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FAST SINGULARITY PRESERVING METHODS FOR INTEGRAL EQUATIONS WITH NON-SMOOTH SOLUTIONS

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ABSTRACT. Fast singularity preserving multiscale Galerkin methods are developed in this paper for solving weakly singular Fredholm integral equations of the second kind with non-smooth solutions. A truncation strategy for the coefficient matrix obtained by using singularity preserving multiscale Galerkin methods is proposed. The multilevel augmentation method is developed for solving the discrete system with the truncated matrix. We prove that the methods preserve the singularities of the solutions and possess optimal order of convergence and linear computational complexity (up to a logarithmic factor). Finally, numerical experiments are presented to confirm theoretical results and demonstrate the efficiency and accuracy of the methods.

1. Introduction. Fast wavelet and multiscale methods for numerical solutions of weakly singular integral equations have attracted much attention recently. The methods are based on the fact that the representation of integral operators by appropriate wavelet and multiscale bases produces numerically sparse matrices. Matrix truncation (compression) techniques are then designed, which lead to fast algorithms for solving the integral equations (see, for example, [**1, 6, 7, 14, 15, 22, 23**] and the references cited therein). Moreover, multilevel augmentation methods are proposed as fast solvers for solving the discrete linear

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systems derived from wavelet and multiscale methods [**7–10**]. These methods were proved to possess nearly linear computational complexity and optimal order of convergence. However, all of these results are based on the regularity assumption of the solutions. It has been shown that solutions of the weakly singular Fredholm integral equations usually have singularities in their derivatives, which reflect the singularities of the kernels (see, for example, [**17, 24–26**]). Some numerical methods are developed based on this fact. The adaptive method is one of the best methods to solve this problem ([**11, 12**] and the references cited therein). The singularity preserving method is a simple and direct method. It allows the approximate spaces to contain some known singular functions that carry the singularities of the exact solution. Another part of the solution is considered as a smooth function. Since singularities of the solution usually reflect important features of practical physical problems, numerical methods preserving singularities of the solution are preferable. Singularity preserving Galerkin and Petrov-Galerkin methods for weakly singular Fredholm integral equations and Hammerstein integral equations were developed respectively (see [**2, 18, 27**]). The computational complexities of these methods are all $\mathcal{O}(N^2)$, where N is the dimension of the approximate subspace of the solution. The paper [**21**] combines the ideas of singularity preserving Galerkin methods and wavelet numerical methods for weakly singular integral equations such that the approximate solution can be obtained by solving a linear system of equation determined by a sparse matrix with $\mathcal{O}(N \log N)$ nonzero entries.

The purpose of this paper is to develop fast singularity preserving Galerkin methods for solving weakly singular Fredholm integral equations of the second kind by using multiscale bases. The solutions of these equations have certain singularities. Singularity preserving Galerkin methods allow us to approximate non-smooth solutions more efficiently. We will develop the corresponding matrix compression technique and multilevel augmentation algorithm to obtain fast solvers of the equations, which possess linear computational complexity (up to a logarithmic factor) and optimal order of convergence.

We organize this paper as follows. In Section 2, we will describe the multiscale (wavelet) bases and present singularity preserving multiscale Galerkin methods for solving weakly singular Fredholm integral equations of the second kind. Section 3 is devoted to proposing a corresponding matrix compression for the coefficient matrix of the discrete system and analyzing the computational complexity of the algorithm and the convergence order of the approximate solutions. In Section 4 we will present a multilevel augmentation algorithm to solve the discrete system derived from the singularity preserving multiscale Galerkin method with a matrix compression scheme. A complete analysis for computational complexity and convergence order is also proposed. Finally, in Section 5, we will give numerical experiments to confirm our theoretical results and illustrate the efficiency of the methods.

2. Singularity preserving multiscale Galerkin methods. In this section we present singularity preserving multiscale Galerkin methods for solving weakly singular Fredholm integral equations of the second kind with non-smooth solutions.

Let $\mathbf{X} := L^2(E)$, with the norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$, where $E \subset \mathbf{R}^d (d \in \mathbf{N} := \{1,2,3,\dots\})$ is a compact domain. Assume that $\mathcal{K} : \mathbf{X} \to \mathbf{X}$ is a compact linear operator defined by

$$(\mathcal{K}x)(s) := \int_E K(s,t)x(t)\,\mathrm{d}t, \quad s \in E,$$

where the function $K : E \times E \to \mathbf{R}$ is a weakly singular kernel satisfying condition (g) mentioned in Section 3. For $f \in \mathbf{X}$, we consider the weakly singular Fredholm integral equation of the second kind

$$x - \mathcal{K}x = f. \tag{2.1}$$

We assume that 1 is not an eigenvalue of $\mathcal{K}$, so that equation (2.1) has a unique solution in $\mathbf{X}$.

As in [**2, 17, 24, 27**], we assume that the solution of equation (2.1) has the singularity decomposition

$$x = v + u, \quad v \in \mathbf{V},\ u \in \mathbf{U}, \tag{2.2}$$

where $\mathbf{V} \subset \mathbf{X}$ is a finite dimensional subspace including known singular functions and $\mathbf{U} \subseteq H^k(E)$ is a subspace of $\mathbf{X}$ consisting of smooth functions.

We denote $\mathbf{N}_0 := \{0,1,2,\dots\}$, $Z_n := \{0,1,2,\dots,n-1\}$, and let $\{\mathbf{U}_n : n \in \mathbf{N}_0\}$ be a sequence of finite dimensional subspaces of $\mathbf{X}$

satisfying $\mathbf{V}\cap\mathbf{U}_n = \{0\}$ and $\overline{\cup_{n\in\mathbf{N}_0}\mathbf{U}_n} \supseteq \mathbf{U}$. The singularity preserving Galerkin method is to find $x_n \in \mathbf{X}_n := \mathbf{V} \oplus \mathbf{U}_n$ such that

$$\langle x_n - \mathcal{K}x_n, y\rangle = \langle f, y\rangle, \quad \text{for all } y \in \mathbf{X}_n, \tag{2.3}$$

where the notation $\mathbf{A} \oplus \mathbf{B}$ stands for the direct sum of two subspaces $\mathbf{A}$ and $\mathbf{B}$.

It has been proved (see, for example, [**2**]) that equation (2.3) has a unique solution x_n when n is sufficiently large and, if x is the solution of (2.1), $x = v + u$ with $v \in \mathbf{V}$, $u \in H^k(E)$, and

$$\inf_{w\in\mathbf{U}_n} \|u - w\| \le ch^k \|u\|_{H^k},$$

then

$$\|x_n - x\| \le ch^k \|u\|_{H^k}, \tag{2.4}$$

where h is the maximal distance of the quasi-uniform mesh.

In order to present multiscale schemes, we assume that there is a family of multiscale partitions $\{E_i : i \in \mathbf{N}_0\}$ such that, for each scale $i \in \mathbf{N}_0$, E_i consists of a family of star-shaped subsets $E_{i,j}$, $j \in Z_{e(i)}$, satisfying that

$$\bigcup_{j\in Z_{e(i)}} E_{i,j} = E, \quad \text{meas}\,(E_{i,j} \cap E_{i,j'}) = 0,\ j, j' \in Z_{e(i)},\ j \neq j',$$

and

$$\text{meas}\,(E_{i,j}) \sim d_i^d, \quad j \in Z_{e(i)},$$

where $e(i)$ denotes the cardinality of set E_i, $d_i := \max\{d(E_{i,j}) : j \in Z_{e(i)}\}$, $d(A)$ denotes the diameter of set A, and the notation $a_i \sim b_i$ means that there are positive constants c_1 and c_2 independent of i such that $c_1 a_i \le b_i \le c_2 a_i$ for all $i \in \mathbf{N}_0$.

We next assume that $\{\mathbf{U}_n : n \in \mathbf{N}_0\}$ are nested, that is, $\mathbf{U}_{n-1} \subset \mathbf{U}_n$, $n \in \mathbf{N}$. Thus, a subspace $\mathcal{W}_n \subset \mathbf{U}_n$ can be defined such that $\mathbf{U}_n$ becomes an orthogonal direct sum of $\mathbf{U}_{n-1}$ and $\mathcal{W}_n$. This leads to a multiscale space decomposition

$$\mathbf{U}_n = \mathcal{W}_0 \oplus^\perp \mathcal{W}_1 \oplus^\perp \cdots \oplus^\perp \mathcal{W}_n,$$

where we denote $\mathcal{W}_0 := \mathbf{U}_0$. We will use denotations $w(i) := \dim \mathcal{W}_i$, $i \in \mathbf{N}_0$ and $s(n) := \dim \mathbf{U}_n = \sum_{i \in Z_{n+1}} w(i)$, $n \in \mathbf{N}_0$.

We associate with the partitions and the subspaces a family of basis functions $\{w_{i,j} : (i,j) \in J\} \subset \mathbf{X}$ such that

$$\mathcal{W}_n = \text{span}\,\{w_{n,j} : j \in Z_{w(n)}\}, \quad n \in \mathbf{N}_0,$$

and

$$\mathbf{U}_n = \text{span}\,\{w_{i,j} : (i,j) \in J_n\},$$

where $J := \{(i,j) : j \in Z_{w(i)}, i \in \mathbf{N}_0\}$ and $J_n := \{(i,j) : j \in Z_{w(i)}, i \in Z_{n+1}\}$. We require that the following properties hold for the partitions, space decomposition and basis functions.

(a) There is a positive integer $\mu > 1$ such that, for $i \in \mathbf{N}_0$,

$$d_i \sim \mu^{-i/d}, \quad w(i) \sim \mu^i, \quad \text{and } s(i) \sim \mu^i. \tag{2.5}$$

Therefore, when using the multiscale scheme, the maximal distance h presented in formula (2.4) has the equivalence property $h \sim (s(n))^{-d}$.

(b) There exist positive integers ρ and r such that, for every $i > r$ and $j \in Z_{w(i)}$ written in the form $j = \nu\rho + s$ where $s \in Z_\rho$ and $\nu \in \mathbf{N}_0$,

$$w_{i,j}(t) = 0, \quad t \notin E_{i-r,\nu}.$$

This means that the support of $w_{i,j}$ is contained in $S_{i,j} := E_{i-r,\nu}$.

(c) Vanishing moment conditions hold such that, for any $(i,j) \in J$ with $i > 0$, and polynomial p of total degree less than k,

$$\langle w_{i,j}, p \rangle = 0.$$

(d) There is a positive constant θ_0 such that, for any $(i,j) \in J$,

$$\|w_{i,j}\| = 1, \quad \text{and} \quad \|w_{i,j}\|_\infty \le \theta_0 \mu^{i/2},$$

where $\|\cdot\|_\infty$ denotes the norm in $L^\infty(E)$.

(e) There is a positive constant θ_1 such that, for all $n \in \mathbf{N}_0$, $v = \sum_{(i,j) \in J_n} v_{i,j} w_{i,j}$,

$$\|\mathbf{v}\|_2 \le \theta_1 \|v\|,$$

where $\mathbf{v} := [v_{i,j} : (i,j) \in J_n]^T$. We remark that, throughout this paper, the notation $\|\mathbf{x}\|_p (1 \le p \le \infty)$ for a vector $\mathbf{x} := [x_j : j \in Z_n]^T$ denotes the ℓ_p-norm defined by

$$\|\mathbf{x}\|_p := \begin{cases} (\sum_{j \in Z_n} |x_j|^p)^{1/p} & 1 \le p < \infty, \\ \max\{|x_j| : j \in Z_n\} & p = \infty. \end{cases}$$

(f) If $\mathcal{P}_n$ is the orthogonal projection of $\mathbf{X}$ onto $\mathbf{U}_n$, then there exists a positive constant c such that, for any $u \in H^k(E)$,

$$\|u - \mathcal{P}_n u\| \le c d_n^k \|u\|_{H^k}. \tag{2.6}$$

The construction of multiscale bases having all properties (a) $\sim$ (f) can be found in much literature (see, for example, [**3, 5, 6, 20**]). Especially, if we choose $\mathbf{U}_n$, $n \in \mathbf{N}_0$, to be spaces of piecewise polynomials of total degree $\le k-1$ with respect to partition E_n, and if we choose $\{w_{i,j} : (i,j) \in J_n\}$ to be a sequence of orthonormal bases for $\mathbf{U}_n$, then property (e) holds with $\|\mathbf{v}\|_2 = \|v\|$, and the vanishing moment property (c) and approximation property (f) hold naturally.

Assume that $\mathcal{W}_{-1} := \mathbf{V} = \operatorname{span}\{w_{-1,j} : j \in Z_l\}$, where $l := w(-1) = \dim \mathcal{W}_{-1}$. Then we have

$$\mathbf{X}_n = \operatorname{span}\{w_{i,j} : (i,j) \in J_n'\},$$

where $J_n' := \{(i,j) : j \in Z_{w(i)}, i \in Z_{n+1}'\}$ with $Z_{n+1}' := \{-1, 0, 1, \ldots, n\}$.

With the multiscale bases described above, the singularity preserving multiscale Galerkin scheme for solving equation (2.1) is to find

$$x_n = \sum_{(i,j) \in J_n'} x_{i,j} w_{i,j} \in \mathbf{X}_n,$$

such that

$$\langle w_{i',j'}, x_n - \mathcal{K} x_n \rangle = \langle w_{i',j'}, f \rangle, \quad \text{for all } (i',j') \in J_n'. \tag{2.7}$$

Let $\mathcal{P}_{-1}$ and $\mathcal{P}_n$ be orthogonal projections of $\mathbf{X}$ onto $\mathbf{W}_{-1}$ and $\mathbf{U}_n$ defined by

$$\langle \mathcal{P}_{-1} x, y \rangle = \langle x, y \rangle, \quad \text{for all } y \in \mathbf{W}_{-1},$$

and

$$\langle \mathcal{P}_n x, y \rangle = \langle x, y \rangle, \quad \text{for all } y \in \mathbf{U}_n,$$

respectively. Equation (2.7) can be formulated as

$$\begin{cases} \mathcal{P}_{-1}(\mathcal{I} - \mathcal{K})x_n = \mathcal{P}_{-1} f, \\ \mathcal{P}_n(\mathcal{I} - \mathcal{K})x_n = \mathcal{P}_n f. \end{cases} \tag{2.8}$$

As in [**5**], we identify the vector $[v, u]^T$ in $\mathbf{V} \times \mathbf{U}_n$ with the sum $v + u$ in $\mathbf{V} \oplus \mathbf{U}_n$. We then introduce the operator $\mathcal{K}_n : \mathbf{X}_n \to \mathbf{X}_n$ by

$$\mathcal{K}_n := \begin{bmatrix} \mathcal{P}_{-1}\mathcal{K}|_{\mathbf{V}} & \mathcal{P}_{-1}\mathcal{K}|_{\mathbf{U}_n} \\ \mathcal{P}_n\mathcal{K}|_{\mathbf{V}} & \mathcal{P}_n\mathcal{K}|_{\mathbf{U}_n} \end{bmatrix}, \tag{2.9}$$

write $x_n \in \mathbf{X}_n$ as $x_n = [v_n, u_n]^T$ with $v_n \in \mathbf{V}$, $u_n \in \mathbf{U}_n$, and define $f_n := [\mathcal{P}_{-1} f, \mathcal{P}_n f]^T$. Equation (2.8) is now written in the form

$$\begin{bmatrix} v_n \\ u_n \end{bmatrix} - \begin{bmatrix} \mathcal{P}_{-1}\mathcal{K}|_{\mathbf{V}} & \mathcal{P}_{-1}\mathcal{K}|_{\mathbf{U}_n} \\ \mathcal{P}_n\mathcal{K}|_{\mathbf{V}} & \mathcal{P}_n\mathcal{K}|_{\mathbf{U}_n} \end{bmatrix} \begin{bmatrix} v_n \\ u_n \end{bmatrix} = \begin{bmatrix} \mathcal{P}_{-1} f \\ \mathcal{P}_n f \end{bmatrix},$$

or the operator equation

$$(\mathcal{I} - \mathcal{K}_n)x_n = f_n. \tag{2.10}$$

We remark that definition (2.9) of the operator $\mathcal{K}_n$ will help us to build a theoretical framework for the analysis of singularity preserving multiscale Galerkin methods.

Denote

$$\mathbf{E}_n := [\langle w_{i',j'}, w_{i,j} \rangle : (i,j), (i',j') \in J'_n]_{s'(n) \times s'(n)},$$

$$\mathbf{K}_n := [\langle w_{i',j'}, \mathcal{K} w_{i,j} \rangle : (i,j), (i',j') \in J'_n]_{s'(n) \times s'(n)},$$

$$\mathbf{f}_n := [\langle w_{i',j'}, f \rangle : (i',j') \in J'_n]^T,$$

and

$$\mathbf{x}_n := [x_{i',j'} : (i',j') \in J'_n]^T,$$

where $s'(n) := l + s(n)$. Equation (2.7) can be written in the matrix form

$$(\mathbf{E}_n - \mathbf{K}_n)\mathbf{x}_n = \mathbf{f}_n. \tag{2.11}$$

3. Matrix compression algorithm. In this section we develop a matrix truncation strategy for singularity preserving multiscale Galerkin methods described in the last section, which will result in matrix compression and lead us to a fast algorithm for approximately solving equation (2.1).

Throughout this paper, we suppose that the following weak singularity conditions on kernel K hold:

(g) For $s, t \in E$, $s \neq t$, K has a continuous partial derivative $D_s^\alpha D_t^\beta K(s,t)$ for $|\alpha| \leq k$, $|\beta| \leq k$. Moreover, there exist constants $\sigma \in [0, d)$ and $c > 0$ such that, for $|\alpha| = |\beta| = k$,

$$|D_s^\alpha D_t^\beta K(s,t)| \leq \frac{c}{|s-t|^{\sigma+2k}}.$$

We denote entries of the matrix $\mathbf{K}_n$ by $K_{i',j';i,j} := \langle w_{i',j'}, \mathcal{K} w_{i,j} \rangle$, $(i,j), (i',j') \in J'_n$. Note that the entries of the coefficient matrix obtained by the singularity preserving multiscale Galerkin method are the same as that by the corresponding multiscale (wavelet) Galerkin method except for the entries with $i = -1$ or $i' = -1$. It has been proved for multiscale Galerkin methods that most of these entries are so small that they can be neglected without affecting the overall accuracy of the approximation scheme. To present and analyze the truncation strategy for the singularity preserving method, we quote the estimate of these entries in the following lemma (cf. [**3, 22**]).

Lemma 3.1. *If conditions* (a)–(d), (g) *hold and there is a constant* $r > 1$ *such that*

$$\operatorname{dist}(S_{i,j}, S_{i',j'}) \geq \max\{rd_i, rd_{i'}\},$$

then there exists a positive constant c independent of i, j, i' and j' such that, for $i, i' > 0$,

$$|K_{i',j';i,j}| \le c(d_i d_{i'})^{k-d/2} \min\left\{ d_{i'}^d \max_{s\in S_{i',j'}} \int_{S_{i,j}} \frac{\mathrm{d}t}{|s-t|^{2k+\sigma}}, \right.$$
$$\left. d_i^d \max_{t\in S_{i,j}} \int_{S_{i',j'}} \frac{\mathrm{d}s}{|s-t|^{2k+\sigma}} \right\}.$$

We now partition matrix $\mathbf{K}_n$ into a block matrix $\mathbf{K}_n = [\mathbf{K}_{i',i}]_{i',i\in Z'_{n+1}}$ with $\mathbf{K}_{i',i} = [K_{i',j';i,j}]_{j'\in Z_{w(i')},j\in Z_{w(i)}}$ and choose truncation parameters $\delta^n_{i',i}$, which will be specified later, to obtain a truncation matrix

$$\widetilde{\mathbf{K}}_n = \left[\widetilde{\mathbf{K}}_{i',i}\right]_{i',i\in Z'_{n+1}},$$

where

$$\widetilde{\mathbf{K}}_{i',i} := \mathbf{K}(\delta^n_{i',i})_{i',i} = \left[\widetilde{K}_{i',j';i,j}\right]_{j'\in Z_{w(i')},\, j\in Z_{w(i)}},$$

with

$$\widetilde{K}_{i',j';i,j} := \begin{cases} K_{i',j';i,j} & \operatorname{dist}(S_{i',j'}, S_{i,j}) \le \delta^n_{i',i}, \\ & \text{or } i=-1, \text{ or } i'=-1, \\ 0 & \text{otherwise.} \end{cases}$$

We remark that if we write $\mathbf{K}_n$ as the block matrix

$$\mathbf{K}_n = \begin{bmatrix} \mathbf{K}_{-1,-1} & \mathbf{K}^0_{-1,n} \\ \mathbf{K}^0_{n,-1} & \mathbf{K}^0_n \end{bmatrix},$$

where $\mathbf{K}^0_{-1,n} := [\mathbf{K}_{-1,0}, \dots, \mathbf{K}_{-1,n}]$, $\mathbf{K}^0_{n,-1} := [\mathbf{K}_{0,-1}, \dots, \mathbf{K}_{n,-1}]^T$, and $\mathbf{K}^0_n := [\mathbf{K}_{i',i}]_{i',i\in Z_{n+1}}$, then we have

$$\widetilde{\mathbf{K}}_n = \begin{bmatrix} \mathbf{K}_{-1,-1} & \mathbf{K}^0_{-1,n} \\ \mathbf{K}^0_{n,-1} & \widetilde{\mathbf{K}}^0_n \end{bmatrix},$$

in which the block $\widetilde{\mathbf{K}}^0_n$ is obtained from $\mathbf{K}^0_n$ by using the truncation strategy as was done in fast multiscale Galerkin methods (cf. [**3, 22**]).

Our approximate algorithm is to find $\widetilde{\mathbf{x}}_n := [\widetilde{x}_{i,j} : (i,j) \in J'_n]^T \in \mathbf{R}^{s'(n)}$ such that

$$(\mathbf{E}_n - \widetilde{\mathbf{K}}_n)\widetilde{\mathbf{x}}_n = \mathbf{f}_n. \tag{3.12}$$

Let $\widetilde{\mathcal{K}}_n : \mathbf{X}_n \to \mathbf{X}_n$ be the linear operator defined by

$$\widetilde{\mathcal{K}}_n := \begin{bmatrix} \mathcal{P}_{-1}\mathcal{K}|_{\mathbf{V}} & \mathcal{P}_{-1}\mathcal{K}|_{\mathbf{U}_n} \\ \mathcal{P}_n\mathcal{K}|_{\mathbf{V}} & \widetilde{\mathcal{K}}_n^0 \end{bmatrix}, \tag{3.13}$$

where $\widetilde{\mathcal{K}}_n^0$ is the operator from $\mathbf{U}_n$ into itself relative to the basis $\{w_{i,j}, (i,j) \in J_n\}$ corresponding to matrix $\widetilde{\mathbf{K}}_n^0$. Then, the approximate algorithm of equation (3.12) in operator form is to find

$$\widetilde{x}_n = \sum_{(i,j)\in J_n'} \widetilde{x}_{i,j} w_{i,j} \in \mathbf{X}_n,$$

such that

$$(\mathcal{I} - \widetilde{\mathcal{K}}_n)\widetilde{x}_n = f_n. \tag{3.14}$$

For any matrix $\mathbf{K}$, we denote by $\mathbf{N}(\mathbf{K})$ the number of nonzero entries in $\mathbf{K}$. Now we estimate the number of nonzero entries of matrix $\widetilde{\mathbf{K}}_n$, which will show that the truncation strategy leads to a fast numerical algorithm for solving equation (2.1).

Theorem 3.2. *Assume that conditions* (a) *and* (b) *hold. Choose the truncation parameters $\delta_{i',i}^n$ such that, for some positive constants a and r with $r > 1$,*

$$\delta_{i',i}^n \le \max\{a\mu^{[-n+\alpha(n-i)+\alpha'(n-i')]/d}, rd_i, rd_{i'}\}, \quad i, i' \in Z_{n+1},$$

where α and α' are any numbers in $(-\infty, 1]$. Then

$$\mathbf{N}(\widetilde{\mathbf{K}}_n) = \mathcal{O}(s(n)\log^{\tau} s(n)),$$

where $\tau = 1$ except for $\alpha = \alpha' = 1$ in which case $\tau = 2$.

Proof. It is obvious that

$$\mathbf{N}(\widetilde{\mathbf{K}}_n) = l^2 + 2ls(n) + \mathbf{N}(\widetilde{\mathbf{K}}_n^0).$$

According to Theorem 3.3 in [**22**],

$$\mathbf{N}(\widetilde{\mathbf{K}}_n^0) = \begin{cases} \mathcal{O}(s(n)\log^2 s(n)) & \alpha = \alpha' = 1, \\ \mathcal{O}(s(n)\log s(n)) & \text{otherwise.} \end{cases}$$

This completes the proof. □

This theorem shows that the singularity preserving multiscale Galerkin method enjoys the same nearly linear computational complexity as general multiscale and wavelet methods. In some literature, this leads to linear computational complexity through the second compression ([**13**] and the references cited therein). Here we will not discuss the second compression.

In order to show the order of convergence of the truncated scheme and stability of the operator equation, we introduce a special projection.

Assume that $\mathcal{W}$ is the complement of $\mathbf{V}$ in $\mathbf{X}$, that is, $\mathbf{X} := \mathbf{V} \oplus \mathcal{W}$ with $\mathbf{U}_n \subseteq \mathcal{W}$. For $n \in \mathbf{N}_0$ we define the linear operator $\mathcal{P}'_n : \mathbf{X} \to \mathbf{X}_n$ by

$$\mathcal{P}'_n := [\mathcal{P}_{-1}|_{\mathbf{V}}, \mathcal{P}_n|_{\mathcal{W}}]^T.$$

Lemma 3.3. $\{\mathcal{P}'_n : n \in \mathbf{N}\}$ *is a sequence of uniformly bounded projections. Moreover, for any* $x = v + u \in \mathbf{X}$ *with* $v \in \mathbf{V}$ *and* $u \in \mathcal{W}$,

$$\|\mathcal{P}'_n x - x\| = \|\mathcal{P}_n u - u\|.$$

Proof. Letting $x = v + w \in \mathbf{X}$ with $v \in \mathbf{V}$ and $w \in \mathcal{W}$, then

$$\mathcal{P}'^{\,2}_n x = \mathcal{P}'_n(\mathcal{P}_{-1}v + \mathcal{P}_n w) = \mathcal{P}_{-1}{}^2 v + \mathcal{P}_n{}^2 w = \mathcal{P}'_n x,$$

which means that $\mathcal{P}'^{\,2}_n = \mathcal{P}'_n$; thus, $\mathcal{P}'_n$ is a linear projection from $\mathbf{X}$ onto $\mathbf{X}_n$. On the other hand, for any $x \in \mathbf{X}$,

$$\mathcal{P}'_n x = v + \mathcal{P}_n w \longrightarrow v + w = x, \quad \text{as } n \to \infty.$$

This means that $\{\mathcal{P}'_n\}$ converges point wisely to the identity operator $\mathcal{I}$ on $\mathbf{X}$. We conclude by using the uniform boundedness theorem that $\mathcal{P}'_n$, $n \in \mathbf{N}$, are uniformly bounded. Finally, we have that $\mathcal{P}'_n x - x = v + \mathcal{P}_n w - (v + w) = \mathcal{P}_n w - w$. This completes the proof. □

With the help of the projection $\mathcal{P}'_n$, we give a useful lemma. To describe it, we assume $\eta := 2k - d + \sigma > 0$, and for real numbers a and b, set

$$\mu[a, b; n] := \sum_{i \in Z_{n+1}} \mu^{ai/d} \sum_{i' \in Z_{n+1}} \mu^{bi'/d}.$$

Lemma 3.4. *Assume that conditions* (a)–(d), (g) *hold. Choose the truncation parameter* $\delta^n_{i',i}$*,* i*,* $i' \in Z_{n+1}$ *such that, for some positive constant* a *and* r *with* $r > 1$*,*

$$\delta^n_{i',i} \geq \max\{a\mu^{[-n+\alpha(n-i)+\alpha'(n-i')]/d}, rd_i, rd_{i'}\}, \quad i, i' \in Z_{n+1},$$

where α *and* α' *are any real numbers. Then, for any* $x = v + u \in \mathbf{X}$ *with* $v \in \mathbf{V}$ *and* $u \in H^m(E)$*,* $0 \leq m \leq k$*,*

$$\|(\mathcal{K}_n - \widetilde{\mathcal{K}}_n)\mathcal{P}'_n x\| \leq c\mu[k + m - \alpha\eta, k - \alpha'\eta; n]\mu^{-(m+d-\sigma)n/d}\|u\|_{H^m},$$

where c *is a constant independent of* n*.*

Proof. It follows from (2.9) and (3.13) that our truncation strategy leads to

$$(\mathcal{K}_n - \widetilde{\mathcal{K}}_n)v = 0, \quad \text{for all } v \in \mathbf{V}.$$

Therefore,

$$(\mathcal{K}_n - \widetilde{\mathcal{K}}_n)\mathcal{P}'_n x = (\mathcal{K}_n - \widetilde{\mathcal{K}}_n)\mathcal{P}'_n u = (\mathcal{K}^0_n - \widetilde{\mathcal{K}}^0_n)\mathcal{P}_n u,$$

where $\mathcal{K}^0_n := \mathcal{P}_n\mathcal{K}|_{\mathbf{U}_n}$. It has been proved (see, for example, [**3, 22**]) that there exists a constant c independent of n such that

$$\|(\mathcal{K}^0_n - \widetilde{\mathcal{K}}^0_n)\mathcal{P}_n u\| \leq c\mu[k + m - \alpha\eta, k - \alpha'\eta; n]\mu^{-(m+d-\sigma)n/d}\|u\|_{H^m}.$$

Thus, the proof of this lemma is completed. ◻

Recall that, for the standard Galerkin scheme, there is a positive constant c_0 such that, when n is sufficiently large,

$$\|(\mathcal{I} - \mathcal{K}_n)x\| \geq c_0\|x\|, \quad \text{for all } x \in \mathbf{X}_n. \tag{3.15}$$

We get the stability of the approximate operator equation (3.14) in the following theorem.

Theorem 3.5. *Assume that conditions* (a)–(e) *and* (g) *hold, and* $\delta^n_{i',i}$ *are chosen as in Lemma* 3.4 *with*

$$\alpha > \frac{1}{2} - \frac{d-\sigma}{2\eta}, \qquad \alpha' > \frac{1}{2} - \frac{d-\sigma}{2\eta}, \quad \alpha + \alpha' > 1.$$

Then there exist a positive constant c *and a positive integer* N *such that, when* $n \geq N$ *and* $x \in \mathbf{X}_n$,

$$\|(\mathcal{I} - \widetilde{\mathcal{K}}_n)x\| \geq c\|x\|.$$

Proof. It follows from the estimate in Lemma 3.4 with $m = 0$, $x \in \mathbf{X}_n$ and $u = x \in \mathbf{X} = H^0(E)$ that

$$\|(\mathcal{K}_n - \widetilde{\mathcal{K}}_n)x\| \leq c\mu[k - \alpha\eta,\ k - \alpha'\eta; n]\mu^{-(d-\sigma)n/d}\|x\|.$$

Note that, for any real numbers a, b and e, $\lim_{n\to\infty} \mu^{-en/d} = 0$ when $e > \max\{0, a, b, a+b\}$. With $a := k - \alpha\eta$, $b := k - \alpha'\eta$ and $e := d - \sigma$, thus the choice of $\delta^n_{i',i}$ ensures that there exists a positive integer N such that, when $n \geq N$,

$$c\mu[k - \alpha\eta, k - \alpha'\eta; n]\mu^{-(d-\sigma)n/d} \leq c_0/2.$$

This along with (3.15) leads to

$$\|(\mathcal{I} - \widetilde{\mathcal{K}}_n)x\| \geq \|(\mathcal{I} - \mathcal{K}_n)x\| - \|(\mathcal{K}_n - \widetilde{\mathcal{K}}_n)x\| \geq \frac{c_0}{2}\|x\|,$$

for any $x \in \mathbf{X}_n$, and the proof is completed. □

The next theorem shows that the convergence order of the matrix compression algorithm is optimal (up to a logarithmic factor).

Theorem 3.6. *Let* x *and* $\widetilde{x}_n$ *be solutions of equations* (2.1) *and* (3.14), *respectively, with a decomposition* $x = v + u$, $v \in \mathbf{V}$ *and* $u \in H^k(E)$. *Assume that conditions* (a)–(g) *hold, and* $\delta^n_{i',i}$ *are chosen as in Lemma* 3.4 *with* α *and* α' *satisfying one of the following conditions*:

(i) $\alpha \ge 1$, $\alpha' > (1/2) - [(d-\sigma)/2\eta]$, $\alpha + \alpha' > 1 + (k/2\eta)$, *or* $\alpha > 1$, $\alpha' \ge [(d-\sigma)/2\eta]$, $\alpha + \alpha' > 1 + (k/2\eta)$, *or* $\alpha > 1$, $\alpha' > [(d-\sigma)/2\eta]$, $\alpha + \alpha' \ge 1 + (k/2\eta)$.

(ii) $\alpha = 1$, $\alpha' = (k/\eta)$, *or* $\alpha = (2k/\eta)$, $\alpha' = (1/2) - [(d-\sigma)/2\eta]$.

Then there exist a positive constant c *and a positive integer* N *such that, when* $n \ge N$,

$$\|x - \widetilde{x}_n\| \le c s(n)^{-k/d} (\log s(n))^{\tau} \|u\|_{H^k},$$

where $\tau = 0$ *in case* (i), *and* $\tau = 1$ *in case* (ii).

Proof. It follows from Theorem 3.5 that there exists a positive constant c independent of n such that

$$\|\mathcal{P}'_n x - \widetilde{x}_n\| \le c \|(\mathcal{I} - \widetilde{\mathcal{K}}_n)(\mathcal{P}'_n x - \widetilde{x}_n)\|.$$

From equations (2.10) and (3.14) we see that

$$(\mathcal{I} - \widetilde{\mathcal{K}}_n)\widetilde{x}_n = (\mathcal{I} - \mathcal{K}_n) x_n.$$

This leads to

$$\begin{aligned}(\mathcal{I} - \widetilde{\mathcal{K}}_n)(\mathcal{P}'_n x - \widetilde{x}_n) &= (\mathcal{I} - \widetilde{\mathcal{K}}_n)\mathcal{P}'_n x - (\mathcal{I} - \mathcal{K}_n) x_n \\ &= (\mathcal{I} - \mathcal{K}_n)(\mathcal{P}'_n x - x_n) + (\mathcal{K}_n - \widetilde{\mathcal{K}}_n)\mathcal{P}'_n x.\end{aligned}$$

It can easily be verified that $\|\mathcal{K}_n\| \le 2\|\mathcal{K}\|$. Therefore, there exists a positive constant c' such that

$$\begin{aligned}\|x - \widetilde{x}_n\| &\le \|x - \mathcal{P}'_n x\| + \|\mathcal{P}'_n x - \widetilde{x}_n\| \\ &\le \|x - \mathcal{P}'_n x\| + c\|(\mathcal{I} - \mathcal{K}_n)(\mathcal{P}'_n x - x_n) + (\mathcal{K}_n - \widetilde{\mathcal{K}}_n)\mathcal{P}'_n x\| \\ &\le c' \left(\|x - \mathcal{P}'_n x\| + \|x_n - x\| + \|(\mathcal{K}_n - \widetilde{\mathcal{K}}_n)\mathcal{P}'_n x\| \right).\end{aligned} \tag{3.16}$$

It follows from Lemma 3.3, properties (a), (f) and (2.4) that

$$\|x - \mathcal{P}'_n x\| = \|u - \mathcal{P}_n u\| \le c s(n)^{-k/d} \|u\|_{H^k}, \tag{3.17}$$

and

$$\|x_n - x\| \le c s(n)^{-k/d} \|u\|_{H^k}. \tag{3.18}$$

From Lemma 3.4, we see that

$$\|(\mathcal{K}_n - \widetilde{\mathcal{K}}_n)\mathcal{P}'_n x\| \le c\mu[2k-\alpha\eta, k-\alpha'\eta; n]\mu^{-(d-\sigma)n/d}\mu^{-kn/d}\|u\|_{H^k}.$$

Observe that

$$\mu[a,b;n]\mu^{-en/d} = \begin{cases} \mathcal{O}(1) & \text{if } e \ge a,\ e > b,\ e > a+b \\ & \text{or } e > a,\ e \ge b,\ e > a+b \\ & \text{or } e > a,\ e > b,\ e \ge a+b, \\ \mathcal{O}(n) & \text{if } e = a,\ b = 0 \text{ or } e = b,\ a = 0, \end{cases}$$

as $n \to \infty$. From this, with $a := 2k - \alpha\eta$, $b := k - \alpha'\eta$, and $e := d - \sigma$, we obtain that

$$\mu[2k-\alpha\eta, k-\alpha'\eta; n]\mu^{-(d-\sigma)n/d} = \begin{cases} \mathcal{O}(1) & \text{in case (i)}, \\ \mathcal{O}(n) & \text{in case (ii)}. \end{cases}$$

Since $s(n) \sim \mu^n$, then

$$\|(\mathcal{K}_n - \widetilde{\mathcal{K}}_n)\mathcal{P}'_n x\| \le cs(n)^{-k/d}(\log s(n))^\tau \|u\|_{H^k}. \tag{3.19}$$

Substituting (3.17)–(3.19) into (3.16) yields the result of this theorem. □

4. Singularity preserving multilevel augmentation methods. In this section, we will develop a fast solver, named the *multilevel augmentation method*, to solve the discrete systems obtained by using the singularity preserving multiscale Galerkin method with the matrix compression scheme described in the last section.

Our multilevel augmentation method is based on the multiscale space decomposition, that is, for a fixed $\ell \in \mathbf{N}_0$ and any $m \in \mathbf{N}_0$,

$$\mathbf{X}_{\ell+m} = \mathbf{V} \oplus \mathbf{U}_\ell \oplus^\perp \mathcal{W}_{\ell+1} \oplus^\perp \cdots \oplus^\perp \mathcal{W}_{\ell+m}. \tag{4.20}$$

We are aiming at approximately solving equation (3.14) with $n = \ell+m$, i.e.,

$$(\mathcal{I} - \widetilde{\mathcal{K}}_{\ell+m})\widetilde{x}_{\ell+m} = f_{\ell+m}. \tag{4.21}$$

To this end, we first denote $\mathcal{Q}_{n+1} = \mathcal{P}_{n+1} - \mathcal{P}_n$, $n \in \mathbf{N}_0$, and write the operator $\widetilde{\mathcal{K}}_{\ell+m} : \mathbf{X}_{\ell+m} \to \mathbf{X}_{\ell+m}$ as the matrix form

(4.22)
$$\widetilde{\mathcal{K}}_{\ell+m} = \begin{bmatrix} \mathcal{P}_{-1}\widetilde{\mathcal{K}}|_{\mathbf{V}} & \mathcal{P}_{-1}\widetilde{\mathcal{K}}|_{\mathbf{U}_\ell} & \mathcal{P}_{-1}\widetilde{\mathcal{K}}|_{\mathcal{W}_{\ell+1}} & \cdots & \mathcal{P}_{-1}\widetilde{\mathcal{K}}|_{\mathcal{W}_{\ell+m}} \\ \mathcal{P}_{\ell}\widetilde{\mathcal{K}}|_{\mathbf{V}} & \mathcal{P}_{\ell}\widetilde{\mathcal{K}}|_{\mathbf{U}_\ell} & \mathcal{P}_{\ell}\widetilde{\mathcal{K}}|_{\mathcal{W}_{\ell+1}} & \cdots & \mathcal{P}_{\ell}\widetilde{\mathcal{K}}|_{\mathcal{W}_{\ell+m}} \\ \mathcal{Q}_{\ell+1}\widetilde{\mathcal{K}}|_{\mathbf{V}} & \mathcal{Q}_{\ell+1}\widetilde{\mathcal{K}}|_{\mathbf{U}_\ell} & \mathcal{Q}_{\ell+1}\widetilde{\mathcal{K}}|_{\mathcal{W}_{\ell+1}} & \cdots & \mathcal{Q}_{\ell+1}\widetilde{\mathcal{K}}|_{\mathcal{W}_{\ell+m}} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{Q}_{\ell+m}\widetilde{\mathcal{K}}|_{\mathbf{V}} & \mathcal{Q}_{\ell+m}\widetilde{\mathcal{K}}|_{\mathbf{U}_\ell} & \mathcal{Q}_{\ell+m}\widetilde{\mathcal{K}}|_{\mathcal{W}_{\ell+1}} & \cdots & \mathcal{Q}_{\ell+m}\widetilde{\mathcal{K}}|_{\mathcal{W}_{\ell+m}} \end{bmatrix},$$

in which we use $\widetilde{\mathcal{K}}$ to denote $\widetilde{\mathcal{K}}_{\ell+m}$ for simplicity. Similarly, the identity operator $\mathcal{I}_{\ell+m} : \mathbf{X}_{\ell+m} \to \mathbf{X}_{\ell+m}$ can be written in the following form

(4.23)
$$\mathcal{I}_{\ell+m} = \begin{bmatrix} \mathcal{P}_{-1}\mathcal{I}|_{\mathbf{V}} & \mathcal{P}_{-1}\mathcal{I}|_{\mathbf{U}_\ell} & \mathcal{P}_{-1}\mathcal{I}|_{\mathcal{W}_{\ell+1}} & \cdots & \mathcal{P}_{-1}\mathcal{I}|_{\mathcal{W}_{\ell+m}} \\ \mathcal{P}_{\ell}\mathcal{I}|_{\mathbf{V}} & \mathcal{P}_{\ell}\mathcal{I}|_{\mathbf{U}_\ell} & 0 & \cdots & 0 \\ \mathcal{Q}_{\ell+1}\mathcal{I}|_{\mathbf{V}} & 0 & \mathcal{Q}_{\ell+1}\mathcal{I}|_{\mathcal{W}_{\ell+1}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathcal{Q}_{\ell+m}\mathcal{I}|_{\mathbf{V}} & 0 & 0 & \cdots & \mathcal{Q}_{\ell+m}\mathcal{I}|_{\mathcal{W}_{\ell+m}} \end{bmatrix},$$

where we have used the equations

$$\mathcal{P}_{\ell}\mathcal{I}|_{\mathcal{W}_{\ell+j}} = 0, \qquad \mathcal{Q}_{\ell+j}\mathcal{I}|_{\mathbf{U}_\ell} = 0, \text{ for } j > 0,$$

and

$$\mathcal{Q}_{\ell+i}\mathcal{I}|_{\mathcal{W}_{\ell+j}} = 0, \quad \text{for } i, j > 0, \ i \neq j,$$

which can easily be verified by using the relation described by (4.20). We next split the operators $\widetilde{\mathcal{K}}_{\ell+m}$ and $\mathcal{I}_{\ell+m}$ as the sums of two operators respectively, that is,

$$\widetilde{\mathcal{K}}_{\ell+m} = \widetilde{\mathcal{K}}^{L}_{\ell,m} + \widetilde{\mathcal{K}}^{H}_{\ell,m}$$

with

$$\widetilde{\mathcal{K}}^{L}_{\ell,m} := \begin{bmatrix} \mathcal{P}_{-1}\widetilde{\mathcal{K}}|_{\mathbf{V}} & \mathcal{P}_{-1}\widetilde{\mathcal{K}}|_{\mathbf{U}_\ell} & \mathcal{P}_{-1}\widetilde{\mathcal{K}}|_{\mathcal{W}_{\ell+1}} & \cdots & \mathcal{P}_{-1}\widetilde{\mathcal{K}}|_{\mathcal{W}_{\ell+m}} \\ \mathcal{P}_{\ell}\widetilde{\mathcal{K}}|_{\mathbf{V}} & \mathcal{P}_{\ell}\widetilde{\mathcal{K}}|_{\mathbf{U}_\ell} & \mathcal{P}_{\ell}\widetilde{\mathcal{K}}|_{\mathcal{W}_{\ell+1}} & \cdots & \mathcal{P}_{\ell}\widetilde{\mathcal{K}}|_{\mathcal{W}_{\ell+m}} \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{bmatrix},$$

and

$$\mathcal{I}_{\ell+m} = \mathcal{I}^M_{\ell,m} + \mathcal{I}^S_{\ell,m}$$

with

$$\mathcal{I}^M_{\ell,m} := \begin{bmatrix} \mathcal{P}_{-1}\mathcal{I}|_{\mathbf{V}} & \mathcal{P}_{-1}\mathcal{I}|_{\mathbf{U}_\ell} & \mathcal{P}_{-1}\mathcal{I}|_{\mathcal{W}_{\ell+1}} & \cdots & \mathcal{P}_{-1}\mathcal{I}|_{\mathcal{W}_{\ell+m}} \\ \mathcal{P}_{\ell}\mathcal{I}|_{\mathbf{V}} & \mathcal{P}_{\ell}\mathcal{I}|_{\mathbf{U}_\ell} & 0 & \cdots & 0 \\ 0 & 0 & \mathcal{Q}_{\ell+1}\mathcal{I}|_{\mathcal{W}_{\ell+1}} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathcal{Q}_{\ell+m}\mathcal{I}|_{\mathcal{W}_{\ell+m}} \end{bmatrix},$$

where $\widetilde{\mathcal{K}}^L_{\ell,m}$ and $\widetilde{\mathcal{K}}^H_{\ell,m}$ correspond to lower and higher frequencies of the operator $\widetilde{\mathcal{K}}_{\ell+m}$, respectively, and $\mathcal{I}^M_{\ell,m}$ and $\mathcal{I}^S_{\ell,m}$ correspond to main and subordinate parts of operator $\mathcal{I}_{\ell+m}$, respectively. Set

$$\mathcal{B}_{\ell,m} := \mathcal{I}^M_{\ell,m} - \widetilde{\mathcal{K}}^L_{\ell,m},$$

and

$$\mathcal{C}_{\ell,m} := \mathcal{I}^S_{\ell,m} - \widetilde{\mathcal{K}}^H_{\ell,m}.$$

Operator equation (4.21) is written as

$$\mathcal{B}_{\ell,m}\widetilde{x}_{\ell+m} = f_{\ell+m} - \mathcal{C}_{\ell,m}\widetilde{x}_{\ell+m}.$$

According to the idea of the multilevel augmentation method (cf. [**7**]), we first solve equation (4.21) at an initial coarse level ℓ and then augment the equation from a coarse level to a finer level one by one. At each level we solve the equation

$$\mathcal{B}_{\ell,m}\widetilde{x}_{\ell,m} = f_{\ell+m} - \mathcal{C}_{\ell,m}\widetilde{x}_{\ell,m-1}$$

to update the solution from the coarse level solution $\widetilde{x}_{\ell,m-1}$ to the finer level solution $\widetilde{x}_{\ell,m}$, where the notations $\widetilde{x}_{\ell,i}$, $i = 1, \ldots, m$, denote the solutions obtained by the augmentation method. The process is repeated until a satisfactory solution is obtained. Since the matrix representation of the operator $\mathcal{B}_{\ell,m}$ has a very simple form, we will show later (the matrix $\mathbf{B}_{\ell,m}$) that the algorithm is fast and efficient. We now describe the method exactly as follows.

Algorithm 4.1 (Operator form of singularity preserving multilevel augmentation methods). *Let $\ell > 0$ be a fixed integer.*

Step 1. *Solve equation* (4.21) *with $m := 0$ for $\widetilde{x}_\ell \in \mathbf{X}_\ell$ exactly.*

Step 2. *Set $\widetilde{x}_{\ell,0} := \widetilde{x}_\ell$ and compute $\mathcal{B}_{\ell,0}$ and $\mathcal{C}_{\ell,0}$.*

Step 3. *For $m \in \mathbf{N}$, suppose that $\widetilde{x}_{\ell,m-1} \in \mathbf{X}_{\ell+m-1}$ has been obtained and do the following*:

- *Augment $\mathcal{B}_{\ell,m-1}$ and $\mathcal{C}_{\ell,m-1}$ to form $\mathcal{B}_{\ell,m}$ and $\mathcal{C}_{\ell,m}$, respectively.*
- *Augment $\widetilde{x}_{\ell,m-1}$ by setting $\overline{x}_{\ell,m} := \begin{bmatrix} \widetilde{x}_{\ell,m-1} \\ 0 \end{bmatrix} \in \mathbf{X}_{\ell+m}$.*
- *Solve $\widetilde{x}_{\ell,m} \in \mathbf{X}_{\ell+m}$ from equation*

$$\mathcal{B}_{\ell,m}\widetilde{x}_{\ell,m} = f_{\ell+m} - \mathcal{C}_{\ell,m}\overline{x}_{\ell,m}. \tag{4.24}$$

To implement Algorithm 4.1 we need to present its matrix form. Corresponding to the form (4.22) of the operator $\widetilde{\mathcal{K}}_{\ell+m}$, we write its matrix representation relative to the singular and multiscale bases $\{w_{i,j} : (i,j) \in J'_{\ell+m}\}$ as:

$$\widetilde{\mathbf{K}}_{\ell+m} = \begin{bmatrix} \widetilde{\mathbf{K}}^\ell_{-1,-1} & \widetilde{\mathbf{K}}^\ell_{-1,0} & \widetilde{\mathbf{K}}^\ell_{-1,1} & \cdots & \widetilde{\mathbf{K}}^\ell_{-1,m} \\ \widetilde{\mathbf{K}}^\ell_{0,-1} & \widetilde{\mathbf{K}}^\ell_{0,0} & \widetilde{\mathbf{K}}^\ell_{0,1} & \cdots & \widetilde{\mathbf{K}}^\ell_{0,m} \\ \widetilde{\mathbf{K}}^\ell_{1,-1} & \widetilde{\mathbf{K}}^\ell_{1,0} & \widetilde{\mathbf{K}}^\ell_{1,1} & \cdots & \widetilde{\mathbf{K}}^\ell_{1,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \widetilde{\mathbf{K}}^\ell_{m,-1} & \widetilde{\mathbf{K}}^\ell_{m,0} & \widetilde{\mathbf{K}}^\ell_{m,1} & \cdots & \widetilde{\mathbf{K}}^\ell_{m,m} \end{bmatrix}.$$

The matrix representations of operators $\mathcal{I}_{\ell+m}$, $\widetilde{\mathcal{K}}^L_{\ell,m}$, $\widetilde{\mathcal{K}}^H_{\ell,m}$, $\mathcal{I}^M_{\ell,m}$, $\mathcal{I}^S_{\ell,m}$, $\mathcal{B}_{\ell,m}$ and $\mathcal{C}_{\ell,m}$ can be given similarly. Especially when we choose $\{w_{i,j} : (i,j) \in J_{\ell+m}\}$ to be orthonormal bases, the matrix representations of operators $\mathcal{B}_{\ell,m}$ and $\mathcal{C}_{\ell,m}$ have the following forms

$$\mathbf{B}_{\ell,m} := \begin{bmatrix} \mathbf{E}^\ell_{-1,-1} - \widetilde{\mathbf{K}}^\ell_{-1,-1} & \mathbf{E}^\ell_{-1,0} - \widetilde{\mathbf{K}}^\ell_{-1,0} & \mathbf{E}^\ell_{-1,1} - \widetilde{\mathbf{K}}^\ell_{-1,1} & \cdots & \mathbf{E}^\ell_{-1,m} - \widetilde{\mathbf{K}}^\ell_{-1,m} \\ \mathbf{E}^\ell_{0,-1} - \widetilde{\mathbf{K}}^\ell_{0,-1} & \mathbf{I} - \widetilde{\mathbf{K}}^\ell_{0,0} & -\widetilde{\mathbf{K}}^\ell_{0,1} & \cdots & -\widetilde{\mathbf{K}}^\ell_{0,m} \\ 0 & 0 & \mathbf{I} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \mathbf{I} \end{bmatrix}$$

and

$$\mathbf{C}_{k,m} := \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \mathbf{E}^{\ell}_{1,-1} - \widetilde{\mathbf{K}}^{\ell}_{1,-1} & -\widetilde{\mathbf{K}}^{\ell}_{1,0} & -\widetilde{\mathbf{K}}^{\ell}_{1,1} & \cdots & -\widetilde{\mathbf{K}}^{\ell}_{1,m} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{E}^{\ell}_{m,-1} - \widetilde{\mathbf{K}}^{\ell}_{m,-1} & -\widetilde{\mathbf{K}}^{\ell}_{m,0} & -\widetilde{\mathbf{K}}^{\ell}_{m,1} & \cdots & -\widetilde{\mathbf{K}}^{\ell}_{m,m} \end{bmatrix},$$

respectively.

Using these notations, the matrix form of Algorithm 4.1 can be described as follows.

Algorithm 4.2 (Matrix form of singularity preserving multilevel augmentation methods). *Let $\ell > 0$ be a fixed integer.*

Step 1. *Solve $\widetilde{\mathbf{x}}_\ell \in \mathbf{R}^{s'(\ell)}$ from the equation $(\mathbf{E}_\ell - \widetilde{\mathbf{K}}_\ell)\widetilde{\mathbf{x}}_\ell = \mathbf{f}_\ell$.*

Step 2. *Set $\widetilde{\mathbf{x}}_{\ell,0} := \widetilde{\mathbf{x}}_\ell$, and compute the matrices $\widetilde{\mathbf{K}}^L_{\ell,0}$, $\widetilde{\mathbf{K}}^H_{\ell,0}$, $\mathbf{E}^M_{\ell,0}$, and $\mathbf{E}^S_{\ell,0}$.*

Step 3. *For $m \in \mathbf{N}$, suppose that $\widetilde{\mathbf{x}}_{\ell,m-1} \in \mathbf{R}^{s'(\ell+m-1)}$ has been obtained and do the following*:

- *Augment the matrices $\widetilde{\mathbf{K}}^L_{\ell,m-1}$, $\widetilde{\mathbf{K}}^H_{\ell,m-1}$, $\mathbf{E}^M_{\ell,m-1}$ and $\mathbf{E}^S_{\ell,m-1}$ to form $\widetilde{\mathbf{K}}^L_{\ell,m}$, $\widetilde{\mathbf{K}}^H_{\ell,m}$, $\mathbf{E}^M_{\ell,m}$ and $\mathbf{E}^S_{\ell,m}$, respectively.*
- *Augment $\widetilde{\mathbf{x}}_{\ell,m-1}$ by setting $\overline{\mathbf{x}}_{\ell,m} := \begin{bmatrix} \widetilde{\mathbf{x}}_{\ell,m-1} \\ 0 \end{bmatrix} \in \mathbf{R}^{s'(\ell+m)}$.*
- *Solve $\widetilde{\mathbf{x}}_{\ell,m} \in \mathbf{R}^{s'(\ell+m)}$ from the linear system*

$$\mathbf{B}_{\ell,m}\widetilde{\mathbf{x}}_{\ell,m} = \mathbf{f}_{\ell+m} - \mathbf{C}_{\ell,m}\overline{\mathbf{x}}_{\ell,m}. \tag{4.25}$$

To derive the detailed computation form and analyze the complexity of Algorithm 4.2, we partition the vectors $\widetilde{\mathbf{x}}_{\ell,m}$ and $\mathbf{f}_{\ell+m}$ as

$$\widetilde{\mathbf{x}}_{\ell,m} = [\widetilde{\mathbf{x}}^{\ell,m}_i : i \in Z'_{m+1}]^T, \quad \text{and} \quad \mathbf{f}_{\ell+m} = [\mathbf{f}^{\ell}_i : i \in Z'_{m+1}]^T$$

respectively, which correspond to the multiscale space decomposition (4.20). With the help of these notations, equation (4.25) can be written as follows: First, compute

$$\widetilde{\mathbf{x}}_i^{\ell,m} := \mathbf{f}_i^\ell - (\mathbf{E}_{i,-1}^\ell - \widetilde{\mathbf{K}}_{i,-1}^\ell)\widetilde{\mathbf{x}}_{-1}^{\ell,m-1} + \sum_{j=0}^{m-1} \widetilde{\mathbf{K}}_{i,j}^\ell \widetilde{\mathbf{x}}_j^{\ell,m-1}, \tag{4.26}$$

to obtain $\widetilde{\mathbf{x}}_i^{\ell,m} \in \mathbf{R}^{w(\ell+i)}$, $i = m, m-1, \dots, 1$. Next, solve $\widetilde{\mathbf{x}}_{-1}^{\ell,m} \in \mathbf{R}^l$ and $\widetilde{\mathbf{x}}_0^{\ell,m} \in \mathbf{R}^{s(\ell)}$ from the equation

$$\begin{bmatrix} \mathbf{E}_{-1,-1}^\ell - \widetilde{\mathbf{K}}_{-1,-1}^\ell & \mathbf{E}_{-1,0}^\ell - \widetilde{\mathbf{K}}_{-1,0}^\ell \\ \mathbf{E}_{0,-1}^\ell - \widetilde{\mathbf{K}}_{0,-1}^\ell & \mathbf{I} - \widetilde{\mathbf{K}}_{0,0}^\ell \end{bmatrix} \begin{bmatrix} \widetilde{\mathbf{x}}_{-1}^{\ell,m} \\ \widetilde{\mathbf{x}}_0^{\ell,m} \end{bmatrix} = \begin{bmatrix} \mathbf{f}_{-1}^\ell - \sum_{j=1}^m (\mathbf{E}_{-1,j}^\ell - \widetilde{\mathbf{K}}_{-1,j}^\ell)\widetilde{\mathbf{x}}_j^{\ell,m} \\ \mathbf{f}_0^\ell + \sum_{j=1}^m \widetilde{\mathbf{K}}_{0,j}^\ell \widetilde{\mathbf{x}}_j^{\ell,m} \end{bmatrix}. \tag{4.27}$$

It can be seen that, at each level m, the higher frequency part of the solution $\widetilde{\mathbf{x}}_{\ell,m}$ is obtained by direct computation, and the lower frequency part is obtained by solving a system with the same coefficient matrix

$$\mathbf{B}_{\ell,0} = \begin{bmatrix} \mathbf{E}_{-1,-1}^\ell - \widetilde{\mathbf{K}}_{-1,-1}^\ell & \mathbf{E}_{-1,0}^\ell - \widetilde{\mathbf{K}}_{-1,0}^\ell \\ \mathbf{E}_{0,-1}^\ell - \widetilde{\mathbf{K}}_{0,-1}^\ell & \mathbf{I} - \widetilde{\mathbf{K}}_{0,0}^\ell \end{bmatrix}.$$

Noting that the order of the matrix $\mathbf{B}_{\ell,0}$ is only $s'(\ell)$, the algorithm reduces the computation cost greatly. In the remainder of this section we will prove that the singularity preserving multilevel augmentation method enjoys the optimal convergence order and linear computational complexity (up to logarithmic factor).

Theorem 4.3. *Let x be the solution of equation* (2.1) *with a decomposition $x = v + u$, $v \in \mathbf{V}$ and $u \in H^k(E)$. Assume that conditions* (a)–(g) *hold, and $\widetilde{x}_{\ell,m}$ is obtained by using Algorithm* 4.1 *with matrix compression described in Section* 3 *and truncation parameters described in Theorem* 3.6*. Then, there exist a positive integer N and a positive constant c such that, for all $\ell \geq N$ and $m \in \mathbf{N}_0$,*

$$\|x - \widetilde{x}_{\ell,m}\| \leq c(s(\ell+m))^{-k/d}(\log s(\ell+m))^\tau \|u\|_{H^k},$$

where τ is the integer described in Theorem 3.6.

Proof. It follows from Theorem 3.5 that the operators $(\mathcal{I} - \widetilde{\mathcal{K}}_n)^{-1} : \mathbf{X}_n \to \mathbf{X}_n$ exist and are uniformly bounded. Moreover, from Theorem 3.6 we see that there exist a positive constant c and a positive integer N such that, when $n \geq N$,

$$\|x - \widetilde{x}_n\| \leq \gamma_n := c(s(n))^{-k/d}(\log s(n))^{\tau}\|u\|_{H^k}.$$

Since $s(n) \sim \mu^n$ as $n \to \infty$, $\gamma_{n+1}/\gamma_n \sim \xi := \mu^{-k/d} > 0$. Therefore, according to Theorem 2.2 in [**8**], the result of this theorem is valid provided the following holds:

$$\lim_{\ell\to\infty} \|\mathcal{C}_{\ell,m}\| = 0, \quad \text{uniformly for } m \in \mathbf{N}. \tag{4.28}$$

To prove (4.28), we observe that

$$\begin{gathered} \mathcal{C}_{\ell,m} = \mathcal{I}^S_{\ell,m} - \tilde{\mathcal{K}}^H_{\ell,m}, \\ \mathcal{I}^S_{\ell,m} = \mathcal{I}_{\ell+m} - \mathcal{I}^M_{\ell,m} = (\mathcal{P}_{\ell+m} - \mathcal{P}_\ell)\mathcal{P}'_{-1}|_{\mathbf{X}_{\ell+m}}, \end{gathered} \tag{4.29}$$

and

$$\begin{aligned} \widetilde{\mathcal{K}}^H_{\ell,m} &= \widetilde{\mathcal{K}}_{\ell+m} - \widetilde{\mathcal{K}}^L_{\ell,m} = (\mathcal{P}_{\ell+m} - \mathcal{P}_\ell)\widetilde{\mathcal{K}}_{\ell+m}|_{\mathbf{X}_{\ell+m}} \\ &= (\mathcal{P}_{\ell+m} - \mathcal{P}_\ell)\mathcal{K}|_{\mathbf{X}_{\ell+m}} + (\mathcal{P}_{\ell+m} - \mathcal{P}_\ell)(\tilde{\mathcal{K}}_{\ell+m} - \mathcal{K}_{\ell+m})|_{\mathbf{X}_{\ell+m}}, \end{aligned}$$

where $\mathcal{P}'_{-1} : \mathbf{X} \to \mathbf{X}$ is a linear operator defined by

$$\mathcal{P}'_{-1}x = v, \quad \text{for } x = v + w \text{ with } v \in \mathbf{V} \text{ and } w \in \mathcal{W}.$$

Since the operator $\mathcal{P}'_{-1}$ has finite dimensional range, it is compact. Noting that the orthogonal projection $\mathcal{P}_n$ pointwisely converges to the identity operator, we conclude that $\|\mathcal{P}_n\mathcal{P}'_{-1} - \mathcal{P}'_{-1}\| \to 0$ as $n \to \infty$. This yields

$$\lim_{\ell\to\infty} \|\mathcal{I}^S_{\ell,m}\| = 0, \quad \text{uniformly for } m \in \mathbf{N}. \tag{4.30}$$

Likewise, the compactness of operator $\mathcal{K}$ leads to

$$\lim_{\ell\to\infty} \|(\mathcal{P}_{\ell+m} - \mathcal{P}_\ell)\mathcal{K}|_{\mathbf{X}_{\ell+m}}\| = 0, \quad \text{uniformly for } m \in \mathbf{N}. \tag{4.31}$$

On the other hand, it follows from Lemma 3.4 that, for any $x \in \mathbf{X}_n$,

$$\|(\widetilde{\mathcal{K}}_n - \mathcal{K}_n)\mathcal{P}'_n x\| \le c\mu[k-\alpha\eta, k-\alpha'\eta; n]\mu^{-(d-\sigma)n/d}\|u\|,$$

which means that, for our choice of the parameters,

$$\|(\widetilde{\mathcal{K}}_n - \mathcal{K}_n)|_{\mathbf{X}_n}\| \longrightarrow 0, \quad \text{as } n \to \infty.$$

This with (4.31) leads to

$$\|\widetilde{\mathcal{K}}^H_{\ell,m}\| \le \|(\mathcal{P}_{\ell+m} - \mathcal{P}_\ell)\mathcal{K}|_{\mathbf{X}_{\ell+m}}\| + 2\|(\widetilde{\mathcal{K}}_{\ell+m} - \mathcal{K}_{\ell+m})|_{\mathbf{X}_{\ell+m}}\| \longrightarrow 0, \quad \text{as } \ell \to \infty, \tag{4.32}$$

uniformly for $m \in \mathbf{N}$.

Combining (4.29), (4.30) and (4.32) yields (4.28), and completes the proof of this theorem. □

We now estimate the computational cost (the total number of multiplications) for obtaining the solution $\widetilde{x}_{k,m}$. To this end, we require an additional condition.

(h) Computing the integrals (entries) that appear in $\mathbf{E}_n$, $\widetilde{\mathbf{K}}_n$ and $\mathbf{f}_n$ requires a constant computational cost per integral.

Lemma 4.4. *The total number of multiplications required for obtaining $\widetilde{\mathbf{x}}_{\ell,m}$ from $\widetilde{\mathbf{x}}_\ell$ is bounded by*

$$(m+1)\mathcal{M}(\ell) + \sum_{i=1}^{m}[\mathcal{N}(\mathbf{E}_{\ell+i}) + \mathcal{N}(\widetilde{\mathbf{K}}_{\ell+i})],$$

where $\mathcal{M}(\ell)$ denotes the number of multiplications required for solving equation (4.27) *with a known right hand side.*

Proof. For fixed ℓ, $m \in \mathbf{N}$, we need $\mathcal{N}(\mathbf{C}_{\ell,m})$ multiplications to obtain the right hand side of (4.25). Since $\mathbf{C}_{\ell,m} = \mathbf{E}^S_{\ell,m} - \widetilde{\mathbf{K}}^H_{\ell,m}$, the number of multiplications to obtain $\widetilde{\mathbf{x}}^{\ell,m}_i$, $i = 1, 2, \dots, m$, from (4.26) is less than or equal to $\mathcal{N}(\mathbf{E}^S_{\ell,m}) + \mathcal{N}(\widetilde{\mathbf{K}}^H_{\ell,m})$. On the other hand, the computation of the right hand side of (4.27) requires $\mathcal{N}(\mathbf{E}^M_{\ell,m}) + \mathcal{N}(\widetilde{\mathbf{K}}^L_{\ell,m})$ numbers of multiplications. Since we have assumed, that solving $\widetilde{\mathbf{x}}_\ell$ from (4.27) with a known right hand side needs $\mathcal{M}(\ell)$ multiplications, the total number of multiplications for computing $\widetilde{\mathbf{x}}_{\ell,m}$ from $\widetilde{\mathbf{x}}_{\ell,m-1}$ is less than or equal to

$$\mathcal{N}_{\ell,m} := \mathcal{M}(\ell) + \mathcal{N}(\mathbf{E}_{\ell+m}) + \mathcal{N}(\widetilde{\mathbf{K}}_{\ell+m}). \tag{4.33}$$

Recall that, to obtain $\widetilde{\mathbf{x}}_{\ell,m}$, we first compute $\widetilde{\mathbf{x}}_\ell$ and then use (4.26)–(4.27) to compute $\widetilde{\mathbf{x}}_{\ell,i}$, $i = 1, 2, \dots, m$, successively. The total number of multiplications required to obtain $\widetilde{\mathbf{x}}_{\ell,m}$ is bounded by

$$\mathcal{M}(\ell) + \sum_{i=1}^{m} \mathcal{N}_{\ell,i} = (m+1)\mathcal{M}(\ell) + \sum_{i=1}^{m} [\mathcal{N}(\mathbf{E}_{\ell+i}) + \mathcal{N}(\widetilde{\mathbf{K}}_{\ell+i})].$$

The proof is complete. □

Theorem 4.5. *Let ℓ, $m \in \mathbf{N}$. Assume that conditions* (a), (b) *and* (h) *hold, and $\widetilde{\mathbf{x}}_{\ell,m}$ is obtained by using Algorithm* 3.2 *with matrix compression described in the last section and truncation parameters described in Theorem* 3.2. *Then, the total number of multiplications required for computing $\widetilde{\mathbf{x}}_{\ell,m}$ is*

$$\mathcal{O}(s(\ell+m)\log^\tau(s(\ell+m))), \quad \text{as } m \to \infty,$$

where τ is the constant appearing in Theorem 3.2.

Proof. According to Lemma 4.4, the total number of multiplications required for computing $\widetilde{\mathbf{x}}_{\ell,m}$ from $\widetilde{\mathbf{x}}_\ell$ is bounded by

$$(m+1)\mathcal{M}(\ell) + \sum_{i=1}^{m} [\mathcal{N}(\mathbf{E}_{\ell+i}) + \mathcal{N}(\widetilde{\mathbf{K}}_{\ell+i})].$$

Noting that $\mathbf{E}_{\ell+i}$ is the matrix representation of $\mathcal{I}_{\ell+i}$, it follows from (4.23), Theorem 3.2 and condition (h) that

$$\mathcal{N}(\mathbf{E}_{\ell+i}) = \mathcal{O}(s(\ell+i)),$$

and

$$\mathcal{N}(\widetilde{\mathbf{K}}_{\ell+i}) = \mathcal{O}(s(\ell+i)\log^{\tau} s(\ell+i)).$$

Since $m \sim \log(s(\ell+m))$ as $m \to \infty$, we conclude

$$\sum_{i=1}^{m}[\mathcal{N}(\mathbf{E}_{\ell+i}) + \mathcal{N}(\widetilde{\mathbf{K}}_{\ell+i})] = \mathcal{O}(s(\ell+m)\log^{\tau}(s(\ell+m))), \quad \text{as } m \to \infty.$$

This with the fact that $\mathcal{M}(\ell)$ is a finite number for any fixed integer ℓ yields the estimate of this theorem. □

5. Numerical experiments. In this section we present the numerical experiments to demonstrate the efficiency and accuracy of the proposed fast singularity preserving methods.

Consider the integral equation

$$x(s) - \int_0^1 K(s,t)x(t)\,\mathrm{d}t = f(s), \quad 0 \le s \le 1, \tag{5.34}$$

where $K(s,t) := \log(|s-t|)m(s,t)$ is a weakly singular kernel with a smooth function $m(s,t) := \exp(2st)$ and $f(s)$ is chosen so that $x(s) = s\log s + (1-s)\log(1-s) + s^2$ is the solution of the equation. That is,

$$f(s) := s\log s + (1-s)\log(1-s) + s^2 \\ - \int_0^1 \log(|s-t|)\exp(2st)[t\log t + (1-t)\log(1-t) + t^2]\,\mathrm{d}t.$$

The solution x of (5.34) has singularities at $s=0$ and $s=1$.

We set $\mathbf{V} := \operatorname{span}\{w_{-1,0}, w_{-1,1}\}$, where (cf. [**2**])

$$w_{-1,0}(s) := s\log s, \quad \text{and } w_{-1,1}(s) := (1-s)\log(1-s).$$

We choose $\mathbf{U}_n$ as the space of piecewise linear polynomials with the knots at $j/2^n$, $j = 1, 2, \ldots, 2^n - 1$. The basis functions of $\mathcal{W}_0$ and $\mathcal{W}_1$ are given by

$$w_{00}(s) := 1, \qquad w_{01}(s) := \sqrt{3}(2s-1),$$

and

$$w_{10}(s) := \begin{cases} 1-6s & s \in [0,1/2], \\ 5-6s & s \in (1/2,1], \end{cases}$$

$$w_{11}(s) := \begin{cases} \sqrt{3}(1-4s) & s \in [0,1/2], \\ \sqrt{3}(4s-3) & s \in (1/2,1], \end{cases}$$

respectively. The basis functions $w_{i,j}$, $(i,j) \in J_n$, are generated recursively by

$$w_{i,j} := \mathcal{T}_\varepsilon w_{i-1,l}, \quad j = \varepsilon + 2l, \ i = 2, 3, \ldots n,$$

where $\mathcal{T}_\varepsilon$ is the isometry defined by $u \in \mathbf{X}$,

$$\mathcal{T}_\varepsilon u := \begin{cases} \sqrt{2}u(2t-\varepsilon) & t \in [\varepsilon/2, (\varepsilon+1)/2], \\ 0 & t \notin [\varepsilon/2, (\varepsilon+1)/2], \end{cases} \quad \varepsilon \in \{0,1\}.$$

The weakly singular integrals appearing in the numerical solution of the example are computed by employing Gauss-type quadrature formulas with a change of variables for the integral and a graded mesh for the integral interval introduced in [**16, 19**]. We denote by $N(n)$ the dimension of the approximate subspace of the solution. The notations C.R. and C.O. stand for the compression rate and convergence order defined, respectively, by

$$\text{C.R.} := \frac{\mathcal{N}(\widetilde{\mathbf{K}}_n)}{\mathcal{N}(\mathbf{K}_n)}, \quad \text{and} \quad \text{C.O.} := \log_2 \frac{\|x - \widetilde{x}_{n-1}\|}{\|x - \tilde{x}_n\|}.$$

The symbol C.T. denotes the computing time for solving the truncated linear system measured in seconds running on the PC with 1.83G CPU and 2G RAM. The results by using the Gauss-Seidel iteration method to solve the truncated discrete equation (3.12) are included in Table 1, where I.N. denotes the iteration number, while Table 2 lists the results of using the multilevel augmentation method described in Section 4 (initial level $\ell = 5$) to solve (3.12). The L^2 error is computed by $(\int_0^1 (x(s) - \widetilde{x}_n(s))^2 ds)^{1/2}$. Figure 1 illustrates the error by the iteration method and the multilevel augmentation method to solve the discrete system. For the piecewise linear scheme, the convergence order of the approximation should be 2, and from the numerical results above we can see that the multilevel augmentation method doesn't ruin the convergence order. Therefore, the numerical experiments confirm the theoretical analysis and illustrate the efficiency of the methods presented in this paper.

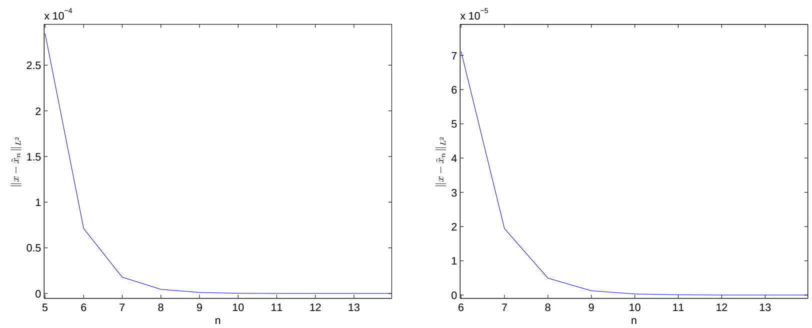

FIGURE 1. (a) errors by the iteration method; (b) errors by MAM.

TABLE 1. Numerical performance of the Gauss-Seidel iteration.

n	$N(n)$	C.R.	C.T	$\|x-\widetilde{x}_n\|_{L^2}$	C.O.	I.N.
5	34	0.834	< 0.01	2.8475e-004	1.999	10
6	66	0.651	< 0.01	7.1120e-005	2.001	11
7	130	0.449	< 0.01	1.7890e-005	1.991	13
8	258	0.300	0.019	4.5665e-006	1.970	14
9	514	0.189	0.038	1.1397e-006	2.002	14
10	1026	0.117	0.115	2.8687e-007	1.990	16
11	2050	7.205e-002	0.289	7.2118e-008	1.992	17
12	4098	4.314e-002	0.684	1.8406e-008	1.970	18
13	8194	2.569e-002	1.687	4.6449e-009	1.986	19
14	16386	1.530e-002	4.136	1.3313e-009	1.803	19

TABLE 2. Numerical performance of the MAM.

n	$N(n)$	C.R.	C.T	$\|x-\widetilde{x}_n\|_{L^2}$	C.O.
6	66	0.651	< 0.01	7.1150e-005	
7	130	0.449	< 0.01	1.9440e-005	1.872
8	258	0.300	< 0.01	4.9542e-006	1.972
9	514	0.189	0.013	1.2622e-006	1.973
10	1026	0.117	0.031	3.1792e-007	1.989
11	2050	7.205e-002	0.063	7.9947e-008	1.992
12	4098	4.314e-002	0.132	1.9995e-008	1.999
13	8194	2.569e-002	0.252	5.0481e-009	1.986
14	16386	1.530e-002	0.605	1.4451e-009	1.805

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