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DISCRETE SUPERCONVERGENCE OF COLLOCATION SOLUTIONS FOR FIRST-KIND VOLTERRA INTEGRAL EQUATIONS

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ABSTRACT. It is known that collocation solutions for first-kind Volterra integral equations based on (discontinuous or continuous) piecewise polynomials cannot exhibit local superconvergence at the points of a uniform mesh. In this paper we present a complete analysis of local superconvergence of such collocation solutions for first-kind Volterra integral equations at non-mesh points. In particular, we discuss (i) the existence of superconvergence points for prescribed collocation points; (ii) the existence of collocation points for prescribed superconvergence points. Numerous examples illustrate the theory.

1. Collocation methods. The convergence properties of collocation and Galerkin solutions in spaces of (discontinuous or continuous) piecewise polynomials for first-kind Volterra integral equations

$$\int_0^t K(t,s)u(s)\,ds = f(t), \quad t \in I := [0,T], \tag{1.1}$$

with bounded (smooth) kernels $K(t,s)$, are now well understood; see [**2–6, 9–12**] and [**7, 8**], respectively. In particular, it was shown in

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[**2, 3, 7**] that local superconvergence of discontinuous collocation or discontinuous Galerkin solutions at the mesh points of a uniform mesh or at the collocation points) is not possible. A similar observation was made in [**12**] for continuous piecewise polynomial solutions (for analogous continuous Galerkin methods, the problem is still open).

However, the discontinuous collocation solution can exhibit local superconvergence (with a gain of one order) at certain points between two mesh points ([**3, 4, 9**]). This holds, for example, if collocation is at the Radau II points and the collocation solution is evaluated at the Gauss points (cf. [**5**, Section 2.4.4]).

It is the aim of this paper to present a complete local superconvergence analysis of discontinuous piecewise polynomial collocation solutions for sets of points $Q_h := \{t_n + d_i h : 0 < d_1 < \cdots < d_m < 1\}$ not belonging to the mesh $I_h := \{t_n := nh : 0 \leq n \leq N \ (Nh = T)\}$. More precisely, we shall systematically study the questions whether, for prescribed collocation points $X_h := \{t_n + c_i h : 0 < c_1 < \cdots < c_m \leq 1\}$, there exist sets Q_h at which the collocation is superconvergent; and whether, for prescribed superconvergence points Q_h (including, for example, the midpoints of the subintervals (t_n, t_{n+1})) there exist sets X_h. Section 2 contains our main results. Their proofs are presented in Sections 3 and 4. The paper concludes with a conjecture (Section 5) and an Appendix containing a number of auxiliary results required in the proofs in Sections 3 and 4.

2. Local superconvergence analysis.

2.1. Discontinuous piecewise polynomial collocation. For a given (uniform) mesh on $I = [0, T]$,

$$I_h := \{t_n := nh : n = 0, 1, \ldots, N \ (t_N = T)\},$$

with $e_n := (t_n, t_{n+1}]$, we denote by

$$S_{m-1}^{(-1)}(I_h) := \{v : v|_{e_n} \in \pi_{m-1} \ (0 \leq n \leq N-1)\} \tag{2.1}$$

the space of (discontinuous) piecewise polynomials of degree not exceeding $m-1$ $(m \geq 1)$. The collocation solution $u_h \in S_{m-1}^{(-1)}(I_h)$ of (1.1) corresponding to the collocation points

$$X_h := \{t_n + c_i h : 0 < c_1 < \cdots < c_m \leq 1 \quad (0 \leq n \leq N-1)\}, \tag{2.2}$$

based on prescribed collocation parameters $\{c_i\}$, is determined by the collocation equation

$$\int_0^t K(t,s)u_h(s)\,ds = f(t), \quad t \in X_h. \tag{2.3}$$

If the given functions K and f in the Volterra integral equation

$$\int_0^t K(t,s)u(s)\,ds = f(t), \quad t \in I := [0,T],$$

are smooth and satisfy $|K(t,t)| > 0$ $(t \in I)$ and $f(0) = 0$, then the collocation equation (2.3) defines a unique collocation approximation u_h to the (unique) solution u of (1.1) whenever $h > 0$ is sufficiently small ([**5**, Section 2.4.1]).

If $K \in C^{m+2}(D)$ (where $D := \{(t,s) : 0 \le s \le t \le T\}$) and $f \in C^{m+2}(I)$ (implying that $u \in C^{m+1}(I)$), then, for all sufficiently small $h > 0$, the collocation error satisfies

$$\begin{aligned} \|u - u_h\|_\infty &:= \sup_{t\in I}\{|u(t) - u_h(t)|\} \\ &\le C \begin{cases} h^m & \text{if } -1 \le \rho_m < 1, \\ h^{m-1} & \text{if } \rho_m = 1, \end{cases} \end{aligned} \tag{2.4}$$

where

$$\rho_m := (-1)^m \prod_{i=1}^m \frac{1-c_i}{c_i}$$

([**2**], [**5**, Section 2.4]). The constant C in (2.4) does not depend on h (but will depend upon the collocation parameters $\{c_i\}$). However, local superconvergence is not possible at the mesh points t_n $(1 \le n \le N)$ (see [**3, 4**]): there does not exist a $p^* \ge m+1$ so that

$$\max_{1\le n\le N} |u(t_n) - u_h(t_n)| \le Ch^{p^*}$$

for some choice of the set $\{c_i\}$. This leads to the obvious question as to whether we can achieve $p^* = m+1$ at certain points *between* mesh points.

In order to make the question more precise, we introduce the set

$$Q_h := \{t_n + d_i h : \ 0 \leq n \leq N-1\},$$

where $\{d_i : 0 < d_1 < \cdots < d_m < 1\}$. We are interested in sets $\{d_i\}$ for which the collocation solution exhibits a higher convergence order than the optimal global order in (2.4). If such a set $\{d_i\}$ exists, then we say that u_h is *locally superconvergent* of order $m+1$ on Q_h.

As indicated in Section 1, it is our aim to solve the following problems:

Problem 1: If the collocation parameters $\{c_i : 0 < c_1 < \cdots < c_m \leq 1\}$ are given, does there exist a set $\{d_i : 0 < d_1 < \cdots < d_m < 1\}$ of *superconvergence parameters* so that the collocation solution satisfies

(2.5)
$$\max_{1\leq i\leq m} |u(t_n + d_i h) - u_h(t_n + d_i h)| \leq C h^{m+1} \quad \text{for all } 0 \leq n \leq N-1?$$

Problem 2: If $\{d_i : 0 < d_1 < \cdots < d_m < 1\}$ are given real numbers so that the $u_h \in S_{m-1}^{(-1)}(I_h)$ has the local superconvergence property (2.5), does there exist a set $\{c_i : 0 < c_1 < \cdots < c_m \leq 1\}$ of *collocation parameters* so that u_h is the collocation solution corresponding to X_h in (2.2) defined by these collocation parameters?

Problem 3: If such sets of superconvergence points $\{d_i\}$ (Problem 1) and collocation points $\{c_i\}$ (Problem 2) exist, are they unique?

2.2. Main results. Recall that the collocation points are given by

$$X_h = \{t_n + c_i h : 0 < c_1 < \cdots < c_m \leq 1 \ (0 \leq n \leq N-1)\},$$

where the collocation parameters $\{c_i\}$ are to be suitably chosen, and that we are interested in sets of *superconvergence points*

$$Q_h := \{t_n + d_i h : 0 < d_1 < \cdots < d_m < 1 \ (0 \leq n \leq N-1)\}$$

for which (2.5) holds.

Case I: $c_m = 1$.

Theorem 2.1. *For given* $\{c_i\}$ *with* $c_m = 1$, *there exists a unique set* $\{d_i\} \subset (0,1)$ *such that local superconvergence* (2.5) *occurs on the corresponding set* Q_h.

Remark. If $\{c_i\}$ are chosen as the zeros of $(s-1)P'_m(2s-1)$ (the $m+1$ Lobatto points in $[0,1]$ minus the point 0) (cf. [**3, 5**]), it is known that the superconvergence occurs at the Gauss points. The above theorem shows that the set of Gauss points is the unique set $\{d_i\}$ corresponding to these collocation parameters.

Theorem 2.2. *Given a set* $\{d_i\}$, *then a set* $\{c_i\}$ *with* $c_m = 1$ *does not always exist. If such a set exists, it is unique.*

In particular, if $\{d_i\}$ *is chosen as the set of Gauss points, or if* m *is odd and* $\{d_i\}$ *is chosen as the zeros of the Chebyshev polynomial of the first kind* $T_m(2s-1)$, *then there exists a unique set of* $\{c_i\}$.

Case II: $c_m < 1$.

Theorem 2.3. *Assume that the given set* $\{c_i\}$ *with* $c_m < 1$ *satisfies*

$$-1 \le (-1)^m \prod_{i=1}^{m} \frac{1-c_i}{c_i} < 1.$$

There exists a set $\{d_i\} \subset (0,1)$ *if and only if* $u^{(m)}(0) = 0$. *If* $(-1)^m \prod_{i=1}^{m}[(1-c_i)/c_i] = 1$, *then such a set* $\{d_i\} \subset (0,1)$ *does not always exist.*

Theorem 2.4. *Given* $\{d_i\}$, *the set* $\{c_i\}$ *may or may not exist. If such a set exists, it may not be unique. In particular, if* $\{d_i\}$ *is chosen as the Gauss points, or if* m *is odd and* $\{d_i\}$ *is chosen as the zeros of the Chebyshev polynomial* $T_m(2s-1)$ *of the first kind, then a set* $\{c_i\}$ *exists but is not unique.*

3. Proofs: Theorems 2.1 and 2.2 ($c_m = 1$)**.** We first analyze the existence and uniqueness of sets $\{d_i\}$ when the collocation parameters $\{c_i\}$ are prescribed.

We choose

$$u_h(t_n + sh) = \sum_{j=1}^{m} \tilde{l}_j(s) U_{n,j}, \quad s \in (0,1], \tag{3.1}$$

with $U_{n,i} := u_h(t_n + d_i h)$ and

$$\tilde{l}_i(s) := \prod_{j=1, j\neq i}^{m} (s - d_j)/(d_i - d_j),$$

as the local representation of $u_h \in S_{m-1}^{(-1)}(I_h)$ on the subinterval e_n. Using a well-known result from interpolation theory (see for example [**1**, page 106–108]), we may write the exact solution $u(t)$ of (1.1) as

$$u(t_n + sh) = \sum_{i=1}^{m} u(t_n + d_i h)\tilde{l}_i(s) + r_n(s) \qquad (s \in (0,1], \quad 0 \le n \le N-1), \tag{3.2}$$

where

$$\begin{aligned} r_n(s) &= h^m \frac{u^{(m)}(t_n + \theta h)}{m!} \prod_{i=1}^{m} (s - d_i) \\ &= h^m \frac{u^{(m)}(t_n)}{m!} \prod_{i=1}^{m} (s - d_i) \\ &\quad + \theta h^{m+1} \frac{u^{(m+1)}(t_n + \xi h)}{m!} \prod_{i=1}^{m} (s - d_i), \end{aligned}$$

with θ, $\xi \in (0,1)$. Thus, by (3.1) we may write the collocation error $e_h(t_n + sh) := u(t_n + sh) - u_h(t_n + sh)$ in the form

$$e_h(t_n + sh) = \sum_{i=1}^{m} e_h(t_n + d_i h)\tilde{l}_i(s) + r_n(s). \tag{3.3}$$

By (1.1) and (2.3), we have at $t_{n,i} := t_n + c_i h$,

$$0 = \int_0^{t_n + c_i h} K(t_{n,i}, s) e_h(s)\, ds = h \sum_{l=0}^{n-1} \int_0^1 K(t_{n,i}, t_l + sh) e_h(t_l + sh)\, ds + h \int_0^{c_i} K(t_{n,i}, t_n + sh) e_h(t_n + sh)\, ds, \tag{3.4}$$

and hence

$$0 = \sum_{l=0}^{n-1} \int_0^1 K(t_{n,i}, t_l + sh) e_h(t_l + sh)\, ds + \int_0^{c_i} K(t_{n,i}, t_n + sh) e_h(t_n + sh)\, ds. \tag{3.5}$$

Rewriting (3.5) with n replaced by $n-1$ and $i = m$, we obtain

$$0 = \sum_{l=0}^{n-2} \int_0^1 K(t_{n-1,m}, t_l + sh) e_h(t_l + sh)\, ds + \int_0^{c_m} K(t_{n-1,m}, t_{n-1} + sh) e_h(t_{n-1} + sh)\, ds, \tag{3.6}$$

and so, by (3.5) and (3.6),

$$\int_0^{c_i} K(t_{n,i}, t_n + sh) e_h(t_n + sh)\, ds = -\sum_{l=0}^{n-1} \int_0^1 K(t_{n,i}, t_l + sh) e_h(t_l + sh)\, ds + \sum_{l=0}^{n-2} \int_0^1 K(t_{n-1,m}, t_l + sh) e_h(t_l + sh)\, ds + \int_0^{c_m} K(t_{n-1,m}, t_{n-1} + sh) e_h(t_{n-1} + sh)\, ds. \tag{3.7}$$

We now distinguish between the two cases $c_m = 1$ and $c_m < 1$.

Case I: $c_m = 1$. Here, (3.7) becomes

$$\begin{aligned}(3.8)\quad &\int_0^{c_i} K(t_{n,i}, t_n + sh)e_h(t_n + sh)\,ds \\ &= -\sum_{l=0}^{n-1}\int_0^1 K(t_{n,i}, t_l + sh)e_h(t_l + sh)\,ds \\ &\quad + \sum_{l=0}^{n-1}\int_0^1 K(t_{n-1,m}, t_l + sh)e_h(t_l + sh)\,ds \\ &= \sum_{l=0}^{n-1}\int_0^1 (K(t_{n-1,m}, t_l + sh) - K(t_{n,i}, t_l + sh))e_h(t_l + sh)\,ds.\end{aligned}$$

Writing

$$(3.9)\quad \begin{gathered} K(t_{n-1,m}, t_l + sh) - K(t_{n,i}, t_l + sh) = -hc_i\partial_1 K(\theta_{n,i}, t_l + sh) \\ (i = 1, \ldots, m),\end{gathered}$$

with $\theta_{n,i} \in (t_{n-1,m}, t_{n,i})$, we are led to

$$\begin{aligned}&\int_0^{c_i} K(t_{n,i}, t_n + sh)e_h(t_n + sh)\,ds \\ &\qquad = -hc_i \sum_{l=0}^{n-1}\int_0^1 \partial_1 K(\cdot, t_l + sh)e_h(t_l + sh)\,ds.\end{aligned}$$

It now follows from (3.3) that

$$\begin{aligned}(3.10)\quad &\sum_{j=1}^{m}\int_0^{c_i} K(t_{n,i}, t_n + sh)\tilde{l}_j(s)\,ds e_h(t_n + d_j h) \\ &= -hc_i \sum_{l=0}^{n-1}\sum_{j=1}^{m}\int_0^1 \partial_1 K(\cdot, t_l + sh)\tilde{l}_j(s)\,ds e_h(t_l + d_j h) + R^1_{n,i},\end{aligned}$$

where

$$\begin{aligned}
R^1_{n,i} &= -hc_i \sum_{l=0}^{n-1} \int_0^1 \partial_1 K(\cdot, t_l + sh)\Big(h^m \frac{u^{(m)}(t_l)}{m!} \prod_{i=1}^m (s-d_i)\Big)\, ds \\
&\qquad - \int_0^{c_i} K(t_{n,i}, t_n + sh)\left(h^m \frac{u^{(m)}(t_n)}{m!} \prod_{i=1}^m (s-d_i)\right) ds + \mathcal{O}(h^{m+1}) \\
&= -h^{m+1} c_i \sum_{l=0}^{n-1} \frac{u^{(m)}(t_l)}{m!} \partial_1 K(t_n, t_l) \int_0^1 \prod_{i=1}^m (s-d_i)\, ds \\
&\qquad - h^m K(t_n, t_n) \frac{u^{(m)}(t_n)}{m!} \int_0^{c_i} \prod_{i=1}^m (s-d_i)\, ds + \mathcal{O}(h^{m+1}).
\end{aligned}$$

The terms involving h^m vanish for $0 \le n \le N-1$ if, and only if,

$$\int_0^{c_i} \prod_{j=1}^m (s-d_j)\, ds = 0 \quad (i = 1, \dots, m). \tag{3.11}$$

If (3.11) holds, then (cf. [**2**–**5**]) $\{d_i\}$ is a set such that $\{t_n + d_i h : 0 \le n \le N-1\}$ is a set of superconvergence points, that is,

$$e_h(t_n + d_i h) = \mathcal{O}(h^{m+1}) \quad (i = 1, \dots, m;\ 0 \le n \le N-1).$$

3.1. Proof of Theorem 2.1. Let $w_m(s) := \prod_{j=1}^m (s-d_j)$. Then

$$\begin{aligned}
\int_0^{c_i} w_m(s)\, ds &= \int_0^{c_i} \prod_{j=1}^m (s-d_j)\, ds \\
&= \int_0^{c_i} (s-d_1)(s-d_2)\cdots(s-d_m)\, ds \\
&= \int_0^{c_i} (s^m - (d_1 + \cdots + d_m) s^{m-1} \\
&\qquad + (d_1 d_2 + \cdots + d_1 d_m + \cdots + d_{m-1} d_m) s^{m-2} \\
&\qquad\qquad + \cdots + (-1)^m d_1 d_2 \cdots d_m)\, ds \\
&= \frac{1}{m+1} c_i^{m+1} - \frac{d_1 + \cdots + d_m}{m} c_i^m \\
&\quad + \frac{d_1 d_2 + \cdots + d_1 d_m + \cdots + d_{m-1} d_m}{m-1} c_i^{m-1} \\
&\quad + \cdots + (-1)^m d_1 d_2 \cdots d_m c_i.
\end{aligned}$$

Therefore, the conditions of (3.11) are equivalent to

$$\begin{aligned}\frac{1}{m+1}c_i^m - \frac{d_1+\cdots+d_m}{m}c_i^{m-1} \\ + \frac{d_1d_2+\cdots+d_1d_m+\cdots+d_{m-1}d_m}{m-1}c_i^{m-2} \\ +\cdots+(-1)^m d_1d_2\cdots d_m = 0 \quad (i=1,2,\dots,m).\end{aligned}$$

Thus, we obtain

$$\begin{pmatrix} c_1^{m-1} & c_1^{m-2} & \cdots & 1 \\ c_2^{m-1} & c_2^{m-2} & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix} \begin{pmatrix} \frac{d_1+\cdots+d_m}{m} \\ -\frac{d_1d_2+\cdots+d_1d_m+\cdots+d_{m-1}d_m}{m-1} \\ \vdots \\ (-1)^{m+1}d_1d_2\cdots d_m \end{pmatrix} = \begin{pmatrix} \frac{c_1^m}{m+1} \\ \frac{c_2^m}{m+1} \\ \vdots \\ \frac{1}{m+1} \end{pmatrix}.$$

The determinant of the coefficient matrix is a Vandermonde determinant. Since $0 < c_1 < \cdots < c_{m-1} < c_m = 1$, its value is different from zero, and thus the solution with the components

$$\tfrac{d_1+\cdots+d_m}{m}, -\tfrac{d_1d_2+\cdots+d_1d_m+\cdots+d_{m-1}d_m}{m-1}, \dots, (-1)^{m+1}d_1d_2\cdots d_m$$

is uniquely determined, as are the numbers

$$d_1+\cdots+d_m, d_1d_2+\cdots+d_1d_m+\cdots+d_{m-1}d_m, \dots, d_1d_2\cdots d_m.$$

Therefore, the polynomial

$$\begin{aligned} w_m(s) &= (s-d_1)(s-d_2)\cdots(s-d_m) \\ &= s^m - (d_1+\cdots+d_m)s^{m-1} \\ &\quad + (d_1d_2+\cdots+d_1d_m+\cdots+d_{m-1}d_m)s^{m-2}+\cdots \\ &+ (-1)^m d_1d_2\cdots d_m \end{aligned}$$

is unique. Because $w_m(s)$ is a polynomial of degree m, it has exactly m (in general complex) zeros. Therefore, there exists a unique set of $\{d_i\}$ in the complex plane.

By [**9**], we know that for a given $\{c_i\}$, there exists a set of $\{d_i\}$ which is unique and given by

$$\prod_{i=1}^{m}(s-d_i) = \frac{(d/ds)\left\{s\prod_{i=1}^{m}(s-c_i)\right\}}{m+1}. \tag{3.12}$$

Since the polynomial $s\prod_{i=1}^{m}(s-c_i)$ has $m+1$ distinct real zeros $0, c_1, \ldots, c_m$, Rolle's theorem implies that between any two consecutive zeros there is (at least) one real zero of $(d/ds)\{s\prod_{i=1}^{m}(s-c_i)$. Thus, there exist m real zeros of $(d/ds)\{s\prod_{i=1}^{m}(s-c_i)$, i.e., of $\prod_{i=1}^{m}(s-d_i)$. Since the latter is a (real) polynomial of degree m, the m real zeros of $\prod_{i=1}^{m}(s-d_i)$ are also unique. This completes the proof of Theorem 2.1.

3.2. Proof of Theorem 2.2. We now look at the existence and uniqueness of sets $\{c_i\}$ (with $c_m = 1$) for a prescribed set $\{d_i\}$. First, we dispose of the following two special cases:

1. If m is odd $\{d_i\}$ is chosen as the zeros of the shifted Legendre polynomial $P_m(2s-1)$, we know ([**3, 5**]) that the set $\{c_i\}$ is unique.

2. If m is odd $\{d_i\}$ is chosen as the zeros of the Chebyshev polynomial of the first kind, $T_m(2s-1)$, then the set $\{c_i\}$ is also unique. In order not to interrupt the flow of the general proof, we give the technical details of the proof of this assertion in the Appendix.

We now turn to the *general case*. If the set of $\{c_i\}$ exists for given $\{d_i\}$, then

$$\int_0^{c_i} w_m(s)\,ds = 0$$

can hold only if $\{c_i\}$ and $\{d_i\}$ interlace (cf. [**3, 5**]). Thus, if there exists another set of $\{\widetilde{c}_i\}$ for given $\{d_i\}$, then $\{\widetilde{c}_i\}$ and $\{d_i\}$ also interlace, and

$$\int_{c_i}^{\widetilde{c}_i} w_m(s)\,ds = 0.$$

This is a contradiction because both c_i and $\widetilde{c}_i$ lie in (d_i, d_{i+1}). Therefore, if the set of $\{c_i\}$ exists, it must be unique.

However, given $\{d_i\}$, a set of $\{c_i\}$ does not always exist, as the following examples show.

Example 1. Let $m = 1$. Then (3.11) becomes

$$\int_0^1 (s - d_1)\,ds = 0,$$

and this yields the unique solution $d_1 = 1/2$. So, if d_1 is not taken as $1/2$, then a $c_1 \in (0, 1]$ does not exist.

Example 2. Let $m = 2$, $d_1 = 1/5$, $d_2 = 7/9$. We readily check that there exists a unique set $\{c_i\}$ with $c_1 = 7/15$, $c_2 = 1$.

Example 3. Let $m = 3$, $d_1 = 1/3$, $d_2 = 1/2$, $d_3 = 2/3$. We can check that a set $\{c_i\}$ does not exist.

This completes the proof of Theorem 2.2. □

Case II: $c_m < 1$**.** Using (3.7) and (3.9), we may write

$$\begin{aligned}\int_0^{c_i} & K(t_{n,i}, t_n + sh) e_h(t_n + sh)\,ds \\ &= -hc_i \sum_{l=0}^{n-2} \int_0^1 \partial_1 K(\cdot, t_l + sh) e_h(t_l + sh)\,ds \\ &\quad - \int_0^1 K(t_{n,i}, t_{n-1} + sh) e_h(t_{n-1} + sh)\,ds \\ &\quad + \int_0^{c_m} K(t_{n-1,m}, t_{n-1} + sh) e_h(t_{n-1} + sh)\,ds.\end{aligned}$$

By (3.3) this becomes

(3.13)
$$\begin{aligned}\sum_{j=1}^{m} & \int_0^{c_i} K(t_{n,i}, t_n + sh) \tilde{l}_j(s)\,ds e_h(t_{n,j}) \\ &= -hc_i \sum_{l=0}^{n-2} \sum_{j=1}^{m} \int_0^1 \partial_1 K(\cdot, t_l + sh) \tilde{l}_j(s)\,ds e_h(t_{l,j})\end{aligned}$$

$$
\begin{aligned}
&- \sum_{j=1}^{m} \int_0^1 K(t_{n,i}, t_{n-1} + sh)\tilde{l}_j(s)\, ds e_h(t_{n-1,j}) \\
&+ \sum_{j=1}^{m} \int_0^{c_m} K(t_{n-1,m}, t_{n-1} + sh)\tilde{l}_j(s)\, ds e_h(t_{n-1,j}) + R^2_{n,i},
\end{aligned}
$$

where

$$
\begin{aligned}
R^2_{n,i} = &-hc_i \sum_{l=0}^{n-2} \int_0^1 \partial_1 K(\cdot, t_l + sh)\left(h^m \frac{u^{(m)}(t_l)}{m!} \prod_{i=1}^{m}(s - d_i)\right) ds \\
&- \int_0^1 K(t_{n,i}, t_{n-1} + sh)\left(h^m \frac{u^{(m)}(t_{n-1})}{m!} \prod_{i=1}^{m}(s - d_i)\right) ds \\
&+ \int_0^{c_m} K(t_{n-1,m}, t_{n-1} + sh)\left(h^m \frac{u^{(m)}(t_{n-1})}{m!} \prod_{i=1}^{m}(s - d_i)\right) ds \\
&- \int_0^{c_i} K(t_{n,i}, t_n + sh)\left(h^m \frac{u^{(m)}(t_n)}{m!} \prod_{i=1}^{m}(s - d_i)\right) ds + \mathcal{O}(h^{m+1}) \\
= &-h^m c_i \sum_{l=0}^{n-2} h\partial_1 K(t_n, t_l) \frac{u^{(m)}(t_l)}{m!} \int_0^1 \prod_{i=1}^{m}(s - d_i)\, ds \\
&- h^m \frac{u^{(m)}(t_{n-1})}{m!} K(t_{n-1}, t_{n-1}) \int_0^1 \prod_{i=1}^{m}(s - d_i)\, ds \\
&+ h^m \frac{u^{(m)}(t_{n-1})}{m!} K(t_{n-1}, t_{n-1}) \int_0^{c_m} \prod_{i=1}^{m}(s - d_i)\, ds \\
&- h^m \frac{u^{(m)}(t_{n-1})}{m!} K(t_{n-1}, t_{n-1}) \int_0^{c_i} \prod_{i=1}^{m}(s - d_i)\, ds + \mathcal{O}(h^{m+1}).
\end{aligned}
$$

The terms involving h^m vanish if, and only if,

$$
(3.14) \quad \int_0^{c_i} \prod_{j=1}^{m}(s - d_j)\, ds + \int_{c_m}^1 \prod_{j=1}^{m}(s - d_j)\, ds = 0 \quad (i = 1, 2, \ldots, m).
$$

Set

$$l_j(s) := \sum_{k=1,k\neq j}^{m} \frac{s-c_k}{c_j-c_k}, \quad b^T := \left(\int_0^1 l_1(s)\,ds, \dots, \int_0^1 l_m(s)\,ds\right),$$

$$A := (a_{ij})_{m\times m}, \quad a_{ij} := \int_0^{c_i} l_j(s)\,ds,$$

$$\widetilde{b}^T := \left(\int_0^1 \tilde{l}_1(s)\,ds, \dots, \int_0^1 \tilde{l}_m(s)\,ds\right),$$

$$\widetilde{A} := (\widetilde{a}_{ij})_{m\times m}, \quad \widetilde{a}_{ij} := \int_0^{c_i} \tilde{l}_j(s)\,ds,$$

and $e := (1, \dots, 1)^T$. If the d_i are mutually distinct, then there exists an invertible matrix $P = (p_{ij})_{m\times m}$, such that

$$\tilde{l}_j(s) := \sum_{k=1}^{m} p_{jk} l_k(s).$$

Thus, since

$$\widetilde{a}_{ij} = \int_0^{c_i} \tilde{l}_j(s)\,ds = \sum_{k=1}^{m} \int_0^{c_i} p_{jk} l_k(s)\,ds = \sum_{k=1}^{m} a_{ik} p_{jk},$$

$$\widetilde{b}_j = \int_0^1 \tilde{l}_j(s)\,ds = \sum_{k=1}^{m} \int_0^1 p_{jk} l_k(s)\,ds = \sum_{k=1}^{m} p_{jk} b_k,$$

we obtain

$$\widetilde{A} = AP^T, \qquad \widetilde{b} = Pb,$$

and hence

$$\widetilde{b}^T \widetilde{A}^{-1} e = (Pb)^T (AP^T)^{-1} e = b^T P^T (P^T)^{-1} A^{-1} e = b^T A^{-1} e. \tag{3.15}$$

If (3.14) holds, we can now state (3.13) in the form

$$\begin{aligned}(K(t_n,t_n) + O(h)) \sum_{j=1}^{m} \widetilde{a}_{ij} e_h(t_{n,j}) &= \sum_{j=1}^{m} K(t_{n-1},t_{n-1})(\widetilde{a}_{mj} - \widetilde{b}_j) e_h(t_{n-1,j}) \\ &\quad + h \sum_{l=0}^{n-1} \sum_{j=1}^{m} \int_0^1 \widehat{K}^{n,i}(\cdot, t_l + sh) \tilde{l}_j(s)\,ds e_h(t_{l,j}) \\ &\quad + \mathcal{O}(h^{m+1}),\end{aligned}$$

with the obvious meaning of $\widehat{K}^{n,i}$. It then follows that

$$\sum_{j=1}^{m} \widetilde{a}_{ij} e_h(t_{n,j}) = \sum_{j=1}^{m} (\widetilde{a}_{mj} - \widetilde{b}_j) e_h(t_{n-1,j}) + h \sum_{l=0}^{n-1} \sum_{j=1}^{m} \int_0^1 \widetilde{K}^{n,i}(\cdot, t_l + sh) \tilde{l}_j(s)\, ds e_h(t_{l,j}) + O(h^{m+1}),$$

where the meaning of $\widetilde{K}^{n,i}$ is again clear. Defining

$$E_n := (e_h(t_{n,1}), \ldots, e_h(t_{n,m}))^T,$$
$$\widetilde{K}^{n,l} := \left(\int_0^1 \widetilde{K}^{n,i}(\cdot, t_l + sh) \tilde{l}_j(s)\, ds \right)_{m \times m},$$

we obtain

$$\widetilde{A} E_n = \widetilde{M_1} E_{n-1} + h \sum_{l=0}^{n-1} \widetilde{K}^{n,l} E_l + \mathcal{O}(h^{m+1}),$$

where $\widetilde{M_1} := e e_m^T \widetilde{A} - e \widetilde{b}^T$ and $e_m := (0, \ldots, 0, 1)^T \in \mathbf{R}^m$. The invertibility of $\widetilde{A}$ implies that

$$E_n = \widetilde{A}^{-1} \widetilde{M_1} E_{n-1} + h \sum_{l=0}^{n-1} \widetilde{A}^{-1} \widetilde{K}^{n,l} E_l + \mathcal{O}(h^{m+1}).$$

The matrix $\widetilde{M_1}$ has rank one with the single non-zero eigenvalue $\sum_{i=1}^m (\widetilde{a}_{mi} - \widetilde{b}_i)$. Thus, the single non-zero eigenvalue of the matrix $\widetilde{A}^{-1}\widetilde{M_1}$ is

$$\begin{aligned}
\lambda = \mathrm{tr}\,(\widetilde{A}^{-1}\widetilde{M_1}) &= \sum_{i=1}^{m} (\widetilde{a}_{mi} - \widetilde{b}_i) \sum_{j=1}^{m} \widetilde{w}_{ij} \\
&= \sum_{i=1}^{m} \sum_{j=1}^{m} \widetilde{a}_{mi} \widetilde{w}_{ij} - \sum_{i=1}^{m} \widetilde{b}_i \sum_{j=1}^{m} \widetilde{w}_{ij} \\
&= 1 - \widetilde{b}^T \widetilde{A}^{-1} e = 1 - b^T A^{-1} e \\
&= R(\infty) = (-1)^m \prod_{i=1}^{m} \frac{1 - c_i}{c_i},
\end{aligned}$$

where the $\widetilde{w}_{ij}$ are the entries of $\widetilde{A}^{-1}$, $R(x) := 1 + xb^T(I - Ax)^{-1}e$, and where we have used (3.15), Lemma 1 and Corollary of [**2**].

If $n = 0$, then we have, by (3.5),

$$\int_0^{c_i} K(t_{0,i}, t_0 + sh)e_h(t_0 + sh)\, ds = 0,$$

and hence

$$\sum_{j=1}^{m} \widetilde{a}_{ij} e_h(t_{0,j}) = -\frac{h^m u^{(m)}(0)}{m!} \int_0^{c_i} \prod_{i=1}^{m}(s - d_i)\, ds + \mathcal{O}(h^{m+1}).$$

This reveals that $e_h(t_0 + d_i h) = \mathcal{O}(h^{m+1})$ if, and only if

$$u^{(m)}(0) = 0, \quad \text{or} \quad \int_0^{c_i} \prod_{j=1}^{m}(s - d_j)\, ds = 0.$$

Motivated by [**7**], we take $u^{(m)}(0) = 0$ (we will explain why we give this restriction in subsection 4.1).

If $1 \le n \le N - 1$, using the techniques of [**5**], we obtain that, under the condition of (3.14) and $u^{(m)}(0) = 0$:

- if $-1 \le (-1)^m \prod_{i=1}^{m}[(1 - c_i)/c_i] < 1$, then $e_h(t_n + d_i h) = \mathcal{O}(h^{m+1})$;
- if $(-1)^m \prod_{i=1}^{m}[(1 - c_i)/c_i] = 1$, then $e_h(t_n + d_i h) = \mathcal{O}(h^m)$.

We now see that the condition in (3.14) is crucial to the superconvergence analysis.

4. Proofs of Theorems 2.3 and 2.4 ($c_m < 1$).

4.1. Proof of Theorem 2.3. We now turn to the analysis of the existence and uniqueness of sets $\{d_i\}$ for prescribed $\{c_i\}$ with $c_m < 1$. Assume that $(-1)^m \prod_{i=1}^{m}[(1 - c_i)/c_i] \ne 1$. Then the desired set of superconvergence points $\{d_i\}$ is determined by the solution of the algebraic equation

$$\prod_{i=1}^{m}(s - d_i) = \frac{1}{m+1} \frac{d}{ds}\left(\left[s - \frac{1}{1 - \prod_{i=1}^{m} \frac{-c_i}{1-c_i}} \right] \prod_{i=1}^{m}(s - c_i) \right).$$

This equation always possesses m real solutions d_i since the polynomial

$$\left(s - \frac{1}{1 - \prod_{i=1}^{m} -c_i/(1-c_i)}\right)\prod_{i=1}^{m}(s - c_i)$$

possesses the $m+1$ real zeros

$$\left(1 - \prod_{i=1}^{m} \frac{-c_i}{1-c_i}\right)^{-1}, \quad c_1, \ldots, c_m.$$

Thus, for a given $\{c_i\}$, a *real* set $\{d_i\}$ exists. In general,

$$\begin{aligned}
&\int_0^{c_i} w_m(s)\,ds + \int_{c_m}^{1} w_m(s)\,ds \\
&= \int_0^{c_i} \prod_{j=1}^{m}(s - d_j)\,ds + \int_{c_m}^{1} \prod_{j=1}^{m}(s - d_j)\,ds \\
&= \int_0^{c_i} (s-d_1)(s-d_2)\cdots(s-d_m)ds \\
&\quad + \int_{c_m}^{1} (s-d_1)(s-d_2)\cdots(s-d_m)\,ds \\
&= \int_0^{c_i} (s^m - (d_1 + \cdots + d_m)s^{m-1} \\
&\qquad + (d_1 d_2 + \cdots + d_1 d_m + \cdots + d_{m-1}d_m)s^{m-2} \\
&\qquad\qquad + \cdots + (-1)^m d_1 d_2 \cdots d_m)\,ds \\
&\quad + \int_{c_m}^{1} (s^m - (d_1 + \cdots + d_m)s^{m-1} \\
&\qquad + (d_1 d_2 + \cdots + d_1 d_m + \cdots + d_{m-1}d_m)s^{m-2} \\
&\qquad\qquad + \cdots + (-1)^m d_1 d_2 \cdots d_m)\,ds \\
&= \frac{1}{m+1}c_i^{m+1} - \frac{d_1 + \cdots + d_m}{m}c_i^m \\
&\quad + \frac{d_1 d_2 + \cdots + d_1 d_m + \cdots + d_{m-1}d_m}{m-1}c_i^{m-1} + \cdots \\
&\quad + (-1)^m d_1 d_2 \cdots d_m c_i + \frac{1}{m+1} - \frac{d_1 + \cdots + d_m}{m}
\end{aligned}$$

$$
\begin{aligned}
&+\frac{d_1d_2+\cdots+d_1d_m+\cdots+d_{m-1}d_m}{m-1}+\cdots\\
&+(-1)^m d_1d_2\cdots d_m-\frac{1}{m+1}c_m^{m+1}\\
&+\frac{d_1+\cdots+d_m}{m}c_m^m-\frac{d_1d_2+\cdots+d_1d_m+\cdots+d_{m-1}d_m}{m-1}c_m^{m-1}\\
&+\cdots+(-1)^{m+1}d_1d_2\cdots d_mc_m.
\end{aligned}
$$

The conditions in (3.14) are equivalent to

$$
\begin{aligned}
&\frac{c_i^{m+1}+1-c_m^{m+1}}{m+1}-\frac{d_1+\cdots+d_m}{m}(c_i^m+1-c_m^m)\\
&\quad+\frac{d_1d_2+\cdots+d_1d_m+\cdots+d_{m-1}d_m}{m-1}(c_i^{m-1}+1-c_m^{m-1})\\
&\quad+\cdots+(-1)^m d_1d_2\cdots d_m(c_i+1-c_m)=0\quad (i=1,2,\dots,m),
\end{aligned}
$$

and they can be expressed in the form

$$
\begin{aligned}
(4.1)\quad &\begin{pmatrix}
c_1^m+1-c_m^m & c_1^{m-1}+1-c_m^{m-1} & \cdots & c_1+1-c_m\\
c_2^m+1-c_m^m & c_2^{m-1}+1-c_m^{m-1} & \cdots & c_2+1-c_m\\
\vdots & \vdots & & \vdots\\
1 & 1 & \cdots & 1
\end{pmatrix}\\
&\times\begin{pmatrix}
\frac{d_1+\cdots+d_m}{m}\\
-\frac{d_1d_2+\cdots+d_1d_m+\cdots+d_{m-1}d_m}{m-1}\\
\vdots\\
(-1)^{m+1}d_1d_2\cdots d_m
\end{pmatrix}
=\begin{pmatrix}
\frac{c_1^{m+1}+1-c_m^{m+1}}{m+1}\\
\frac{c_2^{m+1}+1-c_m^{m+1}}{m+1}\\
\vdots\\
\frac{1}{m+1}
\end{pmatrix}.
\end{aligned}
$$

We see that the determinant of the coefficient matrix,

$$
\begin{aligned}
\det &\begin{pmatrix}
c_1^m+1-c_m^m & c_1^{m-1}+1-c_m^{m-1} & \cdots & c_1+1-c_m\\
c_2^m+1-c_m^m & c_2^{m-1}+1-c_m^{m-1} & \cdots & c_2+1-c_m\\
\vdots & \vdots & & \vdots\\
1 & 1 & \cdots & 1
\end{pmatrix}\\
&=\det\begin{pmatrix}
c_1^m-c_m^m & c_1^{m-1}-c_m^{m-1} & \cdots & c_1-c_m\\
c_2^m-c_m^m & c_2^{m-1}-c_m^{m-1} & \cdots & c_2-c_m\\
\vdots & \vdots & & \vdots\\
1 & 1 & \cdots & 1
\end{pmatrix},
\end{aligned}
$$

can be zero for some c_i. This implies that, in general, a desired set $\{d_i\}$ may not exist if $(-1)^m \prod_{i=1}^m [(1-c_i)/c_i] = 1$.

Remark. If, for $n = 0$, we take

$$\int_0^{c_i} \prod_{j=1}^m (s - d_j)\, ds = 0$$

instead of $u^{(m)}(0) = 0$, then by (3.14), the d_i have to satisfy the $m+1$ conditions

$$\int_0^{c_i} \prod_{j=1}^m (s - d_j)\, ds = 0 \quad (i = 1, 2, \dots, m), \qquad \int_0^1 \prod_{j=1}^m (s - d_j)\, ds = 0,$$

and now (4.1) becomes

$$\begin{pmatrix} c_1^{m-1} & c_1^{m-2} & \cdots & 1 \\ \vdots & \vdots & & \vdots \\ c_m^{m-1} & c_m^{m-2} & \cdots & 1 \\ 1 & 1 & \dots & 1 \end{pmatrix} \begin{pmatrix} \frac{d_1 + \cdots + d_m}{m} \\ -\frac{d_1 d_2 + \cdots + d_1 d_m + \cdots + d_{m-1} d_m}{m-1} \\ \vdots \\ (-1)^{m+1} d_1 d_2 \cdots d_m \end{pmatrix} = \begin{pmatrix} \frac{c_1^m}{m+1} \\ \vdots \\ \frac{c_2^m}{m+1} \\ \frac{1}{m+1} \end{pmatrix}.$$

This system of linear equations has no solutions, so $\{d_i\}$ does not exist. Therefore, in order for the superconvergence results to hold, the restriction $u^{(m)}(0) = 0$ is necessary.

Example 4. Let $m = 1$. Then (3.14) reduces to

$$\int_0^1 (s - d_1)\, ds = 0,$$

implying that $d_1 = 1/2$. So there exists a unique d_1.

In the following, we give two examples showing that if $(-1)^m \prod_{i=1}^m [(1-c_i)/c_i] = 1$, the set of $\{d_i\}$ may not exist.

Example 5. Let $m = 2$, $c_1 = 1/3$, $c_2 = 2/3$. Then the corresponding conditions (3.14) are

$$\int_0^{1/3} (s - d_1)(s - d_2)\, ds + \int_{2/3}^1 (s - d_1)(s - d_2)\, ds = 0$$

and

$$\int_0^1 (s-d_1)(s-d_2)\,ds = 0.$$

Since they are equivalent to

$$\frac{20}{81} - \frac{1}{3}(d_1+d_2) + \frac{2}{3}d_1 d_2 = 0$$

and

$$\frac{1}{3} - \frac{1}{2}(d_1+d_2) + d_1 d_2 = 0,$$

we see that there are no solutions for the unknowns d_1 and d_2.

Example 6. When $m=2$ and $c_1 = [(2-\sqrt{2})/4]$, $c_2 = [(2+\sqrt{2})/4]$ (the zeros of the Chebyshev polynomial $T_2(2s-1)$), the equations (3.14) become

$$\int_0^{[2-\sqrt{2}]/4} (s-d_1)(s-d_2)\,ds + \int_{[2+\sqrt{2}]/4}^1 (s-d_1)(s-d_2)\,ds = 0$$

and

$$\int_0^1 (s-d_1)(s-d_2)\,ds = 0.$$

They are equivalent to

$$\frac{16-7\sqrt{2}}{48} - \frac{2-\sqrt{2}}{4}(d_1+d_2) + \frac{2-\sqrt{2}}{2}d_1 d_2 = 0$$

and

$$\frac{1}{3} - \frac{1}{2}(d_1+d_2) + d_1 d_2 = 0.$$

This reveals that there are no solutions for the unknowns d_1 and d_2.

In the next two examples, we choose $m=3$ and give the sets $\{d_i\}$ when $\{c_i\}$ is either the set of Gauss points or the set of the zeros of the Chebyshev polynomial of the first kind.

Example 7. Let $m=3$ and $c_1 = [(5-\sqrt{15})/10]$, $c_2 = 1/2$, $c_3 = [(5+\sqrt{15})/10]$ (Gauss points). Then we easily verify that we obtain the (unique) solution $d_1 = [(10 - \sqrt{30})/20]$, $d_2 = 1/2$, $d_3 = [(10 + \sqrt{30})/20]$.

Example 8. If $m = 3$ and $c_1 = [(2 - \sqrt{3})/4]$, $c_2 = 1/2$, $c_3 = [(2 + \sqrt{3})/4]$ (the zeros of the Chebyshev polynomial of the first kind), we have the solution $d_1 = [(4 - \sqrt{6})/8]$, $d_2 = 1/2$, $d_3 = [(4 + \sqrt{6})/8]$.

The above examples show that, for a given $\{c_i\}$, a set $\{d_i\}$ may or may not exist. This completes the proof of Theorem 2.3.

4.2. Proof of Theorem 2.4. Theorem 2.4 is concerned with the existence and uniqueness of sets of collocation parameters $\{c_i\}$, with $c_m < 1$, when the set $\{d_i\}$ of superconvergence points Q_h is prescribed.

We have already seen an example showing that the set of $\{c_i\}$ may not exist. Here we add two examples to illustrate that, for a given $\{d_i\}$, there may be many sets $\{c_i\}$ of collocation parameters.

Example 9. Let $m = 2$ and $d_1 = (3 - \sqrt{3})/6$, $d_2 = (3 + \sqrt{3})/6$ (Gauss points). Then the corresponding equations (3.14),

$$\int_0^{c_1} \left(s - \left(\frac{1}{2} - \frac{1}{6}\sqrt{3}\right)\right)\left(s - \left(\frac{1}{2} + \frac{1}{6}\sqrt{3}\right)\right) ds + \int_{c_2}^{1} \left(s - \left(\frac{1}{2} - \frac{1}{6}\sqrt{3}\right)\right)\left(s - \left(\frac{1}{2} + \frac{1}{6}\sqrt{3}\right)\right) ds = 0$$

and

$$\int_0^{1} \left(s - \left(\frac{1}{2} - \frac{1}{6}\sqrt{3}\right)\right)\left(s - \left(\frac{1}{2} + \frac{1}{6}\sqrt{3}\right)\right) ds = 0,$$

yield many sets $\{c_i\}$. We have, for example,

$$c_1 = \frac{2}{3}, \quad c_2 = \frac{1}{12}(5 + \sqrt{33}),$$

and

$$c_1 = \frac{3}{4}, \quad c_2 = \frac{1}{8}(3 + \sqrt{13}).$$

Example 10. The result of Example 9 remains true for Gauss points with $m \geq 3$ Gauss points. Let $P_m(s) := (1/m!)(d^m/ds^m)[(s^2-s)^m]$ be the shifted Legendre polynomial of degree m. Equation (3.14) then becomes

$$\int_0^{c_i} P_m(s)\,ds + \int_{c_m}^1 P_m(s)ds = 0 \quad (i = 1, 2, \ldots, m),$$

or, equivalently,

$$\int_{c_i}^{c_{i+1}} P_m(s)\,ds = 0 \quad (i = 1, 2, \ldots, m-1) \tag{4.2}$$

and

$$\int_0^1 P_m(s)\,ds = 0. \tag{4.3}$$

The last condition (4.3) always holds. So we only need to look for $\{c_i\}$ satisfying (4.2). Due to

$$(4m+2)P_m(s) = \frac{d}{ds}(P_{m+1}(s) - P_{m-1}(s)) \tag{4.4}$$

(see [**7**]), equation (4.2) is equivalent to

$$P_{m+1}(c_{i+1}) - P_{m-1}(c_{i+1}) = P_{m+1}(c_i) - P_{m-1}(c_i) \quad (i = 1, 2, \ldots, m-1).$$

Let $Q_{m+1}(s) := P_{m+1}(s) - P_{m-1}(s)$. Then we need to investigate if there exist m distinct points c_i in $(0, 1)$, such that the values $Q_{m+1}(c_i)$ are all the same. The derivative of $Q_{m+1}(s)$ is

$$Q'_{m+1}(s) = P'_{m+1}(s) - P'_{m-1}(s),$$

which, by (4.4), equals 0 if and only if $P_m(s) = 0$. Thus, the zeros of $Q'_{m+1}(s)$ are the m Gauss points; we will denote them by x_i $(i = 1, 2, \ldots, m)$. It is well known that Bonnet's recursion formula for the shifted Legendre polynomials is

$$(m+1)P_{m+1}(s) = (2m+1)sP_m(s) - mP_{m-1}(s).$$

Thus, at x_i we have $(m+1)P_{m+1}(x_i) = -mP_{m-1}(x_i)$, and this implies that

$$(m+1)[P_{m+1}(x_i) - P_{m-1}(x_i)] = -(2m+1)P_{m-1}(x_i).$$

Since $P_m(0) = (-1)^m$, it follows from the interlacing property of the zeros of P_m and P_{m+1} that the following results hold:

1. If m is odd, then

$$\begin{aligned} Q_{m+1}(x_i) &= P_{m+1}(x_i) - P_{m-1}(x_i)) \\ &= -\frac{2m+1}{m+1}P_{m-1}(x_i)\begin{cases} < 0 & \text{if } i \text{ is odd;} \\ > 0 & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

2. If m is even, then

$$\begin{aligned} Q_{m+1}(x_i) &= P_{m+1}(x_i) - P_{m-1}(x_i) \\ &= -\frac{2m+1}{m+1}P_{m-1}(x_i)\begin{cases} > 0 & \text{if } i \text{ is odd;} \\ < 0 & \text{if } i \text{ is even.} \end{cases} \end{aligned}$$

Also, the identities

$$Q_{m+1}(0) = P_{m+1}(0) - P_{m-1}(0) = \int_0^0 (4m+2)P_m(s)\,ds = 0$$

and

$$Q_{m+1}(1) = P_{m+1}(1) - P_{m-1}(1)) = \int_0^1 (4m+2)P_m(s)\,ds = 0,$$

show that we must have $Q_{m+1}(0) = Q_{m+1}(1) = 0$. Therefore, $Q_{m+1}(s)$ must assume the same value in at least m distinct points in $(0,1)$ (in fact, we can find $m+1$ such points if m is odd).

Example 11. If $\{d_i\}$ is given by the zeros of the Chebyshev polynomial of the first kind, we will show that the set $\{c_i\}$ exists but is not unique.

To do this, we first recall from (A) in the Appendix following Section 5 that a sufficient and necessary condition for

$$\int_0^1 \cos[m \arccos(2s-1)]\,ds = 0$$

is that m be odd; if $m = 1$, then $d_1 = 1/2$, and thus superconvergence occurs for any $c_1 \in (0, 1)$. We will therefore from now on assume that $m > 1$ is odd.

In the following, we investigate the condition for

$$\int_0^{c_i} \cos[m \arccos(2s-1)]\, ds + \int_{c_m}^1 \cos[m \arccos(2s-1)]\, ds = 0 \quad (i = 1, 2, \ldots, m-1)$$

to hold. By (C) of the Appendix, we have

$$\begin{aligned}
&\int_0^{c_i} \cos[m \arccos(2s-1)]\, ds + \int_{c_m}^1 \cos[m \arccos(2s-1)]\, ds \\
&\quad = \frac{1}{4(m+1)}\{\cos[(m+1)\arccos(2c_i - 1)] \\
&\qquad\qquad - \cos[(m+1)\arccos(-1)]\} \\
&\qquad - \frac{1}{4(m-1)}\{\cos[(m-1)\arccos(2c_i - 1)] \\
&\qquad\qquad - \cos[(m-1)\arccos(-1)]\} \\
&\qquad + \frac{1}{4(m+1)}\{\cos[(m+1)\arccos(1)] \\
&\qquad\qquad - \cos[(m+1)\arccos(2c_m - 1)]\} \\
&\qquad - \frac{1}{4(m-1)}\{\cos[(m-1)\arccos(1)] \\
&\qquad\qquad - \cos[(m-1)\arccos(2c_m - 1)]\} \\
&\quad = \frac{1}{4(m+1)}\{\cos[(m+1)\arccos(2c_i - 1)] - 1\} \\
&\qquad - \frac{1}{4(m-1)}\{\cos[(m-1)\arccos(2c_i - 1)] - 1\} \\
&\qquad + \frac{1}{4(m+1)}\{1 - \cos[(m+1)\arccos(2c_m - 1)]\} \\
&\qquad - \frac{1}{4(m-1)}\{1 - \cos[(m-1)\arccos(2c_m - 1)]\} \\
&\quad = \frac{1}{4(m+1)}\{\cos[(m+1)\arccos(2c_i - 1)]
\end{aligned}$$

$$
\begin{aligned}
&\qquad\qquad\qquad\qquad - \cos[(m+1)\arccos(2c_m-1)]\} \\
&- \frac{1}{4(m-1)}\{\cos[(m-1)\arccos(2c_i-1)] \\
&\qquad\qquad\qquad\qquad - \cos[(m-1)\arccos(2c_m-1)]\}.
\end{aligned}
$$

Let $x_i := \arccos(2c_i - 1)$. Then the conditions

$$
\begin{aligned}
&\int_0^{c_i} \cos[m\arccos(2s-1)]\,ds \\
&\qquad + \int_{c_m}^1 \cos[m\arccos(2s-1)]\,ds = 0 \quad (i = 1, 2, \dots, m-1)
\end{aligned}
$$

are equivalent to

$$
\frac{\cos[(m+1)x_i] - \cos[(m+1)x_m]}{m+1} = \frac{\cos[(m-1)x_i] - \cos[(m-1)x_m]}{m-1}
$$

$(i = 1, 2, \dots, m-1)$. Rewriting this as

$$
\begin{aligned}
&m\cos[(m+1)x_i] - \cos[(m+1)x_i] \\
&\qquad - m\cos[(m+1)x_m] + \cos[(m+1)x_m] \\
&\qquad = m\cos[(m-1)x_i] + \cos[(m-1)x_i] \\
&\qquad\quad - m\cos[(m-1)x_m] - \cos[(m-1)x_m]
\end{aligned}
$$

$(i = 1, 2, \dots, m-1)$, we derive the equivalent identities

$$
\begin{aligned}
&m\{\cos[(m+1)x_i] - \cos[(m-1)x_i]\} \\
&\qquad - \{\cos[(m+1)x_i] + \cos[(m-1)x_i]\} \\
&\qquad = m\{\cos[(m+1)x_m] - \cos[(m-1)x_m]\} \\
&\qquad\quad - \{\cos[(m+1)x_m] + \cos[(m-1)x_m]\}
\end{aligned}
$$

or

$$
\begin{aligned}
&-2m\sin(mx_i)\sin(x_i) - 2\cos(mx_i)\cos(x_i) \\
&\qquad = -2m\sin(mx_m)\sin(x_m) - 2\cos(mx_m)\cos(x_m)
\end{aligned}
$$

$(i = 1, 2, \dots, m-1)$. The latter equations can also be written as

$$
\begin{aligned}
&m\sin(mx_i)\sin(x_i) + \cos(mx_i)\cos(x_i) \\
&\qquad = m\sin(mx_m)\sin(x_m) + \cos(mx_m)\cos(x_m)
\end{aligned}
$$

$(i = 1, 2, \dots, m-1)$.

Set $y(x) := m\sin(mx)\sin(x) + \cos(mx)\cos(x)$, $x \in (0, \pi)$. Hence, we need to investigate if there exist m distinct points $x_i \in (0, \pi)$ such that all values $y(x_i)$ coincide.

It follows from

$$\begin{aligned} y'(x) &= m^2\cos(mx)\sin(x) + m\sin(mx)\cos(x) \\ &\quad - m\sin(mx)\cos(x) - \cos(mx)\sin(x) \\ &= (m^2 - 1)\cos(mx)\sin(x) \end{aligned}$$

that $y'(x) = 0$ if and only if $\cos(mx)\sin(x) = 0$. This is equivalent to $\cos(mx) = 0$, i.e.,

$$x = \frac{k\pi + (\pi/2)}{m}, \quad k = 0, 1, \dots, m-1.$$

At these points, we have

$$\begin{aligned} y(x) &= m\sin\left(k\pi + \frac{\pi}{2}\right)\sin\left(\frac{k\pi + (\pi/2)}{m}\right) \\ &\quad + \cos\left(k\pi + \frac{\pi}{2}\right)\cos\left(\frac{k\pi + (\pi/2)}{m}\right) \\ &= m\sin\left(k\pi + \frac{\pi}{2}\right)\sin\left(\frac{k\pi + (\pi/2)}{m}\right) \\ &= (-1)^k m\sin\left(\frac{k\pi + (\pi/2)}{m}\right)\begin{cases} > 0 & \text{if } k \text{ is even;} \\ < 0 & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

It is easy to see that

$$y(0) = 1, \qquad y(\pi) = 1,$$
$$y'\left(\frac{k\pi}{m}\right) = (-1)^k(m^2 - 1)\sin\frac{k\pi}{m}\begin{cases} > 0 & \text{if } k \text{ is even;} \\ < 0 & \text{if } k \text{ is odd.} \end{cases}$$

Therefore, we can find (at least) m points $x_i \in (0, \pi)$ for which

$$\begin{aligned} &m\sin(mx_1)\sin(x_1) + \cos(mx_1)\cos(x_1) \\ &\qquad = m\sin(mx_2)\sin(x_2) + \cos(mx_2)\cos(x_2) \\ &\qquad = \cdots = m\sin(mx_m)\sin(x_m) + \cos(mx_m)\cos(x_m) \end{aligned}$$

holds.

This proves Theorem 2.4. □

5. A conjecture. The sets $\{c_i\}$ of collocation parameters used in Examples 5–8 are *symmetric* in $(0,1)$ and satisfy $c_m < 1$. While unique sets $\{d_i\}$ of superconvergence parameters exist for $m = 1$ and $m = 3$, there are no such sets when $m = 2$. This leads us to state the following conjecture.

Conjecture 5.1. *Assume that the collocation parameters* $\{c_i\}$*, with* $c_m < 1$*, are symmetric in* $(0,1)$ *(that is,* $c_{m+1-i} = 1-c_i$*,* $i = 1, \ldots, m$*). If* m *is even, then a set* $\{d_i\}$ *of superconvergence does not exist.*

If $m = 2k$ is even, $c_m < 1$, and the set $\{c_i\}$ is symmetric in $(0,1)$, then for a given $\{c_i\}$, we can transform the first $k+1$ rows of the coefficient matrix of (4.1) into

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ c_1 - (1-c_1) & c_1^2 - (1-c_1)^2 & \cdots & c_1^{2k} - (1-c_1)^{2k} \\ c_2 - (1-c_2) & c_2^2 - (1-c_2)^2 & \cdots & c_2^{2k} - (1-c_2)^{2k} \\ \vdots & \vdots & & \vdots \\ c_k - (1-c_k) & c_k^2 - (1-c_k)^2 & \cdots & c_k^{2k} - (1-c_k)^{2k} \end{pmatrix},$$

where we have set $m = 2k$. Taking $k = 5$ and applying the following transformation

$$\begin{aligned} \frac{(i)}{2c_i - 1}(i \geq 2) &\longrightarrow (i) - (1)(i \geq 2) \\ &\longrightarrow \frac{(i)}{c_i(c_i - 1)}(i \geq 2) \longrightarrow (i) - (2)(i \geq 3) \\ &\longrightarrow \frac{(i)}{c_i^2 - c_i - (c_1^2 - c_1)}(i \geq 3) \longrightarrow (i) - (3)(i \geq 4) \\ &\longrightarrow \frac{(i)}{c_i^2 - c_i - (c_2^2 - c_2)}(i \geq 4) \longrightarrow (i) - (4)(i \geq 5) \\ &\longrightarrow \frac{(i)}{c_i^2 - c_i - (c_3^2 - c_3)}(i \geq 5) \longrightarrow (i) - (5)(i \geq 6), \end{aligned}$$

where (i) denotes the ith line, we obtain

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2c_1-1 & c_1^2-(1-c_1)^2 & c_1^3-(1-c_1)^3 & c_1^4-(1-c_1)^4 & c_1^5-(1-c_1)^5 & c_1^6-(1-c_1)^6 & c_1^7-(1-c_1)^7 & c_1^8-(1-c_1)^8 & c_1^9-(1-c_1)^9 & c_1^{10}-(1-c_1)^{10} \\ 2c_2-1 & c_2^2-(1-c_2)^2 & c_2^3-(1-c_2)^3 & c_2^4-(1-c_2)^4 & c_2^5-(1-c_2)^5 & c_2^6-(1-c_2)^6 & c_2^7-(1-c_2)^7 & c_2^8-(1-c_2)^8 & c_2^9-(1-c_2)^9 & c_2^{10}-(1-c_2)^{10} \\ 2c_3-1 & c_3^2-(1-c_3)^2 & c_3^3-(1-c_3)^3 & c_3^4-(1-c_3)^4 & c_3^5-(1-c_3)^5 & c_3^6-(1-c_3)^6 & c_3^7-(1-c_3)^7 & c_3^8-(1-c_3)^8 & c_3^9-(1-c_3)^9 & c_3^{10}-(1-c_3)^{10} \\ 2c_4-1 & c_4^2-(1-c_4)^2 & c_4^3-(1-c_4)^3 & c_4^4-(1-c_4)^4 & c_4^5-(1-c_4)^5 & c_4^6-(1-c_4)^6 & c_4^7-(1-c_4)^7 & c_4^8-(1-c_4)^8 & c_4^9-(1-c_4)^9 & c_4^{10}-(1-c_4)^{10} \\ 2c_5-1 & c_5^2-(1-c_5)^2 & c_5^3-(1-c_5)^3 & c_5^4-(1-c_5)^4 & c_5^5-(1-c_5)^5 & c_5^6-(1-c_5)^6 & c_5^7-(1-c_5)^7 & c_5^8-(1-c_5)^8 & c_5^9-(1-c_5)^9 & c_5^{10}-(1-c_5)^{10} \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & c_1^2-c_1+3 & 3c_1^2-3c_1+4 & c_1^4-2c_1^3+7c_1^2-6c_1+5 & 2(2c_1^4-4c_1^3+7c_1^2-5c_1+3) & c_1^6-3c_1^5+13c_1^4-21c_1^3+25c_1^2-15c_1+7 & 5c_1^6-15c_1^5+35c_1^4-45c_1^3+41c_1^2-21c_1+8 \\ 0 & 0 & 0 & 0 & 1 & 3 & 6-c_2+c_2^2-c_1+c_1^2 & 2(5-2c_2+2c_2^2-2c_1+2c_1^2) & 15-2c_2^3+c_2^4-10c_1+11c_1^2-2c_1^3+c_1^4+c_2(-10+c_1-c_1^2)+c_2^2(11-c_1+c_1^2) & 21-10c_2^3+5c_2^4-20c_1+25c_1^2-10c_1^3+5c_1^4-5c_2(4-c_1+c_1^2)+5c_2^2(5-c_1+c_1^2) \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 4 & 10-c_3+c_3^2-c_2+c_2^2-c_1-c_1^2 & 5(4-c_3+c_3^2-c_2+c_2^2-c_1+c_1^2) \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

This reveals that, for $m < 10$, the set of superconvergence points $\{d_i\}$ does not exist. We *conjecture* that, for $m = 2k$ points of c_i, the first $k+1$ rows of the coefficient matrix can be transformed so that

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ 0 & 0 & 1 & 2 & \cdots & * & * \\ \vdots & \vdots & & \vdots & & & \\ 0 & 0 & 0 & 0 & \cdots & 1 & k \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix},$$

and the set of $\{d_i\}$ does not exist either.

APPENDIX

As we have indicated at the beginning of Section 3.2, we now provide the details of the proof that if $\{d_i\}$ is given by the zeros of $T_m(2s-1)$, the set $\{c_i\}$ being unique.

Since $c_m = 1$, and by (3.11), $T_m(s) = \cos[m \arccos(2s-1)]$ has to satisfy

$$\int_0^1 \cos[m \arccos(2s-1)]\, ds = 0.$$

(A) Sufficient and necessary conditions for $\int_0^1 \cos[m \arccos(2s-1)]\, ds = 0$.

(a) If $m > 1$, then

$$\int_0^1 \cos[m \arccos(2s-1)]\, ds = \frac{1+\cos(m\pi)}{2-2m^2}, \tag{A.1}$$

so $\int_0^1 \cos[m \arccos(2s-1)]\, ds = 0$ if and only if $[(1+\cos(m\pi))]/(2-2m^2) = 0$, which is equivalent to $\cos(m\pi) = -1 \Leftrightarrow m$ is odd.

(b) If $m = 1$, then

$$\int_0^1 \cos[\arccos(2s-1)]\, ds = 0.$$

Therefore, $\int_0^1 \cos[m \arccos(2s-1)]\, ds = 0$ if and only if m is odd.

(B) The zeros of the *Chebyshev polynomials* are

$$\frac{\cos((k\pi + (\pi/2))/m) + 1}{2} \quad (k = 0, 1, \ldots, m-1).$$

(C) *Integrals of the Chebyshev polynomials*:

$$\begin{aligned}
\int_\alpha^\beta & \cos[m \arccos(2s-1)]\, ds \\
&= \int_{\arccos(2\alpha-1)}^{\arccos(2\beta-1)} \cos(mx)\Big(-\frac{1}{2}\sin(x)\Big)\, dx \\
&= \frac{1}{4(m+1)}\{\cos[(m+1)\arccos(2\beta-1)] \\
&\qquad - \cos[(m+1)\arccos(2\alpha-1)]\} \\
&\quad - \frac{1}{4(m-1)}\{\cos[(m-1)\arccos(2\beta-1)] \\
&\qquad - \cos[(m-1)\arccos(2\alpha-1)]\}.
\end{aligned}$$

(D) The condition for $\int_0^{c_i} \cos[m \arccos(2s-1)]\, ds = 0$ ($i = 1, 2, \dots, m-1$). In the following, we assume that m is odd and $m > 1$ (for $m = 1$, $c_m = 1$, the superconvergence occurs at $1/2$ (the zero of the Chebyshev polynomial of degree 1)). It follows from (C) that

$$\begin{aligned}
\int_0^{c_i} & \cos[m \arccos(2s-1)]\, ds \\
&= \frac{1}{4(m+1)}\{\cos[(m+1)\arccos(2c_i-1)] \\
&\qquad - \cos[(m+1)\arccos(-1)]\} \\
&\quad - \frac{1}{4(m-1)}\{\cos[(m-1)\arccos(2c_i-1)] \\
&\qquad - \cos[(m-1)\arccos(-1)]\} \\
&= \frac{1}{4(m+1)}\{\cos[(m+1)\arccos(2c_i-1)] - 1\} \\
&\quad - \frac{1}{4(m-1)}\{\cos[(m-1)\arccos(2c_i-1)] - 1\}.
\end{aligned}$$

Let $x_i := \arccos(2c_i - 1)$. Then $\int_0^{c_i} \cos[m \arccos(2s-1)]\, ds = 0$ ($i = 1, 2, \dots, m-1$) is equivalent to

$$\frac{\cos[(m+1)x_i] - 1}{m+1} = \frac{\cos[(m-1)x_i] - 1}{m-1} \quad (i = 1, 2, \dots, m-1),$$

i.e.,

$$m\cos[(m+1)x_i] - \cos[(m+1)x_i] - m + 1$$
$$= m\cos[(m-1)x_i] + \cos[(m-1)x_i] - m - 1$$
$$(i = 1, 2, \dots, m-1),$$

which is equivalent to

$$m\{\cos[(m+1)x_i] - \cos[(m-1)x_i]\}$$
$$- \{\cos[(m+1)x_i] + \cos[(m-1)x_i]\} + 2 = 0$$
$$(i = 1, 2, \dots, m-1),$$

or

$$-2m\sin(mx_i)\sin(x_i) - 2\cos(mx_i)\cos(x_i) + 2 = 0$$
$$(i = 1, 2, \dots, m-1).$$

In other words, we have

$$m\sin(mx_i)\sin(x_i) + \cos(mx_i)\cos(x_i) = 1 \quad (i = 1, 2, \dots, m-1).$$

Let $y(s) := m\sin(ms)\sin(s) + \cos(ms)\cos(s) - 1$. Therefore, we need to investigate if there exist $m-1$ different zeros in $(0, \pi)$. The derivative of $y(s)$ is

$$y'(s) = m^2\cos(ms)\sin(s) + m\sin(ms)\cos(s)$$
$$- m\sin(ms)\cos(s) - \cos(ms)\sin(s)$$
$$= (m^2 - 1)\cos(ms)\sin(s).$$

So $y'(s) = 0$ if and only if $\cos(ms)\sin(s) = 0$, which is equivalent to $\cos(ms) = 0$, i.e.,

$$s = \frac{k\pi + (\pi/2)}{m} \quad (k = 0, 1, \dots, m-1).$$

At these points the function y assumes the values

$$y\left(\frac{k\pi + (\pi/2)}{m}\right) = m\sin\left(k\pi + (\pi/2)\right)\sin\left(\frac{k\pi + (\pi/2)}{m}\right)$$
$$+ \cos\left(k\pi + \frac{\pi}{2}\right)\cos\left(\frac{k\pi + (\pi/2)}{m}\right) - 1$$
$$= m\sin\left(k\pi + \frac{\pi}{2}\right)\sin\left(\frac{k\pi + (\pi/2)}{m}\right) - 1$$
$$= (-1)^k m\sin\left(\frac{k\pi + (\pi/2)}{m}\right) - 1.$$

If k is odd, then $y[(k\pi + (\pi/2))/m] < 0$. In order to investigate the case of even k, we require the following lemma.

Lemma A.1. *Let* $g(x) := \sin((\pi/2)x) - x$, $0 < x \le 1/3$. *Then* $g(x) > 0$.

Proof. The derivative of $g(x)$ is

$$g'(x) = \frac{\pi}{2}\cos\left(\frac{\pi}{2}x\right) - 1,$$

the second derivative of $g(x)$ is

$$g''(x) = -\left(\frac{\pi}{2}\right)^2 \sin\left(\frac{\pi}{2}x\right) < 0.$$

Due to $g'(1/3) = (\sqrt{3}\pi)/4 - 1 > 0$, so $g'(x) > 0$. Moreover, $g(0) = 0$, so $g(x) > 0$.

By Lemma A.1 we have, taking $x = 1/m$, $\sin[\pi/2]/m - (1/m) > 0$, so $m\sin[\pi/2]/m - 1 > 0$. Since $m\sin[\pi/2]/m \le m\sin[k\pi + (\pi/2)]/m$ $(k = 1, \dots, m-1)$, so $m\sin[k\pi + (\pi/2)]/m - 1 > 0$, and

$$y\left(\frac{k\pi + (\pi/2)}{m}\right) \begin{cases} < 0 & \text{if } k \text{ is odd,} \\ > 0 & \text{if } k \text{ is even.} \end{cases}$$

It is clear that $y(0) = y(\pi) = 0$, and

$$\begin{aligned} y'\left(\frac{k\pi}{m}\right) &= (m^2 - 1)\cos\left(m\frac{k\pi}{m}\right)\sin\left(\frac{k\pi}{m}\right) \\ &= (m^2 - 1)\cos(k\pi)\sin\left(\frac{k\pi}{m}\right) \\ &\begin{cases} > 0 & \text{if } k \text{ is even;} \\ < 0 & \text{if } k \text{ is odd.} \end{cases} \end{aligned}$$

Therefore, if $\{d_i\}$ are chosen as the zeros of the Chebyshev polynomials of the first kind, then there exists at least one set of $\{c_i\}$.

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