JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 25, Number 1, Spring 2013

# WELL-POSEDNESS AND ASYMPTOTICS OF SOME NONLINEAR INTEGRO-DIFFERENTIAL EQUATIONS

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Communicated by Stig-Olof Londen

ABSTRACT. We will prove the well-posedness of certain second order ordinary and partial integro-differential equations with an integral term of the form $\int_0^t m(\varphi(t-s))\dot{\varphi}(s)\,ds$, where $m$ is given and $\varphi$ is the solution. The new aspect is the dependence of the kernel on the solution. In addition, for the ordinary integro-differential equation, the asymptotic behavior of the solution is described for some kernels.

**1. Introduction.** In this paper we first will investigate the system of ordinary integro-differential equations (OIDE)

$$\begin{aligned} \lambda\ddot{\varphi}(t) + \dot{\varphi}(t) + \varphi(t) + \int_0^t m(\varphi(t-s))\dot{\varphi}(s)\,ds = f(t),& \\ \varphi(0) = \varphi_0,& \\ \varphi(0) = \varphi_1,& \end{aligned} \tag{1}$$

where $\varphi(t) \in \mathbf{R}^d$, $\lambda > 0$ and the kernel $m$ is a matrix-valued mapping defined on $\mathbf{R}^d$. The function $f$ and the initial values $\varphi_0, \varphi_1$ are given. Here we wish to emphasize that we have a convolution of a function depending upon $\varphi$ with $\dot{\varphi}$.

Equations of this kind appear in the theory of glass-forming systems; they are obtained due to the use of the mode-coupling theory (a derivation can be found in [**4**]) and the components of the solution $\varphi$ are correlation functions. The kernel is mostly assumed as at least a quadratic polynomial function ([**1, 8, 13**]), but in some special cases a linear one is also used ([**12**]). Approaches other than the standard mode-coupling theory lead to an additional explicit time-dependence

Received by the editors on October 24, 2011, and in revised form on April 23, 2012.

DOI:10.1216/JIE-2013-25-1-103 

of the kernel ([**2, 7**]) or to complex valued equations ([**3**]). For these physical applications, $\varphi_0 = 1$ and $\varphi_1 = 0$ are given.

For $\lambda = 0$ and absolute monotone kernels the global existence is shown in [**5**], but the proof cannot be adopted for our system, because it leads to a completely monotone solution and taking the kernel as a sufficiently large constant would cause a contradiction.

In general, the studies of nonlinear OIDEs are concentrated on cases where the integral term contains the solution $\varphi$ or its derivatives only in dependence upon the parameter $s$. The results of those theories may not be applied directly, but we can carry over one of the main ideas: that the convolution is a compact perturbation of a diffeomorphism. Under the assumption that the kernel $m$ and the function $f$ are continuous this is easy to show. If now, in addition, the operator describing the OIDE is a weakly coercive mapping with a pointwise injective derivative, the Fredholm-theory tells us that also the perturbed equation is a diffeomorphism and thus there exists a global solution for any given $\varphi_0, \varphi_1$ and $f$. The injectivity will be given for $C^1$-kernels and an a priori estimate for the solution provides the weakly coerciveness, if the kernel is at most of linear growth. From here on we can show that, for any $C^1$-kernel, there is at least a local solution.

For kernels which behave like $O(|x|^\alpha)$ for $\alpha > 1$ as $x$ tends to zero we will define a sequence of solutions to a linearized equation and deduce the convergence, if some additional smallness condition holds. In contrast to the first method, the exponential stability of the zero solution is also obtained. The long-term asymptotic behavior is important in the theory of glass-transition. If it is zero, the correlation functions belong to a fluid; in the other case, we have a glassy state (values less than zero for the components of $\varphi$ are not expected in the physical application).

The OIDEs appear in the mode-coupling theory after applying a Fourier-transformation to a complicated partial differential equation and carrying out some approximations. This means that there is a partial integro-differential equation (PIDE) linked to the problem and it is of interest to know how to solve such a PIDE, where the kernel also depends upon the solution. To get an idea we consider, without an actual physical background, the equation

$$(2)\quad \lambda u_{tt}(t,x) + u_t(t,x) + u(t,x) - \Delta u(t,x) + \int_0^t m(u(t-s,x))u_s(s,x)\,ds = f(t,x),$$

$$u(0,x) = u_0(x),$$
$$u_t(0,x) = u_1(x),$$

for $t \in (0,T)$, $x \in \mathbf{R}^d$. Again the kernel's dependence upon the solution denies a direct application of the existence result already given for PIDEs as in [**6, 9**]. A Faedo-Galerkin method will work, where we make use of the theory of OIDEs. Therefore, we will need that the multiplication of the equation with a function lying in $H^k$ (the standard Sobolev space for $k \in \mathbf{N}$), where $2k > d$ holds for using Sobolev's embedding theorem, gives an OIDE of which we know the solvability. This implies that $f$ has to be a continuous mapping in time onto $H^k(\mathbf{R}^d)$ and, to get enough regularity, we assume that the kernel is in $C^{k+1}(\mathbf{R},\mathbf{R})$ and that the initial data $u_0$ and $u_1$ are accordingly in $H^{k+1}$ and $H^k$. Under these preliminaries we can show the local in time existence of a solution.

Replacing the $u_s$ in the integral by $\Delta u$ still allows us to use the Faedo-Galerkin method, but there are some technical problems. We will only point out how to solve them without showing all the details.

Trying to carry out this approach for the same equation, but now in a bounded domain with Dirichlet boundary conditions, is not immediately possible because of non-vanishing boundary integrals. Instead we solve a corresponding linear problem and give conditions for a better regularity of the solution. Within the proof we need elliptic regularity, and so we can only handle the convolution with $u_s$. In the one-dimensional case we now can define a sequence, which converges in an initial time interval to a solution of the nonlinear problem.

The paper is organized as follows. In Section 2 we deal with the system of OIDEs and prove the well-posedness and asymptotics under the conditions mentioned above. In Section 3 we will use some of the results for OIDEs to show the local well-posedness of the PIDE. In an appendix we list inequalities needed in Section 3.

**2. Ordinary integro-differential equations.** In this chapter we discuss the system

$$\begin{aligned}\lambda\ddot{\varphi}(t)+\dot{\varphi}(t)+\varphi(t)+\int_0^t m(\varphi(t-s))\dot{\varphi}(s)\,ds&=f(t),\\ \varphi(0)&=\varphi_0,\\ \dot{\varphi}(0)&=\varphi_1,\end{aligned} \tag{3}$$

of OIDEs for $t\in[0,T]$ $(T>0)$. The function $f\in C^0([0,\infty),\mathbf{R}^d)$ and the initial values $\varphi_0,\varphi_1\in\mathbf{R}^d$ are given. Our first theorem on the existence of a solution is

**Theorem 1.** *Let $m\in C^1(\mathbf{R}^d,\mathbf{R}^{d\times d})$. Then, for any given $\varphi_0,\varphi_1\in\mathbf{R}^d$ and $f\in C^0([0,\infty),\mathbf{R}^d)$, there exists some $T>0$ and a unique solution $\varphi\in C^2([0,T],\mathbf{R}^d)$ of (l). $T$ is arbitrary if*

(*V1*) *There exists $c>0$ for all $x,y\in\mathbf{R}^n:|m(x)y|\le c(1+|x|)|y|$*

*holds.*

To prove the well-posedness under condition (V1), the Fredholm-theory as presented in [**15**] is used, for which we need to introduce the operator $\mathcal{A}:C^2([0,T],\mathbf{R}^d)\to C^0([0,T],\mathbf{R}^d)\times\mathbf{R}^d\times\mathbf{R}^d=:X$ for $T>0$ arbitrary as

$$\mathcal{A}(\varphi):=\begin{pmatrix}\lambda\ddot{\varphi}+\dot{\varphi}+\varphi+m(\varphi)*\dot{\varphi}\\ \varphi(0)\\ \varphi(0)\end{pmatrix}. \tag{4}$$

Here $m(\varphi)$ denotes the composition $m\circ\varphi$ and $*$ the convolution of a matrix-valued function with a vector-valued function. We will show that the operator is invertible and that $\mathcal{A}^{-1}$ is continuous.

$\mathcal{A}$ can be divided into a linear part $\mathcal{L}$ and a nonlinear part $\mathcal{N}$:

$$\mathcal{L}(\varphi):=\begin{pmatrix}\lambda\ddot{\varphi}+\dot{\varphi}+\varphi\\ \varphi(0)\\ \dot{\varphi}(0)\end{pmatrix},\qquad \mathcal{N}(\varphi):=\begin{pmatrix}m(\varphi)*\dot{\varphi}\\ 0\\ 0\end{pmatrix}.$$

It is known that $\mathcal{L}$ is a $C^\infty$-diffeomorphism from $C^2([0,T],\mathbf{R}^d)$ onto $X$ (so $\mathcal{L}$ is a Fredholm-operator of index 0), so a perturbation result for Fredholm-operators yields that $\mathcal{A}$ is a $C^k$-diffeomorphism ($1 \le k \le \infty$), if $\mathcal{A}$ is weakly coercive, the derivative $\mathcal{A}'(\varphi)$ is for any $\varphi \in C^2([0,T],\mathbf{R}^d)$ injective and if $\mathcal{N}$ is a compact $C^k$-mapping.

**Lemma 2.** *Let $m \in C^k(\mathbf{R}^d, \mathbf{R}^{d\times d})$ $(1 \le k \le \infty)$. Then $\mathcal{N}$ is a compact $C^k$-operator with the derivative*

$$\begin{aligned} &\mathcal{N}'(\varphi) : C^2([0,T],\mathbf{R}^d) \longrightarrow X, \\ &\mathcal{N}'(\varphi)h = \begin{pmatrix} (m'(\varphi)h) * \dot{\varphi} + m(\varphi) * \dot{h} \\ 0 \\ 0 \end{pmatrix}. \end{aligned} \tag{5}$$

*Proof.* The product- and chainrule for the Fréchet-derivative provide directly that $\mathcal{N}$ is a $C^k$-operator. The compactness follows from the Arzelà-Ascoli theorem. If a sequence $(\varphi_n)_n$ is bounded in $C^2([0,T],\mathbf{R}^d)$, there exists a $C^1([0,T],\mathbf{R}^d)$ converging subsequence $(\varphi_{n'})_{n'}$. Since $\mathcal{N}$ is continuous from $C^1([0,T],\mathbf{R}^d)$ onto $X$, the sequence $(\mathcal{N}(\varphi_{n'}))_{n'}$ converges. □

**Lemma 3.** *Let $m \in C^k(\mathbf{R}^d, \mathbf{R}^{d\times d})$ $(1 \le k \le \infty)$ and $\varphi \in C^2([0,T],\mathbf{R}^d)$. Then $\mathcal{A}'(\varphi)$ is injective.*

*Proof.* We have to show that $\mathcal{A}'(\varphi)h = 0$ for $h \in C^2([0,T],\mathbf{R}^d)$ implies $h = 0$.

$\mathcal{A}'(\varphi)h = 0$ gives, on the one hand $h(0) = 0 = \dot{h}(0)$ and, on the other hand,

$$0 = \lambda\ddot{h} + \dot{h} + h + (m'(\varphi)h) * \varphi + m(\varphi) * \dot{h}.$$

Multiplication with $\dot{h}$ and integration leads, for some constant $c > 0$, to

$$\begin{aligned}\frac{1}{2}h(t)^2 + \frac{1}{2}\lambda\dot{h}(t)^2 \leq & -\int_0^t \dot{h}^2(s)\,ds \\ & + c\Big(\int_0^t\int_0^s |\dot{h}(s)h(s-r)|\,dr\,ds \\ & \qquad + \int_0^t\int_0^s |\dot{h}(s)\dot{h}(r)|\,dr\,ds\Big) \\ \leq & -\int_0^t \dot{h}^2(s)\,ds \\ & + cT\Big(\int_0^t \dot{h}(s)^2 + \int_0^t h(s)^2\,ds\Big).\end{aligned}$$

Applying Gronwall's inequality gives $h = \dot{h} = 0$. ◻

**Lemma 4.** *Let $m \in C^k(\mathbf{R}^d, \mathbf{R}^{d\times d})$ $(1 \leq k \leq \infty)$, and assume that* (V1) *holds. Then $\mathcal{A}$ is weakly coercive.*

*Proof.* Let $\varphi \in C^2([0,T], \mathbf{R}^d)$ and

$$\begin{pmatrix} f \\ \varphi_0 \\ \varphi_1 \end{pmatrix} := \mathcal{A}(\varphi) = \begin{pmatrix} \lambda\ddot{\varphi} + \dot{\varphi} + \varphi + m(\varphi) * \varphi \\ \varphi(0) \\ \varphi(0) \end{pmatrix}.$$

$\mathcal{A}$ is weakly coercive if and only if the image of an unbounded set is unbounded. For this it is sufficient to show that $\varphi$ is bounded by $\mathcal{A}(\varphi)$. Let $\varepsilon > 0$ be arbitrary. The continuity of $\varphi$ and $m$ guarantees the existence of some $T'$ with $|m(\varphi(t))| \leq |m(\varphi(0))| + \varepsilon =: k_1$ for $t \in [0, T']$. With $T_0 := \min\{T', 1/2k_1\}$, we have for $t \in [0, T_0]$,

$$\begin{aligned}\varphi(t)^2 + \lambda\dot{\varphi}(t)^2 \leq & \ \varphi_0^2 + \lambda\varphi_1^2 + \int_0^t f(s)^2\,ds \\ & - \int_0^t \dot{\varphi}(s)^2\,ds + 2\int_0^t\int_0^s |m(\varphi(s-r))\dot{\varphi}(r)\dot{\varphi}(s)|\,dr\,ds \\ \leq & \ \varphi_0^2 + \lambda\varphi_1^2 + \int_0^t f(s)^2\,ds - \int_0^t \dot{\varphi}(s)^2\,ds \\ & + 2k_1T_0\int_0^t \dot{\varphi}(s)^2\,ds\end{aligned}$$

$$\leq \varphi_0^2 + \lambda\varphi_1^2 + \int_0^{T_0} f(s)^2\,ds =: c_1^2.$$

Thus, $\varphi(t)$ and $\dot\varphi(t)$ are bounded by a constant $c_1$ for $t \in [0, T_0]$. For $t \in [T_0, 2T_0]$, it follows that

$$\begin{aligned}
&\int_0^t \int_0^s |m(\varphi(s-r))\dot\varphi(r)\dot\varphi(s)|\,dr\,ds \\
&\quad = \int_0^{T_0} \int_0^s |m(\varphi(s-r))\dot\varphi(r)\dot\varphi(s)|\,dr\,ds \\
&\qquad + \int_{T_0}^t \int_0^s |m(\varphi(s-r))\dot\varphi(r)\dot\varphi(s)|\,dr\,ds \\
&\quad \leq \frac{1}{2}k_1 c_1^2 \int_0^{T_0} \int_0^s \,dr\,ds \\
&\qquad + \int_{T_0}^t \int_0^{T_0} |m(\varphi(s-r))\dot\varphi(r)\dot\varphi(s)|\,dr\,ds \\
&\qquad + \int_{T_0}^t \int_{T_0}^s |m(\varphi(s-r))\dot\varphi(r)\dot\varphi(s)|\,dr\,ds \\
&\quad \leq \frac{1}{2}k_1 c_1^2 T_0^2 + c_1 \int_{T_0}^t \int_0^{T_0} c(1+|\varphi(s-r)|)|\dot\varphi(s)|\,dr\,ds \\
&\qquad + \frac{1}{2}k_1 \int_{T_0}^t \int_{T_0}^s \dot\varphi(r)^2 + \dot\varphi(s)^2\,dr\,ds \\
&\quad \leq \frac{1}{2}k_1 c_1^2 T_0^2 + k_1 \int_{T_0}^t \dot\varphi(s)^2\,ds \\
&\qquad + \frac{1}{2}cc_1(t-T_0)\int_{T_0}^t \dot\varphi(s)^2\,ds + c_1 c\int_{T_0}^t\int_0^{T_0} 1+\varphi(s-r)^2 dr\,ds \\
&\quad \leq \frac{1}{2}k_1 c_1^2 T_0^2 + c_1 c T_0^2 \\
&\qquad + \left(k_1 + \frac{1}{2}cc_1T_0\right)\int_0^t \dot\varphi(s)^2\,ds + c_1 c T_0 \int_0^t \varphi(s)^2\,ds \\
&\quad \Longrightarrow \varphi(t)^2 + \lambda\dot\varphi(t)^2 \\
&\quad \leq \varphi_0^2 + \lambda\varphi_1^2 + \int_0^t f(s)^2\,ds + k_1 c_1^2 T_0^2 + 2c_1 c T_0^2 \\
&\qquad + (2k_1 + cc_1T_0)\int_0^t \dot\varphi(s)^2\,ds + 2c_1 c T_0 \int_0^t \varphi(s)^2\,ds.
\end{aligned}$$

By Gronwall's inequality, we get a bound $c_2$ for $\varphi(t)$, $\dot{\varphi}(t)$ in $[0, 2T_0]$ and a bound $k_2$ for $m(\varphi(t))$. This leads analogously to an estimate for $t \in [0, 4T_0]$ and so successively to a bound for $\varphi$ and $\dot{\varphi}$ in $[0, T]$.

The integro-differential equation provides the boundedness of $\ddot{\varphi}$, so $\mathcal{A}$ is weakly coercive. □

Up to now we have proved for $m \in C^k(\mathbf{R}^d, \mathbf{R}^{d\times d})$ $(1 \leq k \leq \infty)$ with $(V1)$ that $\mathcal{A}$ is for any $T > 0$ a $C^k$-diffeomorphism from $C^2([0,T], \mathbf{R}^d)$ onto $X$; hence, there is a unique solution of (1).

The existence of a local solution for kernels without (V1) can be derived from this global existence result by cutting off the kernel appropriately and solving the equation with a bounded one. This leads to a solution which fulfills the original equation in some initial time interval. Therefore, let $k_1 > |\varphi_0|$ and $k_2 > k_1$ be arbitrary but fixed. Define $\widetilde{m} \in C^1(\mathbf{R}^d, \mathbf{R}^{d\times d})$ by

$$\widetilde{m}(x) := \begin{cases} 0 & |x| \geq k_2 \\ m(x) & |x| \leq k_1 \end{cases}$$

and continuously differentiable extended on $\{x \in \mathbf{R}^d | k_1 \leq |x| \leq k_2\}$.

Then $\widetilde{m}$ is bounded, $\widetilde{m} \in C^1(\mathbf{R}^d, \mathbf{R}^{d\times d})$, and thus there is a global solution $\varphi$ to

$$\lambda\ddot{\varphi}(t) + \dot{\varphi}(t) + \varphi(t) + \int_0^t \widetilde{m}(\varphi(t-s))\dot{\varphi}(s)\, ds = f(t),$$
$$\varphi(0) = \varphi_0,$$
$$\dot{\varphi}(0) = \varphi_1.$$

It is $|\varphi(0)| < k_1$ and $\varphi$ is continuous, so there exists some $T > 0$ with $|\varphi(t)| < k_1$ for all $t \in [0,T]$. This implies $\widetilde{m}(\varphi(t)) = m(\varphi(t))$ for $t \in [0,T]$; hence, $\varphi$ is a local solution to the problem.

To show the uniqueness of such a solution, we only need a locally Lipschitz continuous kernel.

**Lemma 5.** *Let $T > 0$ be arbitrary, and let $m \in C^0(\mathbf{R}^d, \mathbf{R}^{d\times d})$ be locally Lipschitz continuous. Then, for any given $\varphi_0, \varphi_1 \in \mathbf{R}^n$ and $f \in C^0([0,T], \mathbf{R}^d)$, there exists at most one solution $\varphi \in C^2([0,T], \mathbf{R}^d)$ of* (1).

*Proof.* For two solutions $u, v$, let $w = u - v$. Then we have $w(0) = 0 = \dot{w}(0)$ and

$$0 = \lambda \ddot{w}(t) + \dot{w}(t) + w(t) + \int_0^t m(u(t-s))\dot{u}(s) - m(v(t-s))\dot{v}(s)\, ds.$$

This leads to

$$\frac{1}{2}\lambda \frac{\mathrm{d}}{\mathrm{dt}}(\dot{w}(t))^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}(w(t))^2 \\ = -\dot{w}(t)^2 - \dot{w}(t)\int_0^t m(u(t-s))\dot{u}(s) - m(v(t-s))\dot{v}(s)\, ds.$$

By the continuity of $u$ and $v$, we can find some $c > 0$ with $\|u\|_\infty \leq c$ as well as $\|v\|_\infty \leq c$. The constant can be chosen in such a way that, additionally, $\|m(u)\|_\infty \leq c$ holds.

Because of the local Lipschitz continuity of $m$ and the boundedness of $u$ and $v$ there is some $L > 0$ with $|m(u(t)) - m(v(t))| \leq L|u(t) - v(t)|$. Now the integral can be estimated by

$$\int_0^t |m(u(t-s))\dot{u}(s) - m(v(t-s))\dot{v}(s)|\, ds \\ \leq c\int_0^t |\dot{w}(s)|\, ds + cL\int_0^t |w(s)|\, ds.$$

Using this, we obtain

$$\frac{1}{2}\lambda(\dot{w}(t))^2 + \frac{1}{2}(w(t))^2 \\ \leq \left(\frac{1}{2}cLT + cT - 1\right)\int_0^t \dot{w}(s)^2\, ds + \frac{1}{2}cLT\int_0^t w(s)^2\, ds.$$

Gronwall's inequality now gives $w = \dot{w} = 0$, and thus we get $\ddot{w} = 0$. □

Analogously, the continuous dependence on the data follows.

To extend the class of kernels giving a global solution, we will use a different approach. Let $x \in C^0([0,\infty), \mathbf{R}^l)$ ($l \in \mathbf{N}$) be a solution to

$$\begin{aligned} x(t) = \Psi(t)x_0 + \Psi(t)\int_0^t \Psi(-s)\int_0^s M(s-r)x(r)\, dr\, ds \\ + \Psi(t)\int_0^t \Psi(-s)f(s)\, ds, \end{aligned} \tag{6}$$

with given functions $\Psi \in C^0(\mathbf{R}, \mathbf{R}^{l\times l})$, $M \in C^0([0,\infty), \mathbf{R}^{l\times l})$. By carrying out the method above for this equation, we obtain that, for a local Lipschitz continuous $M$, there is a unique global solution, which is in $C^1([0,\infty), \mathbf{R}^l)$ for $\Psi \in C^1(\mathbf{R}, \mathbf{R}^{l\times l})$.

**Theorem 6.** *Let* $|\Psi(t)x_0| \leq e^{-c_0 t}|x_0|$ *for* $t \in [0,\infty)$, $|M(s-r)x(r)| \leq ke^{-c_1(s-r)}|x(r)|$ *for* $s, r \in [0,\infty)$, $s \geq r$, $|\int_0^t \Psi(-s)f(s)\,ds| \leq k_1$ *for some* $k_1 > 0$ *independent of* $t$ *and* $c_1 > c_0$. *Then*

$$|x(t)| \leq (|x_0| + k_1)e^{-(c_0-(k/c_1-c_0))t}$$

*holds for the solution* $x$ *of* (6). *Especially we have in the case* $c_0 - (k/c_1 - c_0) > 0$ *an exponentially decaying solution.*

*Proof.* We have

$$\begin{aligned} e^{c_0 t}|x(t)| &\leq |x_0| + k_1 + k\int_0^t \int_0^s e^{-(c_1-c_0)(s-r)}e^{c_0 r}|x(r)|\,dr\,ds \\ &= |x_0| + k_1 + k\int_0^t \int_r^t e^{-(c_1-c_0)(s-r)}e^{c_0 r}|x(r)|\,ds\,dr \\ &\leq |x_0| + k_1 + \frac{k}{c_1-c_0}\int_0^t e^{c_0 r}|x(r)|\,dr. \end{aligned}$$

Gronwall's inequality gives

$$e^{c_0 t}|x(t)| \leq (|x_0| + k_1)e^{t\cdot k/c_1-c_0}. \qquad \square$$

Now we turn back to nonlinear equation (1). After a transformation to a first order system with

$$A := \begin{pmatrix} 0 & 1 \\ -\frac{1}{\lambda} & -\frac{1}{\lambda} \end{pmatrix}, \qquad M(x(t-s)) := \begin{pmatrix} 0 & 0 \\ 0 & m(x_1(t-s)) \end{pmatrix}$$

and $F(t) = (0, f(t))$ the equation for $x = (x_1, x_2) = (\varphi, \dot\varphi)$ reads as

$$\begin{aligned} \dot x(t) &= Ax(t) - \frac{1}{\lambda}\int_0^t M(x(t-s))x(s)\,ds + F(t), \\ x(0) &= (\varphi_0, \varphi_1) =: x_0. \end{aligned}$$

Variation of constants along with $\Psi(t) := e^{At}$ leads to the integral equation

$$x(t) = \Psi(t)x_0 - \frac{1}{\lambda}\Psi(t)\int_0^t \Psi(-s)\int_0^s M(x(s-r))x(r)\,dr\,ds \\ - \Psi(t)\int_0^t \Psi(-s)f(s)\,ds.$$

We define a sequence $(x_n)_n$ of functions by

(7)
$$x_n(t) = \Psi(t)x_0 - \frac{1}{\lambda}\Psi(t)\int_0^t \Psi(-s)\int_0^s M(x_{n-1}(s-r))x_n(r)\,dr\,ds \\ - \Psi(t)\int_0^t \Psi(-s)F(s)\,ds$$

with $x_0(t) := x_0 e^{-c_1 t}$, where $c_1 > 0$ will be chosen later. For all $n \in \mathbf{N}$, there is a solution $x_n \in C^1([0,\infty), \mathbf{R}^{2d})$ and thus the sequence is well-defined.

By calculating the eigenvalues of $A$ we get $|\Psi(t)x_0| \le e^{-c_0}|x_0|$ with $c_0 = \mathrm{Re}\,(1-\sqrt{1-4\lambda})/2\lambda$.

**Theorem 7.** *Assume that*

$$(V2) \quad \textit{There exist } \alpha > 1,\ v_1 > 0,\ k_1 > 0$$

*with for all* $x, z \in \mathbf{R}^d$, $|z| \le |x_0| + k_1 : |m(z)x| \le v_1|z|^\alpha|x|$ *and for all* $t \in \mathbf{R} : |\int_0^t \Psi(-s)F(s)\,ds| \le k_1$ *holds. Then there is a constant* $k = k(\alpha, c_0) > 0$ *such that, if* $v_1(|x_0| + k_1)^\alpha \le k$ *is fulfilled, we can choose a* $c_1 > 0$ *such that*

(8)
$$|x_n(t)| \le (|x_0| + k_1)e^{-c_1 t}$$

*holds for all* $n \in \mathbf{N}$.

*Proof.* Set $k := \lambda(\alpha-1)^2/4\alpha c_0^2$ and $c_1 := (\alpha + 1/2\alpha)c_0$. For $n = 0$, the statement is trivial.

So let $|x_{n-1}(t)| \le (|x_0| + k_1)e^{-c_1 t}$. We obtain

$$\begin{aligned} |M(x_{n-1}(s-r))x_n(r)| &\le v_1 |x_{n-1}(s-r)|^\alpha |x_n(r)| \\ &\le v_1(|x_0| + k_1)^\alpha e^{-\alpha c_1 (s-r)} |x_n(r)| \\ &\le k e^{-\alpha c_1 (s-r)} |x_n(r)|. \end{aligned}$$

It is $\alpha c_1 = (\alpha + 1/2)c_0 > c_0$ and, applying Theorem 6, yields

$$|x_n(t)| \le (k_1 + |x_0|)e^{-(c_0 - (k/\lambda)(1/\alpha c_1 - c_0))t}$$

with

$$\begin{aligned} c_0 - \frac{k}{\lambda}\frac{1}{\alpha c_1 - c_0} &= c_0 - \frac{k}{\lambda}\frac{2}{(\alpha+1)c_0 - 2c_0} = c_0 - \frac{2(\alpha-1)^2 c_0^2}{4\alpha(\alpha-1)c_0} \\ &= c_0 - \frac{(\alpha-1)c_0}{2\alpha} = \frac{(\alpha+1)c_0}{2\alpha} = c_1. \qquad \square \end{aligned}$$

The theorem ensures that $(x_n)_n$ is, for any $T > 0$, uniformly bounded in $C^0([0,T], \mathbf{R}^{2d})$ and, by differentiating (7), we obtain this in $C^1([0,T], \mathbf{R}^{2d})$. So there is a convergent subsequence in $C^0([0,T], \mathbf{R}^{2d})$ and the derivative of (7) gives the convergence in $C^1([0,T], \mathbf{R}^{2d})$.

**Corollary 8.** *Let $m \in C^1(\mathbf{R}^n, \mathbf{R}^{n\times n})$. Then, for any given $\varphi_0, \varphi_1 \in \mathbf{R}^n$ and $f \in C^0([0,\infty), \mathbf{R}^n)$ with* (V2)*, there is a unique global solution $\varphi \in C^2([0,\infty), \mathbf{R}^n)$ of* (1)*. Additionally,*

$$|\dot\varphi(t)| + |\varphi(t)| \le (|x_0| + k_1)e^{-c_1 t}$$

*holds.*

When dealing with the PIDE, we will need that a local solution of (1), which is uniformly bounded in its interval of existence, is extendable to larger intervals.

**Lemma 9.** *Let $m \in C^1(\mathbf{R}^n, \mathbf{R}^{n\times n})$ and $\varphi \in C^2([0,T], \mathbf{R}^n)$ be a local solution of* (1) *for $\varphi_0, \varphi_1 \in \mathbf{R}^n$, $f \in C^0([0,\infty), \mathbf{R}^n)$. If there exists* *some* $c \geq 0$ *with* $|\varphi(t)| \leq c$ *for* $t \in [0,T]$, *then* $\varphi$ *can be extended to a solution of* (1) *in* $[0, T+\varepsilon]$ *for some* $\varepsilon > 0$.

*Proof.* Let $\delta_2 > \delta_1 > 0$ be arbitrary. Let $\widetilde{m} \in C^1(\mathbf{R}^n, \mathbf{R}^{n\times n})$ be defined as

$$\widetilde{m}(x) := \begin{cases} m(x) & |x| \leq c+\delta_1 \\ 0 & |x| \geq c+\delta_2 \end{cases}$$

and in $\{x \in \mathbf{R}^n | c+\delta_1 \leq |x| \leq c+\delta_2\}$ continuously differentiable. It is $\widetilde{m} \in C^1(\mathbf{R}^n, \mathbf{R}^{n\times n})$ and $\widetilde{m}$ is bounded; thus, there is a unique global solution $\psi$ to

$$\begin{aligned} \lambda\ddot{\psi}(t) + \dot{\psi}(t) + \psi(t) + \int_0^t \widetilde{m}(\psi(t-s))\dot{\psi}(s)\,ds &= f(t), \\ \psi(0) &= \varphi_0, \\ \dot{\psi}(0) &= \varphi_1. \end{aligned} \tag{9}$$

In $[0,T]$, we have $\widetilde{m}(\varphi(t)) = m(\varphi(t))$, so $\varphi$ is also a local solution to (9) and the uniqueness of a solution gives $\varphi = \psi$ in $[0,T]$.

By the continuity of $\psi$ it follows that, for some $\varepsilon > 0$ and $t \in [0, T+\varepsilon]$, $|\psi(t)| \leq c+\delta_1$ holds and this yields $\widetilde{m}(\psi(t)) = m(\psi(t))$, so $\psi$ is the continuation of $\varphi$ onto the interval $[0, T+\varepsilon]$. ◻

*Remark* 10. For kernels also depending continuously upon $t$ and $s$ and for complex kernels we can proceed in the same way as above.

**3. Well-posedness of certain PIDEs.** We now turn to the PIDE

$$\begin{aligned} \lambda u_{tt}(t,x) + u_t(t,x) + u(t,x) - \Delta u(t,x) \\ + \int_0^t m(u(t-s,x))u_s(s,x)\,ds = f(t,x) \end{aligned} \tag{10}$$

$$\begin{aligned} u(0,x) &= u_0(x) \\ u_t(0,x) &= u_1(x) \end{aligned}$$

with $t \in (0,T]$, $x \in \mathbf{R}^d$ $(d \in \mathbf{N})$, and we will show the

**Theorem 11.** *Let $2k > d$ and $m \in C^{k+1}(\mathbf{R},\mathbf{R})$. Then, for any given $u_0 \in H^{k+1}$, $u_1 \in H^k$ and $f \in C^0([0,\infty), H^k)$, there exists some $T > 0$ and a unique solution*

$$u \in C^2([0,T], H^{k-1}) \cap C^1([0,T], H^k) \cap C^0([0,T], H^{k+1})$$

*to* (10).

*Proof* (*Uniqueness*). For $T > 0$ arbitrary, let $u$ and $v$ be solutions with the above regularity, $w := u - v$. Then we have $w \in C^2([0,T], H^{k-1}) \cap C^1([0,T], H^k) \cap C^0([0,T], H^{k+1})$, $w(0) = 0 = w_t(0)$ and

$$\lambda w_{tt}(t) + w_t(t) + w(t) - \Delta w(t) + \int_0^t m(u(t-s))u_s(s) - m(v(t-s))v_s(s)\, ds = 0.$$

Multiplication by $w_t(t)$ in $L^2$ and integration leads for $0 < T' < T$ arbitrary to

$$\begin{aligned}\lambda\|w_t(T')\|_{L^2}^2 + \|w(T')\|_{L^2}^2 + \|\nabla w(T')\|_{L^2}^2 \\ \leq -2\int_0^{T'} \|w_t(t)\|_{L^2}^2\, dt \\ + \int_0^{T'} t\int_0^t \|m(u(t-s))u_s(s) - m(v(t-s))v_s(s)\|_{L^2}^2\, ds \\ + \|w_t(t)\|_{L^2}^2\, dt.\end{aligned}$$

Because of $2k > d$, we have $u(t), v_t(t) \in C_b^0(\mathbf{R}^d, \mathbf{R})$ for $t \in [0,T]$, and thus

$$\begin{aligned}\|m(u(t-s))u_s(s) - m(v(t-s))v_s(s)\|_{L^2} \\ \leq c(\|u_s(s) - v_s(s)\|_{L^2} + \|m(u(t-s)) - m(v(t-s))\|_{L^2}),\end{aligned}$$

where $c > 0$ is independent of $t$. $m(u(s)) - m(v(s))$ can be rewritten as

$$\begin{aligned}\|m(u(s)) - m(v(s))\|_{L^2} \\ = \left\| \int_0^1 m'(ru(s) + (1-r)v(s))(u(s) - v(s))\, dr \right\| \leq c\|w(s)\|_{L^2}.\end{aligned}$$

This gives

$$\lambda\|w_T(T')\|_{L^2}^2 + \|w(T')\|_{L^2}^2 + \|\nabla w(T')\|_{L^2}^2 \leq -\int_0^{T'} \|w_t(t)\|_{L^2}^2\,dt + cT^2\int_0^{T'} \|w(t)\|_{L^2}^2 + \|w_t(t)\|_{L^2}^2\,dt$$

and Gronwall's inequality yields $w = 0$. □

To show the existence of a solution we need a certain kind of regularity of the mapping $u \mapsto m(u) * u_t$.

**Lemma 12.** *Let $2k > d$, $T > 0$ arbitrary and $m \in C^{k+1}(\mathbf{R}, \mathbf{R})$. Let $(u^{(n)})_n \subset C^0([0,T], H^{k+1}) \cap C^1([0,T], H^k)$, $u \in C^0([0,T], H^{k+1}) \cap C^1([0,T], H^k)$, and let $u^{(n)} \to u$ in $C^0([0,T], H^{k+1}) \cap C^1([0,T], H^k)$.*
*$\Rightarrow$. $m(u^{(n)}) * u_t^{(n)} \to m(u) * u_t$ in $L^2((0,T), H^k)$.*

*Proof.* We have

$$\|m(u^{(n)}) * u_t^{(n)} - m(u) * u_t\|_{L^2((0,T),H^k)}^2 \leq 2\int_0^T t\int_0^t \|(m(u^{(n)}(t-s)) - m(u(t-s)))u_s^{(n)}(s)\|_{H^k}^2\,ds\,dt + 2\int_0^T t\int_0^t \|m(u(t-s))(u_s^{(n)}(s) - u_s(s))\|_{H^k}^2\,ds\,dt.$$

In the following, $c$ denotes constants being independent of $t$.

Because of Sobolev's embedding theorem we can use Moser-type inequalities (see Appendix) to estimate the $L^2$-norm of $\nabla^\alpha(m(u(t-s))(u_s^{(n)}(s) - u_s(s)))$ by

$$\begin{aligned}\|\nabla^\alpha(m(u(t-s))(u_s^{(n)}(s) - u_s(s)))\|_{L^2} &\leq c\Big(\|m(u(t-s))\|_\infty\|\nabla^{|\alpha|}(u_s^{(n)}(s) - u_s(s))\|_{L^2} \\ &\quad + \|u_s^{(n)}(s) - u_s(s)\|_\infty\|\nabla^{|\alpha|}m(u(t-s))\|_{L^2}\Big) \\ &\leq c\Big(\|\nabla^{|\alpha|}(u_s^{(n)}(s) - u_s(s))\|_{L^2} \\ &\quad + \|u_s^{(n)}(s) - u_s(s)\|_\infty\|\nabla^{|\alpha|}m(u(t-s))\|_{L^2}\Big).\end{aligned}$$

We get the $L^2$-norm of $\nabla^{|\alpha|}m(u(t-s))$ for $|\alpha| \geq 1$ by

$$\|\nabla^{|\alpha|}m(u(t-s))\|_{L^2} \leq c\|\nabla^{|\alpha|}u(s-r)\|_{L^2}\|u(s-r)\|_\infty^{j-1}.$$

Sobolev's embedding theorem yields $\|u_s^{(n)}(s) - u_s(s)\|_\infty \leq c\|u_s^{(n)}(s) - u_s(s)\|_{H^k}$ and $u \in C^0([0,T], H^{k+1})$ then implies $\|u(t)\|_\infty \leq c$.

$$\Longrightarrow \|m(u(t-s))(u_s^{(n)}(s) - u_s(s))\|_{H^k} \leq c\|u_s^{(n)}(s) - u_s(s)\|_{H^k}.$$

It is $m \in C^{k+1}(\mathbf{R}, \mathbf{R})$; thus, we get

$$\begin{aligned} &m(u^{(n)}(t-s)) - m(u(t-s)) \\ &= (u^{(n)}(t-s) - u(t-s)) \int_0^1 m'(ru^{(n)}(t-s) + (1-r)u(t-s))\,dr, \end{aligned}$$

and similarly to the estimate for $\|\nabla^\alpha(m(u(t-s))(u_s^{(n)}(s) - u_s(s)))\|_{L^2}$, we obtain

$$\begin{aligned} \|(m(u^{(n)}(t-s)) - m(u(t-s)))u_s^{(n)}(s)\|_{H^k} \leq{}& c\|u^{(n)}(t-s) - u(t-s)\|_\infty \\ &+ c \sum_{1 \leq |\alpha| \leq k} \|\nabla^{|\alpha|}((u^{(n)}(t-s) - u(t-s)) \\ &\times \int_0^1 m'(ru^{(n)}(t-s) + (1-r)u(t-s))\,dr)\|_{L^2}. \end{aligned}$$

Moreover, the estimate

$$\begin{aligned} &\left\|\nabla^{|\alpha|}(u^{(n)}(t-s) - u(t-s)) \int_0^1 m'(ru^{(n)}(t-s) + (1-r)u(t-s))\,dr)\right\|_{L^2} \\ &\quad\leq c\|u^{(n)}(t-s) - u(t-s)\|_\infty \\ &\qquad\times \left\|\nabla^{|\alpha|} \int_0^1 m'(ru^{(n)}(t-s) + (1-r)u(t-s))\,dr\right\|_{L^2} \\ &\qquad+ c\left\|\int_0^1 m'(ru^{(n)}(t-s) + (1-r)u(t-s))\,dr\right\|_\infty \\ &\qquad\times \|\nabla^{|\alpha|}(u^{(n)}(t-s) - u(t-s))\|_{L^2} \\ &\quad\leq c\|u^{(n)}(t-s) - u(t-s)\|_{H^k} \\ &\qquad\times \left\|\nabla^{|\alpha|} \int_0^1 m'(ru^{(n)}(t-s) + (1-r)u(t-s))\,dr\right\|_{L^2} \\ &\qquad+ c\|\nabla^{|\alpha|}(u^{(n)}(t-s) - u(t-s))\|_{L^2} \end{aligned}$$

holds. The sequence is convergent; hence, the term $\|\nabla^{|\alpha|}(u^{(n)}(t-s)-u(t-s))\|_{L^2}$ is uniformly bounded. This yields

$$\begin{aligned}\Big\|\nabla^{|\alpha|}\int_0^1 m'(ru^{(n)}(t-s)+(1-r)u(t-s))\,dr\Big\|_{L^2}\\ &\le \int_0^1 c\|\nabla^{|\alpha|}(ru^{(n)}(t-s)+(1-r)u(t-s))\|_{L^2}\\ &\qquad\times\|ru^{(n)}(t-s)+(1-r)u(t-s)\|_\infty^{|\alpha|-1}\,dr\\ &\le c \Longrightarrow \|(m(u^{(n)}(t-s))-m(u(t-s)))u_s^{(n)}(s)\|_{H^k}\\ &\le c\|u^{(n)}(t-s)-u(t-s)\|_{H^k}.\end{aligned}$$

Altogether, we have

$$\begin{aligned}\|m(u^{(n)})*u_t^{(n)}-m(u)*u_t\|_{L^2((0,T),H^k)}\\ &\le c\int_0^T \|u^{(n)}(t)-u(t)\|_{H^k}^2+\|u_t^{(n)}(t)-u_t(t)\|_{H^k}^2\,dt\\ &\qquad\longrightarrow 0\ (n\to\infty).\end{aligned}$$ □

*Proof of Theorem* 13 (*Existence*). We first additionally assume that $m$ and its derivatives up to order $k+1$ are bounded. Let $(\varphi_n)_n\subset H^{k+1}$ be a basis of $H^k$ and $\langle\varphi_i|\varphi_j\rangle_{H^k}=\delta_{ij}$.

Let $V_n := \operatorname{span}\{\varphi_j : 1\le j\le n\}$ with the norm $\|\cdot\|_{H^{k+1}}$. Let $P_n: H^{k+1}\to V_n$ and $\Pi_n: H^k\to V_n$ be the orthogonal projections.

We now construct a sequence $(u^{(n)})_n$ of solutions $u^{(n)}(t)=\sum_{j=1}^n g_{nj}(t)\varphi_j$ with $g_{nj}\in C^2([0,T],\mathbf{R})$ to the equation after projecting it on the finite dimensional subspaces $V_n$. For this, we define

$$\begin{aligned}\Phi_n &:= (\langle\varphi_i,\varphi_j\rangle_{H^k})_{1\le i,j\le n},\\ \Psi_n &:= (\langle\nabla\varphi_i,\nabla\varphi_j\rangle_{H^k})_{1\le i,j\le n},\\ F_n(t) &:= (\langle f(t),\varphi_i\rangle_{H^k})_{1\le i\le n}\end{aligned}$$

and

$$\begin{aligned}g_n(t) := (g_{nj}(t))_{1\le j\le n},\ M_n(g_n(t-s))\\ := \bigg(\bigg\langle m\bigg(\sum_{l=1}^n g_{nl}(t-s)\varphi_l\bigg)\varphi_j,\varphi_i\bigg\rangle_{H^k}\bigg)_{1\le i,j\le n}.\end{aligned}$$

The standard Faedo-Galerkin approach gives for $n \in \mathbf{N}$, an OIDE for $g_n$:

$$\begin{aligned} &\lambda\Phi_n\ddot{g}_n(t) + \Phi_n\dot{g}_n(t) + (\Phi_n + \Psi_n)g_n(t) \\ &\qquad + \int_0^t M_n(g_n(t-s))\dot{g}_n(t) = F_n(t) \qquad (11) \\ &\qquad\qquad + \sum_{j=1}^n g_{nj}(0)\langle\varphi_j,\varphi_i\rangle = \langle P_n u_0, \varphi_i\rangle \quad (1 \le i \le n) \\ &\qquad\qquad \sum_{j=1}^n \dot{g}_{nj}(0)\langle\varphi_j,\varphi_i\rangle = \langle \Pi_n u_1, \varphi_i\rangle \quad (1 \le i \le n). \end{aligned}$$

The mapping $M_n : \mathbf{R}^n \to \mathbf{R}^{n\times n}$ is well defined, since we can estimate

$$\left\|\nabla^\alpha\left(m\left(\sum_{l=1}^n z_l\varphi_l\right)\varphi_j\right)\right\|_{L^2}$$

for any $z = (z_1, \dots, z_n) \in \mathbf{R}^n$ as before. $M_n$ is continuously differentiable, $m \in C^{k+1}(\mathbf{R},\mathbf{R})$ allows estimation of terms of the form

$$\left\|\nabla^\alpha\left(\varphi_i m'\left(\sum_{l=1}^n z_l\varphi_l\right)\varphi_j\right)\right\|_{L^2},$$

and $f \in C^0([0,\infty), H^k)$ provides $F_n \in C^0([0,\infty),\mathbf{R}^n)$.

By using the theory of OIDEs we can conclude that, for any $n \in \mathbf{N}$, the existence of some $T_n > 0$ such that there is a unique solution $g_n \in C^2([0,T_n],\mathbf{R}^n)$, so we have $u^{(n)} \in C^2([0,T_n],H^{k+1})$. If the sequence $(u^{(n)})_n$ is uniformly bounded in $H^1((0,T),H^k) \cap L^2((0,T),H^{k+1})$ for some $T > 0$, we can deduce the weak convergence of a subsequence in that space.

Multiplication of (11) by $\dot{g}_n$ leads to

$$\begin{aligned} &\lambda \sum_{|\alpha|\le k} \|\nabla^\alpha u_t^{(n)}(t)\|_{L^2}^2 + \|\nabla^\alpha u^{(n)}(t)\|_{L^2}^2 + \|\nabla\nabla^\alpha u^{(n)}(t)\|_{L^2}^2 \qquad (12) \\ &\qquad \le \sum_{|\alpha|\le k} \lambda\|\nabla^\alpha u_1\|_{L^2}^2 + \|\nabla^\alpha u_0\|_{H^1}^2 \\ &\qquad\quad + \int_0^t \|\nabla^\alpha f(s)\|_{L^2}^2 + (t-1)\int_0^t \|\nabla^\alpha u_s^{(n)}(s)\|_{L^2}^2\, ds \\ &\qquad\quad + \int_0^t\int_0^s \|\nabla^\alpha(m(u^{(n)}(s-r))u_r^{(n)}(r))\|_{L^2}^2\, dr\, ds. \end{aligned}$$

$C$ denotes different constants independent of $t$ and $n$.

The $L^2$-norm of $\nabla^\alpha(m(u^{(n)}(s-r))u_r^{(n)}(r))$ can be estimated for $1 \le |\alpha| \le k$ by

$$\begin{aligned}
&\|\nabla^\alpha(m(u^{(n)}(s-r))u_r^{(n)}(r))\|_{L^2} \\
&\qquad \le C\left(\|\nabla^{|\alpha|}u_r^{(n)}(r)\|_{L^2} + \|u_r^{(n)}(r)\|_\infty \|\nabla^{|\alpha|}m(u^{(n)}(s-r))\|_{L^2}\right),
\end{aligned}$$

and the boundedness of $m$ in $C^k(\mathbf{R},\mathbf{R})$ gives for $\nabla^{|\alpha|}m(u^{(n)}(s-r))$ $(1 \le |\alpha| \le k)$

$$\|\nabla^{|\alpha|}m(u^{(n)}(s-r))\|_{L^2} \le C\|\nabla^{|\alpha|}u^{(n)}(s-r)\|_{L^2}\|u^{(n)}(s-r)\|_\infty^{|\alpha|-1}.$$

We have $\|u^{(n)}(t)\|_\infty^2 \le C\|u^{(n)}(t)\|_{H^k}^2$ and $\|u_t^{(n)}(t)\|_\infty^2 \le C\|u_t^{(n)}(t)\|_{H^k}^2$, so it follows

$$\begin{aligned}
&\|m(u^{(n)}(s-r))u_r^{(n)}(r)\|_{H^k} \le C\|u_r^{(n)}(r)\|_{L^2} \\
&+ \sum_{1 \le |\alpha| \le k} C\left(\|\nabla^{|\alpha|}u_r^{(n)}(r)\|_{L^2} + \|u_r^{(n)}(r)\|_\infty \|\nabla^m m(u^{(n)}(s-r))\|_{L^2}\right) \\
&\qquad\qquad \le C\left(\|u_r^{(n)}(r)\|_{H^k} + \|u_r^{(n)}(r)\|_{H^k}\|u^{(n)}(s-r)\|_{H^k}^k\right).
\end{aligned}$$

Inserting this into (12), we get

$$\begin{aligned}
&\lambda\|u_t^{(n)}(t)\|_{H^k}^2 + \|u^{(n)}(t)\|_{H^k}^2 + \|\nabla u^{(n)}(t)\|_{H^k}^2 \\
&\quad \le \lambda\|u_1\|_{H^k}^2 + \|u_0\|_{H^k}^2 + \|\nabla u_0\|_{H^k}^2 \\
&\qquad + \int_0^t \|f(s)\|_{H^k}^2 + (t-1)\int_0^t \|u_s^{(n)}(s)\|_{H^k}^2\,ds \\
&\qquad + C\int_0^t\int_0^s \|u_r^{(n)}(r)\|_{H^k}^2 + \|u_r^{(n)}(r)\|_{H^k}^2\|u^{(n)}(s-r)\|_{H^k}^{2k}\,dr\,ds \\
&\quad \le \lambda\|u_1\|_{H^k}^2 + \|u_0\|_{H^k}^2 + \|\nabla u_0\|_{H^k}^2 + \|f\|_{L^2((0,t),H^k)}^2 \\
&\qquad + C\int_0^t \left(1 + \|u_s^{(n)}(s)\|_{H^k}^2 + \|u^{(n)}(s)\|_{H^k}^2\right)^{2k} ds.
\end{aligned}$$

This is an inequality of the kind

$$x(t) \le x_0 + C\int_0^t (1 + x(s))^{2k}\,ds$$

with $x_0 = \lambda\|u_1\|^2_{H^k} + \|u_0{}^2_{H^{k+1}} + \|f\|^2_{L^2((0,T'),H^k)}$ ($T' > 0$ arbitrary), so a nonlinear generalization of Gronwall's inequality (see Appendix) tells us that $x$ is bounded by the solution of

$$\dot{z}(t) = C(z(t)+1)^{2k},$$
$$z(0) = x_0.$$

We easily get, with $K := C(x_0+1)^{2k-1}(2k-1)$,

$$z(t) = -1 + (x_0+1)\left(1 - Kt\right)^{1/1-2k},$$

and $z$ exists in the interval $[0,(1/K))$, so we have for any $T < T'$ with $0 < T < 1/K$ and $C_1 := z(T)$,

$$\|u_t^{(n)}(t)\|^2_{H^k} + \|u^{(n)}(t)\|^2_{H^{k+1}} \le C_1$$

for $t \in [0,T]$. Using $\langle\varphi_i,\varphi_j\rangle_{H^k} = \delta_{ij}$, we get $g_{nj}(t) = \langle g_{nj}(t)\varphi_j,\varphi_j\rangle_{H^k} = \langle u_n(t),\varphi_j\rangle_{H^k}$, and thus we have a bound for $g_{nj}(t)$ in the interval $[0,T]$:

$$|g_{nj}(t)| \le \|u_n(t)\|_{H^k}\|\varphi_j\|_{H^k} \le C_1.$$

With Theorem 9, we can now extend $g_n$ to a solution of (11) in $[0,T]$. Therefore, $(u^{(n)})_n$ is bounded in $C^1([0,T],H^k) \cap C^0([0,T],H^{k+1})$, which carries over to a bound in $H^1((0,T),H^k) \cap L^2((0,T),H^{k+1})$, and we can select a subsequence (without renaming it), which converges weakly to some $u$, $u_n \rightharpoonup u$ in $H^1((0,T),H^k) \cap L^2((0,T),H^{k+1})$. Furthermore, we will conclude the strong convergence in $C^0([0,T],H^{k+1}) \cap C^1([0,T],H^k)$. Without loss of generality, we assume $n \le m$. For $w^{(nm)} := u^{(n)} - u^{(m)}$, we have $w^{(nm)}(0) = P_n u_0 - P_m u_0$, $w^{(nm)}(0) = \Pi_n u_1 - \Pi_m u_1$ and

$$\begin{aligned}
&\langle \lambda w_{tt}^{(nm)}(t), w_t^{(nm)}(t)\rangle_{H^k} + \langle w_t^{(nm)}(t), w_t^{(nm)}(t)\rangle_{H^k}\\
&+ \langle w^{(nm)}(t), w_t^{(nm)}(t)\rangle_{H^k}\\
&+ \left\langle \int_0^t m(u^{(n)}(t-s))u_s^{(n)}(s) - m(u^{(m)}(t-s))u_s^{(m)}(s)\,ds, w_t^{(nm)}(t)\right\rangle_{H^k}\\
&\qquad - \langle \Delta w^{(nm)}(t), w_t^{(nm)}(t)\rangle_{H^k} = \left\langle f(t), \sum_{i=n+1}^{m} \dot{g}_{im}(t)\varphi_i\right\rangle_{H^k}.
\end{aligned}$$

$\langle \varphi_i, \varphi_j \rangle = \delta_{ij}$ allows the rewriting of the right hand side:

$$\left\langle f, \sum_{i=n+1}^{m} \dot{g}_{im}(t)\varphi_i \right\rangle_{H^k} = \left\langle \sum_{j=n+1}^{m} \langle f(t), \varphi_j \rangle \varphi_j, w_t^{(nm)}(t) \right\rangle_{H^k}$$
$$=: \langle f_{nm}(t), w_t^{(nm)}(t) \rangle_{H^k}.$$

Using the boundedness of the sequence we can find some $C = C(T) > 0$ such that, for $t \in [0, T]$,

$$\|w_t^{(nm)}(t)\|_{H^k}^2 + \|w^{(nm)}(t)\|_{H^{k+1}}^2$$
$$\leq C\Big(\|w_t^{(nm)}(0)\|_{H^k}^2 + \|w^{(nm)}(0)\|_{H^{k+1}}^2 + \int_0^T \|f_{nm}(s)\|_{H^k}^2 \, ds\Big)$$

holds. $(\varphi_n)_n$ is a basis in $H^k$, so for all $t \in [0, T]$, it follows that $\|f_{nm}(t)\|_{H^k} \to 0$, and we can deduce $\int_0^T \|f_{nm}(s)\|_{H^k}^2 \, ds \to 0$. Additionally, the projections give $P_n u_0 \to u_0$ in $H^{k+1}$ and $\Pi_n u_1 \to u_1$ in $H^k$ (but not $u(0) = u_0$ yet), so we obtain

$$\sup_{t \in [0, T_0]} \lambda \|w_t^{(nm)}(t)\|_{H^k}^2 + \|w^{(nm)}(t)\|_{H^{k+1}}^2 \longrightarrow 0.$$

Now we have $u \in C^0([0, T], H^{k+1}) \cap C^1([0, T], H^k)$, and we can verify that $u$ is a solution to the problem.

Let $h \in C_0^\infty([0, T], H^{k+1})$ with $h(t) = \sum_{i=1}^{l} h_i(t)\varphi_i$, $l \in \mathbf{N}$, $h_i \in C_0^\infty([0, T_0], \mathbf{R})$, and let $n \geq l$. These functions are dense in $C_0^\infty([0, T], H^{k+1})$, and it follows

$$\lambda \int_0^t \langle u_{ss}^{(n)}(s), h(s) \rangle_{H^k} ds + \int_0^t \langle u_s^{(n)}(s), h(s) \rangle_{H^k} ds$$
$$+ \int_0^t \langle u^{(n)}(s), h(s) \rangle_{H^k} ds - \int_0^t \langle \Delta u^{(n)}(s), h(s) \rangle_{H^k} ds$$
$$+ \int_0^t \left\langle \int_0^s m(u^{(n)}(s - r)) u_r^{(n)}(r) \, dr, h(s) \right\rangle_{H^k} ds$$
$$= \int_0^t \langle f(s), h(s) \rangle_{H^k} ds.$$

Integration by parts leads to

$$\begin{aligned}
&-\lambda\int_0^t \langle u_s^{(n)}(s), h_s(s)\rangle_{H^k}\,ds + \int_0^t \langle u_s^{(n)}(s), h(s)\rangle_{H^k} ds\\
&\quad+\int_0^t \langle u^{(n)}(s), h(s)\rangle_{H^k} ds + \int_0^t \langle \nabla u^{(n)}(s), \nabla h(s)\rangle_{H^k} ds\\
&\quad+\int_0^t \langle (m(u^{(n)}) * u_s^{(n)})(s), h(s)\rangle_{H^k}\,ds = \int_0^t \langle f(s), h(s)\rangle_{H^k}\,ds.
\end{aligned}$$

Applying Lemma 12 gives us that $u_{tt}$ exists in $L^2((0,T),(H^{k+1})')$ ($(H^{k+1})'$ denotes the dual-space of $H^{k+1}$ with respect to the $H^k$-norm) and that $u$ fulfills the integro-differential equation in the weak sense.

On the one hand, we have $P_n u_0 = u^{(n)}(0) \rightharpoonup u(0)$ in $H^k$; on the other hand, the continuity of $P_n$ inherits $P_n u_0 \to u_0$ in $H^{k+1}$, and thus $u_0 = u(0)$. The same arguments ($\Pi_n u_1 \to u_1$ in $H^k$, $\Pi_n u_1 = u_t^{(n)}(0) \rightharpoonup u_t(0)$ in $H^{k-1}$) lead to $u_1 = u_t(0)$.

The integro-differential equation yields

$$\lambda u_{tt}(t) = -u_t(t) - u(t) + \Delta u(t) - \int_0^t m(u(t-s))u_s(s)\,ds + f(t).$$

The right hand side is an element of $C^0([0,T],H^{k-1})$, and thus we have $u \in C^2([0,T],H^{k-1})$. If $m$ is not bounded, we take constants $\delta_2 > \delta_1 > 0$ and define $\widetilde{m}$ by

$$\widetilde{m}(x) := \begin{cases} 0 & |x| \geq \|u_0\|_{H^k} + \delta_2 \\ m(x) & |x| \leq \|u_0\|_{H^k} + \delta_1 \end{cases}$$

and extended into the area $\{x \in \mathbf{R} | \delta_1 < |x| < \delta_2\}$ as a $C^{k+1}(\mathbf{R},\mathbf{R})$-function. To this kernel there is a solution $u \in C^2([0,T_0],H^{k-1}) \cap C^1([0,T_0],H^k) \cap C^0([0,T_0],H^{k+1})$ for some $T_0 > 0$ to (10), and the regularity implies $u \in C^0([0,T_0],C_b^0(\mathbf{R}^d,\mathbf{R}))$. This means that there is some $T$, $0 < T \leq T_0$, with $\|u(t)\|_\infty < \|u_0\|_{H^k} + \delta_1$ for $t \in [0,T]$. In this interval we have the identity $m(u(t)) = \widetilde{m}(u(t))$; hence, $u$ is a solution of the original problem. □

The proofs above can be directly carried over to a convolution with $\partial_j u$ or $u$, but if we look at a convolution with $\Delta u$, some estimates must be modified. In the proof of uniqueness we need to deal with the term

$$\left\langle \int_0^t m(u(t-s))\Delta u(s) - m(v(t-s))\Delta v(s)\, ds, w_t(t) \right\rangle_{L^2} dt,$$

where $u$ and $v$ are solutions to the same data and $w := u-v$. Integration by parts leads to

$$\begin{aligned}
&\int_0^{T'} \left\langle \int_0^t m(u(t-s))\Delta u(s) - m(v(t-s))\Delta v(s)\, ds, w_t(t) \right\rangle_{L^2} dt \\
&= -\int_0^{T'} \left\langle \int_0^t \nabla m(u(t-s))\nabla u(s) - \nabla m(v(t-s))\nabla v(s)\, ds, w_t(t) \right\rangle_{L^2} dt \\
&\quad - \int_0^{T'} \left\langle \int_0^t m(u(t-s))\nabla u(s) - m(v(t-s))\nabla v(s)\, ds, \nabla w_t(t) \right\rangle_{L^2} dt \\
&=: -\mathrm{I1} - \mathrm{I2}.
\end{aligned}$$

$I1$ can be transformed into

$$\begin{aligned}
\mathrm{I1} &= \int_0^{T'} \int_0^t \langle \nabla m(u(t-s))\nabla u(s) - \nabla m(v(t-s))\nabla v(s), w_t(t)\rangle_{L^2}\, ds\, dt \\
&= \int_0^{T'} \int_0^t \langle m'(u(t-s))\nabla u(t-s)\nabla u(s) - m'(v(t-s)) \\
&\qquad\qquad \times \nabla v(t-s)\nabla v(s), w_t(t)\rangle_{L^2}\, ds\, dt \\
&= \int_0^{T'} \int_0^t \langle m'(u(t-s))\nabla u(t-s)\nabla w(s), w_t(t)\rangle_{L^2}\, ds\, dt \\
&\quad + \int_0^{T'} \int_0^t \langle m'(u(t-s))\nabla v(s)\nabla w(t-s), w_t(t)\rangle_{L^2}\, ds\, dt \\
&\quad + \int_0^{T'} \int_0^t \Big\langle \nabla v(t-s)\nabla v(s) w(t-s) \\
&\qquad\qquad \times \int_0^1 m''(ru(t-s) + (1-r)v(t-s))\, dr, w_t(t) \Big\rangle_{L^2} ds\, dt.
\end{aligned}$$

Here we can use the same technique as before to conclude

$$|I1| \le c \int_0^{T'} \|\nabla w(t)\|_{L^2} + \|w(t)\|_{L^2} + \|w_t(t)\|_{L^2}\, dt.$$

A second integration by parts, now with respect to the time-variable, gives for $I2$:

$$\begin{aligned}
I2 &= \left\langle \int_0^{T'} m(u(T'-s))\nabla u(s) - m(v(T'-s))\nabla v(s)\,ds, \nabla w(T')\right\rangle_{L^2} \\
&\quad - \int_0^{T'} \left\langle \frac{\mathrm{d}}{\mathrm{dt}} \int_0^t m(u(t-s))\nabla u(s) - m(v(t-s))\nabla v(s)\,ds, \nabla w(t)\right\rangle_{L^2} dt \\
&= \left\langle \int_0^{T'} m(u(T'-s))\nabla u(s) - m(v(T'-s))\nabla v(s)\,ds, \nabla w(T')\right\rangle_{L^2} \\
&\quad - \int_0^{T'} \Bigg\langle \int_0^t m'(u(t-s))u_s(t-s)\nabla u(s) \\
&\qquad\qquad - m'(v(t-s))v_s(t-s)\nabla v(s)\,ds, \nabla w(t)\Bigg\rangle_{L^2} dt \\
&\quad - \int_0^{T'} \langle m(u(0))\nabla u(t) - m(v(0))\nabla v(t), \nabla w(t)\rangle_{L^2}\,dt \\
&=: I2.1 + I2.2 + I2.3.
\end{aligned}$$

I2.2 can be processed as I1 to get

$$|I2.2| \leq c\int_0^{T'} \|w(t)\|_{H^1} + \|w_t(t)\|_{L^2}\,dt,$$

and, because of $u(0) = v(0)$, we have

$$|I2.3| \leq c\int_0^{T'} \|\nabla w(t)\|_{L^2}^2\,dt.$$

I2.1 is bounded by

$$\begin{aligned}
|I2.1| &\leq \left\| \int_0^{T'} m(u(T'-s))\nabla u(s) - m(v(T'-s))\nabla v(s)\,ds \right\|_{L^2} \|\nabla w(T')\| \\
&\leq \frac{1}{4}\|\nabla w(T')\|_{L^2}^2 + \left\| \int_0^{T'} m(u(T'-s))\nabla u(s) - m(v(T'-s))\nabla v(s) ds \right\|_{L^2}^2 \\
&\leq cT\int_0^{T'} \|w(t)\|_{H^1}^2\,dt + \frac{1}{4}\|\nabla w(T')\|_{L^2}^2.
\end{aligned}$$

From here, the previous proof can be carried out. The idea to do two integration-by-parts to get integrals which only inhabit the function and its first derivatives can also be used to obtain the energy estimate needed for the Faedo-Galerkin method, if the kernel $m$ is bounded. As before, we have

$$\begin{aligned}
&\lambda\|u_t^{(n)}(t)\|_{H^k}^2 + \|u^{(n)}(t)\|_{H^k}^2 + \|\nabla u^{(n)}(t)\|_{H^k}^2 \\
&\quad = \lambda\|u_1\|_{H^k}^2 + \|u_0\|_{H^k}^2 + \|\nabla u_0\|_{H^k}^2 \\
&\qquad - 2\int_0^t \|u_s^{(n)}(s)\|_{H^k}^2\,ds + 2\int_0^t \langle f(s), u_s^{(n)}(s)\rangle_{H^k}\,ds \\
&\qquad - \sum_{|\alpha|\le k} 2\int_0^t \left\langle \nabla^\alpha \int_0^s m(u^{(n)}(s-r))\Delta u^{(n)}(r)\,dr, \nabla^\alpha u_s^{(n)}(s)\right\rangle_{L^2} ds.
\end{aligned}$$

Now we carry out the first integration by parts and get

$$\begin{aligned}
&\int_0^t \left\langle \int_0^s \nabla^\alpha(m(u^{(n)}(s-r))\Delta u^{(n)}(r))\,dr, \nabla^\alpha u_s^{(n)}(s)\right\rangle_{L^2} ds \\
&\quad = -\int_0^t \left\langle \int_0^s \nabla^\alpha(\nabla m(u^{(n)}(s-r))\nabla u^{(n)}(r))\,dr, \nabla^\alpha u_s^{(n)}(s)\right\rangle_{L^2} ds \\
&\qquad - \int_0^t \left\langle \int_0^s \nabla^\alpha(m(u^{(n)}(s-r))\nabla u^{(n)}(r))\,dr, \nabla^\alpha \nabla u_s^{(n)}(s)\right\rangle_{L^2} ds.
\end{aligned}$$

Another integration by parts yields, for the second integral,

$$\begin{aligned}
&\int_0^t \left\langle \int_0^s \nabla^\alpha(m(u^{(n)}(s-r))\nabla u^{(n)}(r))\,dr, \nabla^\alpha \nabla u_s^{(n)}(s)\right\rangle_{L^2} ds \\
&\quad = \left\langle \int_0^t \nabla^\alpha(m(u^{(n)}(t-r))\nabla u^{(n)}(r))\,dr, \nabla^\alpha \nabla u^{(n)}(t)\right\rangle_{L^2} \\
&\qquad\qquad - \int_0^t \langle \nabla^\alpha(m(u^{(n)}(0))\nabla u^{(n)}(s)), \nabla^\alpha\nabla u^{(n)}(s)\rangle_{L^2}\,ds \\
&\qquad + \int_0^t \left\langle \int_0^s \nabla^\alpha(m'(u^{(n)}(s-r))u_s^{(n)}(s-r)\nabla u^{(n)}(r))\,dr,\right. \\
&\qquad\qquad\qquad \left. \nabla^\alpha\nabla u^{(n)}(s)\right\rangle_{L^2} ds \\
&\quad \le \left\| \int_0^t \nabla^\alpha(m(u^{(n)}(t-r))\nabla u^{(n)}(r))\,dr\right\|_{L^2} \|\nabla^\alpha\nabla u^{(n)}(t)\|_{L^2}
\end{aligned}$$

$$+\frac{1}{2}\int_0^t \|\nabla^\alpha(m(u^{(n)}(0))\nabla u^{(n)}(s))\|_{L^2}^2 + \|\nabla^\alpha\nabla u^{(n)}(s)\|_{L^2}^2\,ds$$
$$+\frac{1}{2}\int_0^t \left\|\int_0^s \nabla^\alpha(m'(u^{(n)}(s-r))u_s^{(n)}(s-r)\nabla u^{(n)}(r))\,dr\right\|_{L^2}^2 + \|\nabla^\alpha\nabla u^{(n)}(s)\|_{L^2}^2\,ds.$$

We have

$$\left\|\int_0^t \nabla^\alpha(m(u^{(n)}(t-r))\nabla u^{(n)}(r))\,dr\right\|_{L^2}\|\nabla^\alpha\nabla u^{(n)}(t)\|_{L^2}$$
$$\le t\int_0^t \|\nabla^\alpha(m(u^{(n)}(t-r))\nabla u^{(n)}(r))\|_{L^2}^2\,dr + \frac{1}{4}\|\nabla^\alpha\nabla u^{(n)}(t)\|_{L^2}^2.$$

These inequalities lead to the desired estimate

$$\lambda\|u_t^{(n)}(t)\|_{H^k}^2 + \|u^{(n)}(t)\|_{H^k}^2 + \frac{1}{2}\|\nabla u^{(n)}(t)\|_{H^k}^2$$
$$\le \lambda\|u_1\|_{H^k}^2 + \|u_0\|_{H^k}^2 + \|\nabla u_0\|_{H^k}^2 + \int_0^t \|f(s)\|_{H^k}^2\,ds$$
$$+ c\int_0^t (1 + \|u_t^{(n)}(s)\|_{H^k}^2 + \|u^{(n)}(s)\|_{H^k}^2 + \|\nabla u^{(n)}(s)\|_{H^k}^2)^{2k}\,ds.$$

Hence, the sequence defined within the Faedo-Galerkin method is uniformly bounded in an interval $[0,T]$ and converges weakly to some $u$. As before, we can conclude the strong convergence.

To transfer the previous existence proof we need a similar convergence result for the convolution $u \mapsto m(u) * \Delta u$. It will be weaker than before but still good enough to be used in the Faedo-Galerkin method.

**Lemma 13.** *Let $2k > d$, $T > 0$ arbitrary and $m \in C^{k+2}(\mathbf{R},\mathbf{R})$. Let $(u^{(n)})_n \subset C^0([0,T],H^{k+1}) \cap C^1([0,T],H^k)$, $u \in C^0([0,T],H^{k+1}) \cap C^1([0,T],H^k)$ and $u^{(n)} \to u$ in $C^0([0,T],H^{k+1}) \cap C^1([0,T],H^k)$.*

*$\Rightarrow$. For all $\varphi \in L^2((0,T),H^{k+1})$: $\langle m(u^{(n)}) * \Delta u^{(n)} - m(u^{(n)}) * \Delta u^{(n)}, \varphi\rangle \longrightarrow 0$ $(n \to \infty)$.*

*Proof.* We get

$$
\begin{aligned}
&|\langle m(u^{(n)}) * \Delta u^{(n)} - m(u^{(n)}) * \Delta u^{(n)}, \varphi\rangle| \\
&\leq \left| \int_0^T \left\langle \int_0^t m(u^{(n)}(t-s)) \nabla u^{(n)}(s) - m(u(t-s)) \nabla u(s)\, ds, \nabla\varphi(t) \right\rangle_{H^k} dt \right| \\
&\quad + \left| \int_0^T \left\langle \int_0^t \nabla m(u^{(n)}(t-s)) \nabla u^{(n)}(s) \right.\right. \\
&\qquad\qquad \left.\left. - \nabla m(u(t-s)) \nabla u(s)\, ds, \varphi(t) \right\rangle_{H^k} dt \right|,
\end{aligned}
$$

and thus,

$$
\begin{aligned}
&|\langle m(u^{(n)}) * \Delta u^{(n)} - m(u^{(n)}) * \Delta u^{(n)}, \varphi\rangle| \\
&\quad \leq C \int_0^T (\|u_t(t) - u_t^{(n)}(t)\|_{H^k}^2 + \|u(t) - u^{(n)}(t)\|_{H^{k+1}}^2) \|\varphi(t)\|_{H^k}\, dt,
\end{aligned}
$$

which shows the weak convergence. □

The existence of a local solution for unbounded kernels can be carried over immediately.

If we try to use this Faedo-Galerkin approach for other sets than the whole $\mathbf{R}^d$, we get into trouble because of boundary terms appearing when carrying out an integration by parts. Instead, we first will solve an associated linear problem, in the one-dimensional case and, for a convolution with $u_t$, we then define some sequence which will converge to a solution of the nonlinear problem. We want to find a weak solution to

$$
\begin{aligned}
\lambda u_{tt}(t,x) + u_t(t,x) + u(t,x) - \Delta u(t,x)& \\
+ \int_0^t m(t-s,x) u_s(s,x)\, ds &= f(t,x), \\
u(0,x) &= u_0, \\
u_t(0,x) &= u_1, \\
u(t) &\in H_0^1,
\end{aligned}
\tag{13}
$$

for $x \in \Omega \subset \mathbf{R}^d$ and $t \in [0,T]$, $T > 0$ arbitrary with given data $u_0 \in H_0^1$, $u_1 \in L^2$ and $f \in C^0([0,T], L^2)$. To show existence and uniqueness we follow the Faedo-Galerkin method as presented in [**11**, page 388].

**Theorem 14.** *Let $T > 0$ be arbitrary, $\Omega \subset \mathbf{R}^d$ and $m \in C_b^1([0,T] \times \Omega, \mathbf{R})$. For $u_0 \in H_0^1$, $u_1 \in L^2$ and $f \in C^0([0,T], L^2)$ given, there is a unique weak solution*

$$u \in H^2((0,T),H^{-1}) \cap H^1((0,T),L^2) \cap L^2((0,T),H_0^1) \cap W^{1,\infty}((0,T),L^2)$$

*to* (13).

*Proof.* (*Uniqueness*). For two solutions $u, v$ let $w = u - v$. We have $w(0) = 0 = w_t(0)$,

$$\lambda w_{tt}(t) + w_t(t) + w(t) - \Delta w(t) + \int_0^t m(t-s) w_s(s)\, ds = 0$$

in $L^2((0,T),H^{-1})$ and $w \in H_0^1$. Let $q \in [0,T]$ be arbitrary and $v(t) := \begin{cases} -\int_t^q w(r)\,dr & t<q \\ 0 & t \geq q \end{cases}$. Then it is $v \in L^2((0,T),H^1)$, $v_t = w$ in $[0,q]$ and $v(t) = 0 = v_t(t)$ for $t \geq q$.

$$\begin{aligned} \implies 0 = -\frac{\lambda}{2}\|w(q)\|_{L^2}^2 - \frac{1}{2}\|v(0)\|_{L^2}^2 - \frac{1}{2}\|\nabla v(0)\|_{L^2}^2 \\ - \int_0^q \|w(t)\|_{L^2}^2 + \langle m(0)w(t), v(t)\rangle_{L^2} \\ + \left\langle \int_0^t m'(t-s) w(s)\, ds, v(t) \right\rangle_{L^2} dt. \end{aligned}$$

Let $h(t) = v(t) - v(0) = \int_0^t w(r)\, dr$. This gives $h(q) = v(q) - v(0) = -v(0)$ and, for some $c > 0$, it follows

$$\begin{aligned} &\frac{\lambda}{2}\|w(q)\|_{L^2}^2 + \frac{1}{2}\|h(q)\|_{L^2}^2 + \frac{1}{2}\|\nabla h(q)\|_{L^2}^2 \\ &\leq \int_0^q \left(\frac{1}{2}c - 1\right)\|w(t)\|_{L^2}^2 + 2\|h(t)\|_{L^2}^2 + \frac{1}{2}cT^2\|w(t)\|_{L^2}^2\, dt + 2q\|h(q)\|_{L^2}^2 \\ &\implies \lambda\|w(q)\|_{L^2}^2 + (1-4q)\|h(q)\|_{L^2}^2 + \|\nabla h(q)\|_{L^2}^2 \\ &\leq c\int_0^q \|w(t)\|_{L^2}^2 + \|h(t)\|_{L^2}^2 + \|w(t)\|_{L^2}^2\, dt. \end{aligned}$$

This means $w(q) = h(q) = 0$ for $q \in [0,(1/4))$, and we get

$$0 = \lambda w_{tt}(t) + w_t(t) + w(t) - \Delta w(t) + \int_{1/2}^t m(t-s) w_s(s)\, ds.$$

In the same way, we deduce $w(q) = h(q) = 0$ for $q \in [0, (1/2))$ and, successively, $w(q) = h(q) = 0$ for all $q \in [0, T]$. $\square$

To show the existence, we need the weak continuity of the convolution.

**Lemma 15.** *Let $T > 0$, $\Omega \subset \mathbf{R}^d$ and $m \in C_b^1([0,T] \times \Omega, \mathbf{R})$. Let $(u^{(n)})_n \subset H^1((0,T), L^2)$, $u \in H^1((0,T), L^2)$ with $u^{(n)} \rightharpoonup u$ in $H^1((0,T), L^2)$.*

*$\Longrightarrow$. For all $\varphi \in L^2((0,T), H_0^1) : \langle m * u_t^{(n)}, \varphi \rangle \to \langle m * u_t, \varphi \rangle$.*

*Proof.* We have

$$\begin{aligned}\langle m * u_t^{(n)}, \varphi \rangle &= \int_0^T \left\langle \int_0^t m(t-s) u_s^{(n)}(s)\, ds, \varphi(t) \right\rangle_{L^2} dt \\ &= \int_0^T \int_0^t \langle u_s^{(n)}(s), m(t-s)\varphi(t) \rangle_{L^2}\, ds\, dt.\end{aligned}$$

The weak convergence of $(u^{(n)})_n$ provides that $u^{(n)}(t)$ is almost everywhere weakly convergent, and it is $m(t-s)\varphi(t) \in L^2$ ($m$ bounded). So we get, for $t \in [0,T]$ and for almost every $s \in [0,T]$,

$$\begin{aligned}\langle u_s^{(n)}(s), m(t-s)\varphi(t) \rangle_{L^2} &\longrightarrow \langle u_s(s), m(t-s)\varphi(t) \rangle_{L^2} \\ &\Longrightarrow \langle m * u_t^{(n)}, \varphi \rangle \longrightarrow \langle m * u_t, \varphi \rangle. \qquad \square\end{aligned}$$

*Proof of Theorem* 13. (*Existence*). Let $(\varphi_n)_n \subset H_0^1$ be a basis in $L^2$, $V_n := \operatorname{span}\{\varphi_j : 1 \le j \le n\}$ with the norm $\|\cdot\|_{H^1}$. Let $P_n : H_0^1 \to V_n$ and $\Pi_n : L^2 \to V_n$ be the orthogonal projections. We again construct a sequence of solutions $(u^{(n)})_n$ with $u^{(n)}(t) = \sum_{j=1}^n g_{nj}(t)\varphi_j$ for functions $g_{nj} : [0,T] \to \mathbf{R}$. By defining

$$\begin{aligned}\Phi_n &:= (\langle \varphi_i, \varphi_j \rangle_{L^2})_{1 \le i,j \le n}, \\ \Psi_n &:= (\langle \nabla\varphi_i, \nabla\varphi_j \rangle_{L^2})_{1 \le i,j \le n}, \\ F_n(t) &:= (\langle f(t), \varphi_i \rangle_{L^2})_{1 \le i \le n}\end{aligned}$$

and

$$g_n(t) := (g_{nj}(t))_{1\le j\le n},$$
$$M_n(t-s) := (\langle m(t-s)\varphi_i, \varphi_i\rangle_{L^2})_{1\le i,j\le n},$$

we get

$$\lambda\Phi_n\ddot{g}_n(t) + \Phi_n\dot{g}_n(t) + (\Phi_n+\Psi_n)g_n(t) + \int_0^t M_n(t-s)\dot{g}_n(t) = F_n(t),$$
$$\sum_{j=1}^n g_{nj}(0)\langle\varphi_j,\varphi_i\rangle = \langle P_n u_0, \varphi_i\rangle \quad (1\le i\le n),$$
$$\sum_{j=1}^n \dot{g}_{nj}(0)\langle\varphi_j,\varphi_i\rangle = \langle \Pi_n u_1, \varphi_i\rangle \quad (1\le i\le n).$$

The mapping $M_n : ([0,T], \mathbf{R}^{n\times n})$ is well-defined and continuous since $m$ is bounded and continuous.

For any $n\in\mathbf{N}$ and $T>0$, there is a unique solution $g_n \in C^2([0,T],\mathbf{R}^n)$ to this system, and thus $u^{(n)} = \sum_{j=1}^n g_{nj}\varphi_j \in C^2([0,T], H_0^1)$.

It is easy to show that there is some $c>0$ with

$$\lambda\|u_{(n)}(t)\|_{L^2}^2 + \|u^{(n)}(t)\|_{H^1}^2 \le \Big(\lambda\|u_1\|_{L^2}^2 + \|u_0\|_{H^1}^2 + \int_0^T \|f(s)\|_{L^2}^2\Big)e^{ct},$$

so we have a weakly convergent subsequence to some $u$ in $H^1((0,T),L^2) \cap L^2((0,T),H_0^1)$. That $u_{tt}$ exists in $L^2((0,T),H^{-1})$ and that the integro-differential equation is fulfilled, follows as in [**11**].

The estimate for the sequence $(u^{(n)})_n$ also holds for $u$ and gives $u\in W^{1,\infty}((0,T),L^2)$. □

To use this result when dealing with the nonlinear problem we need a better regularity of the solution.

**Lemma 16.** *Let $T>0$ be arbitrary, $\Omega\subset\mathbf{R}^d$, $k\in\mathbf{N}$ and $m\in C_b^k(\mathbf{R}\times\Omega,\mathbf{R})$. For $u_0\in H_0^{k+1}$, $u_1\in H_0^k$ and $f\in H^k((0,T),H_0^1)\subset C^0([0,T],H_0^1)$, the solution $u$ to* (13) *fulfills*

$$u\in H^{k+2}((0,T),H^{-1})\cap H^{k+1}((0,T),L^2)\cap H^k((0,T),H_0^1).$$

*Proof.* For a solution $u \in H^2((0,T),H^{-1}) \cap H^1((0,T),L^2) \cap L^2((0,T),H^1)$ to (13) and $n \in \mathbf{N}$, let $v^{(n)}$ be the solution to

$$\begin{aligned} v_{tt}^{(n)}(t) + v_t^{(n)}(t) + v^{(n)}(t) - \Delta v^{(n)}(t) + m(0)v^{(n)}(t) \\ = -\int_0^t \frac{\mathrm{d}^n m}{\mathrm{d}t^n}(t-s)u_s(s)\,ds - \sum_{j=1}^{n-1} \frac{\mathrm{d}^j m}{\mathrm{d}t^j}(0)v^{(n-j)}(t) \\ + \frac{\mathrm{d}^n}{\mathrm{d}t^n} f(t) =: f^n(t) \\ v^{(n)}(0) = v_t^{(n-1)}(0) \\ v_t^{(n)}(0) = -v_t^{(n-1)}(0) - v^{(n-1)}(0) + \Delta v^{(n-1)}(0) \\ - \sum_{j=0}^{n-2} \frac{\mathrm{d}^j m}{\mathrm{d}t^j}(0)v^{(n-1-j)}(0) + \frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}} f(0) \\ v^{(n)}(t) \in H_0^1 \end{aligned}$$

with $v^{(0)} = u$; thus, $v^{(0)}(0) = u_0$, $v_t^{(0)}(0) = u_1$. We have $f^n \in L^2((0,T),L^2)$ and, following [**11**, page 389], there is a unique solution

$$v^{(n)} \in H^2((0,T),H^{-1}) \cap H^1((0,T),L^2) \cap L^2((0,T),H^1).$$

By induction, we get $v^{(n)} = v_t^{(n-1)}$ and thus the higher regularity of $u$. □

Using elliptic regularity, we can deduce a better regularity in the space variable.

**Lemma 17.** *Let $T > 0$ be arbitrary, $\Omega \subset \mathbf{R}^d$ bounded with $C^{k+2}$-boundary, $k \in \mathbf{N}$ and $m \in C_b^k([0,T] \times \Omega, \mathbf{R})$. For $u_0 \in H_0^{k+1}$, $u_1 \in H_0^k$ and $f \in H^k((0,T),H_0^1) \cap L^2((0,T),H^k)$ the solution $u$ to* (13) *fulfills*

$$u \in H^2((0,T),H^{k-1}) \cap H^1((0,T),H^k) \cap L^2((0,T),H^{k+1}).$$

*We especially have, for a $C^\infty$-boundary, $m \in C_b^\infty([0,T] \times \Omega, \mathbf{R})$, $u_0 \in C_0^\infty(\Omega)$, $u_1 \in C_0^\infty(\Omega)$ and $f \in C^\infty([0,T] \times \overline{\Omega})$ and therefore $u \in C^\infty([0,T] \times \overline{\Omega}, \mathbf{R})$.*

We now can solve the nonlinear equation for the special case $\Omega = (a,b) \subset \mathbf{R}$ being an interval.

**Theorem 18.** *Let $\Omega = (a,b) \subset \mathbf{R}$ and $m \in C^2(\mathbf{R},\mathbf{R})$. Then, for any given $u_0 \in H_0^2(\Omega)$, $u_1 \in H_0^1$ and $f \in C^0([0,\infty), H_0^1)$, there exists a $T > 0$ and a unique solution*

$$u \in C^2([0,T], L^2) \cap C^1([0,T], H^1) \cap C^0([0,T], H^2 \cap H_0^1)$$

*to*

$$\begin{gathered}\lambda u_{tt}(t) + u_t(t) + u(t) - \Delta u(t) + \int_0^t m(u(t-s))u_s(s)\,ds = f(t),\\ u(0) = u_0,\\ u_t(0) = u_1,\\ u(t) \in H_0^1.\end{gathered}$$

*Proof.* Uniqueness. For two solutions $u, v$ let $w = u - v$. It is $w \in C^2([0,T], L^2) \cap C^1([0,T], H^1) \cap C^0([0,T], H^2 \cap H_0^1)$, $w(0) = 0 = w_t(0)$, $w(t) \in H_0^1$ and

$$\begin{aligned}&\lambda w_{tt}(t) + w_t(t) + w(t) - \Delta w(t)\\ &\qquad + \int_0^t m(u(t-s))u_s(s) - m(v(t-s))v_s(s)\,ds = 0.\end{aligned}$$

This yields

$$\lambda\|w_t(T')\|_{L^2}^2 + \|w(T')\|_{L^2}^2 + \|\nabla w(T')\|_{L^2}^2 \leq -2\int_0^{T'} \|w_t(t)\|_{L^2}^2\,dt$$
$$+\int_0^{T'} t\int_0^t \|m(u(t-s))u_s(s) - m(v(t-s))v_s(s)\|_{L^2}^2\,ds + \|w_t(t)\|_{L^2}^2\,dt.$$

As before, we can conclude, for some $c > 0$,

$$\|m(u(t-s))u_s(s) - m(v(t-s))v_s(s)\|_{L^2} \leq c(\|w(t-s)\|_{L^2} + \|w_s(t-s)\|_{L^2}),$$

and it follows that

$$\lambda\|w_t(T')\|_{L^2}^2 + \|w(T')\|_{H^1}^2 \le -\int_0^{T'} \|w_t(t)\|_{L^2}^2\,dt \\ + cT^2\int_0^{T'} \|w(t)\|_{L^2}^2 + \|w_t(t)\|_{L^2}^2\,dt.$$

This implies $w = 0$.

Existence. We first assume that $m \in C_b^2(\mathbf{R},\mathbf{R})$ and approximate the data and functions $f$ and $m$ by $C^\infty$-functions. Let $T_0 > 0$ be arbitrary. There are sequences $(u_0^{(n)})_n, (u_1^{(n)})_n \subset C_0^\infty(\Omega)$, $(f^{(n)})_n \in C^\infty([0,T_0], C_0^\infty(\Omega))$ with $u_0^{(n)} \to u_0$ in $H_0^2$, $u_1^{(n)} \to u_1$ in $H_0^1$, $f^{(n)} \to f$ in $C^0([0,T_0], H_0^1)$, such that

$$\|u_0^{(n)}\|_{H^2} \le \|u_0\|_{H^2}, \\ \|u_1^{(n)}\|_{H^1} \le \|u_1\|_{H^1}$$

and

$$\|f^{(n)}\|_{L^2((0,T_0),H_0^1)} \le \|f\|_{L^2((0,T_0),H_0^1)}$$

holds. Let $m^{(n)} \in C_b^\infty(\mathbf{R},\mathbf{R})$ with $m^{(n)} \to m$ in $C^1(\mathbf{R},\mathbf{R})$ and

$$\|m^{(n)} - m^{(n-1)}\|_\infty \le \frac{1}{2^n} \quad \text{as well as} \quad \left\|\frac{\mathrm{d}}{\mathrm{dz}}(m^{(n)} - m^{(n-1)})\right\|_\infty \le \frac{1}{2^n}$$

for $n > 1$. We choose $u^{(0)}(t,x) := 0$ and, for $n \ge 1$, let $u^{(n)} \in C^\infty([0,T_0]\times\overline{\Omega},\mathbf{R})$ be the solution to $\lambda u_{tt}^{(n)}(t,x) + u_t^{(n)}(t,x) + u^{(n)}(t,x) - \Delta u^{(n)}(t,x) + \int_0^t m^{(n)}(u^{(n-1)}(t-s,x))u_s^{(n)}(s,x)\,ds = f^{(n)}(t,x)$,

$$u^{(n)}(0,x) = u_0^{(n)}(x), \\ u^{(n)}(0,x) = u_1^{(n)}(x), \\ u^{(n)}(t,a) = 0 = u^{(n)}(t,b).$$

Our previous results show that this is a well-defined sequence (we get $m(u^{(n-1)}) \in C_b^\infty([0,T_0]\times\Omega,\mathbf{R})$ by induction).

We write $\nabla = \partial_x$ and get, for all $\alpha \in \mathbf{N}_0$,

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(\lambda\|\partial_x^\alpha u_t^{(n)}(t)\|_{L^2}^2 + \|\partial_x^\alpha u^{(n)}(t)\|_{L^2}^2\right)&\\ - \langle \partial_x^{\alpha+2} u^{(n)}(t), \partial_x^\alpha u_t^{(n)}(t)\rangle_{L^2}&\\ = -\|\partial_x^\alpha u_t^{(n)}(t)\|_{L^2}^2&\\ - \left\langle \partial_x^\alpha \int_0^t m^{(n)}(u^{(n-1)}(t-s))u_s^{(n)}(s)\,ds, \partial_x^\alpha u_t^{(n)}(t)\right\rangle_{L^2}&\\ + \langle \partial_x^\alpha f^{(n)}(t), \partial_x^\alpha u_t^{(n)}(t)\rangle_{L^2}.&\end{aligned}$$

For all $t$, we have $u^{(n)}(t)$, $u_t^{(n)}(t)$, $u_{tt}^{(n)}(t)$, $f^{(n)}(t) \in H_0^1$, and thus $\partial_x u^{(n)}(t,a) = 0 = \partial_x^2 u^{(n)}(t,b)$. It follows that

$$\begin{aligned}&\langle \partial_x^{\alpha+2} u^{(n)}(t)\partial_x^\alpha u_t^{(n)}(t)\rangle_{L^2}\\ &= -\langle \partial_x^{\alpha+1} u^{(n)}(t), \partial_x^{\alpha+1} u_t^{(n)}(t)\rangle_{L^2}\\ &\Longrightarrow \frac{1}{2}\frac{\mathrm{d}}{\mathrm{dt}}\left(\lambda\|\partial_x^\alpha u_t^{(n)}(t)\|_{L^2}^2 + \|\partial_x^\alpha u^{(n)}(t)\|_{L^2}^2 + \|\partial_x^{\alpha+1} u^{(n)}(t)\|_{L^2}^2\right)\\ &= -\|\partial_x^\alpha u_t^{(n)}(t)\|_{L^2}^2 - \left\langle \partial_x^\alpha \int_0^t m^{(n)}(u^{(n-1)}(t-s))u_s^{(n)}(s)\,ds, \partial_x^\alpha u_t^{(n)}(t)\right\rangle_{L^2}\\ &\quad + \langle \partial_x^\alpha f^{(n)}(t), \partial_x^\alpha u_t^{(n)}(t)\rangle_{L^2}.\end{aligned}$$

From here on, $c$ denotes constants independent of $n$ and $t$. We have

$$\begin{aligned}&\|\partial_x(m^{(n)}(u^{(n-1)}(t-s))u_s^{(n)}(s))\|_{L^2}\\ &\qquad \le c\left(\|\partial_x u_s^{(n)}(s)\|_{L^2} + \|u_s^{(n)}(s)\|_\infty \|\partial_x u^{(n-1)}(t-s)\|_{L^2}\right).\end{aligned}$$

Obviously, $\|m^{(n)}(u^{(n-1)}(t-s))u_s^{(n)}(s)\|_{L^2} \le c\|u_s^{(n)}(s)\|_{L^2}$ holds. Sobolev's embedding theorem yields

$$\|u_s^{(n)}(s)\|_\infty \le c\|u_s^{(n)}(s)\|_{H^1}$$

and

$$\begin{aligned}&\|u^{(n)}(t)\|_\infty \le c\|u^{(n)}(t)\|_{H^1}\\ &\qquad \Longrightarrow \lambda\|\partial_x^\alpha u_t^{(n)}(t)\|_{L^2}^2 + \|\partial_x^\alpha u^{(n)}(t)\|_{L^2}^2\end{aligned}$$

$$
\begin{aligned}
&+ \|\partial_x^{\alpha+1} u^{(n)}(t)\|_{L^2}^2 \\
&\leq \lambda \|\partial_x^\alpha u_1^{(n)}\|_{L^2}^2 + \|\partial_x^\alpha u_0^{(n)}\|_{L^2}^2 + \|\partial_x^{\alpha+1} u_0^{(n)}\|_{L^2}^2 \\
&\quad - \int_0^t \|\partial_x^\alpha u_s^{(n)}(s)\|_{L^2}^2 + \|\partial_x^\alpha f^{(n)}(s)\|_{L^2}^2 \, ds \\
&\quad + 2c \int_0^t \int_0^s \|\partial_x^\alpha u_r^{(n)}(r)\|_{L^2} + \|u_r^{(n)}(r)\|_{H^1} \\
&\quad \times \|u^{(n-1)}(s-r)\|_{H^1} \, dr \|\partial_x^\alpha u_s^{(n)}(s)\|_{L^2} \, ds.
\end{aligned}
$$

By setting

$$\delta^2 := \lambda \|u_1\|_{H^1}^2 + \|\partial_x u_0\|_{H^1}^2 + \|u_0\|_{H^1}^2 + \|f\|_{L^2((0,T_0),H^1)}$$

and

$$T' := \min\{T_0, (c(1 + (1+\delta)^2))^{-1}\},$$

we have $\|u^{(0)}(t)\|_{H^1}^2 \leq \delta^2$. Assuming $\|u^{(n-1)}(t)\|_{H^1}^2 \leq \delta^2$ and summing up the two cases $\alpha = 0$ and $\alpha = 1$ gives

$$
\begin{aligned}
\lambda \|u_t^{(n)}(t)\|_{H^1}^2 + \|u^{(n)}(t)\|_{H^2}^2 &\leq \delta^2 - \int_0^t \|u_s^{(n)}(s)\|_{H^1}^2 \, ds \\
&\quad + c \int_0^t \int_0^s (1+\delta)^2 \|u_r^{(n)}(r)\|_{H^1}^2 + \|u_s^{(n)}(s)\|_{H^1}^2 \, dr \, ds \\
&\leq \delta^2 + (ct(1 + (1+\delta)^2) - 1) \int_0^t \|u_s^{(n)}(s)\|_{H^1}^2 \\
&\leq \delta^2,
\end{aligned}
$$

so the sequence $(u^{(n)})_n$ is bounded in $C^1([0,T'], H^1) \cap C^0([0,T'], H^2)$.

Let $n \in \mathbf{N}$ and $w := u^{(n)} - u^{(n-1)}$.

$$
\begin{aligned}
&\Longrightarrow \lambda \|w_t(t)\|_{H^1}^2 + \|w(t)\|_{H^1}^2 \leq -2 \int_0^t \|w_s(s)\|_{H^1}^2 \, ds \\
&\quad + 2 \int_0^t \Bigg\| \int_0^s m^{(n)}(u^{(n-1)}(s-r)) u_r^{(n)}(r) \\
&\qquad\qquad - m^{(n-1)}(u^{(n-2)}(s-r)) u_r^{(n-1)}(r) \, dr \Bigg\|_{H^1}^2 \, ds.
\end{aligned}
$$

There is a $c > 0$ with $\|m^{(n)}\|_\infty \le c$; thus, we can estimate

$$\begin{aligned}
&\left\| \int_0^s m^{(n)}(u^{(n-1)}(s-r))u_r^{(n)}(r) - m^{(n-1)} \right. \\
&\qquad\qquad \left. (u^{(n-2)}(s-r))u_r^{(n-1)}(r)\,dr \right\|_{H^1} \\
&\le c\int_0^s \|w_r(r)\|_{H^1}\,dr + c\int_0^s \|u^{(n-1)} - u^{(n-2)}\|_{H^1}\,dr \\
&\quad + c\int_0^s \|m^{(n)}(u^{(n-2)}(s-r)) - m^{(n-1)}(u^{(n-2)}(s-r))\|_{H^1}\,dr.
\end{aligned}$$

Let $g^{(n)}(z) := m^{(n)}(z) - m^{(n-1)}(z)$. Then it is $\|g^{(n)}\| \le 1/2^n$, $\|\mathrm{d}g^{(n)}/\mathrm{dz}\|_\infty \le 1/2^n$ for $n > 1$. This yields:

$$\begin{aligned}
&\|m^{(n)}(u^{(n-2)}(s-r)) - m^{(n-1)}(u^{(n-2)}(s-r))\|_{H^1} \\
&\qquad = \|g^{(n)}(u^{(n-1)}(s-r))\|_{H^1} \\
&\qquad \le c\|(u^{(l)}(s-r))\|_{H^1} \max\left\{ \left\|\frac{\mathrm{d}g^{(n)}}{\mathrm{dz}}\right\|_\infty, \|g^{(n)}\|_\infty \right\} \le c\frac{1}{2^n} \\
&\qquad \Longrightarrow \lambda\|w_t(t)\|_{H^1}^2 + \|w(t)\|_{H^1}^2 \le c\int_0^t \|w_s(s)\|_{H^1}^2\,ds \\
&\qquad\quad + c\int_0^t \|u^{(n-1)}(s) - u^{(n-2)}(s)\|_{H^1}^2\,ds + c\frac{1}{2^n}.
\end{aligned}$$

So we get

$$\|w_t(t)\|_{H^1}^2 + \|w(t)\|_{H^1}^2 \le Ct \sup_{s\in[0,t]} \|u^{(n-1)}(s) - u^{(n-2)}(s)\|_{H^1}^2\,ds + C\frac{1}{2^n}$$

for some $C > 0$. Let $T > 0$ such that $q := CT < 1$ and $T < T'$ holds, and let $k \le n$.

$$\begin{aligned}
\Longrightarrow \|u^{(n)}(t) - u^{(k)}(t)\|_{H^1}^2 &+ \|u^{(n)}(t) - u^{(k)}(t)\|_{H^2}^2 \\
&\le \sum_{l=k+1}^{n} \|u^{(l)}(t) - u^{(l-1)}(t)\|_{H^1}^2 \\
&\le \sum_{l=k+1}^{n} (CT_0)^l \|u^{(1)}(t) - u^{(0)}(t)\|_{H^1}^2 + C\frac{1}{2^l} \\
&\le \delta \sum_{l=k+1}^{n} q^l + C\frac{1}{2^l} \longrightarrow 0 \quad (n,k\to\infty).
\end{aligned}$$

Thus, we have the convergence in $C^0([0,T],H^2)\cap C^1([0,T],H^1)$ to some limit $u$ and

$$\left\|\int_0^s m^{(n)}(u^{(n-1)}(s-r))u_r^{(n)}(r)-m(u(s-r))u_r(r)\,dr\right\|_{L^2}\longrightarrow 0\ (n\to\infty),$$

which gives us $u^{(n)}\to u$ in $C^2([0,T],L^2)$ and

$$\lambda u_{tt}(t,x)+u_t(t,x)+u(t,x)-\Delta u(t,x)+\int_0^t m(u(t-s,x))u_s(s,x)\,ds=f(t,x).$$

$u(0)=u_0$ can easily be deduced by

$$u(0)=\lim_{n\to\infty}u^{(n)}(0)=\lim_{n\to\infty}u_0^{(n)}=u_0,$$

analogous for $u_t(0)=u_1$. $H_0^1$ is closed, and we finally have $u(t)\in H_0^1$ for all $t$.

For unbounded kernels we can prove the local existence as in the proof of Theorem 11. □

## APPENDIX

The Moser-type inequalities are proven in [**10**, Lemma 4.8, 4.9] and are stated as follows.

**Lemma 19.** *Let $r\in\mathbf{N}$, $h\in C^r(\mathbf{R},\mathbf{R})$. Then, for any $\Gamma>0$, there is some $c(\Gamma)>0$ such that, for all $u\in H^k(\mathbf{R}^d)\cap L^\infty(\mathbf{R}^d)$ with $\|u\|_{L^\infty(\Omega)}<\Gamma$,*

$$\|\nabla^k h(u)\|_{L^2(\mathbf{R}^d)}\le c(\Gamma)\|\nabla^k u\|_{L^2(\mathbf{R}^d)}\|u\|_\infty^{k-1}$$

*holds. If $h$ is bounded in $C^r(\mathbf{R},\mathbf{R})$, the constant $c$ does not depend on $\Gamma$.*

**Lemma 20.** *Let $k\in\mathbf{N}$. Then there is some $c>0$ such that, for all $f,g\in H^k(\mathbf{R}^d)\cap L^\infty(\mathbf{R}^d)$ and $\alpha\in\mathbf{N}_0^d$, $|\alpha|=k$,*

$$\|\nabla^\alpha(fg)\|_{L^2(\mathbf{R}^d)}\le c\left(\|f\|_\infty\|\nabla^k g\|_{L^2(\mathbf{R}^d)}+\|g\|_\infty\|\nabla^k f\|_{L^2(\mathbf{R}^d)}\right)$$

*holds.*

A nonlinear generalization of Gronwall's inequality is given in [**14**, 1.II].

**Lemma 21.** *Let* $f, h, g \in C^0([0,T], \mathbf{R})$*,* $k \in C^0([0,T]\times[0,T]\times\mathbf{R}, \mathbf{R})$*,* $g, k(t,s,\cdot)$ *nondecreasing with*

$$f(t) \leq g(t) + \int_0^t k(t,s,f(s))\,ds, \qquad h(t) \geq g(t) + \int_0^t k(t,s,h(s))\,ds$$

*for* $t \in [0,T]$*. Then, for* $t \in [0,T]$*,*

$$f(t) \leq h(t)$$

*holds.*

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