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# SUPERCONVERGENT NYSTRÖM AND DEGENERATE KERNEL METHODS FOR INTEGRAL EQUATIONS OF THE SECOND KIND

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ABSTRACT. We propose, in this paper, new methods for approximating the solution of a second kind integral equation with a smooth kernel or kernel having a discontinuity along the diagonal. By using an interpolatory projection at Gaussian points onto the space of (discontinuous) piecewise polynomials of degree $\leq r-1$, we prove that the proposed methods exhibit convergence orders $3r$ and $4r$ for the iterated version. In comparison with Kulkarni's of the same convergence order, we show that our methods are faster and simpler to implement. The theoretical results obtained are illustrated by some numerical examples.

**1. Introduction.** Let us consider the linear integral equation of the second kind

$$u - \mathcal{K}u = f, \tag{1}$$

where $\mathcal{K}$ is the compact linear operator defined on the space $\mathcal{C}[0,1]$ by

$$\mathcal{K}u(s) = \int_0^1 k(s,t)u(t)\,dt, \quad s \in [0,1]$$

with $k(\cdot,\cdot) \in \mathcal{C}([0,1]\times[0,1])$ and $f \in \mathcal{C}[0,1]$.

Assume that the homogenous integral equation $u - \mathcal{K}u = 0$ has, in $C([0,1])$, only the trivial solution; then the operator $(\mathcal{I}-\mathcal{K})$ is invertible and

$$(\mathcal{I}-\mathcal{K})^{-1} = \sum_{n=0}^{\infty} \mathcal{K}^n.$$

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Therefore, equation (1) has a unique solution.

A standard technique to solving (1) approximately is to replace $\mathcal{K}$ by a finite rank operator. The approximate solution is then obtained by solving a system of linear equations. The Galerkin/collocation, Nyström and degenerate kernel methods are commonly used methods for this purpose. They have been extensively studied in the literature (see [**3–5, 13**]). The improvement of the Galerkin solution of a compact operator equation by using iteration techniques was first proposed by Sloan in [**13**]. Similar results for the iterated collocation at Gauss points are given in Chatelin and Lebbar [**6**]. In [**11**], the authors have proposed an interpolation post-processing technique as an alternative to the iteration technique for improving the collocation solution. The same order of convergence as for the iterated collocation method is obtained. More recently, Kulkarni introduced in [**7**] an efficient method, based on projections, that consists of approximating $\mathcal{K}$ by the finite rank operator

$$\pi_n\mathcal{K} + \mathcal{K}\pi_n - \pi_n\mathcal{K}\pi_n$$

where $\pi_n$ is a sequence of projectors onto a space of piecewise polynomials of degree $\leq r-1$. The size of the system of equations that must be solved, in implementing this method, remains the same as for the Galerkin/collocation method, but, for the smooth kernel and right hand side, the order of convergence in the iterated version of the proposed method is twice that of the iterated Galerkin solution and four times that of the Galerkin solution, that is, $4r$. As a direct consequence, this method has been used to solve eigenvalue problems in [**8**]. Moreover, in [**10**], the solution obtained by the proposed method is shown to have an asymptotic series expansion, and by using Richardson extrapolation the order is improved to $4r+2$.

One drawback of Kulkarni's method is that the computation needs some double integrals when generated by the element of the linear system. In order to avoid this drawback we consider, in this paper, the interpolatory projection $\pi_n$ at Gaussian points onto a space of discontinuous piecewise polynomials of degree $\leq r-1$, and we propose to approximate $\mathcal{K}$ by one of the two finite rank operators

$$\mathcal{K}_n := \pi_n\mathcal{K} + \mathcal{K}_{n,i} - \pi_n\mathcal{K}_{n,i}, \quad i = 1, 2,$$
$$\mathcal{K} - \mathcal{K}_n = (\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_{n,i})$$

where $\mathcal{K}_{n,1}$ is the degenerate kernel operator obtained by interpolating the kernel with respect to the second variable and $\mathcal{K}_{n,2}$ is the Nyström operator based upon $\pi_n$. These operators have already been introduced in [**1**] for eigenvalue problems.

In this variant, we have simply replaced $\mathcal{K}\pi_n$ in Kulkarni's operator given above by $\mathcal{K}_{n,i}$, $i = 1, 2$. The common property of these operators is that they are constructed by approximating the function $(s, t) \rightarrow k(s, t)u(t)$ partially or fully by $\pi_n$ with respect to the variable $t$. Thus, as in Kulkarni's method, the extra factors of $(\mathcal{I} - \pi_n)$ in $\mathcal{K} - \mathcal{K}_n$ exhibit superconvergence orders of $3r$ and $4r$ for the iterated version. Moreover, it is shown that the system of equations to be solved, in implementing this method for $\mathcal{K}_{n,2}$, is simpler than that of Kulkarni's method since we have only simple integrals to calculate.

The paper is organized in the following way. In Section 2, the proposed method for the solution of equation (1) is defined along with relevant notations. In Sections 3 and 4 the systems of linear equations which need to be solved to obtain approximations to the solution are discussed. In Section 5, error estimates are given, and precise orders of convergence are obtained there. A discrete version of the proposed method is also defined along with its implementation and the evaluation of computational cost. Numerical validation is given in Section 6. Section 7 is devoted to considering the case when the kernel $k(.,.)$ may be discontinuous on the diagonal. Finally, we give, in Section 8, a numerical example which illustrates the precise orders of convergence obtained in Section 7.

**2. Method and notations.** Consider the following partition of $[0, 1]$

$$0 = x_0 < x_1 < \cdots < x_n = 1. \tag{2}$$

Let $I_i = [x_{i-1}, x_i]$, $h_i = x_i - x_{i-1}$, $i = 1, 2, \ldots, n$, and $h = \max_{1 \le i \le n} h_i$ be the norm of the partition. We assume that $h \mapsto 0$ as $n \mapsto \infty$. For $r \geq 1$, we denote by $\mathcal{P}_r$ the space of all polynomials of degree $\leq r - 1$. Let

$$\mathcal{S}_{r,n} = \{u : [0, 1] \mapsto \mathbf{R} : u|_{I_i} \in \mathcal{P}_r,\ 1 \leq i \leq n\}$$

be the space of piecewise polynomials of degree $\leq r - 1$, with breakpoints at $x_1, x_2, \ldots, x_{n-1}$. We assume no continuity conditions at the breakpoints. Let $B_r = \{\tau_1, \ldots, \tau_r\}$ be the set of $r$ Gaussian points, i.e., the zeros of the Legendre polynomials $p_r(t) = (d^r/dt^r)(t^2-1)^r$ in the interval $[-1,1]$. Define $f_i : [-1,1] \mapsto [x_{i-1}, x_i]$ as follows:

$$f_i(t) = \frac{1-t}{2} x_{i-1} + \frac{1+t}{2} x_i, \quad t \in [-1,1].$$

Then

$$A = \bigcup_{i=1}^{n} f_i(B_r) = \{\tau_{ij} = f_i(\tau_j) : 1 \le i \le n, \ 1 \le j \le r\}$$

is the set of $N_h = nr$ Gaussian points on $[0,1]$. Let

$$\ell_i(x) = \prod_{\substack{k=1 \\ k \neq i}}^{r} \frac{x - \tau_k}{\tau_i - \tau_k}, \quad i = 1,2,\dots,r, \ x \in [-1,1],$$

be the Lagrangian polynomials of degree $r-1$ on $[-1,1]$, which satisfies $\ell_i(\tau_j) = \delta_{ij}$.

Define

$$\phi_{kp}(x) := \begin{cases} \ell_k(f_p^{-1}(x)), & x \in [x_{p-1}, x_p], \\ 0, & \text{otherwise.} \end{cases}$$

Then $\phi_{kp} \in \mathcal{S}_{r,n}$ and $\phi_{kp}(\tau_{ij}) = \delta_{jk}\delta_{ip}$, $i,p = 1,2,\dots,r$, $j,k = 1,2,\dots,n$.

For $j = 1,2,\dots,r$ and $k = 1,2,\dots,n$, we introduce the following notation:

$$t_{(k-1)r+j} = \tau_{jk}, \quad \psi_{(k-1)r+j} = \phi_{jk}.$$

Then $A = \{t_i : i = 1,2,\dots,N_h\}$ is the set of $nr$ Gauss points on $[0,1]$ and $\{\psi_i : i = 1,2,\dots,N_h\}$ is a basis for $\mathcal{S}_{r,n}$.

Let $\pi_n : \mathcal{C}[0,1] \to \mathcal{S}_{r,n}$ be the interpolatory operator defined by

$$\pi_n u(x) := \sum_{i=1}^{N_h} u(t_i)\psi_i(x),$$

satisfying

$$\pi_n u \in \mathcal{S}_{r,n}, \qquad \pi_n u(t_i) = u(t_i), \quad i = 1,2,\dots,N_h.$$

Then $\pi_n u \to u$ as $n \to \infty$ for each $u \in \mathcal{C}[0,1]$. By using a result of [**2**], $\pi_n$ can be extended to $\mathcal{L}^\infty[0,1]$, and then $\pi_n : \mathcal{L}^\infty[0,1] \to \mathcal{S}_{r,n}$ is a projection.

Defining the following degenerate kernel

$$\pi_n k(s,\cdot) = \kappa_n(s,t) = \sum_{i=1}^{N_h} k(s,t_i)\psi_i(t),$$

the associated degenerate kernel operator is given by

$$\mathcal{K}_{n,1}(u)(s) := \int_0^1 k_n(s,t)u(t)\,dt. \tag{3}$$

On the other hand, the Nyström operator based on $\pi_n$ is defined by

$$\mathcal{K}_{n,2}(u)(s) := \sum_{i=1}^{N_h} w_i k(s,t_i)u(t_i), \tag{4}$$

where $w_i := \int_0^1 \psi_i(t)\,dt$, $i = 1,2,\dots,N_h$.

We propose to approximate $\mathcal{K}$ by the following two finite rank operators:

$$\mathcal{K}_n = \pi_n\mathcal{K} + \mathcal{K}_{n,i} - \pi_n\mathcal{K}_{n,i}, \quad i = 1,2.$$

The corresponding approximation of (1) becomes

$$u_{n,i} - (\pi_n\mathcal{K} + \mathcal{K}_{n,i} - \pi_n\mathcal{K}_{n,i})u_{n,i} = f, \tag{5}$$

while the iterated solution is defined by

$$\widetilde{u}_{n,i} = \mathcal{K}u_{n,i} + f. \tag{6}$$

Now, we have the following result.

**Theorem 1.** *For $i = 1,2$,*

$$\|\mathcal{K} - \mathcal{K}_n\|_\infty = \|(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_{n,i})\|_\infty \longrightarrow 0$$
*when $n \to \infty$.*

*Proof.* For $i = 1$, we have

$$
\begin{aligned}
&\|(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_{n,i})u\|_\infty \\
&\qquad = \max_{s\in[0,1]} \left|(\mathcal{I} - \pi_n) \int_0^1 [\kappa_n(\cdot, t) - k(\cdot, t)](s)u(t)\, dt\right| \\
&\qquad \le \|\mathcal{I} - \pi_n\| \max_{s\in[0,1]} \left| \int_0^1 [\kappa_n(\cdot, t) - k(\cdot, t)](s)u(t)\, dt\right| \\
&\qquad \le \|\mathcal{I} - \pi_n\|_\infty \|u\|_\infty \max_{s\in[0,1]} \int_0^1 |[(\mathcal{I} - \pi_n)k(s, \cdot)](t)|\, dt.
\end{aligned}
$$

Then

$$
\begin{aligned}
&\|(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_{n,i})\|_\infty \\
&\qquad \le \|\mathcal{I} - \pi_n\|_\infty \max_{s,t\in[0,1]} |[(\mathcal{I} - \pi_n)k(s, \cdot)](t)| \to 0, \quad n \to \infty.
\end{aligned}
$$

For $i = 2$, the proof is similar. □

In the next section we consider the reduction of (5) to a system of linear equations, and we give some details on the numerical implementation and the computational cost of the proposed method. In Section 5 we prove, under some conditions, that $\widetilde{u}_{n,i}$ converges to $u$ faster than $u_{n,i}$.

## 3. Approximate solution for the operator $\mathcal{K}_{n,1}$.

**Theorem 2.** *Let $a$ and $b$ be vectors with the components*

$$a_j := \mathcal{K}f(t_j) \quad \text{and} \quad b_j := \langle f, \psi_j\rangle,$$

*and let $A, B, C, D, E$ be matrices with the entries*

$$
\begin{gathered}
A_{ij} := \widetilde{\psi}_j(t_i), \qquad B_{ij} := k_j(t_i), \qquad C_{ij} := \langle \psi_i, \psi_j\rangle, \\
D_{ij} := \widetilde{k}_j(t_i), \qquad E_{ij} := \langle k_j, \psi_i\rangle
\end{gathered}
$$

*where $k_j := k(., t_j)$, $\widetilde{k}_j := \mathcal{K}k_j$ and $\widetilde{\psi}_j := \mathcal{K}\psi_j$. The approximate solution is given by*

$$u_{n,1} = f + \sum_{i=1}^{N_h} X_i \psi_i + \sum_{j=1}^{N_h} Y_j k_j,$$

*where* $Z = [X\ Y]^T$ *is the solution of the following linear system of size* $2N_h$:

$$(I - F)Z = c \tag{7}$$

*where*

$$F := \begin{bmatrix} A & D-B \\ C & E \end{bmatrix} \qquad c := \begin{bmatrix} a \\ b \end{bmatrix}.$$

*Proof.* We successively obtain

$$\begin{aligned} \pi_n \mathcal{K} u &= \sum_{i=1}^{N_h} \left( \int_0^1 k(t_i, t) u(t)\, dt \right) \psi_i = \sum_{i=1}^{N_h} W_i \psi_i. \\ \mathcal{K}_{n,1} u &= \sum_{j=1}^{N_h} k(\cdot, t_j) \int_0^1 \psi_j(t) u(t)\, dt = \sum_{j=1}^{N_h} Y_j k(\cdot, t_j). \end{aligned} \tag{8}$$

Then

$$\pi_n \mathcal{K}_{n,1} u = \sum_{i=1}^{N_h} \left( \sum_{j=1}^{N_h} Y_j k(t_i, t_j) \right) \psi_i.$$

Now, by using (5), the approximate solution can be written as

$$u_{n,1} = f + \sum_{i=1}^{N_h} X_i \psi_i + \sum_{j=1}^{N_h} Y_j k_j \tag{9}$$

with $X_i = W_i - \sum_{j=1}^{N_h} Y_j k(t_i, t_j)$. The coefficients $X_i, Y_i$, $i = 1, \ldots, N_h$, are obtained by substituting $u_{n,1}$ from equation (9) into equation (4). Then, we successively have

$$\begin{aligned} \pi_n \mathcal{K} u_{n,1} &= \sum_{i=1}^{N_h} \mathcal{K} u_{n,1}(t_i) \psi_i \\ &= \sum_{i=1}^{N_h} \left( \mathcal{K} f(t_i) + \sum_{k=1}^{N_h} X_k \widetilde{\psi}_k(t_i) + \sum_{\ell=1}^{N_h} Y_\ell \widetilde{k}_\ell(t_i) \right) \psi_i, \\ \mathcal{K}_{n,1} u_{n,1} &= \sum_{j=1}^{N_h} k_j \int_0^1 \psi_j(t) u_{n,1}(t)\, dt \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{N_h} k_j \langle u_{n,1}, \psi_j \rangle \\
&= \sum_{j=1}^{N_h} \Big( \langle f, \psi_j \rangle + \sum_{k=1}^{N_h} X_k \langle \psi_k, \psi_j \rangle + \sum_{\ell=1}^{N_h} Y_\ell \langle k_\ell, \psi_j \rangle \Big) k_j, \\
\pi_n \mathcal{K}_{n,1} u_{n,1} &= \sum_{i=1}^{N_h} \mathcal{K}_{n,1} u_{n,1}(t_i) \psi_i \\
&= \sum_{i=1}^{N_h} \Big( \sum_{j=1}^{N_h} \Big( \langle f, \psi_j \rangle + \sum_{k=1}^{N_h} X_k \langle \psi_k, \psi_j \rangle \\
&\qquad + \sum_{\ell=1}^{N_h} Y_\ell \langle k_\ell, \psi_j \rangle \Big) k_j(t_i) \Big) \psi_i.
\end{aligned}
$$

Unless, in some very specific situations, the family of functions $\{\psi_i, k_j\}$ is linearly independent, we therefore can identify the coefficients of $\psi_i$ and $k_j$, respectively, and obtain

$$
\begin{aligned}
X_i &= \mathcal{K} f(t_i) + \sum_{k=1}^{N_h} X_k \widetilde{\psi}_k(t_i) + \sum_{\ell=1}^{N_h} Y_\ell \widetilde{k}_\ell(t_i) \\
&\quad - \sum_{j=1}^{N_h} \Big( \langle f, \psi_j \rangle + \sum_{k=1}^{N_h} X_k \langle \psi_k, \psi_j \rangle + \sum_{\ell=1}^{N_h} Y_\ell \langle k_\ell, \psi_j \rangle \Big) k_j(t_i), \\
Y_j &= \langle f, \psi_j \rangle + \sum_{k=1}^{N_h} X_k \langle \psi_k, \psi_j \rangle + \sum_{\ell=1}^{N_h} Y_\ell \langle k_\ell, \psi_j \rangle.
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
X &= a + AX + DY - B(b + CX + EY), \\
Y &= b + CX + EY.
\end{aligned}
$$

Replacing $Y$ by its value in the first equation, we get

$$
\begin{aligned}
X &= a + AX + (D - B)Y, \\
Y &= b + CX + EY,
\end{aligned}
$$

which completes the proof. □

*Remark* 1. In practice, the following integrals need to be evaluated numerically:

$$\begin{aligned} a_j &:= \mathcal{K}f(t_j) = \int_0^1 k(t_j, s) f(s)\, ds, \\ b_j &:= \langle f, \psi_i \rangle = \int_0^1 \psi_i(s) f(s)\, ds, \\ A_{ij} &:= \widetilde{\psi}_j(t_i) = \mathcal{K}\psi_j(t_i) = \int_0^1 k(t_i, s)\psi_j(s)\, dt, \\ D_{ij} &:= \widetilde{k}_j(t_i) = \mathcal{K}k_j(t_i) = \int_0^1 k(t_i, s) k(s, t_j)\, ds, \\ E_{ij} &:= \langle k_j, \psi_i \rangle = \int_0^1 k(s, t_j)\psi_i(s)\, dt. \end{aligned}$$

For this purpose we define in subsection 5.2 a discrete version of the proposed method. Note that, since $\psi_i$ and $\psi_j$, $i, j = 1, 2, \dots N_h$, are basis functions having small support in $[0, 1]$ and a simple structure, the integrals

$$\int_0^1 \psi_i(t)\psi_j(t)\, dt, \quad i, j = 1, 2, \dots N_h$$

appearing in matrix $C$ can be evaluated exactly.

**4. Approximate solution for the operator $\mathcal{K}_{n,2}$.**

**Theorem 3.** *Let $a$ and $b$ be vectors with the components*

$$a_j := \mathcal{K}f(t_j), \quad b_j := f(t_j),$$

*and let $A, \overline{B}, \overline{C}$ be matrices with the entries*

$$A_{ij} := \widetilde{\psi}_j(t_i), \qquad \overline{B}_{ij} := w_j k_j(t_i), \qquad \overline{C}_{ij} := w_j \widetilde{k}_j(t_i),$$

*where $k_j := k(., t_j)$, $\widetilde{k}_j := \mathcal{K}k_j$ and $\widetilde{\psi}_j := \mathcal{K}\psi_j$.*

*The approximate solution is given by*

$$u_{n,2} = f + \sum_{i=1}^{N_h} (X_i - b_i)\psi_i + \sum_{i=1}^{N_h} X_i w_i k_i - \sum_{i=1}^{N_h}\sum_{j=1}^{N_h} X_i \overline{B}_{ij}\psi_j, \tag{10}$$

*where $X$ is the solution of the following linear system of size $N_h$:*

$$(I - \overline{C} + A(\overline{B} - I))X = a + (I - A)b. \tag{11}$$

*Proof.* Applying $\pi_n$ and $(\mathcal{I} - \pi_n)$ to equation (5), we obtain

$$\pi_n u_{n,2} - \pi_n \mathcal{K} u_{n,2} = \pi_n f, \tag{12}$$

$$(\mathcal{I} - \pi_n) u_{n,2} - (\mathcal{I} - \pi_n)\mathcal{K}_{n,2} u_{n,2} = (\mathcal{I} - \pi_n) f. \tag{13}$$

By writing

$$\mathcal{K} u_{n,2} = \mathcal{K}(\mathcal{I} - \pi_n) u_{n,2} + \mathcal{K}\pi_n u_{n,2} \tag{14}$$

and replacing $(\mathcal{I} - \pi_n)u_{n,2}$ by its expression from equation (13), equation (14) becomes

$$\mathcal{K} u_{n,2} = \mathcal{K}(\mathcal{I} - \pi_n)\mathcal{K}_{n,2} u_{n,2} + \mathcal{K}\pi_n u_{n,2} + \mathcal{K}(\mathcal{I} - \pi_n) f. \tag{15}$$

On the other hand, since $\mathcal{K}_{n,2} u_{n,2} = \mathcal{K}_{n,2}\pi_n u_{n,2}$, we get

$$\mathcal{K} u_{n,2} = \mathcal{K}(\mathcal{I} - \pi_n)\mathcal{K}_{n,2}\pi_n u_{n,2} + \mathcal{K}\pi_n u_{n,2} + \mathcal{K}(\mathcal{I} - \pi_n) f.$$

Now, by replacing $\mathcal{K} u_{n,2}$ in equation (12), we obtain

$$\pi_n u_{n,2} - (\pi_n \mathcal{K} + \pi_n \mathcal{K}(\mathcal{I} - \pi_n)\mathcal{K}_{n,2})\pi_n u_{n,2} = \pi_n f + \pi_n \mathcal{K}(\mathcal{I} - \pi_n) f,$$

and then for $i = 1, \dots, N_h$, we have

$$\begin{aligned} u_{n,2}(t_i) - (\mathcal{K}\pi_n u_{n,2})(t_i) - \left(\mathcal{K}(\mathcal{I} - \pi_n)\mathcal{K}_{n,2}\pi_n u_{n,2}\right)(t_i) \\ = f(t_i) + (\mathcal{K}(\mathcal{I} - \pi_n) f)(t_i), \end{aligned} \tag{16}$$

which is linear system (11). Now, from equation (13), the approximate solution is given by

$$\begin{aligned} u_{n,2} &= \pi_n u_{n,2} + (\mathcal{I} - \pi_n)\mathcal{K}_{n,2} w_n + (\mathcal{I} - \pi_n) f \\ &= f + \sum_{i=1}^{N_h} (X_i - b_i)\psi_i + \sum_{i=1}^{N_h} X_i w_i k_i - \sum_{i=1}^{N_h}\sum_{j=1}^{N_h} X_i \overline{B}_{ij}\psi_j, \end{aligned}$$

which completes the proof. $\square$

*Remark* 2. A comparison of system (11) with that given by Kulkarni in [**7**]

$$(I - S + A(A - I))X = a + (I - A)b \tag{17}$$

where

$$S_{ij} := \mathcal{K}^2\psi_j(t_i) = \mathcal{K}\widetilde{\psi}_j(t_i) = \int_0^1 \int_0^1 k(t_i, y)k(y, t)\psi_j(t)\, dy\, dt,$$

shows that the two systems have the same right hand side, but the matrix in (11) is simpler than that in (17) since there are only simple integrals to evaluate. On the other hand, the approximate solution obtained by Kulkarni's method is given by

$$u_n = f + \sum_{i=1}^{N_h}(X_i - b_i)\psi_i + \sum_{i=1}^{N_h} X_i\widetilde{\psi}_i - \sum_{i=1}^{N_h}\sum_{j=1}^{N_h} X_i A_{ij}\psi_j.$$

It can be seen that the solution given by (10) is simpler to obtain since one has just to evaluate the functions $k_i$ instead of the integrals $\widetilde{\psi}_i = \mathcal{K}\psi_i$.

In the next section, we make the convergence orders of the new method precise for both operators $\mathcal{K}_{n,1}$ and $\mathcal{K}_{n,2}$.

## 5. Orders of convergence.

**5.1. Error estimates.** Note that, throughout this paper, $C_1$ and $C_2$ denote generic constants, which may take different values at their different occurrences but will be independent of $n$. The error estimates for $u_{n,i}$ and $\widetilde{u}_{n,i}$, $i = 1, 2$, can be summarized as follows.

**Theorem 4.** *For all integers $n$ large enough and $i = 1, 2$,*

$$\|u - u_{n,i}\|_\infty \le C_1 \|(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_{n,i})u\|_\infty \tag{18}$$

*and*

$$\begin{aligned} \|u - \widetilde{u}_{n,i}\|_\infty \le \|(\mathcal{I} - \mathcal{K})^{-1}\|_\infty \\ \times (\|\mathcal{K}(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_{n,i})u\|_\infty \\ + \|\mathcal{K}(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_{n,i})\|_\infty \|u - u_{n,i}\|_\infty) \end{aligned} \tag{19}$$

*where $C_1$ is a constant independent of $n$.*

*Proof.* Since $\|\mathcal{K}-\mathcal{K}_n\|_\infty \to 0$, when $n \to \infty$, for all large $n$, $(\mathcal{I}-\mathcal{K}_n)$ is invertible and $\|(\mathcal{I}-\mathcal{K}_n)^{-1}\|_\infty \le C_1$, $C_1$ is a constant independent of $n$. For $i=1,2$, we have

$$\begin{aligned} u-u_{n,i} &= [(\mathcal{I}-\mathcal{K})^{-1}-(\mathcal{I}-\mathcal{K}_n)^{-1}]f \\ &= (\mathcal{I}-\mathcal{K}_n)^{-1}(\mathcal{K}-\mathcal{K}_n)u. \end{aligned}$$

Thus,

$$\begin{aligned} \|u-u_{n,i}\|_\infty &\le \|(\mathcal{I}-\mathcal{K}_n)^{-1}\|_\infty \|(\mathcal{I}-\pi_n)(\mathcal{K}-\mathcal{K}_{n,i})u\|_\infty \\ &\le C_1\|(\mathcal{I}-\pi_n)(\mathcal{K}-\mathcal{K}_{n,i})u\|_\infty, \end{aligned}$$

which completes the proof of (18). Moreover, we have

$$\begin{aligned} u-\widetilde{u}_{n,i} &= \mathcal{K}(u-u_{n,i}) \\ &= \mathcal{K}(I-\mathcal{K})^{-1}(\mathcal{K}-\mathcal{K}_n)(\mathcal{I}-\mathcal{K}_n)^{-1}f \\ &= (\mathcal{I}-\mathcal{K})^{-1}\mathcal{K}(\mathcal{I}-\pi_n)(\mathcal{K}-\mathcal{K}_n^i)(u+u_{n,i}-u), \end{aligned}$$

and estimate (19) follows. ◻

Let $v_{n,1}$ and $v_{n,2}$ be the approximate solutions of equation (1) using degenerate kernel and Nyström methods, respectively. Then, for $i=1,2$

$$v_{n,i}-\mathcal{K}_{n,i}v_{n,i}=f.$$

We quote the following error estimate from Theorems 2.1.1 and 4.1.2 of [**3**]:

$$\|v-v_{n,i}\|_\infty \le C_1\|(\mathcal{K}-\mathcal{K}_{n,i})u\|_\infty. \tag{20}$$

Thus, it is clear that $u_{n,i}$ is more accurate than $v_{n,i}$.

Choose $r \ge 1$, and assume that $k(\cdot,\cdot) \in \mathcal{C}^{2r,2r}([0,1]\times[0,1])$, where $\mathcal{C}^{l,m}([0,1]\times[0,1])$ is the space of kernels $k(s,t)$ defined on the square $[0,1]\times[0,1]$ and having continuous derivatives $\partial^{i+j}/(\partial s^i\partial t^j)k(s,t)$ for $i \le l$, $j \le m$. Then the range $R(\mathcal{K}) \subset \mathcal{C}^{2r}[0,1]$. Thus, if $k(\cdot,\cdot) \in \mathcal{C}^{2r}([0,1]\times[0,1])$ and $f \in \mathcal{C}^{2r}[0,1]$, then $u \in \mathcal{C}^{2r}[0,1]$. We set

$$D^{i,j}k = \frac{\partial^{i+j}}{\partial s^i\partial t^j}k(s,t), \quad s,t \in [0,1],$$

$$\|k\|_{2r,\infty} = \sum_{i=0}^{2r}\sum_{j=0}^{2r}\|D^{i,j}k\|_\infty$$

and

$$\|u\|_{2r,\infty} = \sum_{i=0}^{2r} \|u^{(i)}\|_\infty .$$

Let $u_n$ and $\widetilde{u}_n$, respectively, be the solutions obtained by Kulkarni's method and its iterated version. By [**7**, Theorem 3.1], we have, for all large $n$,

$$\|u - u_n\|_\infty \le C_2 \|(\mathcal{I} - \pi_n)\mathcal{K}(\mathcal{I} - \pi_n)u\|_\infty \tag{21}$$

and

$$\begin{aligned} \|u - \widetilde{u}_n\|_\infty \le \|(\mathcal{I} - \mathcal{K})^{-1}\|_\infty \Big( &\|\mathcal{K}(\mathcal{I} - \pi_n)\mathcal{K}(\mathcal{I} - \pi_n)u\|_\infty \\ &+ \|\mathcal{K}(\mathcal{I} - \pi_n)\mathcal{K}(\mathcal{I} - \pi_n)\|_\infty \|u - u_{n,i}\|_\infty \Big). \end{aligned} \tag{22}$$

By comparing estimates (18), (19) and (21), (22), we see that the term $\mathcal{K}(\mathcal{I} - \pi_n)$ in the first estimates is replaced by $(\mathcal{K} - \mathcal{K}_{n,i})$, $i = 1, 2$, in the second estimates. Then, in order to recover the superconvergence of $3r$ and $4r$ of Kulkarni's method and its iterated version by our variation, it suffices to prove that

**Proposition 1.** *For $y \in \mathcal{C}^r[0,1]$,*

$$\|(\mathcal{K} - \mathcal{K}_{n,1})y\|_{2r,\infty} \le C_2(2r+1)\|k\|_{2r,\infty}\|y\|_{r,\infty} h^{2r} \tag{23}$$

*and, for $y \in \mathcal{C}^{2r}[0,1]$, we have*

$$\|(\mathcal{K} - \mathcal{K}_{n,2})y\|_{2r,\infty} \le (2r+1)^2 C_2 \|k\|_{2r,\infty}\|y\|_{2r,\infty} h^{2r}. \tag{24}$$

*Proof.* We give here the proof of (24). For (23) the proof is similar.

For a fixed $j$ such that $0 \le j \le 2r$, we have

$$[(\mathcal{K} - \mathcal{K}_{n,2})y](s) = \int_0^1 (\mathcal{I} - \pi_n)(k(s,\cdot)y(\cdot))(t)\,dt.$$

Then

$$[(\mathcal{K} - \mathcal{K}_{n,2})y]^{(j)}(s) = \int_0^1 (\mathcal{I} - \pi_n)(\ell_s(\cdot)y(\cdot))(t)\,dt,$$

where, for a fixed $s \in [0,1]$,

$$\ell_s(t) = \ell(s,t), \quad t \in [0,1],$$

and

$$\ell(s,t) = \frac{\partial^j}{\partial s^j} k(s,t), \quad s,t \in [0,1].$$

Then, for each $s \in [0,1]$,

$$|[(\mathcal{K} - \mathcal{K}_{n,2})y]^{(j)}(s)| \le C_2 \|\ell_s y\|_{2r,\infty} h^{2r}.$$

On the other hand, we have

$$\|\ell_s y\|_{2r,\infty} = \sum_{k=0}^{2r} \|(\ell_s y)^{(k)}\|_\infty,$$

and since

$$\|(\ell_s y)^{(k)}\|_\infty \le \|k\|_{2r,\infty} \|y\|_{k,\infty},$$

we deduce that

$$\|[(\mathcal{K} - \mathcal{K}_{n,2})y]^{(j)}\|_\infty \le C_2(2r+1)\|k\|_{2r,\infty}\|y\|_{2r,\infty} h^{2r}. \tag{25}$$

Therefore,

$$\begin{aligned}\|(\mathcal{K} - \mathcal{K}_{n,2})y\|_{2r,\infty} &= \sum_{j=0}^{2r} \|[(\mathcal{K} - \mathcal{K}_{n,2})y]^{(j)}\|_\infty \\ &\le (2r+1)^2 C_2 \|k\|_{2r,\infty}\|y\|_{2r,\infty} h^{2r}. \qquad \square\end{aligned}$$

Now, by using the above estimates, the following theorem may be proved in much the same way as in [**7**, Section 4]. It suffices to replace $\mathcal{K}(\mathcal{I} - \pi_n)$ by $(\mathcal{K} - \mathcal{K}_{n,i})$, $i = 1,2$.

**Theorem 5.** *Let $u_{n,i}$ and $\widetilde{u}_{n,i}$, $i = 1,2$, be the approximate solutions of equation* (1) *defined by* (5) *and* (6), *respectively. In the case of the degenerate kernel operator, we assume that $k(\cdot,\cdot) \in \mathcal{C}^{2r,2r}([0,1]\times[0,1])$*

*and $f \in \mathcal{C}^r[0,1]$, while in the case of the Nyström operator, we assume that $k(\cdot,\cdot) \in \mathcal{C}^{2r,2r}([0,1]\times[0,1])$ and $f \in \mathcal{C}^{2r}[0,1]$. Then*

$$\|u - u_{n,i}\|_\infty = \mathcal{O}(h^{3r}) \tag{26}$$

*and*

$$\|u - \widetilde{u}_{n,i}\|_\infty = \mathcal{O}(h^{4r}). \tag{27}$$

*Remark* 3. i) Since

$$\|\mathcal{K}(\mathcal{I}-\pi_n)(\mathcal{K}-\mathcal{K}_{n,i})\|_\infty \le \|\mathcal{K}\|\left[\|(\mathcal{I}-\pi_n)\mathcal{K}\| + \|(\mathcal{I}-\pi_n)\mathcal{K}_{n,i}\|\right]$$

and

$$\|(\mathcal{I}-\pi_n)\mathcal{K}\| = \mathcal{O}(h^r), \qquad \|(\mathcal{I}-\pi_n)\mathcal{K}_{n,i}\| = \mathcal{O}(h^{4r}),$$

it follows that

$$\|\mathcal{K}(\mathcal{I}-\pi_n)(\mathcal{K}-\mathcal{K}_{n,i})\|_\infty = \mathcal{O}(h^r). \tag{28}$$

Now, by combining (19), (26) and (28) with the following estimate

$$\|\mathcal{K}(\mathcal{I}-\pi_n)(\mathcal{K}-\mathcal{K}_{n,i})u\|_\infty = O(h^{4r}),$$

we get (27).

ii) Note that, in the case of the degenerate kernel operator $\mathcal{K}_{n,1}$, the condition imposed on the solution is that $u \in \mathcal{C}^r[0,1]$ instead of $u \in \mathcal{C}^{2r}[0,1]$ in Kulkarni's method.

**5.2. Discrete methods.** In the discretized version of the proposed method, the operator $\mathcal{K}_n$ is replaced by

$$\mathcal{K}_n^D = \pi_n\mathcal{K}_{m,2} + \mathcal{K}_{n,i} - \pi_n\mathcal{K}_{n,i}, \quad i = 1,2,$$

where $\mathcal{K}_{m,2}$ is the Nyström operator based on $\pi_m$ and given by (4)

$$\mathcal{K}_{m,2}(u)(s) := \sum_{i=1}^{M_{\tilde{h}}} w_i k(s,t_i)u(t_i), \tag{29}$$

for some $m \geq n$, with $M_{\tilde{h}} = mr$, $\widetilde{h} = 1/m$. Let

$$(\mathcal{I} - \mathcal{K}_n^D)u_{n,i}^D = f$$

and

$$\widetilde{u}_{n,i}^D = \mathcal{K}_{m,2}u_{n,i}^D + f.$$

Let $\overline{u}_m$ be the solution of the Nyström equation $(\mathcal{I} - \mathcal{K}_{m,2})\overline{u}_m = f$. It is easy to show that

$$\|u - \overline{u}_m\|_\infty = \mathcal{O}(\widetilde{h}^{2r}).$$

On the other hand, the estimates (23)–(25) are valid when $\mathcal{K}$ is replaced by $\mathcal{K}_{m,2}$. Hence,

$$\|\overline{u}_m - u_{n,i}^D\|_\infty = \mathcal{O}(h^{3r})$$

and

$$\|\overline{u}_m - \widetilde{u}_{n,i}^D\|_\infty = \mathcal{O}(h^{4r})$$

with $h = 1/n$. Under the assumptions of Theorem 4, we get

$$\|u - u_{n,i}^D\|_\infty = \mathcal{O}(\max\{\widetilde{h}^{2r}, h^{3r}\})$$

and

$$\|u - \widetilde{u}_{n,i}^D\|_\infty = \mathcal{O}(\max\{\widetilde{h}^{2r}, h^{4r}\}).$$

Thus, if $m \geq n^{3/2}$, respectively $m \geq n^2$, then the order of convergence in (26), respectively in (27), is retained.

In what follows, we look at the number of arithmetic operations used in computing the approximate solutions $u_{n,i}^D$, $i = 1, 2$, $u_n^D$ obtained, respectively, by discretized collocation and Kulkarni's method on a point $t \in [0, 1]$.

**5.3. Computational cost.** • The calculation of each one of the vectors $a$ and $b$ requires approximately $3N_h M_{\tilde{h}}$ flops.

• The calculation of each one of the matrices $A, D, E$ requires approximately $3N_h^2 M_{\tilde{h}}$ flops, while the calculation of the matrices $\overline{B}, \overline{C}$, requires, respectively, $N_h^2, 4N_h^2 M_{\tilde{h}}, 5N_h^2 M_{\tilde{h}}^2$ flops.

• Evaluation of the matrix $F$ in linear system (7) requires approximately $N_h^2$ flops, while the evaluation of each one of the matrices on the left hand side of linear systems (11) and (17) requires approximately $3N_h^2 + 2N_h^3$ flops.

• The calculation of the right hand side of (11) requires approximately $3N_h^2 + N_h$ flops.

• The LU-factorization of matrix $F$ requires $(2/3)(2N_h)^3$ flops. The LU-factorization of each one of the matrices on the left hand side of linear systems (11) and (17) requires approximately $(2/3)(N_h)^3$ flops.

• The solution of the linear systems requires approximately $2(2N_h)^3$ flops, while the solution of each one of linear systems (11) and (17) requires approximately $2(N_h)^3$ flops.

• The final step is the evaluation of $u_{n,1}^D(t)$, $u_{n,2}^D(t)$ and $u_n^D(t)$, which requires, respectively, $4N_h, 8N_h + 2N_h^2, 3N_h M_{\tilde{h}} + 2N_h^2 + 7N_h$ flops.

Thus, the total cost in operations in the three methods is given in the following table:

| | |
|---|---|
| Collocation 1 | $3M_{\tilde{h}}(3N_h^2 + 2N_h) + (16/3)N_h^3 + 9N_h^2 + 4N_h$ |
| Collocation 2 | $M_{\tilde{h}}(7N_h^2 + 3N_h) + (8/3)N_h^3 + 11N_h^2 + 9N_h$ |
| Kukarni | $5M_{\tilde{h}}^2 N_h^2 + 3M_{\tilde{h}}(N_h^2 + N_h) + (8/3)N_h^3 + 9N_h^2 + 8N_h$ |

*Remark* 4. For $m \gg n$, collocation methods 1 and 2 have, respectively, the costs of approximately $3M_{\tilde{h}}(3N_h^2+2N_h)$ and $M_{\tilde{h}}(7N_h^2+3N_h)$ arithmetic operations, while Kulkarni's method has a cost of approximately $5M_{\tilde{h}}^2 N_h^2 + 3M_{\tilde{h}}(N_h^2 + N_h)$, which is more expensive.

**6. Numerical results.** We choose the following Fredholm integral equation of the second kind

$$u(s) - \int_0^1 \frac{1}{1 + e^{(s^2 - t)}} u(t)\, dt = 1 + s^{5/2},$$

where the exact solution is unknown. Since $k(\cdot,\cdot) \in \mathcal{C}^\infty([0,1] \times [0,1])$ and $f \in \mathcal{C}^2[0,1]$, we deduce that $u \in \mathcal{C}^2[0,1]$, but $u$ does not belong to $\mathcal{C}^3[0,1]$. Let $\mathcal{X}_n$ be the space of piecewise constant functions ($r = 1$) with respect to the uniform partition of $[0,1]$ on $n$ subintervals with mesh length $h = 1/n$:

$$0 = \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n}{n} = 1,$$

and let $\pi_n : \mathcal{C}[0,1] \to \mathcal{X}_n$ be the interpolatory projection at the $N_h = nr = n$ midpoints

$$t_k^{(n)} = \frac{2k-1}{n}, \quad k = 1, \dots, n.$$

In implementing these methods, we replace the integral operator $\mathcal{K}$ by the operator $\overline{\mathcal{K}} \equiv \mathcal{K}_{m,2}$ with $m = 1024$. Then we have

$$\overline{\mathcal{K}}x(s) = \sum_{k=1}^{m} \frac{1}{m}\left(\frac{1}{1+e^{(s^2-t_k^{(m)})}}\right)x(t_k^{(m)}), \quad s \in [0,1].$$

The solution obtained by using $\overline{\mathcal{K}}$ instead of $\mathcal{K}$ is denoted by $\overline{u} = \overline{u}_m$ and considered as a reference solution. We denote by $u_n$ and $\widetilde{u}_n$, respectively, the solutions obtained by Kulkarni's method and its iterated version. The quantities $\|\overline{u} - u_{n,i}^D\|_\infty$, $\|\overline{u} - \widetilde{u}_{n,i}^D\|_\infty$, $i = 1, 2$, $\|\overline{u} - u_n^D\|_\infty$ and $\|\overline{u} - \widetilde{u}_n^D\|_\infty$ are computed for $n = 2, 4, 8, 16, 32$. Using two successive values of $n$, the orders of convergence of the three methods and their iterated versions are computed and denoted by $\alpha$ and $\beta$, respectively. We also give the time in seconds denoted by $\mathcal{T}$ spanned by the machine for calculating the norm of the errors corresponding to the three methods. Note that, for $r = 1$, the theoretically predicted values of $\alpha$ and $\beta$ are 3 and 4, respectively; it can be seen from Tables 1 and 2 that the computed values of $\alpha$ and $\beta$ match well with the expected values. On the other hand, the three methods give comparable results since they have the same convergence order, but the methods proposed here are easier to implement and faster. The lower computational cost of these methods will appear more clearly in two-dimensional equations where we will only have to evaluate double integrals instead of integrals of order 4 in Kulkarni's method, which is more expensive.

TABLE 1. New collocation and Kulkarni's methods.

| $n$ | $\|\overline{u}-u_{n,1}^D\|_\infty$ | $\mathcal{T}$ | $\alpha$ | $\|\overline{u}-u_{n,2}^D\|_\infty$ | $\mathcal{T}$ | $\alpha$ | $\|\overline{u}-u_n^D\|_\infty$ | $\mathcal{T}$ | $\alpha$ |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 1.44(-03) | 0.7 | - | 1.74(-04) | 0.7 | — | 1.45(-03) | 1.2 | — |
| 4 | 1.73(-04) | 0.7 | 3.06 | 2.25(-05) | 0.8 | 2.96 | 1.91(-04) | 1.4 | 2.93 |
| 8 | 2.18(-05) | 1.6 | 2.99 | 3.02(-06) | 0.9 | 2.89 | 2.44(-05) | 2.7 | 2.96 |
| 16 | 2.67(-06) | 2.6 | 3.03 | 4.24(-07) | 2.4 | 2.84 | 3.09(-06) | 7.7 | 2.98 |
| 32 | 3.36(-07) | 5.2 | 2.99 | 5.82(-08) | 3.6 | 2.86 | 3.88(-07) | 26.7 | 2.99 |

TABLE 2. Iterated new collocation and Kulkarni's methods.

| $n$ | $\|\overline{u}-\widetilde{u}_{n,1}^D\|_\infty$ | $\mathcal{T}$ | $\beta$ | $\|\overline{u}-\widetilde{u}_{n,2}^D\|_\infty$ | $\mathcal{T}$ | $\beta$ | $\|\overline{u}-\widetilde{u}_n^D\|_\infty$ | $\mathcal{T}$ | $\beta$ |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 1.13(-04) | 1.2 | - | 7.25(-06) | 1.5 | — | 1.24(-04) | 50.0 | — |
| 4 | 6.56(-06) | 1.7 | 4.11 | 8.79(-07) | 2.2 | 3.04 | 7.67(-06) | 107.3 | 4.02 |
| 8 | 4.02(-07) | 2.1 | 4.03 | 5.96(-08) | 5.1 | 3.88 | 4.77(-07) | 152.0 | 4.01 |
| 16 | 2.50(-08) | 4.2 | 4.01 | 3.76(-09) | 10.1 | 3.99 | 2.97(-08) | 408.9 | 4.00 |
| 32 | 1.56(-09) | 7.5 | 4.01 | 2.34(-10) | 38.4 | 4.01 | 1.85(-09) | 571.5 | 4.01 |

**7. Case of discontinuous kernels.** Let $\alpha$ and $\gamma$ be two integers such that $\alpha \geq \gamma$, $\alpha \geq 0$ and $\gamma \geq -1$. We assume that kernel $k$ is of the following form

$$k(s,t) = \begin{cases} k_1(s,t), & 0 \leq s \leq t \leq 1, \\ k_2(s,t), & 0 \leq t \leq s \leq 1, \end{cases}$$

with $k_1 \in \mathcal{C}^\alpha(\{0 \leq s \leq t \leq 1\})$, $k_2 \in \mathcal{C}^\alpha(\{0 \leq t \leq s \leq 1\})$. If $\gamma \geq 0$, then it is assumed that $k \in \mathcal{C}^\gamma([0,1]\times[0,1])$ and, if $\gamma = -1$, then kernel $k$ may have a discontinuity of the first kind along the line $s = t$. Following Chatelin-Lebbar [**6**], the class of kernels of the above form is denoted by $\mathcal{C}(\alpha,\gamma)$. Then the operator $\mathcal{K} : \mathcal{C}[0,1] \to \mathcal{C}[0,1]$ is compact. In fact, the range of $\mathcal{K}$, $R(\mathcal{K})$, is contained in $\mathcal{C}^{\gamma_1}[0,1]$, where $\gamma_1 = \min\{\gamma+1, \alpha\}$ (see [**6, 9, 12**]).

For $\nu \geq 0$, set

$$\mathcal{C}^\nu_{\mathcal{I}_n} = \{y \in L^\infty : y|_{\mathcal{I}_i} \in \mathcal{C}^\nu(\mathcal{I}_i),\ 1 \leq i \leq n\},$$

where $\mathcal{I}_n$ is the nonuniform subdivision given by (2) and $\mathcal{I}_i = [x_{i-1}, x_i]$.

According to [**6**], $\mathcal{K}$ is a continuous map from $\mathcal{C}^\alpha_{\mathcal{I}_n}$ to $\mathcal{C}^\alpha_{\mathcal{I}_n}$, and if $u \in \mathcal{C}_{\mathcal{I}_n}$, we have

$$\|(\mathcal{K}u)^{(\mu)}\|_\infty \leq C\|u\|_\infty, \quad 0 \leq \mu \leq \gamma_1 + 1. \tag{30}$$

We recall that $\pi_n$ is the interpolatory projector on the space $\mathcal{S}_{r,n}$ and $\mathcal{K}_{n,i}$, $i = 1, 2$, are the operators given by (3) and (4). Then, if we put

$$\begin{aligned}
\beta &= \min\{\alpha, r\},\\
\beta_1 &= \min\{\beta, \gamma+1\} = \min\{\alpha, r, \gamma+1\},\\
\beta_2 &= \min\{\beta, \gamma+2\} = \min\{\alpha, r, \gamma+2\},\\
\beta_3 &= \min\{\alpha, 2r, r+\gamma+2\},\\
\beta_4 &= \min\{\alpha, 2r, r+\gamma+1\},\\
\beta_5 &= \min\{\alpha, 2r, r+\gamma\},
\end{aligned}$$

we quote the following estimates from Chatelin-Lebbar [**6**]. There exists a generic constant $C$ such that, for any $u \in \mathcal{C}^{\alpha}_{\mathcal{I}_n}$,

$$\|(\mathcal{I} - \pi_n)u\|_\infty \le C\|u^{(\beta)}\|_\infty h^{\beta}. \tag{31}$$

Similarly, if $u \in \mathcal{C}^{\beta_1}_{\mathcal{I}_n}$, then

$$\|(\mathcal{I} - \pi_n)u\|_\infty \le C\|u^{(\beta_1)}\|_\infty h^{\beta_1}. \tag{32}$$

If the kernel $k \in \mathcal{C}(\alpha, \gamma)$ with $\alpha \ge r+1$, then

$$\|\mathcal{K}(\mathcal{I} - \pi_n)u\|_\infty \le C\|u\|_{\infty,\beta_3} h^{\beta_3}, \tag{33}$$

and for $i = 1, 2$,

$$\|(\mathcal{K} - \mathcal{K}_{n,i})u\|_\infty \le C\|u\|_{\infty,\beta_i} h^{\beta_3}. \tag{34}$$

**Theorem 6.** *Let $\mathcal{K}$ be an integral operator with the kernel $k \in \mathcal{C}(2\alpha, \gamma)$, $\alpha \ge r+1$, and let $\mathcal{K}_{n,i}$ be the operators defined by* (3) *and* (4)*. Assume that the integral equation $u - \mathcal{K}u = f$, $f \in \mathcal{C}^{\alpha}_{\mathcal{I}_n}$ is uniquely solvable. Then, for $i = 1, 2$, we have*

$$\|(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_{n,i})u\| = \mathcal{O}(h^{\beta_1+\beta_4}), \tag{35}$$

$$\|(\mathcal{I} - \pi_n)\mathcal{K}_{n,i}\| = \mathcal{O}(h^{\beta_2}), \tag{36}$$

$$\|\mathcal{K}(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_{n,i})u\| = \mathcal{O}(h^{\beta_3+\beta_{4+i-1}}). \tag{37}$$

*Proof.* Using (32),

$$\|(\mathcal{I} - \pi_n)(\mathcal{K} - \mathcal{K}_{n,i})u\|_\infty \le C\|[(\mathcal{K} - \mathcal{K}_{n,i})u]^{(\beta_1)}\|_\infty h^{\beta_1}.$$

Since, by assumption, the kernel $k \in \mathcal{C}(2\alpha, \gamma)$, the kernel $\ell(s,t) = (\partial^{\beta_1})/(\partial s^{\beta_1})k(s,t) \in \mathcal{C}(\alpha, \gamma - \beta_1)$, we deduce by (34) that

$$\|[(\mathcal{K} - \mathcal{K}_{n,i})u]^{(\beta_1)}\|_\infty \le C\|u\|_{\infty,\beta_i} h^{\beta_4} \tag{38}$$

which gives (35). Now, by writing

$$(\mathcal{I} - \pi_n)\mathcal{K}_{n,i} = (\mathcal{I} - \pi_n)(\mathcal{K}_{n,i} - \mathcal{K}) + (\mathcal{I} - \pi_n)\mathcal{K}$$

and combining with (35) and $(\mathcal{I} - \pi_n)\mathcal{K} \le Ch^{\beta_2}$ (see [**9**]), estimate (36) follows. For $k = 0, 1 \ldots, \beta_3$ and $i = 1, 2$,

$$\|[(\mathcal{K} - \mathcal{K}_{n,i})u]^{(k)}\|_\infty \le C\|u\|_{\infty,\beta_i} h^{\beta_{4+i-1}},$$

which implies

$$\begin{aligned}\|(\mathcal{K} - \mathcal{K}_{n,i})u\|_{\infty,\beta_3} &= \sum_{k=0}^{\beta_3} \|[(\mathcal{K} - \mathcal{K}_{n,i})u]^{(k)}\|_\infty \\ &\le C(\beta_3 + 1)\|u\|_{\infty,\beta_i} h^{\beta_{4+i-1}}.\end{aligned}$$

Hence, by (33) estimate (37) follows. □

Now, from Theorems 3 and 5, we obtain the following convergence orders.

**Theorem 7.** *For all large $n$, we have for the operator $\mathcal{K}_{n,i}$, $i = 1, 2$,*

$$\|u - u_{n,i}\|_\infty = \mathcal{O}(h^{\beta_1 + \beta_{4+i-1}}), \tag{39}$$

$$\|u - \widetilde{u}_{n,i}\|_\infty = \mathcal{O}(h^{\beta_3 + \beta_{4+i-1}}). \tag{40}$$

*Remark* 5. As for the smooth case, the operator $\mathcal{K}_n$ in the discretized version of this method is replaced by

$$\mathcal{K}_n^D = \pi_n \mathcal{K}_{m,2} + \mathcal{K}_{n,i} - \pi_n \mathcal{K}_{n,i}, \quad i = 1, 2,$$

for some integer $m$. Let $\overline{u}_m$ be the solution of the Nyström equation $(\mathcal{I} - \mathcal{K}_{m,2})\overline{u}_m = f$. Then

$$\|u - \overline{u}_m\|_\infty = \mathcal{O}(\widetilde{h}^{\beta_3})$$

with $\widetilde{h} = 1/m$. Thus, if $m \geq n^{(\beta_1+\beta_{4+i-1})/\beta_3}$, respectively $m \geq n^{1+(\beta_{4+i-1})/\beta_3}$, $i = 1, 2$, then the order of convergence in (39), respectively in (40), is retained.

**8. Numerical example.** Let us consider the integral equation

$$u(s) - \int_0^1 k(s,t)u(t)\,dt = f(s), \quad 0 \leq s \leq 1,$$

where

$$k(s,t) = \begin{cases} s(1-t), & 0 \leq s \leq t \leq 1, \\ e^{s-t}, & 0 \leq t \leq s \leq 1, \end{cases}$$

and the right hand side is chosen so that the exact solution is $u(s) = s^{7/2}$. Thus, $u \in \mathcal{C}^3[0,1]$, but $u$ does not belong to $\mathcal{C}^4[0,1]$. For this example we have $\alpha = \infty$, $\gamma = -1$ (discontinuity along the line $s = t$). Thus, for the collocation at midpoints ($r = 1$), we have $\beta_1 = 0$, $\beta_3 = 2$ and $\beta_4 = 1$. Hence, by estimates (39) and (40), we have for the operator $\mathcal{K}_{n,1}$,

$$\|u - u_{n,1}\|_\infty = \mathcal{O}(h), \quad \|u - \widetilde{u}_{n,1}\|_\infty = \mathcal{O}(h^3).$$

In Table 3 we give the error approximation and the computed convergence orders obtained by the new method using the degenerate kernel operator $\mathcal{K}_{n,1}$.

It can be seen from Table 3 that the computed convergence orders match well with the expected values.

**Conclusion.** In this paper we have proposed new superconvergent Nyström and degenerate kernel methods for the approximate solution of integral equations with a smooth kernel, and we have extended these results to the case where the kernel may have a discontinuity

TABLE 3. Collocation at midpoints.

| $n$ | $\|u - u_{n,1}\|_\infty$ | | $\|u - \widetilde{u}_{n,1}\|_\infty$ | |
|---|---|---|---|---|
| 2 | 1.27(-01) | – | 3.43(-02) | – |
| 4 | 7.44(-02) | 0.77 | 5.28(-03) | 2.70 |
| 8 | 4.18(-02) | 0.83 | 7.09(-04) | 2.89 |
| 16 | 2.18(-02) | 0.94 | 8.97(-05) | 2.98 |
| 32 | 1.15(-07) | 0.99 | 1.11(-05) | 3.01 |

along the diagonal. We have proved that these methods exhibit the same orders as Kulkarni's methods. Further, we have shown that the methods introduced here are faster and simpler to implement. Finally, we have presented some numerical tests illustrating the theoretical results obtained.

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