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SYSTEMS OF SINGULARLY PERTURBED FRACTIONAL INTEGRAL EQUATIONS

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ABSTRACT. The solution of a singularly perturbed type of a system of fractional integral (differential) equations is studied in this paper. The formal asymptotic solution is derived and proved to be asymptotically correct. Basic matrix algebra is used to prove the asymptotic decay in the inner layer solution.

1. Introduction. Consider the singularly perturbed system

$$\varepsilon \mathbf{u}(t) = \mathbf{g}(t) + {}_0J_t^{\alpha} A(t)\mathbf{u}(t), \quad 0 \le t \le T,\ 0 < \alpha < 1,\ \mathbf{g}(0) = \mathbf{0}. \tag{1.1}$$

The vector valued function $\mathbf{g}(t)$ is continuous for $0 \le t \le T$; the matrix valued function $A(t)$ is also continuous on $0 \le t \le T$ for $T > 0$. The positive parameter ε is considered to be very small, nearly zero.

The operator ${}_\varsigma J_t^{\gamma}$, and later ${}_\varsigma D_t^{\gamma}$ are defined using Riemann-Liouville definition. That, for a continuous function ϕ and for $\varsigma < \gamma < 1$,

(1.2a)
$$ {}_\varsigma J_t^{\gamma}\phi(t) := \frac{1}{\Gamma(\gamma)} \int_\varsigma^t (t-s)^{\gamma-1}\phi(s)\, ds, \quad t \ge \varsigma,$$

(1.2b)
$$ {}_\varsigma D_t^{\gamma}\phi(t) := \frac{1}{\Gamma(1-\gamma)} \frac{d}{dt} \int_\varsigma^t (t-s)^{-\gamma}\phi(s)\, ds, \quad t > \varsigma.$$

Mathematical modeling of real life processes using differential and integral equations, has recently been in favor of fractional order models

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over integer order models. For a detailed discussion on the examples of immune response fractional order models, see [**1–3, 17**]. For transportation fractional order models, see [**26**] and the references therein. Chen [**13**] investigates memory and chaos in simulations of financial systems using a fractional order model. For applications in physics, see [**18**] and for general applications of fractional calculus, see [**16, 20, 24, 25**]. This paper is motivated by the recent surge of interest in fractional order (integral and differential) models.

The existence and uniqueness of solutions of system (1.1) for all $t \geq 0$ and all $\varepsilon > 0$, under some given conditions, has been proved by several researchers in applied mathematics, especially those seeking numerical solutions. Some of the existence and uniqueness theorems can be found in [**20, 24, 25**]. This investigation is interested in the solution of (1.1) for the case when ε approached zero under some given conditions on the vector function $\mathbf{g}$ and a square matrix A.

The additive decomposition method of study of singularly perturbed equations is applied here. The method was first successfully applied to study singularly perturbed Volterra integral equations by Angell and Olmstead in [**4, 5**]. They investigated the singularly perturbed scalar Volterra integral and integro-differential equations. Imposing some smoothness conditions on the forcing function, and stability conditions on the kernel, Angell and Olmstead derived the first few terms of the formal solution–inner and outer equations. Their approach lacked the general form of the formal solution, and the results given were not proved to be asymptotically valid. Skinner in [**27**] applied the additive decomposition, in a slightly complicated way, to develop a methodology of generating all the terms of the formal solution and showed that the formal solution is indeed an asymptotic solution. His work was based on that of Smith [**28**, Chapter 6], O'Malley [**22**, Chapter 4] and O'Malley [**23**, Chapter 2] on singularly perturbed initial value problems for nonlinear ordinary differential equations. Since then, the interest in asymptotic solutions of singularly perturbed Volterra equations rose, see [**19**] for a comprehensive survey of the literature on singularly perturbed Volterra integral and integro-differential equations, up to the year 1997.

Singularly perturbed Volterra integral equations with weakly singular kernels received less attention though. This is due to the following fact: In applying the additive decomposition method, the inner layer solution is assumed to be negligible in the outer region. Exponential decay exhibited by Volterra equations with continuous kernels simplifies the analysis. However, equations with singular kernels have inner layer solutions that decay algebraically, and this impedes the analysis. Some of the recent articles that discuss asymptotic solutions of scalar Volterra integral and integro-differential equations with weakly singular kernels include [**6–12**]. The current research is for $n \times n$ systems of Volterra equations with weakly singular kernels or fractional integral (differential) equations of order α, $0 < \alpha < 1$. The novelty of this current work is that, interestingly, the algebraic decay (behavior) of the inner layer formal solution is proved using simple matrix algebra.

In the next section, mathematical preliminaries of notations, symbols and important results employed in the paper are introduced and explained. Section 3 contains the main work; the derivation of the formal approximate solution and proofs to show that it has the required properties. The last section, Section 4, contains a discussion of two examples, a theoretical example, and a practical example that is discussed in [**21**]. These examples demonstrate the methodology outlined in Section 3.

2. Mathematical preliminaries.

The following hypotheses are used:

- H_1. $0 < \alpha < 1$.
- H_2. The functions ${}_0D_t^\alpha \mathbf{g} : [0, \infty) \to \mathbf{R}^n$, $\mathbf{A} : [0, \infty) \to \mathbf{R}^{n\times n}$ are both C^∞.
- H_3. There exists a number $\eta > 0$ such that

$$\max_{\lambda\in\sigma(A(0))} \{\mathrm{Re}\,(\lambda)\} \le -\eta.$$

- H_4. For every $\lambda \in \sigma(A(0))$, the algebraic multiplicity of λ is equal to the dimension of its eigenspace.

The following theorem from [**14, 29**] will be used to establish the asymptotic behavior of the inner layer correction functions in Section 3.

Theorem 2.1. *Let $\Psi(s)$ denote the Laplace transform of the function $\psi(t)$. If $\Psi(s)$ can be expanded in a neighborhood of α_0 in an absolutely* *convergent power series with arbitrary exponents:*

$$\Psi(s) = \sum_{\nu=0}^{\infty} c_\nu (s-\alpha_0)^{\lambda_\nu}, \quad -N < \lambda_0 < \lambda_1 <\to \infty,$$

then the following asymptotic expansion for $\psi(t)$ *is valid for* $t \to \infty$*,*

$$\psi(t) \approx \mathrm{e}^{\alpha_0 t} \sum_{\nu=0}^{\infty} \frac{c_\nu}{\Gamma(-\lambda_\nu)} t^{-\lambda_\nu - 1}.$$

Similarly, if

$$\Psi(s) = \sum_{\upsilon=0}^{\infty} c_\upsilon s^{-\mu_\upsilon}, \quad -1 < \mu_0 < \mu_1 <\to \infty,$$

as $s \to \infty$*, then as* $t \to 0$*,*

$$\psi(t) \approx \sum_{\upsilon=0}^{\infty} \frac{c_\upsilon}{\Gamma(\mu_\upsilon)} t^{\mu_\upsilon - 1}.$$

Finally, The Laplace transform pair:

(2.1)

$$t^{\varrho i+\delta-1} \mathrm{E}\, \varrho, \delta^{(i)}(\pm\lambda t^{\varrho}) \circ \frac{i!\, s^{\varrho-\delta}}{(s^{\varrho} \mp \lambda)^{i+1}}, \quad \mathrm{Re}\ s > |\lambda|^{1/\varrho},\ \varrho, \delta \in \mathbf{R}^+,\ \lambda \in \mathbf{C},$$

with the usual notation for the ith derivative, will be employed. See [**24**] for details.

3. Heuristic analysis and formal solution.

3.1. Derivation of the formal solution. Since the vector valued function $\mathbf{g}(0) = \mathbf{0}$, without loss of generality, we can write equation (1.1) as

$$\varepsilon \mathbf{u}(t) = {}_0J_t^{\alpha} \left\{ \mathbf{g}^{\star}(t) + A(t)\mathbf{u}(t) \right\}, \quad 0 \le t \le T,\ 0 < \alpha < 1, \tag{3.1}$$

where

$$\mathbf{g}^{\star}(t) = {}_0D_t^{\alpha} \mathbf{g}(t).$$

Then one seeks a formal solution in the form of the formal sum,

$$\mathbf{U}(t;\varepsilon) = \mathbf{x}(t;\varepsilon) + \mu(\varepsilon)\mathbf{y}(\tau;\varepsilon),$$

where the vector function,

$$\mathbf{x}(t;\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \mathbf{x}_n(t),$$

is the outer solution and

$$\mathbf{y}(\tau;\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^{\gamma n} \mathbf{y}_n(\tau),$$

is the inner layer solution.

Since the outer solution is adequate outside the layer region, the region where $t > 0$, the inner layer functions satisfy

$$\mathbf{y}_i(\tau) \to \mathbf{0}, \quad \tau \to \infty, \ i = 0, 1, \dots .$$

Thus, $\mathbf{y}(\tau;\varepsilon)$ is assumed (and will be proved in Section 3) to be negligible in the outer layer region.

Forming the partial sum

$$\mathbf{U}_N(t;\varepsilon) = \sum_{n=0}^{N} \varepsilon^n \mathbf{x}_n(t) + \mu(\varepsilon) \sum_{n=0}^{N} \varepsilon^{\gamma n} \mathbf{y}_n\left(\frac{t}{\varepsilon^\gamma}\right),$$

substituting it into (3.1) and applying the dominant balance technique, it follows that

$$\mu(\varepsilon) = \mathrm{O}(1), \quad \varepsilon \to 0 \text{ and } \gamma = \frac{1}{\alpha}, \ 0 < \alpha < 1.$$

Therefore,

$$\mathbf{u}_N(t;\varepsilon) = \sum_{n=0}^{N} \varepsilon^n \mathbf{x}_n(t) + \sum_{n=0}^{N} \varepsilon^{\gamma n} \mathbf{y}_n\left(\frac{t}{\varepsilon^\gamma}\right), \quad \gamma = \frac{1}{\alpha}. \tag{3.2}$$

To derive the formal solution, substitute (3.2) into (3.1), giving

$$\sum_{n=0}^{N}\varepsilon^{n+1}\mathbf{x}_n(t)+\sum_{n=0}^{N}\varepsilon^{\gamma n+1}\mathbf{y}_n(t/\varepsilon^{\alpha}) = {}_0J_t^{\alpha}\left\{\mathbf{g}^{\star}(t)+A(t)\sum_{n=0}^{N}\varepsilon^{n}\mathbf{x}_n(t)+A(t)\sum_{n=0}^{N}\varepsilon^{\gamma n}\mathbf{y}_n(t/\varepsilon^{\alpha})\right\}. \tag{3.3}$$

3.1.1. Outer equation. For the outer solution, take the limit as $\varepsilon \to 0$ keeping $t > 0$ fixed. Since the inner layer function, $\mathbf{y}_n(t/\varepsilon^{\gamma})$, is negligible in the outer region,

$$\lim_{\substack{\varepsilon\to 0\\ t>0}}\mathbf{y}_n(t/\varepsilon^{\gamma})=\mathbf{0},\quad \text{for all } n\ge 0.$$

This implies that

$$\sum_{n=1}^{N}\varepsilon^{n}\mathbf{x}_{n-1}(t) = {}_0J_t^{\alpha}\left\{\mathbf{g}^{\star}(t)+A(t)\sum_{n=0}^{N}\varepsilon^{n}\mathbf{x}_n(t)\right\}+O(\varepsilon^{N}). \tag{3.4}$$

Equating coefficients of equal powers of ε, one obtains

$$\mathbf{0} = {}_0J_t^{\alpha}\left\{\mathbf{g}^{\star}(t)+A(t)\mathbf{x}_0(t)\right\} \tag{3.5}$$

and

$$\mathbf{x}_{n-1}(t) = {}_0J_t^{\alpha}\left\{A(t)\mathbf{x}_n(t)\right\},\quad n\ge 1. \tag{3.6}$$

3.1.2. Inner layer correction equation. For the inner layer solution, express (3.3) in terms of $\tau = t/\varepsilon^{\gamma}$, and collect terms together to get

$$\sum_{n=1}^{N}\varepsilon^{n}\sum_{m=0}^{n-1}\frac{1}{m!}\mathbf{x}_{n-1-m\gamma}^{(m)}(0)\tau^{m}+\sum_{n=0}^{N}\varepsilon^{\gamma n+1}\mathbf{y}_{n}(\tau)$$
$$=\sum_{n=0}^{N}\varepsilon^{\gamma n+1}\frac{1}{n!}\mathbf{g}^{\star(n)}(0)_{0}J_{\tau}^{\alpha}\tau^{n}+\sum_{n=1}^{N}\varepsilon^{n}\sum_{m=0}^{n-1}\frac{1}{m!}\mathbf{x}_{n-1-m\gamma}^{(m)}(0)A(0)_{0}J_{\tau}^{\alpha}\tau^{m}+\sum_{n=0}^{N}\varepsilon^{\gamma n+1}A(0)_{0}J_{\tau}^{\alpha}\mathbf{y}_{n}(\tau).$$

Here $\mathbf{x}_i$ and $\mathbf{y}_j$ are only defined when i and j are non-negative integers.

Equating coefficients of equal powers of ε gives

$$\mathbf{x}_0(0)+\mathbf{y}_0(\tau)={}_0J_{\tau}^{\alpha}\left\{\mathbf{g}^{\star}(0)+A(0)(\mathbf{x}_0(0)+\mathbf{y}_0(\tau))\right\},\quad \tau\geq 0,\ \mathbf{x}_0(0)+\mathbf{y}_0(0)=\mathbf{0}. \tag{3.7}$$

Higher order terms depend on the actual value of α and, hence, that of γ. See [**11**] for a detailed discussion on formal solutions of singularly perturbed equations in which the initial layer thickness is not of order of magnitude of $O(\varepsilon)$, as $\varepsilon\to 0$.

3.2. Properties of the formal solution.

3.2.1. Outer solution. Solving (3.5) gives

$$\mathbf{g}^{\star}(t)+A(t)\mathbf{x}_0(t)=\mathbf{0},\quad t\geq 0. \tag{3.8}$$

This implies that

$$\mathbf{x}_0(t)=-[A(t)]^{-1}\mathbf{g}^{\star}(t),\quad t\geq 0. \tag{3.9}$$

For the higher order terms,

$$\mathbf{x}_n(t)=[A(t)]^{-1}{}_0D_{\tau}^{\alpha}\mathbf{x}_{n-1}(t),\quad t\geq 0,\ \mathbf{x}_{n-1}(0)=\mathbf{0}. \tag{3.10}$$

Conditions H_1, H_2 and H_3 guarantee the existence of the solutions $\mathbf{x}_0$ and $\mathbf{x}_n$ for $0 \le t \le T$, $T > 0$, for all $n \ge 1$.

3.2.2. Inner layer solution. The solution of the leading order outer equation implies that

$$\mathbf{g}^{\star}(0) + A(0)\mathbf{x}_0(0) = \mathbf{0}.$$

This reduces the leading order inner layer equation (3.7) into

$$\mathbf{x}_0(0) + \mathbf{y}_0(\tau) = {}_0J_\tau^\alpha A(0)\mathbf{y}_0(\tau), \quad \tau \ge 0. \tag{3.11}$$

In particular, $\mathbf{x}_0(0) + \mathbf{y}_0(0) = \mathbf{0}$. It can be shown that system (3.11), which is equivalent to the fractional differential equation of the same order, has a unique solution $\mathbf{y}_0(\tau)$ for all $\tau \ge 0$. For details, see, for example, [**24**].

To prove that $\mathbf{y}_0(\tau)$ decays to zero as τ goes to infinity, apply the Laplace transform on both sides of (3.11). This yields

$$\frac{1}{s}\mathbf{x}_0(0) + \mathbf{Y}_0(s) = s^{-\alpha}A(0)\mathbf{Y}_0(s),$$

where $\mathbf{Y}_0$ is the Laplace transform of $\mathbf{y}_0$. Rearranging and collecting terms together gives

$$\mathbf{Y}_0(s) = -R^{-1}(s)s^{\alpha-1}\mathbf{x}_0(0), \tag{3.12}$$

where the matrix $R(s) = [s^\alpha I - A(0)]$. Let $-\lambda_i$ for $\lambda_i > 0$ be the ith eigenvalue of $A(0)$. Assuming H_4 holds, the matrix $R(s)$ and hence $R^{-1}(s)$ are diagonalizable. From elementary linear algebra, one writes (3.12) as

$$\mathbf{Y}_0(s) = PDP^{-1}s^{\alpha-1}\mathbf{x}_0(0).$$

Matrix D above is the diagonal matrix,

$$D = \begin{pmatrix} 1/s^\alpha + \lambda_1 & 0 & \cdots & \cdots & 0 \\ 0 & 1/s^\alpha + \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & \cdots & 1/s^\alpha + \lambda_n \end{pmatrix},$$

where

$$\frac{1}{s^\alpha + \lambda_i}$$

is the ith eigenvalue of $R(s)^{-1}$. The matrices P and P^{-1} are independent of s. Therefore, for each ith component of $\mathbf{Y}_0$, $Y_{0i} \in \mathbf{Y}_0$,

$$Y_{0i}(s) \approx \xi_0 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{{\lambda_i}^k} s^{\alpha k-1}, \quad s \to 0,$$

for some number ξ_0 which depends upon the matrices P, P^{-1} and the vector $\mathbf{x}(0)$.

Then Theorem 2.1 implies that

$$y_{0i}(\tau) \approx \xi_0 \sum_{k=1}^{\infty} \frac{(-1)^{k-1}{\lambda_i}^{-k}}{\Gamma(1-\alpha k)} \tau^{-\alpha k}, \quad \tau \to \infty,\ 1 \le i \le n, \tag{3.13}$$

as required, where y_{0i} is the ith component in $\mathbf{y}_0$.

3.2.3. The remainder term. Suppose that vector $\mathbf{u}_N(t;\varepsilon)$ satisfies (1.1) approximately with a remainder, $\mathbf{r}_N(t;\varepsilon)$. Then

$$\varepsilon \mathbf{u}_N(t;\varepsilon) = \mathbf{g}(t) + {}_0J_t^\alpha A(t)\mathbf{u}_N(t;\varepsilon) - \mathbf{r}_N(t;\varepsilon).$$

It follows that

$$\mathbf{r}_N(t;\varepsilon) = \mathbf{g}(t) + {}_0J_t^\alpha A(t)\mathbf{u}_N(t;\varepsilon) - \varepsilon \mathbf{u}_N(t;\varepsilon). \tag{3.14}$$

Theorem 3.1. *Suppose that* H_1, H_2 *and* H_3 *hold. Then for* $N = 0$, *the residual* $\mathbf{r}_0(t;\varepsilon)$ *given in* (3.14) *satisfies*

$$|\mathbf{r}_0(t;\varepsilon)| \le \kappa\varepsilon, \quad \varepsilon \to 0,$$

uniformly for all $0 \le t \le T$, *for some fixed positive constant* κ *which does not depend upon* ε.

Proof. From (3.14),

$$\mathbf{r}_0(t;\varepsilon) = \mathbf{g}(t) + {}_0J_t^\alpha A(t)\mathbf{u}_0(t;\varepsilon) - \varepsilon \mathbf{u}_0(t;\varepsilon).$$

Substituting $\mathbf{u}_0(t;\varepsilon) = \mathbf{x}_0(t) + \mathbf{y}_0(t/\varepsilon^\gamma)$, gives

$$\mathbf{r}_0(t;\varepsilon) = \mathbf{g}(t) + {}_0J_t^\alpha A(t)\left\{\mathbf{x}_0(t) + \mathbf{y}_0\left(\frac{t}{\varepsilon^\gamma}\right)\right\} - \varepsilon\left\{\mathbf{x}_0(t) + \mathbf{y}_0\left(\frac{t}{\varepsilon^\gamma}\right)\right\}.$$

Equivalently,

$$\begin{aligned}\mathbf{r}_0(t;\varepsilon) = \mathbf{g}(t) &+ {}_0J_t^\alpha A(t)\mathbf{x}_0(t)\\ &+ {}_0J_t^\alpha A(0)\mathbf{y}_0\left(\frac{t}{\varepsilon^\gamma}\right) - \varepsilon\left\{\mathbf{x}_0(0) + \mathbf{y}_0\left(\frac{t}{\varepsilon^\gamma}\right)\right\}\\ &+ {}_0J_t^\alpha\left\{A(t) - A(0)\right\}\mathbf{y}_0\left(\frac{t}{\varepsilon^\gamma}\right) - \varepsilon\left\{\mathbf{x}_0(t) - \mathbf{x}_0(0)\right\}.\end{aligned}$$

Using (3.8) and (3.11), the above equation simplifies into

$$\mathbf{r}_0(t;\varepsilon) = {}_0J_t^\alpha\left\{A(t) - A(0)\right\}\mathbf{y}_0\left(\frac{t}{\varepsilon^\gamma}\right) - \varepsilon\left\{\mathbf{x}_0(t) - \mathbf{x}_0(0)\right\}.$$

Changing variables yields

$$\begin{aligned}\mathbf{r}_0(t;\varepsilon) = &\frac{t^\alpha}{\Gamma(\alpha)}\int_0^1 (1-\sigma)^{\alpha-1}\left\{A(t\sigma) - A(0)\right\}\mathbf{y}_0\left(\frac{t\sigma}{\varepsilon^\gamma}\right)d\sigma\\ &- \varepsilon\left\{\mathbf{x}_0(t) - \mathbf{x}_0(0)\right\}.\end{aligned}$$

The asymptotic behavior of $\mathbf{y}_0(\tau)$ as $\tau \to \infty$ in (3.13) implies that, for each component y_{0i} of $\mathbf{y}_0$,

$$y_{0i}\left(\frac{t\sigma}{\varepsilon^\gamma}\right) \sim \frac{\xi_0}{\lambda_i\Gamma(1-\alpha)}(t\sigma)^{-\alpha}\varepsilon^{\alpha\gamma} + o(\varepsilon), \quad \varepsilon \to 0.$$

It then follows that, for ε sufficiently small and, for all $0 \le t \le T$, $T > 0$,

$$|\mathbf{r}_0(t;\varepsilon)| \le \frac{\varepsilon|\xi_0||A|}{\lambda_i\Gamma(\alpha)\Gamma(1-\alpha)}\int_0^1 (1-\sigma)^{\alpha-1}\sigma^{-\alpha}\,d\sigma + \varepsilon|\mathbf{x}|, \tag{3.15}$$

where

$$|A| = \max_{1\le i,j\le n}\{A_{ij}(t) - A_{ij}(0)\}$$

and

$$|\mathbf{x}| = \max_{1\le i\le n}\{x_{0i}(t) - x_{0i}(0)\}, \quad 0 \le t \le T.$$

Evaluating the integral using the properties of the beta function gives

$$|\mathbf{r}_0(t;\varepsilon)| \le \varepsilon\kappa,$$

for all $0 \le t \le T$ and all ε sufficiently small. In this case, $\kappa = |\xi_0||A|/\lambda_i + |\mathbf{x}|$ is independent of ε.

4. Examples. In this section two examples are given to demonstrate the methodology developed in the previous section.

4.1. A theoretical example. Consider the singularly perturbed fractional equation

$$\varepsilon\mathbf{u}(t) = \mathbf{g}(t) + {}_0J_\tau^\alpha A\mathbf{u}(t), \quad t \ge 0, \tag{4.1}$$

where

$$A = \begin{pmatrix} -1 & 1 \\ 0 & -2 \end{pmatrix} \quad \text{and} \quad \mathbf{g}(t) = \begin{pmatrix} t^{\alpha+1} \\ t^\alpha \end{pmatrix}.$$

Both of these functions satisfy the conditions given in Section 2.

The leading outer equation and its solution follow from (3.9) as

$$\begin{aligned}\mathbf{x_0}(\mathbf{t}) &= -A^{-1}\mathbf{g}^\star(t)\\ &= \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}\begin{pmatrix} {}_0D_t^\alpha t^{\alpha+1} \\ {}_0D_t^\alpha t^\alpha \end{pmatrix}.\end{aligned}$$

Using the Riemann-Liouville definition (1.2b), one simplifies the leading outer solution in the form

$$\mathbf{x}_0(t) = \begin{pmatrix} 2t/\Gamma(1-\alpha)\int_0^1 (1-\sigma)^{-\alpha}\sigma^{\alpha+1}\,d\sigma + 1/2a \\ 1/2a \end{pmatrix}, \quad t \ge 0, \tag{4.2}$$

where

$$\begin{aligned} a &= \frac{1}{\Gamma(1-\alpha)}\int_0^1 (1-\sigma)^{-\alpha}\sigma^\alpha\,d\sigma,\\ &= \frac{1}{\Gamma(1-\alpha)}\mathcal{B}(1-\alpha, 1+\alpha),\\ &= \Gamma(1+\alpha).\end{aligned} \tag{4.3}$$

In particular,

$$\mathbf{x}_0(0) = \begin{pmatrix} 1/2a \\ 1/2a \end{pmatrix} \neq \mathbf{0},$$

and therefore,

$$\mathbf{x}_n(t) = \mathbf{0}, \quad \text{for all } n \geq 1, \text{ for all } t \geq 0.$$

Equation (3.11) implies that the leading order inner layer equation is

$$\mathbf{y}_0(\tau) = -\begin{pmatrix} 1/2a \\ 1/2a \end{pmatrix} + {}_0J_\tau^\alpha A\mathbf{y}_0(\tau).$$

Applying the Laplace transform on both sides of the above equation above yields

$$\mathbf{Y}_0(s) = -\frac{1}{s}\begin{pmatrix} 1/2a \\ 1/2a \end{pmatrix} + s^{-\alpha}A\mathbf{Y}_0(s), \quad s > 0.$$

Substituting A and rearranging the terms, one gets

$$\mathbf{Y}_0(s) = -\frac{1}{2}a\begin{pmatrix} s^{\alpha-1}/(s^\alpha+1) + s^{\alpha-1}/(s^\alpha+1)(s^\alpha+2) \\ s^{\alpha-1}/(s^\alpha+2) \end{pmatrix}, \tag{4.4}$$

where a is as defined in (4.3).

Using the Laplace transform pair (2.1), one writes the inner layer solution as

$$\mathbf{y}_0(\tau) = -\frac{1}{2}a\begin{pmatrix} \mathrm{E}\,\alpha(-\tau^\alpha) + \int_0^\tau (\tau-\sigma)^{\alpha-1}\mathrm{E}\,\alpha,\alpha(-2(\tau-\sigma)^\alpha)\mathrm{E}\,\alpha,1(-\sigma^\alpha)\,d\sigma \\ \mathrm{E}\,\alpha(-2\tau^\alpha) \end{pmatrix}. \tag{4.5}$$

Integrating yields

$$\mathbf{y}_0(\tau) = -\frac{1}{2}a\begin{pmatrix} \mathrm{E}\,\alpha(-\tau^\alpha) + \tau^\alpha\{2\mathrm{E}\,\alpha,\alpha+1(-2\tau^\alpha) - \mathrm{E}\,\alpha,\alpha+1(-\tau^\alpha)\} \\ \mathrm{E}\,\alpha(-2\tau^\alpha) \end{pmatrix}. \tag{4.6}$$

To show that $\mathbf{y}_0(\tau) = O(\tau^{-\alpha})$ as $\tau \to \infty$, the following property of the Mittag-Leffler function is used:

$$z\mathrm{E}\,\varepsilon,\varepsilon+\varrho(z) = \mathrm{E}\,\varepsilon,\varrho(z) - \frac{1}{\Gamma(\varrho)}, \quad \varepsilon,\varrho > 0, \ z \in \mathbf{C}. \tag{4.7}$$

Thus (4.6) is equivalent to:

$$\mathbf{y}_0(\tau) = -\frac{1}{2}a \begin{pmatrix} 2\mathrm{E}\,\alpha(-\tau^\alpha) - \mathrm{E}\,\alpha(-2\tau^\alpha) \\ \mathrm{E}\,\alpha(-2\tau^\alpha) \end{pmatrix}. \tag{4.8}$$

Clearly, $\mathbf{y}_0(\tau) \to 0$, $\tau \to \infty$. The algebraic decay of $\mathbf{y}_0$ follows from the asymptotic properties of the Mittag-Leffler function developed in [**15**], and later in several research articles including [**6, 24**].

Therefore, to the leading order, the asymptotic solution of (4.1) is

$$\mathbf{u}(t;\varepsilon) = \begin{pmatrix} 1/2a+(2t/\Gamma(1-\alpha))\int_0^1(1-\sigma)^{-\alpha}\sigma^{\alpha+1}\,d\sigma-1/2a\{2\mathrm{E}\,\alpha(-t^\alpha/\varepsilon)-\mathrm{E}\,\alpha(-2(t^\alpha/\varepsilon))\} \\ 1/2a-1/2a\mathrm{E}\,\alpha(-2(t^\alpha/\varepsilon)) \end{pmatrix},$$

for all $t \geq 0$ and all $\varepsilon > 0$.

4.2. A practical example. Consider the singularly perturbed system resulting from the equation of motion of a particle through a liquid, discussed in [**21**].

$$\varepsilon \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix} = \begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} + {}_0J_t^{1/2} \begin{pmatrix} 0 & 1 \\ -c_2 & -c_1 \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}, \tag{4.9}$$

where c_1 and c_2 are positive constants.

Let $\mathbf{g}(t) = \begin{pmatrix} 0 \\ 2/\sqrt{\pi}t^{1/2} \end{pmatrix}$, which implies that $\mathbf{g}^\star(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. It follows that $\mathbf{g}$ satisfies condition H_2 stated in Section 2. For A to fully satisfy H_3 and H_4, one requires

$$c_1^2 - 4c_2 \neq 0. \tag{4.10}$$

The exact solution of (4.9) when $\varepsilon = 0$ is $\mathbf{u} = \begin{pmatrix} 1/c_2 \\ 0 \end{pmatrix}$. This solution is qualitatively different from the exact solution when $\varepsilon \neq 0$, in particular, because $\mathbf{u}(0, \varepsilon) = \mathbf{0}$. Thus, problem (4.9) is singularly perturbed.

Equation (4.9) can be compared with equation (2.17) on page 306 in [**21**] with $\alpha = 1/2$, the so-called ordinary Basset problem. Constants c_1 and c_2 are such that $c_1 = a$ and $c_2 = 1$, and $u_1(t) = V(t)$. Solving (4.9) is equivalent to solving (2.17) with the Basset force (term) approaching zero and the particle's initial velocity being zero.

The outer solution is given from (3.9) by

$$\begin{aligned}\mathbf{x}_0(t) &= A^{-1}\mathbf{g}^\star(t)\\ &= \begin{pmatrix} c_1/c_2 & 1/c_2\\ -1 & 0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}1/c_2\\0\end{pmatrix}.\end{aligned}$$

Therefore, the leading order outer solution is

$$\mathbf{x}_0(t) = \begin{pmatrix}1/c_2\\0\end{pmatrix}, \quad \text{for all } t \geq 0. \tag{4.11}$$

Since $\mathbf{x}_0(0) \neq \mathbf{0}$, $\mathbf{x}_i(t) = \mathbf{0}$ for all $i \geq 1$ and all $t \geq 0$.

To derive the leading order inner layer solution, one may use (3.12) to get

$$\mathbf{Y}_0(s) = \begin{pmatrix} s^{1/2} & -1\\ c_2 & s^{1/2}+c_1\end{pmatrix}^{-1} s^{-1/2}\begin{pmatrix}-1/c_2\\0\end{pmatrix}.$$

Further computation leads to

$$\begin{aligned}&\mathbf{Y}_0(s)\\ &\quad= \begin{pmatrix} 1/c_2/[(s^{1/2}+a^\star)(s^{1/2}+b^\star)]-c_1/c_2 s^{-1/2}/[(s^{1/2}+a^\star)(s^{1/2}+b^\star)]\\ s^{-1/2}/[(s^{1/2}+a^\star)(s^{1/2}+b^\star)]\end{pmatrix},\end{aligned}$$

where

$$a^\star + b^\star = c_1 \tag{4.12}$$

$$a^\star b^\star = c_2. \tag{4.13}$$

This implies that both $a^\star$ and $b^\star$ are positive constants.

The Laplace transform pair (2.1) and the convolution theorem (of the Laplace transform) imply that

$$\begin{aligned}&\mathbf{y}_0(\tau)\\ &= \begin{pmatrix} -1/c_2\int_0^\tau(\tau-\sigma)^{-1/2}\mathrm{E}_{1/2,1/2}(-a^\star(\tau-\sigma)^{1/2})\sigma^{-1/2}\mathrm{E}_{1/2,1/2}(-b^\star\sigma^{1/2})\,d\sigma\\ -(c_1/c_2)\int_0^\tau(\tau-\sigma)^{-1/2}\mathrm{E}_{1/2,1/2}(-a^\star(\tau-\sigma)^{1/2})\sigma^{-1/2}\mathrm{E}_{1/2,1}(-b^\star\sigma^{1/2})\,d\sigma\\ \int_0^\tau(\tau-\sigma)^{-1/2}\mathrm{E}_{1/2,1/2}(-a^\star(\tau-\sigma)^{1/2})\sigma^{-1/2}\mathrm{E}_{1/2,1}(-b^\star\sigma^{1/2})\,d\sigma\end{pmatrix}.\end{aligned}$$

Integration gives

$$\begin{aligned}&\mathbf{y}_0(\tau)\\ &\quad= \begin{pmatrix} -1/c_2(a^\star-b^\star)\{a^\star\mathrm{E}_{1/2}(-a^\star\tau^{1/2})-b^\star\mathrm{E}_{1/2}(-b^\star\tau^{1/2})\}\\ -c_1\tau^{1/2}/c_2(a^\star-b^\star)\{a^\star\mathrm{E}_{1/2,3/2}(-a^\star\tau^{1/2})-b^\star\mathrm{E}_{1/2,3/2}(-b^\star\tau^{1/2})\}\\ \tau^{1/2}/c_2(a^\star-b^\star)\{a^\star\mathrm{E}_{1/2,3/2}(-a^\star\tau^{1/2})-b^\star\mathrm{E}_{1/2,3/2}(-b^\star\tau^{1/2})\}\end{pmatrix}.\end{aligned}$$

The diagonalizability condition (4.10) guarantees that $a^\star \neq b^\star$. It then follows that

$$\mathbf{y}_0(0) = \begin{pmatrix} -1/c_2 \\ 0 \end{pmatrix}$$

as required, since $\mathbf{x}_0(0) + \mathbf{y}_0(0) = \mathbf{0}$.

To see that $y_0(\tau) \to \mathbf{0}$ as $\tau \to \infty$, one applies (4.7) to express $\mathbf{y}_0(\tau)$ as

$$\mathbf{y}_0(\tau) = \begin{pmatrix} -1/c_2(a^\star - b^\star)\left\{a^\star \mathrm{E}_{1/2}(-a^\star \tau^{1/2}) - b^\star \mathrm{E}_{1/2}(-b^\star \tau^{1/2})\right\} \\ -c_1/c_2(a^\star - b^\star)\left\{\mathrm{E}_{1/2}(-b^\star \tau^{1/2}) - \mathrm{E}_{1/2}(-a^\star \tau^{1/2})\right\} \\ 1/(a^\star - b^\star)\left\{\mathrm{E}_{1/2}(-b^\star \tau^{1/2}) - \mathrm{E}_{1/2}(-a^\star \tau^{1/2})\right\} \end{pmatrix}.$$

To the leading order, the asymptotic solution of (4.9) is given by

$$(4.14)\quad \mathbf{u}(t;\varepsilon) = \begin{pmatrix} (1/c_2) - (1/c_2(a^\star - b^\star))\{a^\star \mathrm{E}_{1/2}(-a^\star t^{1/2}/\varepsilon) - b^\star \mathrm{E}_{1/2}(-b^\star t^{1/2}/\varepsilon)\} \\ -(c_1/c_2(a^\star - b^\star))\{\mathrm{E}_{1/2}(-b^\star t^{1/2}/\varepsilon) - \mathrm{E}_{1/2}(-a^\star t^{1/2}/\varepsilon)\} \\ 1/(a^\star - b^\star)\{\mathrm{E}_{1/2}(-b^\star t^{1/2}/\varepsilon) - \mathrm{E}_{1/2}(-a^\star t^{1/2}/\varepsilon)\} \end{pmatrix}.$$

Further calculations on the higher order terms using

$$\mathbf{u}(t;\varepsilon) = \sum_{n=0}^{\infty} \varepsilon^n \mathbf{x}_n(t) + \sum_{n=0}^{\infty} \varepsilon^{2n} \mathbf{y}_n\left(\frac{t}{\varepsilon^2}\right),$$

reveal that

$$\mathbf{x}_n = \mathbf{0}, \quad n \geq 1 \text{ and } \mathbf{y}_n = \mathbf{0},\ n \geq 1.$$

This then implies that the asymptotic solution of the singularly perturbed system (4.9), to all orders, is given by (4.14).

To see specific solutions, consider the case of a light particle, and let χ as defined in [**21**] be equal to 0.5. Then $a = c_1 = 3/\sqrt{2}$. If $c_2 = 1$, it follows that

$$a^\star = \sqrt{2}, \qquad b^\star = \frac{1}{\sqrt{2}} \text{ or } a^\star = \frac{1}{\sqrt{2}}, \qquad b^\star = \sqrt{2}.$$

Taking $a^\star = \sqrt{2}$, $b^\star = 1/\sqrt{2}$, for example, and $\varepsilon = 0.2$, the solution $u_1(t;0.2)$ which is equivalent to $V(t)$ in [**21**] approaches 1, the solution without the Basset term, as desired, when t increases. See Figure 4.1.

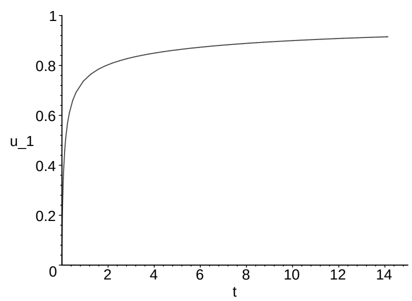

FIGURE 4.1. $u_1(t)$ when $\varepsilon = 0.2$, $c_1 = 3/\sqrt{2}$, $c_2 = 1$.

Note that, for cases when $\chi > 0.625$, the positive constants $a^\star$ and $b^\star$ are complex numbers. This leads to the argument of the Mittag-Leffler function being complex. These cases have also been discussed in [**21**].

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