra integral equation

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and the Hammerstein integral equation on an infinite interval, see [**3**, pages 151–160], [**8**]. The integral equation (1.1) is also called an integral equation of mixed type, see [**2**].

In the case $E = \mathbf{R}^d$, some types of (1.1) have been studied by Avramescu and Vladimirescu [**1, 2**]. The authors have proved the existence of asymptotically stable solutions to the following integral equations

$$x(t) = Q(t) + f(t, x(t)) + \int_0^t V(t,s)x(s)\,ds + \int_0^t G(t,s,x(s))\,ds, \quad t \in \mathbf{R}_+, \tag{1.2}$$

or

$$x(t) = Q(t)+\int_0^t K(t,s,x(s))\,ds+\int_0^\infty G(t,s,x(s))\,ds, \quad t \in \mathbf{R}_+, \tag{1.3}$$

under suitable hypotheses. In the proofs, a fixed point theorem of Krasnosel'skii type is used, (see [**1, 2**]).

Applying a fixed point theorem of Krasnosel'skii type and giving the suitable assumptions, Dhage and Ntouyas [**4**] and Purnaras [**8**] also obtained some results on the existence of solutions to the following nonlinear functional integral equation

$$\begin{aligned} x(t) &= Q(t) + \int_0^{\mu(t)} k(t,s)f(s,x(\theta(s)))\,ds \\ &\quad + \int_0^{\sigma(t)} v(t,s)g(s,x(\eta(s)))\,ds, \quad t \in [0,1], \end{aligned} \tag{1.4}$$

where $E = \mathbf{R}$, $0 \le \mu(t) \le t$, $0 \le \sigma(t) \le t$, $0 \le \theta(t) \le t$, $0 \le \eta(t) \le t$, for all $t \in [0,1]$.

In the case where E is the general Banach space, the existence of asymptotically stable solutions of the integral equation

$$\begin{aligned} x(t) &= Q(t) + f(t, x(t), x(\pi(t))) \\ &\quad + \int_0^t V(t,s,x(s),x(\sigma(s)))\,ds \\ &\quad + \int_0^t G(t,s,x(s),x(\chi(s)))\,ds, \end{aligned} \tag{1.5}$$

$t \in \mathbf{R}_+$, was proved in [**7**] by using the fixed point theorem of Krasnosel'skii type as follows.

Theorem 1.1. *Let $(X, |\cdot|_n)$ be a Fréchet space, and let $U, C : X \to X$ be two operators. Assume that*:

(i) *U is a k-contraction operator, $k \in [0,1)$ (depending on n), with respect to a family of seminorms $\|\cdot\|_n$ equivalent with the family $|\cdot|_n$*;

(ii) *C is completely continuous*;

(iii) $\lim_{|x|_n \to \infty} |Cx|_n/|x|_n = 0$, *for all $n \in \mathbf{N}$.*

Then $U + C$ has a fixed point.

Applying Theorem 1.1 while adding some suitable conditions, similarly to (1.5), we get the same results for (1.1). These results may be considered to be generalizations of [**2**], by combination of the proofs in [**7**] and arguments of density and some techniques in [**2**]. The paper consists of four sections and the existence of solutions, the existence of asymptotically stable solutions for (1.1) will be presented in Sections 2 and 3. Finally, we give two illustrated examples.

2. Existence of solutions. Let $X = C(\mathbf{R}_+; E)$ be the space of all continuous functions on $\mathbf{R}_+$ to E which are equipped with the numerable family of seminorms

$$|x|_n = \sup_{t\in[0,n]} |x(t)|, \quad n \geq 1.$$

Then $(X, |\cdot|_n)$ is complete in the metric

$$d(x,y) = \sum_{n=1}^{\infty} 2^{-n} \frac{|x-y|_n}{1+|x-y|_n},$$

and X is the Fréchet space. In X we also consider the family of seminorms defined by

$$\|x\|_n = |x|_{\gamma_n} + |x|_{h_n}, \quad n \geq 1,$$

where

$$|x|_{\gamma_n} = \sup_{t\in[0,\gamma_n]} |x(t)|, \ |x|_{h_n} = \sup_{t\in[\gamma_n, n]} e^{-h_n(t-\gamma_n)} |x(t)|,$$

$\gamma_n \in (0,n)$ and $h_n > 0$ are arbitrary numbers, which is equivalent to $|\cdot|_n$, since

$$e^{-h_n(n-\gamma_n)}\left|x\right|_n \leq \left\|x\right\|_n \leq 2\left|x\right|_n, \quad \text{for all } x \in X, \text{ for all } n \geq 1.$$

We make the following assumptions.

(A_1) There exists a constant $L \in [0,1)$ such that

$$|f(t,u_1,u_2) - f(t,v_1,v_2,)| \leq \frac{L}{2}\left(|u_1 - v_1| + |u_2 - v_2|\right), \\ \text{for all } u_1,u_2,v_1,v_2 \in E, \text{ for all } t \in \mathbf{R}_+;$$

(A_2) There exists a continuous function $\omega_1 : \Delta_\mu \to \mathbf{R}_+$ such that

$$|V(t,s,u_1,\dots,u_p) - V(t,s,v_1,\dots,v_p)| \leq \omega_1(t,s)\sum_{i=1}^{p}|u_i - v_i|,$$

for all $(u_1,\dots,u_p)$, $(v_1,\dots,v_p) \in E^p$ and $(t,s) \in \Delta_\mu$;

(A_3) G is completely continuous such that, for all bounded subsets I_1, I_2 of $[0,\infty)$ and for any bounded subsets J of E^q, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $t_1,t_2 \in I_1$,

$$|t_1 - t_2| < \delta \Longrightarrow |G(t_1,s,u_1,\dots,u_q) - G(t_2,s,u_1,\dots,u_q)| < \varepsilon,$$

for all $(u_1,\dots,u_q) \in J$ and $s \in I_2$;

(A_4) There exists a continuous function $\omega_2 : \mathbf{R}_+ \times \mathbf{R}_+ \to \mathbf{R}_+$ such that, for each bounded subset I of $\mathbf{R}_+$,

$$\int_0^\infty \sup_{t\in I}\omega_2(t,s)\,ds < \infty,$$

and

$$|G(t,s,u_1,\dots,u_q)| \leq \omega_2(t,s), \\ \text{for all } (t,s) \in I \times \mathbf{R}_+, \text{ for all } (u_1,\dots,u_q) \in E^q;$$

(A_5) $0 \leq \mu(t) \leq t$, $0 \leq \pi(t) \leq t$, $0 \leq \sigma_i(t) \leq t$, $\chi_j(t) \geq 0$, for all $t \in \mathbf{R}_+$, $i = 1,\dots,p$, $j = 1,\dots,q$.

By hypothesis, (A_1) and $0 \le \pi(t) \le t$, it is easily seen that the operator $\Phi : X \to X$, with $\Phi x(t) = Q(t) + f(t, x(t), x(\pi(t)))$, $x \in X$, $t \in \mathbf{R}_+$, is the L-contraction mapping on the Fréchet space, $(X, |\cdot|_n)$, so the following lemma is obtained thanks to the known Banach's contraction principle.

Lemma 2.1. *Let* (A_1) *hold and* $0 \le \pi(t) \le t$. *Then the equation*

$$x(t) = Q(t) + f(t, x(t), x(\pi(t))), \quad t \in \mathbf{R}_+ \tag{2.1}$$

has a unique solution $x = \xi$.

On the other hand, the following lemma for the relative compactness of a subset in X is useful in order to prove our main results.

Lemma 2.2. *Let* $X = C(\mathbf{R}_+; E)$ *be the Fréchet space defined as above, and let* A *be a subset of* X. *For each* $n \in \mathbf{N}$, *let* $X_n = C([0, n]; E)$ *be the Banach space of all continuous functions* $u : [0, n] \to E$ *with the norm* $|u|_n = \sup_{t \in [0,n]} |u(t)|$ *and* $A_n = \{x|_{[0,n]} : x \in A\}$.

The set A *in* X *is relatively compact if and only if, for each* $n \in \mathbf{N}$, A_n *is equicontinuous in* X_n *and, for every* $s \in [0, n]$, *the set* $A_n(s) = \{x(s) : x \in A_n\}$ *is relatively compact in* E.

This condition was stated in [**5**] and proved in detail in [**7**]. The proof follows from Ascoli-Arzela's theorem, (see [**6**, page 211]).

Theorem 2.3. *Let* (A_1)–(A_5) *hold. Then* (1.1) *has a solution on* $\mathbf{R}_+$.

Proof. The proof consists of four steps.

Step 1. By the transformation $x = y + \xi$, where ξ is the unique solution of equation (2.1), we can write equation (1.1) in the form

$$y(t) = Ay(t) + By(t) + Cy(t), \quad t \in \mathbf{R}_+, \tag{2.2}$$

where

(2.3)
$$\begin{cases} Ay(t) = Q(t) + f(t, (y+\xi)(t), (y+\xi)(\pi(t))) - \xi(t), & t \in \mathbf{R}_+, \\ By(t) = \int_0^{\mu(t)} V(t, s, (y+\xi)(\sigma_1(s)), \dots, (y+\xi)(\sigma_p(s)))\, ds, \\ Cy(t) = \int_0^{\infty} G(t, s, (y+\xi)(\chi_1(s)), \dots, (y+\xi)(\chi_q(s)))\, ds. \end{cases}$$

Step 2. Put $U = A + B$. Then, U is a k_n-contraction operator, $k_n \in [0, 1)$ (depending on n), with respect to a family of seminorms $\|\cdot\|_n$. Indeed, fix an arbitrary positive integer $n \in \mathbf{N}$.

For all $t \in [0, \gamma_n]$, with $\gamma_n \in (0, n)$ to be chosen later, we have

$$\begin{aligned} |Uy(t) - U\widetilde{y}(t)| &\leq \frac{L}{2}\,|y(t) - \widetilde{y}(t)| + \frac{L}{2}\,|y(\pi(t)) - \widetilde{y}(\pi(t))| \\ &\quad + \int_0^{\mu(t)} \omega_1(t,s) \sum_{i=1}^{p} |y(\sigma_i(s)) - \widetilde{y}(\sigma_i(s))|\, ds \\ &\leq (L + p\widetilde{\omega}_{1n}\gamma_n)\,|y - \widetilde{y}|_{\gamma_n}, \end{aligned} \tag{2.4}$$

where

$$\begin{aligned} \widetilde{\omega}_{1n} &= \sup\{\omega_1(t,s) : (t,s) \in \Delta_n\}, \\ \Delta_n &= \{(t,s) : 0 \leq s \leq \mu(t), 0 \leq t \leq n\}. \end{aligned} \tag{2.5}$$

This implies that

$$|Uy - U\widetilde{y}|_{\gamma_n} \leq (L + p\widetilde{\omega}_{1n}\gamma_n)\,|y - \widetilde{y}|_{\gamma_n}. \tag{2.6}$$

For all $t \in [\gamma_n, n]$, similarly, we also have

$$\begin{aligned} |Uy(t) - U\widetilde{y}(t)| &\leq \frac{L}{2}\,|y(t) - \widetilde{y}(t)| + \frac{L}{2}\,|y(\pi(t)) - \widetilde{y}(\pi(t))| \\ &\quad + \widetilde{\omega}_{1n} \int_0^{\gamma_n} \sum_{i=1}^{p} |y(\sigma_i(s)) - \widetilde{y}(\sigma_i(s))|\, ds \\ &\quad + \widetilde{\omega}_{1n} \int_{\gamma_n}^{\mu(t)} \sum_{i=1}^{p} |y(\sigma_i(s)) - \widetilde{y}(\sigma_i(s))|\, ds. \end{aligned} \tag{2.7}$$

By the inequalities

(2.8)
$$0 < e^{-h_n(t-\gamma_n)} \le e^{-h_n(\pi(t)-\gamma_n)} < 1, \quad \text{for all } t \in [\gamma_n, n],$$
$$0 < e^{-h_n(t-\gamma_n)} \le e^{-h_n(\sigma_i(t)-\gamma_n)} < 1, \quad \text{for all } t \in [\gamma_n, n],\ i = 1, \dots, p,$$

in which $h_n > 0$ is also chosen later, we get

(2.9)
$$\begin{aligned}
&|Uy(t) - U\tilde{y}(t)|\, e^{-h_n(t-\gamma_n)} \\
&\le \frac{L}{2}\, |y(t) - \widetilde{y}(t)|\, e^{-h_n(t-\gamma_n)} \\
&\quad + \frac{L}{2}\, |y(\pi(t)) - \widetilde{y}(\pi(t))|\, e^{-h_n(\pi(t)-\gamma_n)} + p\widetilde{\omega}_{1n}\gamma_n\, |y - \widetilde{y}|_{\gamma_n} \\
&\quad + \widetilde{\omega}_{1n} \int_{\gamma_n}^{\mu(t)} \sum_{i=1}^{p} |y(\sigma_i(s)) - \widetilde{y}(\sigma_i(s))|\, e^{-h_n(t-\gamma_n)} ds \\
&\le \frac{L}{2}\, |y - \widetilde{y}|_{h_n} + \frac{L}{2}\, \|y - \widetilde{y}\|_n \\
&\quad + p\widetilde{\omega}_{1n}\gamma_n\, |y - \widetilde{y}|_{\gamma_n} + p\widetilde{\omega}_{1n}\, \|y - \widetilde{y}\|_n \int_{\gamma_n}^{t} e^{h_n(s-t)}\, ds \\
&\le \frac{L}{2}\, |y - \widetilde{y}|_{h_n} + p\widetilde{\omega}_{1n}\gamma_n\ \ |y - \widetilde{y}|_{\gamma_n} + \left(\frac{L}{2} + \frac{p\widetilde{\omega}_{1n}}{h_n}\right) \|y - \widetilde{y}\|_n .
\end{aligned}$$

We get

$$\begin{aligned}
|Uy - U\widetilde{y}|_{h_n} &\le \frac{L}{2}\, |y - \widetilde{y}|_{h_n} + p\widetilde{\omega}_{1n}\gamma_n\, |y - \widetilde{y}|_{\gamma_n} \\
&\quad + \left(\frac{L}{2} + \frac{p\widetilde{\omega}_{1n}}{h_n}\right) \|y - \widetilde{y}\|_n .
\end{aligned} \tag{2.10}$$

Combining (2.6) and (2.10), we deduce that

(2.11)
$$\begin{aligned}
\|Uy - U\widetilde{y}\|_n &= |Uy - U\widetilde{y}|_{\gamma_n} + |Uy - U\widetilde{y}|_{h_n} \\
&\le (L + p\widetilde{\omega}_{1n}\gamma_n)\, |y - \widetilde{y}|_{\gamma_n} + \frac{L}{2}\, |y - \widetilde{y}|_{h_n} + p\widetilde{\omega}_{1n}\gamma_n\, |y - \widetilde{y}|_{\gamma_n} \\
&\quad + \left(\frac{L}{2} + \frac{p\widetilde{\omega}_{1n}}{h_n}\right) \|y - \widetilde{y}\|_n
\end{aligned}$$

$$\begin{aligned}
&\leq (L + 2p\widetilde{\omega}_{1n}\gamma_n)\left(|y-\widetilde{y}|_{\gamma_n} + |y-\widetilde{y}|_{h_n}\right)\\
&\quad + \left(\frac{L}{2} + \frac{p\widetilde{\omega}_{1n}}{h_n}\right)\|y-\widetilde{y}\|_n\\
&\leq (L + 2p\widetilde{\omega}_{1n}\gamma_n)\|y-\widetilde{y}\|_n + \left(\frac{L}{2} + \frac{p\widetilde{\omega}_{1n}}{h_n}\right)\|y-\widetilde{y}\|_n\\
&\leq \widetilde{k}_n \|y-\widetilde{y}\|_n,
\end{aligned}$$

where $\widetilde{k}_n = \max\{L + 2p\widetilde{\omega}_{1n}\gamma_n, (L/2) + (p\widetilde{\omega}_{1n})/h_n\}$. Choose

$$0 < \gamma_n < \min\left\{\frac{1-L}{2p\widetilde{\omega}_{1n}}, n\right\}, \qquad h_n > \frac{p\widetilde{\omega}_{1n}}{1-(L/2)}; \tag{2.12}$$

then we have $\widetilde{k}_n < 1$ by (2.11), U is a $\widetilde{k}_n$-contraction operator with respect to a family of seminorms $\|\cdot\|_n$.

Step 3. We show that $C : X \to X$ is completely continuous. We first show that C is continuous. For any $y_0 \in X$, let $(y_m)_m$ be a sequence in X such that $\lim_{m\to\infty} y_m = y_0$.

Let $n \in \mathbf{N}$ be fixed. For any given $\varepsilon > 0$, by $\int_0^\infty \sup_{t\in[0,n]} \omega_2(t,s)\,ds < \infty$, there exists a $T_n \in \mathbf{N}$ (T_n is big enough) such that

$$\int_{T_n}^\infty \omega_2(t,s)\,ds \leq \int_{T_n}^\infty \sup_{t\in[0,n]} \omega_2(t,s)\,ds < \frac{\varepsilon}{8}, \quad \text{for all } t \in [0,n]. \tag{2.13}$$

Put

$$\begin{aligned}
&K_1 = \{(y_m + \xi)(\chi_1(s)) : s \in [0,T_n],\ m \in \mathbf{Z}_+\},\\
&\quad\vdots\\
&K_q = \{(y_m + \xi)(\chi_q(s)) : s \in [0,T_n],\ m \in \mathbf{Z}_+\}.
\end{aligned} \tag{2.14}$$

Then $K_1, \dots, K_q$ are compact in E since $\lim_{m\to\infty} |y_m - y_0|_{T_n} = 0$. Indeed, let $\{(y_{m_j} + \xi)(\chi_1(s_j))\}_j$ be a sequence in K_1. We can assume that $\lim_{j\to\infty} s_j = s_0$ and that $\lim_{j\to\infty} y_{m_j} + \xi = y_0 + \xi$. We have

$$\begin{aligned}
&\left|(y_{m_j} + \xi)(\chi_1(s_j)) - (y_0 + \xi)(\chi_1(s_0))\right|\\
&\quad\leq \left|(y_{m_j} + \xi)(\chi_1(s_j)) - (y_0 + \xi)(\chi_1(s_j))\right|\\
&\qquad + \left|(y_0 + \xi)(\chi_1(s_j)) - (y_0 + \xi)(\chi_1(s_0))\right|\\
&\quad\leq \left|y_{m_j} - y_0\right|_{T_n} + \left|(y_0 + \xi)(\chi_1(s_j)) - (y_0 + \xi)(\chi_1(s_0))\right|,
\end{aligned} \tag{2.15}$$

which shows that $\lim_{j\to\infty}(y_{m_j}+\xi)(\chi_1(s_j)) = (y_0+\xi)(\chi_1(s_0))$ in E. It means that K_1 is compact in E and, in a similar manner, the same holds true for $K_2, \dots, K_q$.

By G continuous on the compact set $[0,n]\times[0,T_n]\times K_1\times\cdots\times K_q$, there exists a $\delta>0$ such that, for every $(u_1,\dots,u_q)$, $(v_1,\dots,v_q)\in K_1\times\cdots\times K_q$, $|u_i-v_i|<\delta$, $i=1,\dots,q$,

$$|G(t,s,u_1,\dots,u_q)-G(t,s,v_1,\dots,v_q)|<\frac{\varepsilon}{4T_n}, \quad \text{for all } (t,s)\in[0,n]\times[0,T_n]. \tag{2.16}$$

With $i=1,\dots,q$, by

$$\sup_{0\le s\le T_n}|(y_m+\xi)(\chi_i(s))-(y_0+\xi)(\chi_i(s))| \le \sup_{0\le s\le T_n}|(y_m+\xi)(s)-(y_0+\xi)(s)| = |y_m-y_0|_{T_n}\longrightarrow 0,$$

as $m\to\infty$; hence, there exists an m_0 such that, for $m>m_0$,

$$|(y_m+\xi)(\chi_i(s))-(y_0+\xi)(\chi_i(s))|<\delta, \tag{2.17}$$

for all $s\in[0,T_n]$, for all $i=1,\dots,q$.

This implies that, for all $t\in[0,n]$, for all $m>m_0$,

(2.18)

$$\begin{aligned}|Cy_m(t)-Cy_0(t)| \le \int_0^{T_n} &|G(t,s,(y_m+\xi)(\chi_1(s)),\dots,(y_m+\xi)(\chi_q(s))) \\ &- G(t,s,(y_0+\xi)(\chi_1(s)),\dots,(y_0+\xi)(\chi_q(s)))|\,ds \\ &+2\int_{T_n}^{\infty}\omega_2(t,s)\,ds < T_n\frac{\varepsilon}{4T_n}+2\frac{\varepsilon}{8}=\frac{\varepsilon}{2},\end{aligned}$$

so $|Cy_m-Cy|_n<\varepsilon$, for all $m>m_0$, and the continuity of C is proved.

It remains to show that C maps bounded sets into relatively compact sets. We use Lemma 2.2.

Now, let Ω be a bounded subset of X. We have to prove that, for $n\in\mathbf{N}$,

(a) The set $(C\Omega)_n$ is equicontinuous in X_n.

(b) For every $t \in [0, n]$, the set $(C\Omega)_n(t) = \{Cy|_{[0,n]}(t) : y \in \Omega\}$ is relatively compact in E.

Let $n \in \mathbf{N}$ be fixed. Consider any $\varepsilon > 0$ given. Then, there exists a $T_n \in \mathbf{N}$ (T_n big enough) such that (2.13) is valid.

Proof of (a). For any $y \in \Omega$, for all t_1, $t_2 \in [0, n]$,

(2.19)
$$|Cy(t_1) - Cy(t_2)| \leq \int_0^{T_n} |G(t_1, s, (y+\xi)(\chi_1(s)), \ldots, (y+\xi)(\chi_q(s))) - G(t_2, s, (y+\xi)(\chi_1(s)), \ldots, (y+\xi)(\chi_q(s)))|\, ds + \int_{T_n}^{\infty} (\omega_2(t_1, s) + \omega_2(t_2, s))\, ds.$$

According to (2.13), (2.19) and the hypothesis (A_3), $(C\Omega)_n$ is equicontinuous on X_n.

Proof of (b). Let $\{Cy_k|_{[0,n]}(t)\}_k$, $y_k \in \Omega$, be a sequence in $(C\Omega)_n(t)$. We have to show that there exists a convergent subsequence of $\{Cy_k|_{[0,n]}(t)\}_k$. Put

(2.20)
$$\begin{aligned} S_1 &= \{(y+\xi)(\chi_1(s)) : y \in \Omega,\ s \in [0, T_n]\}, \\ &\vdots \\ S_q &= \{(y+\xi)(\chi_q(s)) : y \in \Omega,\ s \in [0, T_n]\}. \end{aligned}$$

Then $S_1, \ldots, S_q$ are bounded in E and, consequently, the set $G([0, n] \times [0, T_n] \times S_1 \times \cdots \times S_q)$ is relatively compact in E, since G is completely continuous.

Let $\widehat{Q}$ be the set of rational numbers in $[0, T_n]$; it means $\widehat{Q} = \mathbf{Q} \cap [0, T_n]$. It is known that the set of rational numbers $\mathbf{Q}$ is countable and $\overline{\mathbf{Q}} = \mathbf{R}$. So, $\widehat{Q}$ is countable and has form $\widehat{Q} = \{s_m\}$.

For $m = 1$, the sequence $\{G(t, s_1, (y_k+\xi)(\chi_1(s_1)), \ldots, (y_k+\xi)(\chi_q(s_1)))\}_k$ belongs to $G([0, n] \times [0, T_n] \times S_1 \times \cdots \times S_q)$, that is relatively compact in E, so there exists a subsequence of $\{y_k\}$, denoted by $\{y_k^{(1)}\}_k$, such that

$$\left\{G\left(t, s_1, (y_k^{(1)} + \xi)(\chi_1(s_1)), \ldots, (y_k^{(1)} + \xi)(\chi_q(s_1))\right)\right\}_k \text{ converges in } E.$$

For $m = 2$, similarly, there exists a subsequence of $\{y_k^{(1)}\}_k$, denoted by $\{y_k^{(2)}\}_k$, such that

$$\left\{G\left(t, s_2, (y_k^{(2)} + \xi)(\chi_1(s_2)), \dots, (y_k^{(2)} + \xi)(\chi_q(s_2))\right)\right\}_k \text{ converges in } E.$$

Therefore, for all $m \in \mathbf{N}$, by induction, we can establish a subsequence $\{y_k^{(m+1)}\}_k$ of $\{y_k^{(m)}\}_k$, such that

$$\left\{G\left(t, s_{m+1}, (y_k^{(m+1)} + \xi)(\chi_1(s_{m+1})), \dots, (y_k^{(m+1)} + \xi)(\chi_q(s_{m+1}))\right)\right\}_k$$

converges in E.

Put $z_k = y_k^{(k)}$. Then $\{z_k\}_k$ is a subsequence of $\{y_k\}_k$ and $\{G(t, s_m, (z_k + \xi)(\chi_1(s_m)), \dots, (z_k + \xi)(\chi_q(s_m)))\}_k$ converges in E, for all $s_m \in \widehat{Q}$. Then, there exists a $k_0 \geq 1$ (depending only on ε) such that for all $k, l \geq k_0$,

$$\begin{aligned} \text{(2.21)} \quad & |G(t, s_m, (z_k + \xi)(\chi_1(s_m)), \dots, (z_k + \xi)(\chi_q(s_m))) \\ & - G(t, s_m, (z_l + \xi)(\chi_1(s_m)), \dots, (z_l + \xi)(\chi_q(s_m)))| < \frac{\varepsilon}{8T_n}, \end{aligned}$$

for all $s_m \in \widehat{Q}$.

For each $s \in [0, T_n]$, the sequence $\{s_m\}$, $s_m \in \widehat{Q}$, $m = 1, 2, \dots$, exists such that $\lim_{m \to \infty} s_m = s$.

By continuity of the functions G, ξ, z_k, z_l, $\chi_1, \dots, \chi_q$, passing (2.21) to the limit, we obtain that, for all $k, l \geq k_0$,

$$\begin{aligned} \text{(2.22)} \quad & |G(t, s, (z_k + \xi)(\chi_1(s)), \dots, (z_k + \xi)(\chi_q(s))) \\ & -G(t, s, (z_l + \xi)(\chi_1(s)), \dots, (z_l + \xi)(\chi_q(s)))| < \frac{\varepsilon}{8T_n}, \end{aligned}$$

for all $s \in [0, T_n]$.

It follows that, for every $t \in [0,n]$, for all $k, l \geq k_0$, we have

(2.23)
$$\begin{aligned}|Cz_k(t) - Cz_l(t)| \leq & \int_0^{T_n} |G(t,s,(z_k+\xi)(\chi_1(s)),\dots,(z_k+\xi)(\chi_q(s))) \\ & -G(t,s,(z_l+\xi)(\chi_1(s)),\dots,(z_l+\xi)(\chi_q(s)))|\, ds \\ & + \int_{T_n}^{\infty} |G(t,s,(z_k+\xi)(\chi_1(s)),\dots,(z_k+\xi)(\chi_q(s))) \\ & -G(t,s,(z_l+\xi)(\chi_1(s)),\dots,(z_l+\xi)(\chi_q(s)))|\, ds \\ & \leq \frac{3\varepsilon}{8} + \frac{2\varepsilon}{8} < \varepsilon.\end{aligned}$$

It implies that $\{Cz_k|_{[0,n]}(t)\}_k$ is the Cauchy sequence in the Banach E; the convergence of $\{Cz_k|_{[0,n]}(t)\}_k$ follows. Note that $\{Cz_k|_{[0,n]}(t)\}_k$ is a subsequence of $\{Cy_k|_{[0,n]}(t)\}_k$. Then, $(C\Omega)_n(t)$ is relatively compact in E.

In view of Lemma 2.2, $C(\Omega)$ is relatively compact in X.

Therefore, C is completely continuous. Step 3 is proved.

Step 4. Finally, we show that, for all $n \in \mathbf{N}$,

$$\lim_{|y|_n \to \infty} \frac{|Cy|_n}{|y|_n} = 0. \tag{2.24}$$

By the assumption (A_4), for all $t \in [0,n]$, we get

$$\begin{aligned}|Cy(t)| &\leq \int_0^{\infty} |G\left(t,s,(y+\xi)(\chi_1(s)),\dots,(y+\xi)(\chi_q(s))\right)|\, ds \\ &\leq \int_0^{\infty} \omega_2(t,s)\, ds < \infty.\end{aligned} \tag{2.25}$$

It follows that

$$\lim_{|y|_n \to \infty} \frac{|Cy|_n}{|y|_n} = 0. \tag{2.26}$$

By applying Theorem 1.1, operator $U + C$ has a fixed point y in X. Then equation (1.1) has a solution $x = y + \xi$ on $\mathbf{R}_+$. Theorem 2.3 is proved. □

3. Asymptotically stable solutions. We now consider the asymptotically stable solutions for (1.1) defined as follows.

Definition. A function x is said to be an *asymptotically stable solution* of (1.1) if, for any solution $\widetilde{x}$ of (1.1),

$$\lim_{t\to\infty} |x(t) - \widetilde{x}(t)| = 0.$$

In this section, we assume (A_1)–(A_5) hold and assume, in addition,

(A_6) $\pi(t) = t$, for all $t \in \mathbf{R}_+$,

(A_7) $V(t, s, 0, \dots, 0) = 0$, for all $(t, s) \in \Delta_\mu$.

Then, by Theorem 2.3, equation (1.1) has a solution on $[0, \infty)$.

On the other hand, if x is a solution of (1.1), then, as Step 1 in the proof of Theorem 2.3, $y = x - \xi$ satisfies (2.2). This implies that, for all $t \in \mathbf{R}_+$,

$$|y(t)| \le |Ay(t)| + |By(t)| + |Cy(t)|, \tag{3.1}$$

where $Ay(t)$, $By(t)$ and $Cy(t)$ are as in (2.3). Using (A_1), (A_2), (A_5)–(A_7) and noting that $A0 = 0$, we obtain for all $t \in \mathbf{R}_+$,

(3.2)
$$\begin{aligned}|y(t)| &\le L\,|y(t)| + \int_0^{\mu(t)} \omega_1(t,s) \sum_{i=1}^p |(y+\xi)(\sigma_i(s))|\,ds + \int_0^\infty \omega_2(t,s)\,ds \\ &\le L\,|y(t)| + \int_0^{t} \omega_1(t,s) \sum_{i=1}^p |(y+\xi)(\sigma_i(s))|\,ds + \int_0^\infty \omega_2(t,s)\,ds.\end{aligned}$$

It follows that

(3.3)
$$|y(t)| \le \frac{1}{1-L} \int_0^t \omega_1(t,s) \sum_{i=1}^p |y(\sigma_i(s))|\,ds + a(t), \quad \text{for all } t \in \mathbf{R}_+,$$

where

$$a(t) = \frac{1}{1-L} \int_0^t \omega_1(t,s) \sum_{i=1}^p |\xi(\sigma_i(s))|\,ds + \frac{1}{1-L} \int_0^\infty \omega_2(t,s)\,ds. \tag{3.4}$$

For all $j = 1, \dots, p$, (3.3) leads to:

(3.5)
$$\begin{aligned}|y(\sigma_j(t))| &\le \frac{1}{1-L}\int_0^{\sigma_j(t)} \omega_1(\sigma_j(t), s)\sum_{i=1}^{p} |y(\sigma_i(s))|\, ds + a(\sigma_j(t))\\ &\le \frac{1}{1-L}\int_0^{t} \omega_1(\sigma_j(t), s)\sum_{i=1}^{p} |y(\sigma_i(s))|\, ds\\ &\quad + a(\sigma_j(t)), \quad \text{for all } t \in \mathbf{R}_+.\end{aligned}$$

From (3.5), summing up with respect to $j = 1, \dots, p$, afterwards, adding to (3.3), we obtain

(3.6)
$$\begin{aligned}&|y(t)| + \sum_{j=1}^{p} |y(\sigma_j(t))|\\ &\le \frac{1}{1-L}\int_0^{t} \left[\omega_1(t, s) + \sum_{j=1}^{p} \omega_1(\sigma_j(t), s)\right]\sum_{i=1}^{p} |y(\sigma_i(s))|\, ds\\ &\quad + a(t) + \sum_{j=1}^{p} a(\sigma_j(t))\\ &\le \frac{1}{1-L}\int_0^{t} \bar{\omega}_1(t, s)\left(|y(s)| + \sum_{i=1}^{p} |y(\sigma_i(s))|\right) ds + \bar{a}(t),\end{aligned}$$

where

$$\overline{\omega}_1(t, s) = \omega_1(t, s) + \sum_{j=1}^{p} \omega_1(\sigma_j(t), s),$$

$$\overline{a}(t) = a(t) + \sum_{j=1}^{p} a(\sigma_j(t)).$$

Using the inequality $(a+b)^2 \le 2(a^2 + b^2)$, for all $a, b \in \mathbf{R}$, (3.6) leads to:

(3.7)
$$\begin{aligned}&\left(|y(t)| + \sum_{i=1}^{p} |y(\sigma_i(t))|\right)^2\\ &\le \frac{2}{(1-L)^2}\int_0^{t} \overline{\omega}_1^2(t, s)\, ds \int_0^{t}\left(|y(s)| + \sum_{i=1}^{p} |y(\sigma_i(s))|\right)^2 ds + 2\overline{a}^2(t).\end{aligned}$$

Put $v(t) = (|y(t)| + \sum_{i=1}^{p} |y(\sigma_i(t))|)^2$, $b(t) = 2/(1-L)^2 \int_0^t \overline{\omega}_1^2(t,s)\,ds$; (3.7) is rewritten as follows:

$$v(t) \le b(t) \int_0^t v(s)\,ds + 2\overline{a}^2(t). \tag{3.8}$$

By (3.8), based on classical estimates, we obtain

$$v(t) \le 2\overline{a}^2(t) + 2b(t) \int_0^t \overline{a}^2(s) \exp\left(\int_s^t b(u)\,du \right) ds, \quad \text{for all } t \in \mathbf{R}_+. \tag{3.9}$$

Then we have the following theorem about asymptotically stable solutions.

Theorem 3.1. *Let* (A_1)–(A_7) *hold. If*

$$\lim_{t\to\infty} \left[\widehat{a}^2(t) + b(t) \int_0^t \widehat{a}^2(s) \exp\left(\int_s^t b(u)\,du \right) ds \right] = 0, \tag{3.10}$$

where

$$\begin{cases} \widehat{a}(t) = \frac{1}{(1-L)^2} \int_0^t \overline{\omega}_1(t,s) \sum_{i=1}^{p} |Q(\sigma_i(s)) + f(\sigma_i(s),0,0)|\,ds \\ \qquad + \frac{1}{1-L} \int_0^\infty \overline{\omega}_2(t,s)\,ds, \\ \overline{\omega}_2(t,s) = \omega_2(t,s) + \sum_{j=1}^{p} \omega_2(\sigma_j(t),s), \\ b(t) = \frac{2}{(1-L)^2} \int_0^t \overline{\omega}_1^2(t,s)\,ds, \end{cases} \tag{3.11}$$

then every solution x *to* (1.1) *is an asymptotically stable solution. Furthermore,*

$$\lim_{t\to\infty} |x(t) - \xi(t)| = 0. \tag{3.12}$$

Proof. We first note

$$\begin{aligned} |\xi(t)| &= |Q(t) + f(t,\xi(t),\xi(t))| \\ &= |Q(t) + f(t,0,0) + f(t,\xi(t),\xi(t)) - f(t,0,0)| \\ &\le |Q(t) + f(t,0,0)| \\ &\quad + |f(t,\xi(t),\xi(t)) - f(t,0,0)| \\ &\le |Q(t) + f(t,0,0)| + L\,|\xi(t)|, \quad t \in \mathbf{R}_+. \end{aligned} \tag{3.13}$$

Then

$$|\xi(t)| \leq \frac{1}{1-L} \left| Q(t) + f(t,0,0) \right|, \quad t \in \mathbf{R}_+; \tag{3.14}$$

hence,

$$\begin{aligned} a(t) &\leq \frac{1}{(1-L)^2} \int_0^t \omega_1(t,s) \sum_{i=1}^p \left| Q(\sigma_i(s)) + f(\sigma_i(s),0,0) \right| \, ds \\ &\quad + \frac{1}{1-L} \int_0^\infty \omega_2(t,s)\, ds \\ &\equiv \frac{1}{(1-L)^2} \int_0^t \omega_1(t,s)\lambda(s)\, ds + \frac{1}{1-L} \int_0^\infty \omega_2(t,s)\, ds, \end{aligned} \tag{3.15}$$

where $\lambda(s) = \sum_{i=1}^p |Q(\sigma_i(s)) + f(\sigma_i(s),0,0)|$;

$$\begin{aligned} a(\sigma_j(t)) &\leq \frac{1}{(1-L)^2} \int_0^{\sigma_j(t)} \omega_1(\sigma_j(t),s)\lambda(s)\, ds \\ &\quad + \frac{1}{1-L} \int_0^\infty \omega_2(\sigma_j(t),s)\, ds \\ &\leq \frac{1}{(1-L)^2} \int_0^t \omega_1(\sigma_j(t),s)\lambda(s)\, ds \\ &\quad + \frac{1}{1-L} \int_0^\infty \omega_2(\sigma_j(t),s)\, ds. \end{aligned} \tag{3.16}$$

Hence,

$$\begin{aligned} \overline{a}(t) &= a(t) + \sum_{j=1}^p a(\sigma_j(t)) \\ &\leq \frac{1}{(1-L)^2} \int_0^t \overline{\omega}_1(t,s)\lambda(s)\, ds + \frac{1}{1-L} \int_0^\infty \overline{\omega}_2(t,s)\, ds \\ &\equiv \widehat{a}(t), \end{aligned} \tag{3.17}$$

where

$$\begin{cases} \overline{\omega}_1(t,s) = \omega_1(t,s) + \sum_{j=1}^p \omega_1(\sigma_j(t),s), \\ \overline{\omega}_2(t,s) = \omega_2(t,s) + \sum_{j=1}^p \omega_2(\sigma_j(t),s). \end{cases}$$

Combining (3.9)–(3.11) and (3.15)–(3.17), we obtain

(3.18) $$\lim_{t\to\infty} |y(t)| = \lim_{t\to\infty} |x(t) - \xi(t)| = 0.$$

Theorem 3.1 is completely proved. □

4. Examples. Let us give two examples illustrating the results obtained. Let $E = C([0,1];\mathbf{R})$ be the Banach space of all continuous functions $u : [0,1] \to \mathbf{R}$ with the norm

$$\|u\| = \sup_{0\leq\eta\leq 1} |u(\eta)|, \quad u \in E.$$

Then, for all $x \in X = C(\mathbf{R}_+; E)$, for any $t \in \mathbf{R}_+$, $x(t)$ is an element of E and we denote

$$x(t)(\eta) = x(t,\eta), \quad 0 \leq \eta \leq 1.$$

Example 1. Consider (1.1) in the following with $p = q = 2$, $\pi(t) = t/2$, $\mu(t) = t^3/(1+t^2)$, $\sigma_1(s) = s$, $\sigma_2(s) = s/3$, $\chi_1(s) = s$, $\chi_2(s) = 4s$,

(4.1)
$$x(t) = Q(t) + f\left(t, x(t), x\left(\frac{t}{2}\right)\right) + \int_0^{t^3/(1+t^2)} V\left(t, s, x(s), x\left(\frac{s}{3}\right)\right) ds + \int_0^\infty G(t, s, x(s), x(4s))\, ds,$$

$t \in \mathbf{R}_+$, where Q, f, V and G are continuous functions defined, respectively, as follows:

(i) Function Q. $Q(t)(\eta) = Q(t,\eta) = 2(1 - k_1 - k_2)(1/(e^t + \eta))$, $0 \leq \eta \leq 1$, $t \geq 0$,with k_1, k_2 are given constants such that

$$\max\{|k_1|, |k_2|\} < \frac{1}{2}.$$

(ii) Function f. $\mathbf{R}_+ \times E^2 \to E$,

$$f(t,u_1,u_2)(\eta) = k_1 |u_1(\eta)| + k_2 \frac{e^{t/2} + \eta}{e^t + \eta} u_2(\eta),$$
$$0 \leq \eta \leq 1, \quad (t,u_1,u_2) \in \mathbf{R}_+ \times E^2.$$

(iii) Function V. $\Delta_\mu \times E^2 \to E$, $\Delta_\mu = \{(t,s) \in \mathbf{R}_+^2 : 0 \le s \le \mu(t), t \ge 0\}$,

$$V(t,s,u_1,u_2)(\eta) = \frac{1}{e^t+\eta}e^{-2s}\left[\sin\left(\frac{\pi}{2}(e^s+\eta)u_1(\eta)\right) + \sin\left(\frac{3\pi}{2}(e^{s/3}+\eta)u_2(\eta)\right)\right],$$

$0 \le \eta \le 1$, $(t,s,u_1,u_2) \in \Delta_\mu \times E^2$.

(iv) Function G. $\mathbf{R}_+^2 \times E^2 \to E$,

$$G(t,s,u_1,u_2)(\eta) = \frac{k}{e^t+\eta}e^{-2s}\left[\sin\left(\frac{\pi}{2}\int_0^1 (e^s+\zeta)u_1(\zeta)\,d\zeta\right) - \sin\left(\frac{3\pi}{2}\int_0^1 (e^{4s}+\zeta)u_2(\zeta)\,d\zeta\right)\right],$$

with $k = k_1 + k_2 - 1$, $0 \le \eta \le 1$, $(t,s,u_1,u_2) \in \mathbf{R}_+^2 \times E^2$.

It is clear that (A_5) holds. We can show that the functions Q, f, V and G satisfy (A_1)–(A_4).

Assumption (A_1) is valid, since, for all (u_1,u_2), $(\widetilde{u}_1,\widetilde{u}_2) \in E^2$, for all $t \ge 0$, for all $\eta \in [0,1]$,

$$\begin{aligned}
&|f(t,u_1,u_2)(\eta) - f(t,\widetilde{u}_1,\widetilde{u}_2)(\eta)| \\
&\qquad \le |k_1|\,|u_1(\eta)-\widetilde{u}_1(\eta)| + |k_2|\,\frac{e^{t/2}+\eta}{e^t+\eta}\,|u_2(\eta)-\widetilde{u}_2(\eta)| \\
&\qquad \le |k_1|\,\|u_1-\widetilde{u}_1\| + |k_2|\,\|u_2-\widetilde{u}_2\| \\
&\qquad \le \frac{L}{2}\left[\|u_1-\widetilde{u}_1\| + \|u_2-\widetilde{u}_2\|\right],
\end{aligned}$$

in which

$$0 \le L = 2\max\{|k_1|\,,\,|k_2|\} < 1.$$

Assumption (A_2) holds since, for all (u_1,u_2), $(\widetilde{u}_1,\widetilde{u}_2) \in E^2$ and $(t,s) \in \Delta_\mu$, for all $\eta \in [0,1]$,

$$\begin{aligned}
&V(t,s,u_1,u_2)(\eta) - V(t,s,\widetilde{u}_1,\widetilde{u}_2)(\eta) \\
&= \frac{1}{e^t+\eta}e^{-2s}\left[\sin\left(\frac{\pi}{2}(e^s+\eta)u_1(\eta)\right) - \sin\left(\frac{\pi}{2}(e^s+\eta)\widetilde{u}_1(\eta)\right)\right] \\
&\quad + \frac{1}{e^t+\eta}e^{-2s}\left[\sin\left(\frac{3\pi}{2}(e^{s/3}+\eta)u_2(\eta)\right) - \sin\left(\frac{3\pi}{2}(e^{s/3}+\eta)\tilde{u}_2(\eta)\right)\right],
\end{aligned}$$

note that

$$\frac{1}{e^t+\eta}e^{-2s}(e^s+\eta) \le 2e^{-t-s}, \qquad \frac{1}{e^t+\eta}e^{-2s}(e^{s/3}+\eta) \le 2e^{-t-s},$$

so

$$\begin{aligned}
&|V(t,s,u_1,u_2)(\eta) - V(t,s,\widetilde{u}_1,\widetilde{u}_2)(\eta)| \\
&\qquad \le \frac{1}{e^t+\eta}e^{-2s}\frac{\pi}{2}(e^s+\eta)\,|u_1(\eta)-\widetilde{u}_1(\eta)| \\
&\qquad\quad + \frac{1}{e^t+\eta}e^{-2s}\frac{3\pi}{2}(e^{s/3}+\eta)\,|u_2(\eta)-\widetilde{u}_2(\eta)| \\
&\qquad \le \pi e^{-t-s}\,\|u_1-\widetilde{u}_1\| + 3\pi e^{-t-s}\,\|u_2-\widetilde{u}_2\| \\
&\qquad \le \omega_1(t,s)\left[\|u_1-\widetilde{u}_1\| + \|u_2-\widetilde{u}_2\|\right],
\end{aligned}$$

where

$$\omega_1(t,s) = 3\pi e^{-t-s}.$$

Assumption (A_3) is also fulfilled; the proof is as below.

First, we show $G : \mathbf{R}_+^2 \times E^2 \to E$ is continuous. For all (t,s,u_1,u_2), $(\widetilde{t},\widetilde{s},\widetilde{u}_1,\widetilde{u}_2) \in \mathbf{R}_+^2 \times E^2$,

$$\begin{aligned}
&G(t,s,u_1,u_2)(\eta) - G(\widetilde{t},\widetilde{s},\widetilde{u}_1,\widetilde{u}_2)(\eta) \\
&\quad = k\biggl(\frac{1}{e^t+\eta}e^{-2s} - \frac{1}{e^{\tilde{t}}+\eta}e^{-2\widetilde{s}}\biggr) \\
&\qquad \times \left[\sin\left(\frac{\pi}{2}\int_0^1 (e^s+\zeta)u_1(\zeta\,)d\zeta\right) - \sin\left(\frac{3\pi}{2}\int_0^1 (e^{4s}+\zeta)u_2(\zeta)\,d\zeta\right)\right] \\
&\qquad + \frac{k}{e^{\tilde{t}}+\eta}e^{-2\tilde{s}}\left[\sin\left(\frac{\pi}{2}\int_0^1 (e^s+\zeta)u_1(\zeta)\,d\zeta\right)\right. \\
&\qquad\qquad\qquad\qquad \left. - \sin\left(\frac{\pi}{2}\int_0^1 (e^{\tilde{s}}+\zeta)\widetilde{u}_1(\zeta)\,d\zeta\right)\right] \\
&\qquad - \frac{k}{e^{\tilde{t}}+\eta}e^{-2\tilde{s}}\left[\sin\left(\frac{3\pi}{2}\int_0^1 (e^{4s}+\zeta)u_2(\zeta)\,d\zeta\right)\right. \\
&\qquad\qquad\qquad\qquad \left. - \sin\left(\frac{3\pi}{2}\int_0^1 (e^{4\tilde{s}}+\zeta)\widetilde{u}_2(\zeta)\,d\zeta\right)\right].
\end{aligned}$$

Then,

$$\begin{aligned}
&\left|G(t,s,u_1,u_2)(\eta) - G(\widetilde{t},\widetilde{s},\widetilde{u}_1,\widetilde{u}_2)(\eta)\right| \\
&\quad \leq 2\,|k| \left|\frac{1}{e^t+\eta}e^{-2s} - \frac{1}{e^{\tilde{t}}+\eta}e^{-2\tilde{s}}\right| \\
&\qquad + \frac{\pi}{2}\,|k|\,e^{-2\tilde{s}-\tilde{t}} \int_0^1 \left|(e^s+\zeta)u_1(\zeta) - (e^{\tilde{s}}+\zeta)\widetilde{u}_1(\zeta)\right| d\zeta \\
&\qquad + \frac{3\pi}{2}\,|k|\,e^{-2\tilde{s}-\tilde{t}} \int_0^1 \left|(e^{4s}+\zeta)u_2(\zeta) - (e^{4\tilde{s}}+\zeta)\widetilde{u}_2(\zeta)\right| d\zeta \\
&\quad \leq 2\,|k| \left|\frac{1}{e^t+\eta}\left(e^{-2s}-e^{-2\tilde{s}}\right) + \frac{e^{\tilde{t}}-e^t}{(e^t+\eta)(e^{\tilde{t}}+\eta)}e^{-2\tilde{s}}\right| \\
&\qquad + \frac{\pi}{2}\,|k|\,e^{-2\tilde{s}-\tilde{t}} \int_0^1 \left|\left(e^s-e^{\tilde{s}}\right)u_1(\zeta)\right. \\
&\qquad \left. +(e^{\tilde{s}}+\zeta)\left(u_1(\zeta)-\widetilde{u}_1(\zeta)\right)\right| d\zeta \\
&\qquad + \frac{3\pi}{2}\,|k|\,e^{-2\tilde{s}-\tilde{t}} \int_0^1 \left|(e^{4s}-e^{4\tilde{s}})u_2(\zeta)\right. \\
&\qquad \left. +(e^{4\tilde{s}}+\zeta)\left(u_2(\zeta)-\widetilde{u}_2(\zeta)\right)\right| d\zeta \\
&\quad \leq 4\,|k|\left(|s-\widetilde{s}| + \left|t-\widetilde{t}\right|\right) \\
&\qquad + \frac{\pi}{2}\,|k|\,e^{-\tilde{s}-\tilde{t}}\left[e^s\,|s-\widetilde{s}|\,\|u_1\| + 2\,\|u_1-\widetilde{u}_1\|\right] \\
&\qquad + 3\pi\,|k|\,e^{2\tilde{s}-\tilde{t}}\left[2e^{4s}\,|s-\widetilde{s}|\,\|u_2\| + \|u_2-\widetilde{u}_2\|\right] \\
&\quad \leq 4\,|k|\left(|s-\widetilde{s}| + \left|t-\widetilde{t}\right|\right) \\
&\qquad + \frac{\pi}{2}\,|k|\left[e^s\,|s-\widetilde{s}|\,\|u_1\| + 2\,\|u_1-\widetilde{u}_1\|\right] \\
&\qquad + 3\pi\,|k|\,e^{2\widetilde{s}}\left[2e^{4s}\,|s-\widetilde{s}|\,\|u_2\| + \|u_2-\widetilde{u}_2\|\right].
\end{aligned}$$

So

$$\begin{aligned}
&\left\|G(t,s,u_1,u_2) - G(\widetilde{t},\widetilde{s},\widetilde{u}_1,\widetilde{u}_2)\right\| \\
&\qquad \leq 4\,|k|\left(|s-\widetilde{s}| + \left|t-\widetilde{t}\right|\right) \\
&\qquad\quad + \frac{\pi}{2}\,|k|\left[e^s\,|s-\widetilde{s}|\,\|u_1\| + 2\,\|u_1-\widetilde{u}_1\|\right] \\
&\qquad\quad + 3\pi\,|k|\,e^{2\tilde{s}}\left[2e^{4s}\,|s-\widetilde{s}|\,\|u_2\| + \|u_2-\widetilde{u}_2\|\right],
\end{aligned}$$

and the continuity of G is proved.

Next, we show $G : \mathbf{R}_+^2 \times E^2 \to E$ is compact. Let B be bounded in $\mathbf{R}_+^2 \times E^2$. We have:

$$\|G(t,s,u_1,u_2)\| \le 2\,|k|\,e^{-t-2s} \le 2\,|k| \equiv M, \quad \text{for all } (t,s,u_1,u_2) \in B,$$

which implies that $G(B)$ is uniformly bounded in E. For all $\eta_1, \eta_2 \in [0,1]$, for all $(t,s,u_1,u_2) \in B$,

$$\begin{aligned} &G(t,s,u_1,u_2)(\eta_1) - G(t,s,u_1,u_2)(\eta_2) \\ &= k\frac{\eta_2-\eta_1}{(e^t+\eta_1)\,(e^t+\eta_2)} \\ &\quad \times e^{-2s}\bigg[\sin\bigg(\frac{\pi}{2}\int_0^1 (e^s+\zeta)u_1(\zeta)\,d\zeta\bigg) - \sin\bigg(\frac{3\pi}{2}\int_0^1 (e^{4s}+\zeta)u_2(\zeta)\,d\zeta\bigg)\bigg]; \end{aligned}$$

hence,

$$\begin{aligned} |G(t,s,u_1,u_2)(\eta_1) - G(t,s,u_1,u_2)(\eta_2)| &\le 2\,|k|\,e^{-2s}\frac{|\eta_2-\eta_1|}{(e^t+\eta_1)\,(e^t+\eta_2)} \\ &\le 2\,|k|\;|\eta_2-\eta_1|\,. \end{aligned}$$

Consequently, $G(B)$ is equicontinuous.

Finally, for all bounded subsets I_1, I_2 of $\mathbf{R}_+$ and for any bounded subsets J of E^2, for all $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $t_1, t_2 \in I_1$,

$$|t_1 - t_2| < \delta \Longrightarrow |G(t_1,s,u_1,u_2) - G(t_2,s,u_1,u_2)| < \varepsilon,$$

for all $(u_1,u_2) \in J$ and $s \in I_2$. Indeed, we get the above property since

$$\begin{aligned} &G(t_1,s,u_1,u_2)(\eta) - G(t_2,s,u_1,u_2)(\eta) \\ &= k\bigg(\frac{1}{e^{t_1}+\eta} - \frac{1}{e^{t_2}+\eta}\bigg) \\ &\quad \times e^{-2s}\bigg[\sin\bigg(\frac{\pi}{2}\int_0^1 (e^s+\zeta)u_1(\zeta)\,d\zeta\bigg) - \sin\bigg(\frac{3\pi}{2}\int_0^1 (e^{4s}+\zeta)u_2(\zeta)\,d\zeta\bigg)\bigg]. \end{aligned}$$

Thus,

$$\begin{aligned} |G(t_1,s,u_1,u_2)(\eta) - G(t_2,s,u_1,u_2)(\eta)| &\le 2\,|k|\left|\frac{1}{e^{t_1}+\eta} - \frac{1}{e^{t_2}+\eta}\right|e^{-2s} \\ &= 2\,|k|\,\frac{|e^{t_2}-e^{t_1}|}{(e^{t_1}+\eta)\,(e^{t_2}+\eta)}e^{-2s} \\ &\le 2\,|k|\;|t_1-t_2|\,. \end{aligned}$$

Assumption (A_4) is also clearly seen by the facts that, for all $\eta \in [0,1]$, for all $(t, s, u_1, u_2) \in \mathbf{R}_+^2 \times E^2$,

$$\begin{aligned}
|G(t,s,u_1,u_2)(\eta)| &\le \frac{2\,|k|}{e^t+\eta} e^{-2s} \le 2\,|k|\, e^{-t-2s} \equiv \omega_2(t,s),\\
\int_0^\infty \sup_{t\in I} \omega_2(t,s)\,ds &= 2\,|k| \int_0^\infty \sup_{t\in I} e^{-t-2s} ds \le 2\,|k| \int_0^\infty e^{-2s} ds\\
&= |k| < \infty.
\end{aligned}$$

We conclude that the result of Theorem 2.3 holds true for equation (4.1).

For more details, let us consider

$$x : \mathbf{R}_+ \longrightarrow E; \qquad x(t)(\eta) = x(t,\eta) = \frac{1}{e^t+\eta}, \quad \text{for all } \eta \in [0,1].$$

It is clear that x defined as above is the solution of (4.1).

Example 2. Consider (1.1) in the following with $p = q = 2$, $\pi(t) = t$, $\mu(t) = t^3/(1+t^2)$, $\sigma_1(s) = s$, $\sigma_2(s) = s/3$, $\chi_1(s) = s$, $\chi_2(s) = 4s$,

(4.2)
$$\begin{aligned}
x(t) &= Q(t) + f_1(t, x(t), x(t))\\
&\quad + \int_0^{t^3/(1+t^2)} V\left(t, s, x(s), x\left(\frac{s}{3}\right)\right) ds + \int_0^\infty G(t, s, x(s), x(4s))\, ds,
\end{aligned}$$

$t \in \mathbf{R}_+$, where V, G and Q are continuous functions defined as in Example 1, and f_1 is given as follows:

$$\begin{aligned}
f_1 : \mathbf{R}_+ \times E^2 &\longrightarrow E\\
(t, u_1, u_2) &\longmapsto f_1(t, u_1, u_2),\\
f_1(t,u_1,u_2)(\eta) &= k_1\,|u_1(\eta)| + k_2 u_2(\eta),
\end{aligned}$$

$0 \le \eta \le 1$, $t \ge 0$, $(u_1, u_2) \in E^2$, with k_1 and k_2 being constants such that

$$\max\{|k_1|\,,\,|k_2|\} < \frac{1}{2}.$$

It is obvious that f_1 satisfies (A_1) and that (A_2)–(A_7) hold. On the other hand, (3.10) is also valid. Indeed,

(i) Estimate $\widehat{a}(t) = 1/(1-L)^2 \int_0^t \overline{\omega}_1(t,s) \sum_{i=1}^p \|Q(\sigma_i(s))+ f_1(\sigma_i(s), 0,0)\| \, ds + (1/(1-L)) \int_0^\infty \overline{\omega}_2(t,s) \, ds$.

We have

$$\begin{aligned}
&\sum_{i=1}^p \|Q(\sigma_i(s)) + f_1(\sigma_i(s),0,0)\| \\
&\qquad\qquad = \sum_{i=1}^2 \|Q(\sigma_i(s))\| = 2\,|k| \left(e^{-s} + e^{-s/3}\right) \le 4\,|k|\,, \\
&\overline{\omega}_1(t,s) = \omega_1(t,s) + \sum_{j=1}^2 \omega_1(\sigma_j(t),s) \\
&\qquad\qquad = 3\pi e^{-s}\left(2e^{-t} + e^{-t/3}\right) \le 9\pi e^{-s} e^{-t/3}, \\
&\overline{\omega}_2(t,s) = \omega_2(t,s) + \sum_{j=1}^p \omega_2(\sigma_j(t),s) \\
&\qquad\qquad = 2\,|k|\, e^{-2s}\left(2e^{-t} + e^{-t/3}\right) \le 6\,|k|\, e^{-2s} e^{-t/3},
\end{aligned}$$

$$\begin{aligned}
\int_0^t \overline{\omega}_1(t,s) \sum_{i=1}^p |Q(\sigma_i(s)) + f(\sigma_i(s),0,0)| \, ds &\le 36\,|k|\,\pi e^{-t/3} \int_0^t e^{-s} ds \\
&\le 36\,|k|\,\pi e^{-t/3} = \text{Const.} e^{-t/3},
\end{aligned}$$

$$\int_0^\infty \bar{\omega}_2(t,s)\, ds = 6\,|k|\, e^{-t/3} \int_0^\infty e^{-2s} ds \le 3\,|k|\, e^{-t/3}.$$

So

$$\widehat{a}(t) \le C_1 e^{-t/3}. \tag{4.3}$$

(ii) Estimate $b(t) = 2/(1-L)^2 \int_0^t \overline{\omega}_1^2(t,s)\, ds$. By

$$\begin{aligned}
\int_0^t \overline{\omega}_1^2(t,s)\, ds &\le 81\pi^2 e^{-2t/3} \int_0^t e^{-2s} ds \\
&= \frac{81\pi^2}{2} e^{-2t/3}\left(1 - e^{-2t}\right) \le \text{Const.} e^{-2t/3}.
\end{aligned}$$

Hence,

$$b(t) \leq C_2 e^{-2t/3}.$$

(iii) Estimate $\int_s^t b(u)\,du$, $s \leq t$.

$$\int_s^t b(u)\,du \leq C_2 \int_s^t e^{-2u/3} du = \frac{3C_2}{2}\left(e^{-2s/3} - e^{-2t}/3\right) \leq \frac{3C_2}{2}.$$

(iv) Estimate $b(t)\int_0^t \widehat{a}^2(s)\exp(\int_s^t b(u)\,du)\,ds$.

$$\begin{aligned}(4.4) \quad & b(t)\int_0^t \widehat{a}^2(s)\exp\left(\int_s^t b(u)\,du\right)ds \\ & \leq C_2 C_1^2 e^{-2t/3}\int_0^t e^{-2s/3}\exp\left(\frac{3C_2}{2}\right)ds \\ & = \frac{3}{2}C_2 C_1^2 \exp\left(\frac{3C_2}{2}\right)e^{-2t/3}\left(1 - e^{-2t/3}\right) \leq \text{Const.}e^{-2t/3}.\end{aligned}$$

Combining (4.3) and(4.4), (3.10) follows. We conclude that the result of Theorem 3.1 holds true for equation (4.2).

For more details, the following equation

$$\xi(t) = Q(t) + f_1\left(t, \xi(t), \xi(t)\right), \quad t \geq 0,$$

has a unique solution ξ defined by

$$\xi : \mathbf{R}_+ \longrightarrow E; \qquad \xi(t)(\eta) = \xi(t,\eta) = \frac{2}{e^t + \eta}, \quad \text{for all } \eta \in [0,1].$$

And we can compute to assert that

$$x : \mathbf{R}_+ \longrightarrow E; \qquad x(t)(\eta) = x(t,\eta) = \frac{1}{e^t + \eta}, \quad \text{for all } \eta \in [0,1],$$

is the solution of (4.1). Furthermore,

$$\lim_{t\to\infty} |x(t) - \xi(t)| = \lim_{t\to\infty} e^{-t} = 0.$$

So, it is clear that x, ξ are asymptotically stable solutions of (4.2).

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