JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 24, Number 3, Fall 2012

COMPACTNESS OF LINEAR INTEGRAL OPERATORS IN IDEAL SPACES OF VECTOR FUNCTIONS

MARTIN VÄTH

Communicated by Paul Martin

ABSTRACT. Estimates for the measure of noncompactness of linear integral operators of vector functions in ideal spaces are obtained. When the kernel function is compact, no additional uniformity or measurability hypotheses are needed; however, noncompact nonmeasurable kernel functions are also treated.

1. Introduction. Throughout this paper, let T and S be σ-finite measure spaces. Under some continuity or growth assumptions for f, it is well known that

$$Ax(t) := \int_S f(t,s,x(s))\,ds \quad (t \in T) \tag{1}$$

is continuous and compact in C (if T and S are compact subsets of $\mathbf{R}^n$) or in L_p or, more generally, in ideal spaces, respectively, see e.g., [**6**, Part I, Theorems 3.1, 3.2], [**7**, Sections 5, 19] or [**18**].

It is natural to conjecture that, in the case of vector functions x and f, i.e., if $f\colon T \times S \times U \to V$ with Banach spaces U and V, one obtains similar results. More precisely, one might conjecture that if $f(t,s,\cdot)$ is a compact operator for almost all $(t,s) \in T \times S$ and if f is a Carathéodory function (i.e., $u \mapsto f(t,s,u)$ is continuous for almost all (t,s) and $(t,s) \mapsto f(t,s,u)$ is (strongly) measurable for each $u \in U$) then under natural additional hypotheses the corresponding

2010 AMS *Mathematics subject classification.* Primary 45P05, 45N05, 46E30, Secondary 47H30, 45G10, 47G10, 47B34.
Keywords and phrases. Integral operator, kernel operator, Urysohn operator, measure of noncompactness, non-measurable kernel, Carathéodory function, ideal space, Banach function space, spaces of measurable functions.
This paper was written in the framework of a Heisenberg fellowship (Az. VA 206/1-1 and VA 206/1-2). Financial support by the DFG is gratefully acknowledged.
Received by the editors on November 20, 2010, and in revised form on December 17, 2010.
DOI:10.1216/JIE-2012-24-3-393

operator A will be continuous and compact in the corresponding spaces of continuous vector functions or in ideal spaces of vector functions.

For the space of continuous functions, this is indeed easy to see, if one assumes in addition that the compactness is uniform in a certain sense with respect to (t,s), e.g., that $f(t,s,B)$ is for bounded $B \subseteq U$ contained in a compact set which is independent of t and s. Actually, under reasonably mild equicontinuity hypotheses, the independence of such a set (for the integrals) with respect to t follows from a vector-valued version of the Arzelà-Ascoli theorem. However, the independence of a compact set as above from s (which follows, e.g., from uniform continuity of f) remained an artificial assumption in several publications until Mönch (see, e.g., [**10**, Proposition 1.4] or [**11**, Proposition 2]) succeeded in showing that, for each compactly generated space V and each bounded sequence $f_n \in C(S,V)$,

$$\chi_V\left\{\int_S f_n(s)\,ds : n = 1,2,\dots\right\} \leq \int_S \chi_V\{f_n(s) : n = 1,2,\dots\}\,ds. \tag{2}$$

Here, $\chi_V(M)$ denotes the *Hausdorff measure of noncompactness* of $M \subseteq V$, i.e., the infimum of all $\varepsilon > 0$ such that M has a finite ε-net in V. For later usage, let us also introduce the related notation $\chi^\circ(M) := \chi_M(M)$ for the *inner Hausdorff measure of noncompactness*, i.e., the infimum of all numbers $\varepsilon > 0$ such that M has a finite ε-net in itself, the *Kuratowski measure of noncompactness* $\alpha(M)$ which is the infimum of all $\delta > 0$ such that M can be divided into finitely many sets of diameter at most δ, and the *Istrăţescu measure of noncompactness* $\beta(M)$ which is the supremum of all $d \geq 0$ such that M contains a sequence whose elements have pairwise distance at least d.

Mönch's original proof of (2) contained a small mistake, but this was fixed later (see e.g., [**5**], [**1**, Section 4.2] or [**14**, Proposition 11.12]), and moreover, the result was generalized to measurable functions with equicontinuous L_1-norm, uncountable sets of functions, and for general Banach spaces V (in the last two cases one should insert the factor 2 or 4 into the right-hand side of the above equation), see e.g., [**14**, Corollary 11.19]. Roughly speaking, these results, combined with a vector-valued Arzelà-Ascoli theorem, allow us to prove the required compactness of operator A in spaces of continuous vector functions without requiring any artificial "uniformity" hypotheses. Thus, in the space of continuous vector functions with compact $f(t,s,\cdot)$, this problem can be considered solved.

However, in L_p spaces or, more generally, in ideal spaces, the problem about getting rid of the "uniformity" hypotheses remained. Another way to implement such "uniformity" hypotheses is to consider instead of Carathéodory functions only functions which satisfy this Carathéodory condition "uniformly" in a sense. By measurability we always mean strong (Bochner) measurability.

Definition 1. A function $f: T \times S \times U \to V$ is a *strict Carathéodory* function if $(t,s) \mapsto f(t,s,\cdot)$ is measurable as a function from $T \times S$ into $C(U,V)$ where the latter space is equipped with the topology of uniform convergence on bounded sets.

The name is explained by the fact that each strict Carathéodory function is a Carathéodory function; the converse holds if $\dim U < \infty$, see [**14**, Theorem 8.15] (cf. also [**2**]).

For linear $k(t,s) = f(t,s,\cdot)$, the function f is a strict Carathéodory function if and only if k is measurable considered as a function from $T \times S$ into the space $\mathfrak{L}(U,V)$ of bounded linear operators from U into V, equipped with the operator norm.

If one assumes that f is a strict Carathéodory function, then a certain variant of a compactness proof for the scalar case can be modified to show that (under some growth hypotheses for f) operator A is continuous and compact in ideal spaces of vector functions if $f(t,s,\cdot)$ is compact, see [**14**, Theorem 9.10].

We point out that the crucial difference between a Carathéodory function and a strict Carathéodory function is the separability of the (essential) range of the latter; actually this is exactly the difference if U is separable, see [**14**, Theorem 8.5]. Unfortunately, for infinite-dimensional U, the space $C(U,V)$ is never separable, and so the hypothesis that f is a strict Carathéodory function is usually only satisfied in very special situations. (However, it is typically satisfied in the important situation when $f(t,s,\cdot)$ is itself a Urysohn operator, see [**14**, Theorem 10.4].)

The strict Carathéodory condition cannot be omitted, in general. In [**16**] even a linear example of a compact (jointly in all variables) kernel

$k(t,s) = f(t,s,\cdot)$ was given where

$$Ax(t) := \int_S k(t,s)x(s)\,ds \quad (t \in T) \tag{3}$$

fails to be compact from $L_p(S,U)$ into every $L_q(T,\mathbf{R})$. In this example, U^* is nonseparable. The latter is not surprising, since it was also shown in [**16**] that if the natural candidate for the adjoint operator

$$A'x(s) := \int_T k(t,s)^* x(t)\,dt \quad (s \in S) \tag{4}$$

has a kernel generated by a Carathéodory function (which would be the case for separable U^*) and k is "uniformly" compact, then (3) is compact in ideal spaces. However, even for separable U^*, the uniformity of compactness remained a severe restriction.

The main aim of the current paper is to obtain compactness results *without* assuming that f is a *strict* Carathéodory function. As remarked, this hypothesis is very restrictive, and even in those cases where it is satisfied, this is often very hard to see.

We will show that, in the linear case $k(t,s)u = f(t,s,u)$, the situation is much better as one might expect by the above-mentioned results. In fact, the purpose of this paper is two-fold: In Section 2, we show that, in the compact case, i.e., if $k(t,s)$ is linear and compact, the above-mentioned uniformity hypothesis of [**16**] is actually superfluous in a sense, i.e., actually the separability of U^* is already sufficient for the compactness. Although the proof is rather simple, this came to the author as a big surprise. However, roughly speaking, this is due to the fact that the space $\mathfrak{K}(U,V)$ of linear compact operators from U into V is surprisingly small compared to the nonlinear or noncompact situation. For the case that $k(t,s)$ is noncompact, we cannot get rid of some uniformity hypothesis, but in Section 3 we show for this case a quantitative variant of [**16**] in terms of measures of noncompactness.

2. The compact case. In order to keep the notation analogous to the familiar notation of integral operators of scalar functions, we will throughout use the notation $|\cdot|$ for the norms in U and V. The same symbol will be used for the norm of bounded functionals or of bounded linear operators $U \to V$. In contrast, the norm of function spaces or of operators in function spaces will be denoted by $\|\cdot\|$.

By $\mathfrak{M}(T,V)$, we denote the space of all (equivalence classes of) measurable functions $y\colon T \to V$. Recall that a pre-ideal space (sometimes also called Köthe space) $Y \subseteq \mathfrak{M}(T,V)$ is a normed space with the property that the relations $x \in Y$, $y \in \mathfrak{M}(T,V)$ and $|y(t)| \le |x(t)|$ almost everywhere imply that $y \in Y$ and $\|y\| \le \|x\|$. If Y is also complete, it is called an ideal space (or sometimes also Banach function space). For the general theory of pre-ideal spaces, we refer to, e.g., [**13, 19, 20**] or the last section of [**17**]. Each pre-ideal space has a corresponding real form $Y_{\mathbf{R}} \subseteq \mathfrak{M}(T,\mathbf{R})$, defined in the obvious way by means of the relation $\||y|\|_{Y_{\mathbf{R}}} = \|y\|$, see e.g., [**13**, Section 2.1].

The *regular part* of Y is the set of those $y \in Y$ for which $\|P_{E_n}y\| \to 0$ whenever $E_n \subseteq T$ are measurable with $E_n \downarrow \varnothing$ where $P_E y(t) := \chi_E(t)y(t)$. The regular part is always a closed subspace of Y.

Together with (3), we also consider the operator

$$|A|x(t) := \int_S |k(t,s)|x(s)\,ds \quad (t \in T);$$

recall that $|k(t,s)|$ denotes the operator norm of $k(t,s)$. Note that, under the hypothesis of the following theorem, since the separability of U is one of the hypotheses, the kernel function $|k|$ of this operator is automatically measurable, see e.g., [**16**].

Recall that A is said to be *regular* from a pre-ideal space $X \subseteq \mathfrak{M}(S,U)$ into a pre-ideal space $Y \subseteq \mathfrak{M}(T,V)$ if $|A|$ acts from $X_{\mathbf{R}}$ into $Y_{\mathbf{R}}$.

Motivated by this notion, we call the integral operator (3) *regularizing* if there is a sequence $x_n \in \mathfrak{M}(S,\mathbf{R})$ with $0 \le x_n(s) \to \infty$ $(n \to \infty)$ with the property that, for each n and each measurable set $D \subseteq S$ with $P_D x_n \in X_{\mathbf{R}}$ we have that $|A|P_D x_n$ belongs to the regular part of $Y_{\mathbf{R}}$. This is satisfied, in particular, if $|A|$ maps integrable simple functions from $X_{\mathbf{R}}$ into the regular part of $Y_{\mathbf{R}}$.

Recall that our given measure space S is more verbosely a triple (S,Σ,μ) with Σ being the σ-algebra of measurable sets and $\mu\colon\Sigma \to [0,\infty]$ the measure. A *normalized* measure ν on this measure space is a measure $\nu\colon\Sigma \to [0,1]$ which has the same null sets as μ but satisfies

$\nu(S) = 1$. Since we assume that (S, Σ, μ) is σ-finite, such a normalized measure always exists.

Theorem 1. *Let U^* be separable, and let $k: T \times S \to \mathfrak{K}(U, V)$ be such that, for each $u \in U$, the function $(t, s) \mapsto k(t, s)u$ is measurable as a function from $T \times S$ into V.*

Let $X \subseteq \mathfrak{M}(S, U)$ and $Y \subseteq \mathfrak{M}(T, V)$ be pre-ideal spaces such that the corresponding integral operator (3) *is regularizing. Then, for each bounded set $B \subseteq X$ and each normalized measure ν on S, we have the estimate*

$$\alpha(A(B)) \leq 2 \cdot \limsup_{\delta \to 0} \sup_{\nu(D) \leq \delta} \sup_{x \in B} \|AP_D x\|. \tag{5}$$

In particular, $A(B)$ is precompact if the right-hand side of (5) *vanishes.*

Recall that α in (5) denotes the Kuratowski measure of noncompactness of $A(B) \subseteq Y$.

The hypothesis that $(t, s) \mapsto k(t, s)u$ is measurable is equivalent to the statement that the kernel $f(t, s, u) := k(t, s)u$ in (1) is a Carathéodory function. Recall that this is the least reasonable requirement under which one can expect to find useful statements about operator A, since otherwise one could not apply Fubini-Tonelli's theorem, and thus even the measurability of Ax ($x \in B$) would be a very delicate problem. For scalar functions, it is even known that every linear integral operator in L_p can be written in the form (3) where $f(t, s, u) = k(t, s)u$ is a Carathéodory function, see e.g, [**12**], but it seems to be unknown whether an analogous result holds for vector-valued functions.

Remark 1. An analogous result holds even if X and Y are only quasinormed, i.e., if instead of the triangle inequality we have only $\|x + y\| \leq q(\|x + \|y)$ with a finite constant q: One just has to insert the factor q^2 (with q corresponding to the space Y) on the right-hand side of (5). Moreover, B need not necessarily be bounded in X but only in the space $\mathfrak{M}(S, U)$ of measurable functions, considered as a topological vector space with the topology of convergence in measure on sets of finite measure.

The term on the right-hand side of (5) should not be too surprising: It measures in a sense how "singular" the kernel k is. Theorem 1 and various generalizations thereof are contained in [**3**] under the additional hypothesis that $k\colon T \times S \to \mathfrak{K}(U, V)$ is measurable. The following result states that, surprisingly, this hypothesis is satisfied automatically.

Theorem 2. *Let U^* be separable, and let $k\colon T \times S \to \mathfrak{K}(U, V)$ be such that $(t, s) \mapsto k(t, s)u$ is measurable for each $u \in U$, i.e., $f(t, s, u) := k(t, s)u$ defines a Carathéodory function $f\colon T \times S \times U \to V$. Then f is a strict Carathéodory function, i.e., k is measurable as a function with values in $\mathfrak{K}(U, V)$* (*equipped with the operator norm*).

Note in this connection that, in view of, e.g., [**14**, Corollary 2.4], the measurability of k depends actually only upon the topology, i.e., the measurability of $k\colon T \times S \to \mathfrak{K}(U, V)$ is equivalent to that of $k\colon T \times S \to \mathfrak{L}(U, V)$ or of $k\colon T \times S \to C(U, V)$ (with the topology of uniform convergence on bounded sets).

The crucial point in the proof of Theorem 2 is the following observation which was pointed out to the author by D. Werner (with a more cumbersome proof).

Lemma 1. *Let U and V both be nontrivial. Then $\mathfrak{K}(U, V)$ is separable if and only if U^* and V are both separable.*

Proof. If $\mathfrak{K}(U, V)$ is separable, fix $g_0 \in U^*$ and $v_0 \in V$ with $|g_0| = |v_0| = 1$. Then the subspaces $\{g_0(\cdot)v \in \mathfrak{K}(U, V) : v \in V\}$ and $\{g(\cdot)v_0 \in \mathfrak{K}(U, V) : g \in U^*\}$ are separable and isometric to V and U^*, respectively.

For the converse, let U^* and V be separable. Since V is a separable normed space, it is isometric to a subspace of $C([0, 1])$. Hence, increasing V if necessary, we may assume without loss of generality that V has the approximation property. Then the finite rank operators are dense in $\mathfrak{K}(U, V)$. It now suffices to observe that the separability of U^* and of V obviously implies that the subset of finite rank operators is separable. $\square$

In view of Lemma 1, one might conjecture that it is also necessary to assume in Theorem 2 that V is separable. However, the range of Carathéodory functions is automatically separable if U is separable. In fact, this holds even in a more general setting of metric spaces.

Lemma 2. *Let U and V be metric spaces, and let $f\colon T \times S \times U \to V$ be a Carathéodory function. If U is separable, then there is a null set $N \subseteq T \times S$ such that $f(((T \times S) \setminus N) \times U)$ is separable.*

Proof. Let $N_0 \subseteq T \times S$ be a null set such that $f(t, s, \cdot)$ is continuous for $(t, s) \notin N_0$, and let $\{u_1, u_2, \dots\}$ be dense in U. Since $f(\cdot, \cdot, u_n)$ is measurable, there are null sets $N_n \subseteq T \times S$ such that $f((T \times S) \setminus N_n, u_n)$ is separable, i.e., contained in the closure of a countable set $V_n := \{v_{n,1}, v_{n,2}, \dots\} \subseteq V$. Put $N := N_0 \cup \bigcup_n N_n$. Then $\bigcup_n V_n$ is dense in $f(((T \times S) \setminus N) \times U)$. Indeed, for each $(t, s, u) \in (T \times S) \setminus N) \times U$ and each $\varepsilon > 0$, we find by the continuity of $f(t, s, \cdot)$ some n with $d(f(t, s, u_n), f(t, s, u)) < \varepsilon/2$ and some k with $d(f(t, s, u_n), v_{n,k}) < \varepsilon/2$, and so $d(f(t, s, u), v_{n,k}) < \varepsilon$. $\square$

Proof of Theorem 2. In view of Lemma 2, there are a separable closed subspace $V_0 \subseteq V$ and a null set N such that $k((T \times S) \setminus N)(U) \subseteq V_0$. Hence, modifying k on the null set N and replacing V by V_0, we may assume without loss of generality that V is separable. Lemma 1 thus implies that $k(T \times S)$ is a separable subset of $\mathfrak{L}(U, V)$. Theorem 2 now follows from [**14**, Theorem 8.5]. $\square$

3. The noncompact case. For noncompact $k(t, s)$, one cannot expect a result which is similarly simple as that of Section 2. In fact, the crucial point of Section 2 was the separability of $\mathfrak{K}(U, V)$. For $\mathfrak{L}(U, V)$ no analogous result is available under reasonable hypotheses for U and V, as we make clear by the following Proposition 1.

Recall that a Schauder basis $(e_n)_n$ in a Banach space U is called *unconditional* if, for each $u \in U$, the corresponding expansion $u = \sum_{n=1}^{\infty} \lambda_n e_n$ converges unconditionally; most classical Banach spaces possess such a basis.

Proposition 1. *If U is an infinite-dimensional Banach space with an unconditional basis, then $\mathfrak{L}(U, U)$ fails to be separable* (*with respect to the operator norm*).

Proof. If $(e_n)_n$ is an unconditional basis, we can define for each $N \subseteq \mathbf{N}$ a corresponding projection operator $P_N \in \mathfrak{L}(U,U)$ by

$$P_N \sum_{n=1}^{\infty} \lambda_n e_n := \sum_{\substack{n=1 \\ n \in N}}^{\infty} \lambda_n e_n.$$

If $N, M \subseteq \mathbf{N}$ differ, say $n \in N \setminus M$, then we have $(P_N - P_M)e_n = e_n$, and so $\|P_N - P_M\| \geq 1$. Hence, we have found an uncountable family of elements of $\mathfrak{L}(U,U)$ with pairwise distance at least 1 which implies that $\mathfrak{L}(U,U)$ fails to be separable. $\square$

We now give a positive result which, however, will require some "uniformity" in a compactness estimate with respect to t. We consider the situation of Section 2 with the difference that we now assume only $k\colon T \times S \to \mathfrak{L}(U,V)$. We assume that $f(t,s,u) = k(t,s)u$ is a Carathéodory function, i.e., $(t,s) \mapsto k(t,s)u$ is measurable for each $u \in U$. We also assume that $(t,s) \mapsto |k(t,s)|$ and $(t,s) \mapsto k(t,s)^* v^*$ (for each $v^* \in V^*$) are measurable (these are not additional requirements if U or U^* are separable, respectively, see [**16**]).

Let $X \subseteq \mathfrak{M}(S,U)$ and $Y \subseteq \mathfrak{M}(T,V)$ be pre-ideal spaces. Since S and T are σ-finite, these spaces have a support, i.e., there is an (up to null sets) smallest set $\operatorname{supp} X$ which contains the support of all functions from X. Recall that the associate space $X'_{\mathbf{R}}$ is the pre-ideal (actually even ideal) space of all functions $y \in \mathfrak{M}(S,\mathbf{R})$ vanishing outside $\operatorname{supp} X$ for which the norm

$$\|y\|_{X'_{\mathbf{R}}} := \sup_{\|x\|_{X_{\mathbf{R}}} \leq 1} \int_S |y(s)||x(s)|\, ds$$

is finite. Together with (3), we consider the associate operator (4) as well as

$$|A|' x(s) := \int_T |k(t,s)| x(t)\, dt \quad (s \in S)$$

and the nonlinear operator

$$|A|_0 x(t) := \int_S |k(t,s)x(s)|\, ds \quad (t \in T)$$

and introduce for bounded $B \subseteq X$ the numbers

$$\begin{aligned}
\gamma_S^0(k,B) &:= \sup_{S \supseteq D_n \downarrow \varnothing} \limsup_{n\to\infty} \sup_{x\in B} \||A|_0 P_{D_n} x\|_{Y_{\mathbf{R}}},\\
\gamma_T^0(k,B) &:= \sup_{T \supseteq E_n \downarrow \varnothing} \limsup_{n\to\infty} \sup_{x\in B} \|P_{E_n}|A|_0 x\|_{Y_{\mathbf{R}}},\\
\gamma_S(k) &:= \sup_{S \supseteq D_n \downarrow \varnothing} \limsup_{n\to\infty} \||A|P_{D_n}\|_{\mathfrak{L}(X_{\mathbf{R}},Y_{\mathbf{R}})},\\
\gamma_T(k) &:= \sup_{T \supseteq E_n \downarrow \varnothing} \limsup_{n\to\infty} \|P_{E_n}|A|\|_{\mathfrak{L}(X_{\mathbf{R}},Y_{\mathbf{R}})},\\
\gamma_S'(k) &:= \sup_{S \supseteq D_n \downarrow \varnothing} \limsup_{n\to\infty} \|P_{D_n}|A|'\|_{\mathfrak{L}(Y_{\mathbf{R}}',X_{\mathbf{R}}')},\\
\gamma_T'(k) &:= \sup_{T \supseteq E_n \downarrow \varnothing} \limsup_{n\to\infty} \||A|'P_{E_n}\|_{\mathfrak{L}(Y_{\mathbf{R}}',X_{\mathbf{R}}')}.
\end{aligned}$$

The above characteristics have been introduced in [**16**]. Their occurrence should not be too surprising since it is well-known that they vanish if and only if $|A|$ and $|A|'$ are compact (if all involved spaces are regular), see e.g., [**9**]. Roughly speaking, the above quantities measure in a sense how singular the kernel k is, and they are actually rather related.

To make the latter more precise, we recall that Y is called c_Y*-almost perfect* if $Y_{\mathbf{R}}$ has the property that the relations $0 \le y_n(t) \uparrow y(t)$ almost everywhere and $y \in Y_{\mathbf{R}}$ imply that $\|y\| \le c_Y \sup_n \|y_n\|$. Most ideal spaces occurring in applications are c_Y-almost perfect with $c_Y = 1$; for instance, all Orlicz and Lebesgue-spaces (even weighted) and all regular pre-ideal spaces have this property (see e.g., [**13**, Corollary 3.3.4] for the latter). If A is regular, i.e., $|A| \in \mathfrak{L}(X_{\mathbf{R}}, Y_{\mathbf{R}})$, then (under our measurability hypotheses) automatically $A \in \mathfrak{L}(X, Y)$ and $|A|' \in \mathfrak{L}(Y_{\mathbf{R}}', X_{\mathbf{R}}')$, see e.g., [**16**]; more precisely,

$$\||A|'\|_{\mathfrak{L}(Y_{\mathbf{R}}',X_{\mathbf{R}}')} \le \||A|\|_{\mathfrak{L}(Y_{\mathbf{R}},X_{\mathbf{R}})}, \tag{6}$$

and if Y is c_Y-almost perfect also the converse estimate

$$\||A|_{\mathfrak{L}(Y_{\mathbf{R}},X_{\mathbf{R}})} \le c_Y \||A|'\|_{\mathfrak{L}(Y_{\mathbf{R}}',X_{\mathbf{R}}')} \tag{7}$$

holds. Applying these estimates with the auxiliary integral operator with kernel function $\widetilde{k}(t,s) := |k(t,s)|\chi_D(s)$ or $\widetilde{k}(t,s) := \chi_E(t)|k(t,s)|$, we obtain the estimates

$$\gamma_{S'}(k) \le \gamma_S(k') \quad \text{and} \quad \gamma_{T'}(k) \le \gamma_T(k), \tag{8}$$

respectively and, if Y is c_Y-almost perfect, we obtain similarly the converse estimates

$$\gamma_S(k) \le c_Y \gamma_{S'}(k') \quad \text{and} \quad \gamma_T(k) \le c_Y \gamma_{T'}(k). \tag{9}$$

Putting $B_X := \{x \in X : \|x\| \le 1\}$, one can easily verify that

$$\gamma_S^0(k, B) \le \gamma_S(k) \quad \text{and} \quad \gamma_T^0(k, B) \le \gamma_T(k) \quad \text{if } B \subseteq B_X, \tag{10}$$

with equality at least if k is a scalar function and $B = B_X$. If Y is regular, then the quantities $\gamma_S(k)$, $\gamma_T(k)$, $\gamma_{S'}(k)$ and $\gamma_{T'}(k)$ vanish if and only if $|A|$ is compact; more precisely, several estimates exist (from above and below) which relate these characteristics to various measures of noncompactness of $|A|(B_X)$, see e.g., [**3, 16**] for such estimates. In some cases, it is possible to calculate the above quantities explicitly, e.g., for the Hardy operator:

Example 1. Let $X = Y = L_p([0,1])$ $(1 < p < \infty)$ and $|k(t,s)| = (1/t)\chi_{[0,t]}(s)$, i.e., $|A|x(t) = (1/t)\int_0^t x(s)\,ds$. Then

$$\begin{aligned}\gamma_S(k) &= \gamma_T(k) = \gamma_{S'}(k) = \gamma_{T'}(k)\\ &= \| |A| \|_{\mathfrak{L}(X_{\mathbf{R}}, Y_{\mathbf{R}})} = \| |A|' \|_{\mathfrak{L}(Y'_{\mathbf{R}}, X'_{\mathbf{R}})} = \frac{p}{1-p},\end{aligned} \tag{11}$$

and by the remark after (10) also $\gamma_S^0(k, B_X) = \gamma_T^0(k, B_X) = p/(1-p)$. Indeed, the estimate $\| |A| \|_{\mathfrak{L}(X_{\mathbf{R}}, Y_{\mathbf{R}})} \le p/(1-p)$ is the well-known Hardy inequality. Note that, trivially, $\gamma_S(k), \gamma_T(k) \le \| |A| \|_{\mathfrak{L}(X_{\mathbf{R}}, Y_{\mathbf{R}})}$. For the converse, we consider the sequences $D_n = E_n = [0, 1/n]$ and $x_n(t) := \chi_{D_n}(t) t^{-c_n}$ with $0 < c_n \uparrow 1/p$. Then $|A| P_{D_n} x_n(t) \ge P_{E_n} |A| x_n(t) = 1/(1-c_n) x_n(t)$ which implies $\| |A| P_{D_n} \|, \| P_{E_n} |A| \| \ge 1/(1-c_n) \to p/(p-1)$. Hence, $\gamma_S(A), \gamma_T(A) \ge p/(p-1)$. Combining the estimates obtained with (6), (7), (8) and (9) (with $c_Y = 1$), we obtain (11).

Note that the fourth equality sign in (11) is somewhat exceptional: in general, the characteristics are smaller than the norm. In fact, in a sense, only the "singularities" of k are responsible for the size of these quantities:

Example 2. We modify the previous example by assuming $|k(t,s)| = (1/t)\chi_{[0,t]}(s) + C$ with $C > p/(p-1)$. Then $\|\|A\|\|_{\mathfrak{L}(X_{\mathbf{R}},Y_{\mathbf{R}})}$, $\|\|A|'\|_{\mathfrak{L}(Y'_{\mathbf{R}},X'_{\mathbf{R}})} \geq C$ while all the other characteristics are the same as for the kernel function $k_0(t,s) := (1/t)\chi_{[0,t]}(s)$. Indeed, putting $k_C(t,s) := C$, we have $\gamma^0_S(k,B) \leq \gamma^0_S(k_0,B) + \gamma^0_S(k_C,B)$, and the last summand vanishes for bounded $B \subseteq X$ since Hölder's inequality (with $(1/p)+(1/p')=1$) implies that $\|CP_{D_n}x\|_{L_1} \leq C\|\chi_{D_n}\|_{L_{p'}}\|x\|_X \to 0$ uniformly in $x \in B$ as $D_n \downarrow \varnothing$. The argument for the other characteristics is analogous.

Now we are in a position to formulate a result which yields an estimate for the measure of noncompactness of $A(B)$. This estimate depends on the one hand on the quantities considered above (i.e., on the "singularities" of $|k|$). On the other hand, the estimate for the measure of noncompactness of $A(B)$ also depends upon, roughly speaking, the measure of noncompactness of the set of all $\int_S k(t,s)x(s)\,ds$ (in V) where t and x vary. To deal with singularities of k and x, for the latter actually some auxiliary "weight" functions $\lambda_{n,x}$ and sets $R_n \subseteq T \times S$ are allowed in this integral in the formulation of the theorem. (And t varies only in T_j where (T_j) is a partition of $\operatorname{supp} Y$.)

We will discuss in a moment how this technical hypothesis can be replaced by a much simpler hypothesis, namely, by a hypothesis which requires only knowledge of the measure of noncompactness for $k(t,s)x(s)$ (when t and x vary but s is fixed) instead of their integrals: The latter is possible by using a Mönch type theorem.

Theorem 3. *Let Y be c_Y-almost perfect, and assume the measurability hypotheses mentioned after Proposition 1. Assume in addition that $A\colon X \to Y$ is regular and that $|A|'$ sends $Y'_{\mathbf{R}}$ into the regular part of $X'_{\mathbf{R}}$. Let $B \subseteq X$ be bounded. For an at most countable index set J, let $T_j \subseteq \operatorname{supp} Y$ $(j \in J)$ be measurable with $\bigcup_j T_j \supseteq \operatorname{supp} A(B)$. Let $P_{n,j} \subseteq V$ $(j \in J)$ be such that there is a sequence $R_n \uparrow T \times S$ (up to null sets) and for each $x \in B$ and each n a measurable function $\lambda_{n,x}\colon S \to [0,2]$ satisfying*

$$\lambda_{n,x}(s) = 1 \text{ a.e. on } \{s : |x(s)| \leq n\} \tag{12}$$

and such that, for all $j \in J$,

$$\int_S \chi_{R_n}(t,s)\lambda_{n,x}(s)k(t,s)x(s)\,ds \in P_{n,j} \quad \text{for almost all } t \in T_j. \tag{13}$$

If $\chi_j, \rho_j \in [0, \infty)$ *satisfy*

$$\rho_j > 2\chi_j, \tag{14}$$

$$\sup_n \chi_{\{v \in V : |v| \geq \rho_j\}}(\{v \in P_{n,j} - P_{n,j} : |v| \geq \rho_j\}) \leq \chi_j, \tag{15}$$

and $\rho := \sum_j \rho_j \chi_{T_j} \in Y''_{\mathbf{R}}$, *then we have the estimates*

$$\begin{aligned} \chi_Y(A(B)) &\leq \gamma_S^0(k, B) + \gamma_T^0(k, B) \\ &\qquad + \sup_{j \in J} \frac{c_Y \operatorname{diam} B}{1 - 2\rho_j^{-1}\chi_j}(\gamma'_S(k) + \gamma'_T(k)) + c_Y \|\rho\|_{Y''_{\mathbf{R}}}. \end{aligned} \tag{16}$$

If J *is finite and if even the choice* $\lambda_{n,x} = 1$ *and* $R_n = T \times S$ *is possible for some* n*, i.e.,* $Ax(t) \in P_{n,j}$ *almost everywhere on* T_j *for each* $x \in B$*, then we even have the stronger estimate*

$$\chi^\circ(A(B)) \leq \max_{j \in J} \frac{c_Y \operatorname{diam} B}{1 - 2\rho_j^{-1}\chi_j}(\gamma'_S(k) + \gamma'_T(k)) + c_Y \|\rho\|_{Y''_{\mathbf{R}}}. \tag{17}$$

Before we prove Theorem 3, we make some remarks.

First, we will show in Proposition 2 below that we can get rid of the strange dependency of χ_j on ρ_j in (15) when we use a different (although in a sense less optimal) measure of noncompactness instead.

The second remark concerns what we mentioned before. Let us assume the hypothesis (19) described below (this is a mild hypothesis which is satisfied if, roughly speaking, $|k|$ is not "extremely singular"). Then we can apply a Mönch type result to verify all hypotheses (12)–(15) with small constants χ_j, ρ_j. More precisely, in the subsequent Proposition 2 we will show that, instead of requiring estimates for the measure of noncompactness of integrals, it suffices to know for fixed s estimates $r_{n,j}(s)$ for the measure of noncompactness of the set of points $k(t, s)x(s)$ (where $t \in T_j$ and x vary), and to integrate $r_{n,j}$. Here, singularities of $|x|$ are treated by ignoring points s with $|x(s)| > n$, and the sets R_n can still be used to treat singularities of $|k|$.

Recall that for $M \subseteq V$ the quantities $\beta(M)$, $\chi_V(M)$, and $\chi^\circ(M)$ denote the Istrăţescu, Hausdorff, or inner Hausdorff measure of noncompactness of M, respectively.

Proposition 2. *Hypotheses* (14) *and* (15) *in Theorem* 3 *are satisfied if*

$$\chi_j \geq 2 \sup_n \beta(P_{n,j}) \quad \text{and} \quad \rho_j > 2\chi_j. \tag{18}$$

Moreover, given $R_n \uparrow T \times S$ (up to null sets), this and all hypotheses (12)–(15) *hold with appropriate $P_{n,j}$ and $\lambda_{n,x}$ under the following assumptions.*

(1) *For each $j \in J$, each n and each sequence of measurable sets $D_k \downarrow \varnothing$:*

$$\lim_{k\to\infty} \sup_{x\in B} \sup_{t\in T_j} \int_{D_k \cap \{s:|x(s)|\leq n\}} \chi_{R_n}(t,s)|k(t,s)x(s)|\, ds = 0. \tag{19}$$

(2) *The estimates*

$$\chi_j \geq 4 \sup_n \int_S r_{n,j}(s)\, ds \quad \text{and} \quad \rho_j > 2\chi_j \tag{20}$$

hold with measurable functions $r_{n,j}\colon S \to [0,\infty]$ satisfying

$$r_{n,j}(s) \geq \chi^\circ(\{k(t,s)x(s) : (x,t) \in C,\ |x(s)| \leq n,\ (t,s) \in R_n\}) \quad \text{for almost all } s \in S \tag{21}$$

for all countable $C \subseteq B \times T_j$. If V is separable, one can replace χ° in (21) *by χ_V.*

We understand the estimate (21) of course in the sense of choosing one function x for every equivalence class considered. Since we consider only countable sets C, the estimate (21) is (up to null sets) independent of the choice of the representing functions.

Remark 2. Our proof will show that, if V has the L-retraction property in the sense of [**14**, Definition 11.8], one can replace χ° in (21) by $L \cdot \chi_V$. Using the axiom of choice, it can be shown that every weakly compactly generated space has the 1-retraction property [**14**, Theorem 11.10]; in particular, also for reflexive spaces V, one can replace χ° in (21) by χ_V.

Proof of Proposition 2. For the first assertion, it suffices to note that the left-hand side of (15) is bounded from above by

$$\begin{aligned}\chi^\circ(\{v \in P_{n,j} - P_{n,j} : |v| \geq \rho_j\}) &\leq \beta(\{v \in P_{n,j} - P_{n,j} : |v| \geq \rho_j\}) \\ &\leq 2\beta(P_{n,j}).\end{aligned}$$

For the second assertion, we put $\lambda_{n,x} := \chi_{\{s:|x(s)|\leq n\}}$ and $P_{n,j} := \{\int_S y(s)\,ds : y \in B_{n,j}\}$ with

$$\begin{aligned}B_{n,j} := \{&y_{n,x,t} \in \mathfrak{M}(S,V) : y_{n,x,t}(s) \\ &= \chi_{R_n}(t,s)\lambda_{n,x}(s)k(t,s)x(s),\ x \in B,\ t \in T_j\}.\end{aligned}$$

Then (18) follows from [**14**, Theorem 11.17] with $f(s,u) = u$. □

Note that the particular choices $\chi_j := 2\sup_n \beta(P_{n,j})$ in (18) or $\chi_j := 4\sup_n \int_S r_{n,j}(s)\,ds$ in (20), respectively, are independent of ρ_j. Hence, for this (in a sense almost optimal) choice of χ_j, the corresponding ρ_j in Theorem 3 is only subject to condition (14). On the other hand, the estimates (16), (17) obtained consist essentially of two summands which can be weighted to some extent by this choice of ρ_j. Roughly speaking, the first summand measures the singularity of $|k|$, and the second summand measures the noncompactness of the operators $k(t,s)$ (which is somewhat weighted on T_j by $\|\cdot\|_{Y''_{\mathbf{R}}}$). Thus, not too surprisingly, the singularity and noncompactness of k determine the measure of noncompactness of $A(B)$, and by our choice of ρ_j, we can define to some extent which of these two properties is more important to us for estimates (16), (17).

Our final remark is that one can formulate the estimates in Theorem 3 also without using quantities which refer to the associate operator $|A|'$.

Remark 3. Estimate (16) implies in the case $B \subseteq B_X$ that

$$\chi_Y(A(B)) \leq \sup_{j\in J}\left(1 + \frac{c_Y \operatorname{diam} B}{1 - 2\rho_j^{-1}\chi_j}\right)(\gamma_S(k) + \gamma_T(k)) + c_Y\|\rho\|_{Y''_{\mathbf{R}}}. \tag{22}$$

This follows from (8), (10).

Proof of Theorem 3. First we show the last assertion. Let $d_j > \chi_j$, and let $e_j \in (0,1)$ be such that $(1 + e_j^{-1})d_j < \rho_j$. Put $P_j := P_{n,j} - P_{n,j}$

(with n as in the last claim), and let $\{v_{j,k} : k = 1, \dots, k_j\}$ be a finite d_j-net for $M_j := \{v \in P_j : |v| \geq \rho_j\}$ such that $|v_{j,k}| \geq \rho_j$. For each j,k, there are $g_{k,j} \in V^*$ with $|g_{j,k}| = 1$ and $|g_{j,k}(v_{j,k})| \geq e_j |v_{j,k}|$. Let $V' \subseteq V^*$ be the finite-dimensional subspace spanned by all $g_{j,k}$. For each $v \in M_j$, we find some k with $|v - v_{j,k}| \leq d_j$, and so

$$\begin{aligned} |g_{j,k}(v)| &\geq |g_{j,k}(v_{j,k})| - |v - v_{j,k}| \\ &\geq e_j|v| - (1+e_j)|v - v_{j,k}| \\ &\geq e_j|v| - (1+e_j)d_j. \end{aligned}$$

Since $(1+e_j)d_j \leq (1+e_j)d_j|v|/\rho_j$, we conclude for $c_j := e_j - \rho_j^{-1}(1+e_j)d_j > 0$ that

$$\sup_{\substack{g \in V' \\ |g| \leq 1}} |g(v)| \geq c_j |v| \quad (v \in P_j,\ |v| \geq \rho_j).$$

If $i\colon V \to (V')^*$ denotes the canonical evaluation embedding, the above inequality means $|i(v)| \geq c_j|v|$, and so we have for all $y \in A(B-B)$ with $c_0 := \min_j c_j$ that

$$|i(y(t))| + c_0\rho(t) \geq \max\{|i(y(t))|, c_0\rho(t)\} \geq c_0|y(t)|.$$

Hence, proceeding completely analogous to the proof of the main result of [**16**], essentially just replacing $\rho_0\chi_E$ in that proof by the function ρ, we can conclude that, for each $c_1 \in (0,1)$,

$$\chi^\circ(A(B)) \leq \max\left\{\frac{c_Y \operatorname{diam} B}{c_0 c_1}(\gamma'_S(k) + \gamma'_T(k)), \frac{c_Y\|\rho\|_{Y''_{\mathbf{R}}}}{1-c_1}\right\}.$$

Since $\inf\{\max\{a/c_1, b/(1-c_1)\} : 0 < c_1 < 1\} = a + b$ for $a, b \geq 0$ (the infimum being attained for $c_1 = a/(a+b)$ in case $a, b > 0$), we conclude

$$\chi^\circ(A(B)) \leq \max_{j \in J} \frac{c_Y \operatorname{diam} B}{e_j - \rho_j^{-1}(1+e_j)d_j}(\gamma'_S(k) + \gamma'_T(k)) + c_Y\|\rho\|_{Y''_{\mathbf{R}}}.$$

Observing that we can choose d_j and e_j arbitrarily close to χ_j and 1 respectively, we obtain (17).

For the proof of the first assertion, observe that it may be assumed without loss of generality that each R_n is contained in a set of the form

$T_0 \times S$ where T_0 is a union of only finitely many of the sets T_j. Indeed, if $J = \{j_1, j_2, \dots\}$ is countably infinite, one can just replace R_n by

$$\widetilde{R}_n := R_n \cap ((T_{j_1} \cup \dots \cup T_{j_n}) \times S)$$

to arrange this. Hence, (16) follows from (17) in the same manner as in [**16**], i.e., by using the approximation theorem of nonlinear Urysohn operators from [**15**] and applying (17) to the approximating operators. Indeed, in view of our additional hypothesis on R_n, we can really apply (17) for the approximating operators (since we can choose a corresponding finite set J). $\square$

Remark 4. All results in this paper hold if the axiom of choice (AC) is replaced by the so-called axiom of dependent choices (which, roughly speaking, allows countably many recursive choices) under the following small changes: in Lemma 1 we assume $U^* \neq \{0\}$ instead of $U \neq \{0\}$ (thanks to a referee for pointing this out), in Theorems 1 and 2, besides the separability of U^*, we also require the separability of U, and in Theorem 3, we additionally require

$$|v| \leq \alpha \sup_{|g|_{V^*} \leq 1} |g(v)| \quad (v \in V) \tag{23}$$

with some constant $\alpha \geq 1$; in the proof of Theorem 3, we must then require $e_j \in (0, 1/\alpha)$. Correspondingly, the requirement $\rho_j > 2\chi_j$ in (14), (18) and (20) must be replaced by $\rho_j > (1+\alpha)\chi_j$, and in the estimates obtained, (16), (17) and (22), the fractions $[(c_Y \operatorname{diam} B)/(1 - 2\rho_j^{-1}\chi_j)]$ have to be replaced by $[(c_Y \alpha \operatorname{diam} B)/(1 - \rho_j^{-1}(1+\alpha)\chi_j)]$.

In the presence of AC, all of these additional hypotheses (with $\alpha = 1$) are satisfied automatically by the Hahn-Banach theorem. However, without AC, it cannot be excluded in general that there are nonseparable Banach spaces U with $U^* = \{0\}$ (see [**8**]), and for every $\alpha_0 > 1$, Banach spaces V for which (23) holds exactly for $\alpha \geq \alpha_0$ (see, e.g., [**4**]).

Acknowledgments. The author wants to thank Chur-Jen Chen, Dirk Werner and the referees for valuable comments and suggestions.

REFERENCES

1. R.R. Akhmerov, M.I. Kamenskiĭ, A.S. Potapov, A.E. Rodkina and B.N. Sadovskiĭ, *Measures of noncompactness and condensing operators*, Birkhäuser, Basel, 1992.

2. J. Appell and M. Väth, *The space of Carathéodory functions*, Trudy Inst. Mat. NAN Belarus. Minsk. **2** (1999), 39–43.

3. ———, *Misure di non compattezza di insiemi ed operatori integrali in spazi di funzioni misurabili*, Rend. Mat. Appl. **21** (2001), 159–190.

4. G. Dirr and M. Väth, *Continuity of near-duality maps and characterizations of ideal spaces of measurable functions*, in *Recent trends in nonlinear analysis*, J. Appell, ed., Birkhäuser, Basel, 2000.

5. H.-P. Heinz, *On the behaviour of measures of noncompactness with respect to differentiation and integration of vector-valued functions*, Nonlinear Anal. **7** (1983), 1351–1371.

6. M.A. Krasnoselskiĭ, *Topological methods in the theory of nonlinear integral equations*, Gostehizdat, Moscow, 1956 (in Russian); Pergamon Press, Oxford 1964, (in English).

7. M.A. Krasnoselskiĭ, P.P. Zabreĭko, E.I. Pustylnik and P.E. Sobolevskiĭ, *Integral operators in spaces of summable functions*, Nauka, Moscow, 1958 (in Russian); Noordhoff, Leyden, 1976 (in English).

8. W.A.J. Luxemburg and M. Väth, *The existence of non-trivial bounded functionals implies the Hahn-Banach extension theorem*, Z. Anal. Anwend. **20** (2001), 267–279.

9. W.A.J. Luxemburg and A.C. Zaanen, *Compactness of integral operators in Banach function spaces*, Math. Ann. **149** (1963), 150–180.

10. H. Mönch, *Boundary value problems for nonlinear ordinary differential equations in Banach spaces*, Nonlinear Anal. **4** (1980), 985–999.

11. H. Mönch and G.-F. von Harten, *On the Cauchy problem for ordinary differential equations in Banach spaces*, Arch. Math. (Basel) **39** (1982), 153–160.

12. W. Schachermayer, *Integral operators on L^p spaces*. I, Indiana Univ. Math. J. **30** (1981), 123–140.

13. M. Väth, *Ideal spaces*, Lect. Notes Math. **1664**, Springer, Berlin, 1997.

14. ———, *Volterra and integral equations of vector functions*, Marcel Dekker, New York, 2000.

15. ———, *Convergence theorems and measures of noncompactness for noncompact Urysohn operators in ideal spaces*, J. Integral Equations Appl. **16** (2004), 67–82.

16. ———, *Compactness estimates for integral operators of vector functions with nonmeasurable kernels*, J. Integral Equations Appl. **18** (2006), 59–81.

17. A.C. Zaanen, *Integration*, North-Holland Publishing Company, Amsterdam, 1967.

18. P.P. Zabreĭko, *Nonlinear integral operators*, Voronezh. Gos. Univ. Trudy Sem. Funktsional. Anal. **8** (1966), 1–148.

19. P.P. Zabreĭko, *Ideal spaces of functions* I, Vestnik Jaroslav. Univ. **8** (1974), 12–52.

20. P.P. Zabreĭko and P. Obradovich, *On the theory of Banach vector function spaces*, Voronezh. Gos. Univ. Trudy Sem. Funk. Anal. (1968), 431–440.

University of Würzburg, Math. Institut, Am Hubland, D-97074, Würzburg, Germany
Email address: vaeth@mathematik.uni-wuerzburg.de