JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 24, Number 1, Spring 2012

ASYMPTOTIC EXPANSIONS FOR APPROXIMATE SOLUTIONS OF FREDHOLM INTEGRAL EQUATIONS WITH GREEN'S FUNCTION TYPE KERNELS

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Communicated by Kendall Atkinson

ABSTRACT. Asymptotic expansions at the node points for approximate solutions of the second kind Fredholm integral equation with a kernel of Green's type function are obtained in the Nyström method based on the composite midpoint, the composite Simpson and the composite modified Simpson rules. Similar expansions are also obtained for the iterated collocation method associated with piecewise constant, piecewise linear and piecewise quadratic functions. Richardson extrapolation is used to obtain approximate solutions with higher order of convergence at the node/partition points. Numerical examples are given to illustrate various results.

1. Introduction. Asymptotic expansions of approximate solutions of second kind Fredholm integral equations with a smooth kernel have been extensively studied in the research literature. Some of the important methods for finding an approximate solution are Nyström methods defined by replacing the integral in the integral operator by a convergent quadrature formula and projection related methods such as the classical Galerkin method and its variants. Asymptotic expansions in the case of the Nyström method associated with various composite quadrature rules have been obtained by Baker [**4**] and McLean [**11**]. They also consider the iterated Galerkin and the iterated collocation methods associated with projections onto a piecewise polynomial space with respect to a uniform partition. These expansions are based on

2010 AMS *Mathematics subject classification.* Primary 45B05, 65R20.
Keywords and phrases. Fredholm integral equation, Green's function type kernels, Nyström method, iterated collocation method, asymptotic expansion.
The work of the first author was partially supported by CEFIPRA, project no. 4101-1.
Received by the editors on November 1, 2010, and in revised form on May 18, 2011.
DOI:10.1216/JIE-2012-24-1-39

the Euler-MacLaurin summation formula. The case of the iterated collocation at Gauss points has also been considered in Lin, Sloan and Xie [**10**]. In [**10**] the range of the interpolatory projection is chosen to be a piecewise polynomial space with respect to a non-uniform partition. In this case, a different technique than the Euler-MacLaurin summation formula needs to be developed. In Kulkarni and Grammont [**8**] a modified projection method has been considered.

For an integral operator with a kernel of the type of Green's function, only two cases seem to have been studied. Baker [**3, 4**] considered the case of the Nyström operator associated with the composite trapezoidal rule and obtained asymptotic expansions for approximate solutions at the node points. As the node points in the composite trapezoidal rule are the end-points of subintervals, the technique of the Euler-MacLaurin summation formula is still applicable. In Lin and Liu [**9**] the case of the iterated collocation method based on continuous piecewise linear polynomials with collocation points as the partition points is considered. It is shown that one step of Richardson extrapolation provides a globally superconvergent solution. The improvement in the order of convergence is from h^2 to h^4, where h is the length of the subinterval in a uniform partition of $[0, 1]$. The results in this paper are based on a careful and clever manipulation of the terms. However, it is not evident how to generalize them so as to be applicable to other cases of piecewise polynomials or to obtain an asymptotic expansion. The case of non-linear operators for the composite trapezoidal rule has been studied by Sidi and Pennline [**12**] and Ford et al. [**6**].

In a more general setting, the case of an integral operator T with kernel of the type of Green's function is considered in Frammartino [**7**]. The operator T is approximated by the iterated collocation operator $T\pi_n$, where $\pi_n f$ is the interpolating polynomial at n zeros of the Jacobi polynomial. If the right hand side is sufficiently smooth, then the error in W^1 norm is shown to be on the order of $n^{-(r+1)}$.

In this paper we consider the case of an integral operator T with a kernel of the type of Green's function:

$$(Tx)(s) = \int_0^1 k(s,t)x(t)\,dt.$$

Let x be a smooth function. We first consider the Nyström method associated with the composite midpoint and the composite Simpson rules. Let T_n denote the approximating operator. As the kernel $k(s,t)$ is of the type of Green's function, it lacks differentiability along the diagonal, that is, when $s = t$. In order to obtain asymptotic expansions at the node points in the composite midpoint rule, the first variable s is fixed to be a midpoint of a subinterval of a uniform partition, and hence the corresponding function $k(s,t)x(t)$ fails to be differentiable at an interior point. Thus, the Euler-MacLaurin formula is not applicable. A similar situation arises in the composite Simpson rule. In order to take care of this difficulty, we first prove an extension of the Euler-MacLaurin formula for a function which fails to be differentiable at an interior point. Asymptotic expansions at the node points, $(T_nx - Tx)(r_i)$, are then obtained both in the cases of the composite midpoint and the composite Simpson rules. We would like to use the Richardson extrapolation to obtain approximations of higher order at the node points. In the case of the composite midpoint rule, if we refine the partition by subdividing each subinterval into two equal parts, then the new set of nodes does not contain the old nodes, and thus extrapolation is not possible. However, it is possible to obtain asymptotic expansions also at the partition points, which allows us to use the Richardson extrapolation.

Since the kernel is not smooth, in the composite Simpson rule, the order of convergence at the node points which are not partition points gets reduced from h^4 to h^2, where h is the length of the subinterval of an uniform partition. In Atkinson and Shampine [**2**] and Cubillos [**5**], a modified Simpson rule is proposed to restore the order of convergence to h^4. We obtain an asymptotic expansion in the case of the Nyström method with modified Simpson rule as well.

Next we obtain asymptotic expansions for $(T_nx - Tx)(r_i)$ in the iterated collocation methods based on piecewise constant polynomials, continuous piecewise linear polynomials and continuous piecewise quadratic polynomials, and show that the result of Lin and Liu [**9**] is a special case of our result.

The main result of this paper is an asymptotic expansion for $(T_nx - Tx)(r_i)$, where r_i are the node points in Nyström methods and the collocation/partition points in iterated collocation methods. Let u denote the exact solution and u_n an approximate solution. The asymptotic expansions $(u_n - u)(r_i)$ are then obtained essentially by the technique described in Ford et al. [**6**]. The Richardson extrapolation allows us to construct higher order approximations to u by refinement of partitions. We illustrate our results by numerical examples and show that they match well with the theoretical predictions.

The above results can be extended to nonlinear operator equations which will be discussed in another paper.

2. Preliminaries. Consider the following integral operator

$$Tx(s) = \int_0^1 k(s,t)x(t)\,dt, \quad s \in [0,1], \tag{2.1}$$

where the kernel $k(\cdot,\cdot) \in C([0,1]\times[0,1])$. In addition, we assume that, for every $s \in [0,1]$, the function $k(s,t)$ is $m+2$ times differentiable with respect to t in $[0,s)\cup(s,1]$, where m is an even positive integer. An example of such a kernel with $m=\infty$ is given below:

$$k(s,t) = \begin{cases} s(1-t) & \text{if } s \le t, \\ t(1-s) & \text{if } t < s. \end{cases}$$

The operator $T : C[0,1] \to C[0,1]$ is compact. Assume that 1 is in the resolvent set of T so that the following equation

$$u(s) - (Tu)(s) = f(s), \quad s \in [0,1], \tag{2.2}$$

has a unique solution.

Let T_n be a sequence of operators such that either T_n converges to T in the norm or T_n converges to T pointwise and the set $\{T_n : n \ge 1\}$ is collectively compact, that is, the set $\{T_n x : n \ge 1,\ \|x\| \le 1\}$ has a compact closure in $C[0,1]$. Thus,

$$\|(T-T_n)T\| \longrightarrow 0, \qquad \|(T-T_n)T_n\| \longrightarrow 0, \quad \text{as } n \to \infty.$$

In both cases, for all n large enough, 1 is in the resolvent set of T_n and

$$u_n(s) - (T_n u_n)(s) = f(s), \quad s \in [0,1], \tag{2.3}$$

has a unique solution. (See Atkinson [**1**].)

Let $n \in \mathbf{N}$ and $h = 1/n$. Throughout this paper we consider the following uniform partition of $[0,1]$:

$$0 < \frac{1}{n} < \frac{2}{n} < \cdots < \frac{n-1}{n} < 1. \tag{2.4}$$

Let $t_i = (i-1)h$, $i = 1, 2, \ldots, n+1$, be the partition points.

For $\nu \geq 1$, let $B_\nu(t)$ be the Bernoulli polynomial of degree ν and $B_\nu = B_\nu(0)$ be the Bernoulli numbers. Let

$$B_0(t) = 1.$$

We have

$$B_\nu(1-t) = (-1)^\nu B_\nu(t) \tag{2.5}$$

and, for $\nu \geq 2$,

$$B_\nu(1) = B_\nu(0). \tag{2.6}$$

It then follows from (2.5) and (2.6) that, for ν odd, $\nu \geq 3$,

$$B_\nu = B_\nu(0) = 0, \qquad B_\nu\left(\frac{1}{2}\right) = 0. \tag{2.7}$$

On the other hand, since $B_1(t) = t - (1/2)$,

$$B_1 = B_1(0) = -\frac{1}{2}, \qquad B_1(1) = \frac{1}{2}, \qquad B_1\left(\frac{1}{2}\right) = 0.$$

We define $\overline{B}_\nu$ as a periodic function on $\mathbf{R}$ with period 1:

$$\overline{B}_\nu(t) = B_\nu(t), \quad 0 \leq t < 1, \qquad \overline{B}_\nu(t+1) = \overline{B}_\nu(t), \quad t \in \mathbf{R}.$$

Then since $B_1(0) \neq B_1(1)$, $\overline{B}_1$ is discontinuous at the integers, whereas for $\nu = 0$ and $\nu \geq 2$, it follows from (2.6) that $\overline{B}_\nu$ is continuous on $\mathbf{R}$. Also, for $\nu = 1, 2, \ldots,$

$$(\overline{B}_\nu)'(t) = \nu \overline{B}_{\nu-1}(t), \quad t \in \mathbf{R}/\mathbf{Z},$$

where $(\overline{B}_\nu)'$ denotes the derivative of $(\overline{B}_\nu)$.

Euler-MacLaurin series expansion (Steffensen [**14**], McLean [**11**]): Let $f : [0,1] \to \mathbf{R}$ be m times differentiable on $[0,1]$. Let $0 \leq \tau \leq 1$. Then

$$f(\tau) = \int_0^1 f(t)\,dt + \sum_{\nu=1}^{m} \frac{B_\nu(\tau)}{\nu!}[f^{(\nu-1)}(1-) - f^{(\nu-1)}(0+)] + R_m, \tag{2.8}$$

where

$$R_m = -\int_0^1 \frac{\overline{B}_m(\tau - t)}{m!} f^{(m)}(t)\, dt. \tag{2.9}$$

3. Euler-MacLaurin series expansion for non-smooth functions. We first prove an extension of the Euler-MacLaurin series expansion for functions which are m times differentiable on $(0,1)$ except at one point. The following proposition provides a basis for various results which follow.

Proposition 3.1. *Fix $s \in (0,1)$. Let $f : [0,1] \to \mathbf{R}$ be continuous and m times differentiable on $[0,s) \cup (s,1]$. Then for $0 \le \tau \le 1$,*

$$\begin{aligned} f(\tau) = \int_0^1 f(t)\, dt &+ \sum_{\nu=1}^{m} \frac{B_\nu(\tau)}{\nu!}[f^{(\nu-1)}(1-) - f^{(\nu-1)}(0+)] \\ &- \sum_{\nu=2}^{m} \frac{\overline{B}_\nu(\tau - s)}{\nu!}[f^{(\nu-1)}(s+) - f^{(\nu-1)}(s-)] + R_m, \end{aligned} \tag{3.1}$$

where

$$R_m = -\int_0^1 \frac{\overline{B}_m(\tau - t)}{m!} f^{(m)}(t)\, dt. \tag{3.2}$$

Proof. Let $m \ge 2$. Recall that, for $m \ge 2$, $\overline{B}_m$ is continuous on $\mathbf{R}$. Consider

$$\begin{aligned} R_m &= -\int_0^1 \frac{\overline{B}_m(\tau - t)}{m!} f^{(m)}(t)\, dt \\ &= -\int_0^s \frac{\overline{B}_m(\tau - t)}{m!} f^{(m)}(t)\, dt \\ &\quad - \int_s^1 \frac{\overline{B}_m(\tau - t)}{m!} f^{(m)}(t)\, dt. \end{aligned}$$

On integration by parts, we obtain

$$
\begin{aligned}
R_m &= -\left[\frac{\overline{B}_m(\tau-t)}{m!} f^{(m-1)}(t)\right]_{t=0+}^{t=s-} - \int_0^s \frac{\overline{B}_{m-1}(\tau-t)}{(m-1)!} f^{(m-1)}(t)\,dt \\
&\quad - \left[\frac{\overline{B}_m(\tau-t)}{m!} f^{(m-1)}(t)\right]_{t=s+}^{t=1-} - \int_s^1 \frac{\overline{B}_{m-1}(\tau-t)}{(m-1)!} f^{(m-1)}(t)\,dt \\
&= -\frac{B_m(\tau)}{m!}\left[f^{(m-1)}(1-) - f^{(m-1)}(0+)\right] \\
&\quad + \frac{\overline{B}_m(\tau-s)}{m!}\left[f^{(m-1)}(s+) - f^{(m-1)}(s-)\right] \\
&\quad - \int_0^1 \frac{\overline{B}_{m-1}(\tau-t)}{(m-1)!} f^{(m-1)}(t)\,dt \\
&= -\frac{B_m(\tau)}{m!}\left[f^{(m-1)}(1-) - f^{(m-1)}(0+)\right] \\
&\quad + \frac{\overline{B}_m(\tau-s)}{m!}\left[f^{(m-1)}(s+) - f^{(m-1)}(s-)\right] \\
&\quad + R_{m-1}.
\end{aligned}
$$

Continuing in this fashion, we get

$$
\begin{aligned}
R_m &= -\sum_{\nu=2}^{m} \frac{B_\nu(\tau)}{\nu!}[f^{(\nu-1)}(1-) - f^{(\nu-1)}(0+)] \\
&\qquad + \sum_{\nu=2}^{m} \frac{\overline{B}_\nu(\tau-s)}{\nu!}[f^{(\nu-1)}(s+) - f^{(\nu-1)}(s-)] + R_1.
\end{aligned} \tag{3.3}
$$

Consider

$$
R_1 = -\int_0^1 \overline{B}_1(\tau-t) f'(t)\,dt.
$$

Since $\overline{B}_1$ is discontinuous at integers, the function $\overline{B}_1(\tau-t)$ is discontinuous at $t=\tau$. By assumption, the derivative of f, f', is discontinuous at $t=s$. Let $0 < s \le \tau$. Then

$$
\begin{aligned}
R_1 &= -\int_0^\tau \overline{B}_1(\tau-t) f'(t)\,dt - \int_\tau^1 \overline{B}_1(\tau-t) f'(t)\,dt \\
&= -\int_0^\tau \left(\tau - t - \frac{1}{2}\right) f'(t)\,dt - \int_\tau^1 B_1(\tau-t+1) f'(t)\,dt
\end{aligned}
$$

$$
\begin{aligned}
&= -\int_0^\tau \left(\tau - t - \frac{1}{2}\right) f'(t)\,dt - \int_\tau^1 \left(\tau - t + \frac{1}{2}\right) f'(t)\,dt \\
&= \int_0^1 t f'(t)\,dt - \left(\tau - \frac{1}{2}\right) \int_0^1 f'(t)\,dt - \int_\tau^1 f'(t)\,dt \\
&= \int_0^s t f'(t)\,dt + \int_s^1 t f'(t)\,dt \\
&\quad - \left(\tau - \frac{1}{2}\right)(f(1-) - f(0+)) - (f(1-) - f(\tau)) \\
&= -\int_0^1 f(t)\,dt - B_1(\tau)(f(1-) - f(0+)) + f(\tau).
\end{aligned}
$$

Thus,

$$
f(\tau) = \int_0^1 f(t)\,dt + B_1(\tau)(f(1-) - f(0+)) + R_1,
$$

and, using (3.3), we obtain (3.1).

If $\tau < s < 1$, then in an exactly similar manner we can show that the expansion (3.1) is valid. □

By an affine change of variables, we deduce the following analogous result on an interval $[a, b]$.

Corollary 3.2. *Fix $v \in (a, b)$. Let $g : [a, b] \to \mathbf{R}$ be continuous and m times differentiable on $[a, v) \cup (v, b]$. Then, for $0 \le \tau \le 1$,*

(3.4)

$$
\begin{aligned}
(b-a)\, g(a + \tau(b-a)) = \int_a^b g(t)\,dt &+ \sum_{\nu=1}^{m} \frac{(b-a)^\nu}{\nu!} B_\nu(\tau)\Big[g^{(\nu-1)}(b-) - g^{(\nu-1)}(a+)\Big] \\
&- \sum_{\nu=2}^{m} \frac{(b-a)^\nu}{\nu!} \overline{B}_\nu\left(\tau - \frac{v-a}{b-a}\right) \\
&\qquad \times \Big[g^{(\nu-1)}(v+) - g^{(\nu-1)}(v-)\Big] + \widetilde{R}_m,
\end{aligned}
$$

where

$$
\widetilde{R}_m = -\frac{(b-a)^m}{m!} \int_a^b \overline{B}_m\left(\tau - \frac{t-a}{b-a}\right) g^{(m)}(t)\,dt. \tag{3.5}
$$

Proof. Let $\phi : [0,1] \to [a,b]$ be defined as

$$\phi(t) = a + t(b-a).$$

Then ϕ is one-to-one, onto and affine. Let

$$v = \phi(s) = a + s(b-a), \quad \text{for some } s \in (0,1).$$

Then, for $t \in (0,s) \cup (s,1)$ and $\nu = 2, 3, \ldots, m+1$,

$$(g \circ \phi)^{(\nu-1)}(t) = (b-a)^{(\nu-1)} g^{(\nu-1)}(\phi(t)).$$

By putting $f = g \circ \phi$ in (3.1) and (3.2), we obtain (3.4) and (3.5). □

Remark 3.3. If g is m times differentiable on $[a,b]$, then using (2.8), we obtain

$$(3.6) \quad (b-a)\, g(a + \tau(b-a)) = \int_a^b g(t)\, dt + \sum_{\nu=1}^{m} \frac{(b-a)^\nu}{\nu!} B_\nu(\tau) \left[g^{(\nu-1)}(b-) - g^{(\nu-1)}(a+) \right] + \widetilde{R}_m,$$

where $\widetilde{R}_m$ is given by (3.5).

The following proposition will be used for obtaining asymptotic expansions $(T_n x - Tx)(r_i)$ for various choices of approximating operators T_n.

Consider the uniform partition of $[0,1]$ defined by (2.4). The partition points are given by $t_i = (i-1)h$, $i = 1, 2, \ldots, n+1$, with $h = 1/n$.

Proposition 3.4. *Fix* $v \in (0,1)$. *Let* $f : [0,1] \to \mathbf{R}$ *be a continuous function and* m *times differentiable on* $[0,v) \cup (v,1]$. *Then, for* $0 \le \tau \le 1$,

$$(3.7) \quad h \sum_{j=1}^{n} f((j-1+\tau)h) = \int_0^1 f(t)\, dt + \sum_{\nu=1}^{m} \frac{h^\nu}{\nu!} B_\nu(\tau) \left[f^{(\nu-1)}(1) - f^{(\nu-1)}(0) \right] - \sum_{\nu=2}^{m} \frac{h^\nu}{\nu!} \overline{B}_\nu \left(\tau - \frac{v}{h} \right) \left[f^{(\nu-1)}(v+) - f^{(\nu-1)}(v-) \right] + \widetilde{R}_m,$$

where

$$\widetilde{R}_m = -\frac{h^m}{m!}\int_0^1 \overline{B}_m\left(\tau - \frac{t}{h}\right) f^{(m)}(t)\,dt. \tag{3.8}$$

$f : [0,1] \to \mathbf{R}$ is m times differentiable on $[0,1]$.

(3.9)

$$h\sum_{j=1}^{n} f((j-1+\tau)h) = \int_0^1 f(t)\,dt + \sum_{\nu=1}^{m} \frac{h^\nu}{\nu!} B_\nu(\tau)\left[f^{(\nu-1)}(1) - f^{(\nu-1)}(0)\right] + \widetilde{R}_m.$$

Proof. Case 1: Let $v = (i-1)h$ for some i, $1 \le i \le n+1$.

Then, using (3.6) for the intervals $[(j-1)h, jh]$, $j = 1, 2, \ldots, n$, we obtain

(3.10)

$$hf((j-1+\tau)h) = \int_{(j-1)h}^{jh} f(t)\,dt + \sum_{\nu=1}^{m} \frac{h^\nu}{\nu!} B_\nu(\tau)\left[f^{(\nu-1)}(jh-) - f^{(\nu-1)}((j-1)h+)\right] + \widetilde{R}_m^j,$$

where

$$\widetilde{R}_m^j = -\frac{h^m}{m!}\int_{(j-1)h}^{jh} \overline{B}_m\left(\tau - \frac{t-(j-1)h}{h}\right) f^{(m)}(t)\,dt.$$

Since $\overline{B}_m$ is a periodic function with period 1,

$$\widetilde{R}_m^j = -\frac{h^m}{m!}\int_{(j-1)h}^{jh} \overline{B}_m(\tau - \frac{t}{h})\, f^{(m)}(t)\,dt.$$

Hence, taking the sum from $j = 1$ to n, we obtain

$$(3.11)\quad h\sum_{j=1}^{n} f((j-1+\tau)h) = \int_0^1 f(t)\,dt + \sum_{\nu=1}^{m} \frac{h^\nu}{\nu!}\,B_\nu(\tau)\left[f^{(\nu-1)}(1) - f^{(\nu-1)}(0)\right] - \sum_{\nu=2}^{m} \frac{h^\nu}{\nu!}\,B_\nu(\tau)\left[f^{(\nu-1)}((i-1)h+) - f^{(\nu-1)}((i-1)h-)\right] + \widetilde{R}_m,$$

which proves (3.7) for $v = (i-1)h$.

Case 2: Let $v \in ((i-1)h, ih)$ for some i, $1 \le i \le n$.

Then, using (3.4), we obtain

$$hf((i-1+\tau)h) = \int_{(i-1)h}^{ih} f(t)\,dt + \sum_{\nu=1}^{m} \frac{h^\nu}{\nu!}\,B_\nu(\tau)\left[f^{(\nu-1)}(ih-) - f^{(\nu-1)}((i-1)h+)\right] - \sum_{\nu=2}^{m} \frac{h^\nu}{\nu!}\,\overline{B}_\nu\left(\tau - \frac{v}{h}\right)\left[f^{(\nu-1)}(v+) - f^{(\nu-1)}(v-)\right] + \widetilde{R}^i_m.$$

If $j \ne i$, then (3.10) holds. Hence,

$$h\sum_{j=1}^{n} f((j-1+\tau)h) = \int_0^1 f(t)\,dt + \sum_{\nu=1}^{m} \frac{h^\nu}{\nu!}\,B_\nu(\tau)\left[f^{(\nu-1)}(1) - f^{(\nu-1)}(0)\right] - \sum_{\nu=2}^{m} \frac{h^\nu}{\nu!}\,\overline{B}_\nu\left(\tau - \frac{v}{h}\right)\left[f^{(\nu-1)}(v+) - f^{(\nu-1)}(v-)\right] + \widetilde{R}_m,$$

which proves (3.7) for $v \in ((i-1)h, ih)$.

If f is m times differentiable on $[0, 1]$, then (3.10) holds for all j and, by summing up from $j = 1$ to n, we obtain (3.9). □

Remark 3.5. The case $v = (i-1)h$ in the above proposition has been considered by Baker [**4**]. We include it for the sake of completeness.

Let $f : [0, 1] \to \mathbf{R}$ be a continuous function. For a fixed $v \in (0, 1)$, assume that f is $m+2$ times differentiable in $[0, v) \cup (v, 1]$.

Consider the following remainder term given by (3.8):

$$\begin{aligned} \widetilde{R}_m &= -\frac{h^m}{m!} \int_0^1 \overline{B}_m\left(\tau - \frac{t}{h}\right) f^{(m)}(t)\, dt \\ &= -\frac{h^m}{m!} \left[\int_0^v \overline{B}_m\left(\tau - \frac{t}{h}\right) f^{(m)}(t)\, dt + \int_v^1 \overline{B}_m\left(\tau - \frac{t}{h}\right) f^{(m)}(t)\, dt \right]. \end{aligned}$$

Since, for $\nu \geq 1$ and $t \in \mathbf{R}/\mathbf{Z}$,

$$\overline{B}_\nu'(t) = \nu \overline{B}_{\nu-1}(t),$$

we obtain the following result by integration by parts:

$$\begin{aligned} \widetilde{R}_m = \frac{h^{m+1}}{(m+1)!} &\left(\left[\overline{B}_{m+1}\left(\tau - \frac{t}{h}\right) f^{(m)}(t) \right]_{t=0}^{t=v} \right. \\ &\left. + \left[\overline{B}_{m+1}\left(\tau - \frac{t}{h}\right) f^{(m)}(t) \right]_{t=v}^{t=1} \right) \\ &- \frac{h^{m+1}}{(m+1)!} \int_0^1 \overline{B}_{m+1}\left(\tau - \frac{t}{h}\right) f^{(m+1)}(t)\, dt. \end{aligned}$$

Hence,

$$\widetilde{R}_m = O\left(h^{m+1}\right). \tag{3.12}$$

Thus, if f is $m+1$ times differentiable in $[0, v) \cup (v, 1]$, then the remainder terms in (3.7) and (3.9) are of the order of h^{m+1}.

If $\tau = 0$ or $\tau = 1/2$, and $v = t_i = (i-1)h$ or $v = s_i = (i - (1/2))h$, then, since m is even, from (2.7)

$$B_{m+1}(\tau) = 0 \quad \text{and} \quad \overline{B}_{m+1}\left(\tau - \frac{v}{h}\right) = 0.$$

Then further integration by parts gives

$$\widetilde{R}_m = \frac{h^{m+2}}{(m+2)!}\left(\left[\overline{B}_{m+2}\left(\tau - \frac{t}{h}\right) f^{(m+1)}(t)\right]_{t=0}^{t=v} + \left[\overline{B}_{m+2}\left(\tau - \frac{t}{h}\right) f^{(m+1)}(t)\right]_{t=v}^{t=1}\right) - \frac{h^{m+2}}{(m+2)!}\int_0^1 \overline{B}_{m+2}\left(\tau - \frac{t}{h}\right) f^{(m+2)}(t)\,dt.$$

As a consequence,

$$\widetilde{R}_m = O\left(h^{m+2}\right). \tag{3.13}$$

4. Numerical quadrature. In this section, we consider the composite midpoint and the composite Simpson rules. The case of the composite trapezoidal rule has already been discussed in Baker [**3, 4**].

Let $f : [0,1] \to \mathbf{R}$ be a continuous function. Consider the uniform partition of $[0,1]$ defined by (2.4). We have $t_i = (i-1)/n = (i-1)h$, $i = 1, 2, \dots, n+1$, as the partition points. Define

$$s_i = \frac{t_i + t_{i+1}}{2} = \left(i - \frac{1}{2}\right)h, \quad i = 1, 2, \dots, n.$$

Composite midpoint rule: Let

$$\mathcal{M}_n(f) = h\sum_{j=1}^{n} f\left(\left(j - \frac{1}{2}\right)h\right) = h\sum_{j=1}^{n} f(s_j).$$

Theorem 4.1. *Fix $v \in (0,1)$, and let f be continuous and $m+2$ times differentiable on $[0,v) \cup (v,1]$. If $v = t_i$, $i = 2, 3, \dots, n$, then*

$$\mathcal{M}_n(f) = \int_0^1 f(t)\,dt + \sum_{\substack{\nu=2 \\ \nu \text{ even}}}^{m} \frac{h^\nu}{\nu!} B_\nu\left(\frac{1}{2}\right)\left[f^{(\nu-1)}(1) - f^{(\nu-1)}(0)\right] - \sum_{\substack{\nu=2 \\ \nu \text{ even}}}^{m} \frac{h^\nu}{\nu!} B_\nu\left(\frac{1}{2}\right)\left[f^{(\nu-1)}(t_i+) - f^{(\nu-1)}(t_i-)\right] + O\left(h^{m+2}\right). \tag{4.1}$$

If $v = s_i$, $i = 1, 2, \ldots, n$, then
(4.2)

$$\mathcal{M}_n(f) = \int_0^1 f(t)\,dt + \sum_{\substack{\nu=2 \\ \nu \text{ even}}}^{m} \frac{h^\nu}{\nu!} B_\nu\left(\frac{1}{2}\right) \left[f^{(\nu-1)}(1) - f^{(\nu-1)}(0)\right] - \sum_{\substack{\nu=2 \\ \nu \text{ even}}}^{m} \frac{h^\nu}{\nu!} B_\nu(0) \left[f^{(\nu-1)}(s_i+) - f^{(\nu-1)}(s_i-)\right] + O\left(h^{m+2}\right).$$

If f is $m+2$ times differentiable on $[0,1]$, then

$$\mathcal{M}_n(f) = \int_0^1 f(t)\,dt + \sum_{\substack{\nu=2 \\ \nu \text{ even}}}^{m} \frac{h^\nu}{\nu!} B_\nu\left(\frac{1}{2}\right) \left[f^{(\nu-1)}(1) - f^{(\nu-1)}(0)\right] + O\left(h^{m+2}\right). \tag{4.3}$$

Proof. Putting $\tau = 1/2$ in (3.7) and using the fact that, for ν odd, $B_\nu(1/2) = 0$, we obtain
(4.4)

$$\mathcal{M}_n(f) = \int_0^1 f(t)\,dt + \sum_{\substack{\nu=2 \\ \nu \text{ even}}}^{m} \frac{h^\nu}{\nu!} B_\nu\left(\frac{1}{2}\right) \left[f^{(\nu-1)}(1) - f^{(\nu-1)}(0)\right] - \sum_{\nu=2}^{m} \frac{h^\nu}{\nu!} \overline{B}_\nu\left(\frac{1}{2} - \frac{v}{h}\right) \left[f^{(\nu-1)}(v+) - f^{(\nu-1)}(v-)\right] + \widetilde{R}_m,$$

where

$$\widetilde{R}_m = -\frac{h^m}{m!} \int_0^1 \overline{B}_m\left(\frac{1}{2} - \frac{t}{h}\right) f^{(m)}(t)\,dt. \tag{4.5}$$

If $v = t_i = (i-1)h$, $i = 1, 2, \cdots, n+1$, then for ν odd,

$$\overline{B}_\nu\left(\frac{1}{2} - \frac{v}{h}\right) = \overline{B}_\nu\left(\frac{1}{2} - (i-1)\right) = B_\nu\left(\frac{1}{2}\right) = 0.$$

Similarly, if $v = s_i = (i - (1/2))h$, $i = 1, 2, \ldots, n$, then for ν odd,

$$\overline{B}_\nu\left(\frac{1}{2} - \frac{v}{h}\right) = \overline{B}_\nu\left(-(i-1)\right) = B_\nu\left(0\right) = 0.$$

Hence from (3.13) it follows that

$$\widetilde{R}_m = O\left(h^{m+2}\right).$$

The required expansions (4.1) and (4.2) follow by putting, respectively, $v = t_i$ and $v = s_i$ in (4.4).

In a similar fashion, by putting $\tau = 1/2$ in (3.9), we obtain (4.3). □

Composite Simpson rule: Let

$$\mathcal{S}_n(f) = h \sum_{j=1}^{n} \frac{f((j-1)h) + 4f\left((j-(1/2))h\right) + f\left(jh\right)}{6}.$$

Theorem 4.2. *Fix $v \in (0,1)$, and let f be continuous and $m+2$ times differentiable on $[0,v) \cup (v,1]$. If $v = t_i$, $i = 2, 3, \dots, n$, then*

(4.6)

$$\begin{aligned} S_n(f) = \int_0^1 f(t)\,dt &+ \sum_{\substack{\nu=4 \\ \nu \text{ even}}}^{m} \frac{h^\nu}{\nu!}\left(\frac{B_\nu(0) + 4B_\nu(1/2) + B_\nu(1)}{6}\right) \\ &\qquad \times \left(f^{(\nu-1)}(1) - f^{(\nu-1)}(0)\right) \\ &- \sum_{\substack{\nu=4 \\ \nu \text{ even}}}^{m} \frac{h^\nu}{\nu!}\left(\frac{B_\nu(0) + 4B_\nu(1/2) + B_\nu(1)}{6}\right) \\ &\qquad \times \left(f^{(\nu-1)}(t_i+) - f^{(\nu-1)}(t_i-)\right) \\ &+ O\left(h^{m+2}\right). \end{aligned}$$

If $v = s_i$, $i = 1, 2, \dots, n$, then

(4.7)

$$\begin{aligned} \mathcal{S}_n(f) = \int_0^1 f(t)\,dt &+ \sum_{\substack{\nu=4 \\ \nu \text{ even}}}^{m} \frac{h^\nu}{\nu!}\left(\frac{B_\nu(0) + 4B_\nu(1/2) + B_\nu(1)}{6}\right) \\ &\qquad \times \left(f^{(\nu-1)}(1) - f^{(\nu-1)}(0)\right) \\ &- \sum_{\substack{\nu=2 \\ \nu \text{ even}}}^{m} \frac{h^\nu}{\nu!}\left(\frac{B_\nu(1/2) + 4B_\nu(0) + B_\nu(1/2)}{6}\right) \end{aligned}$$

$$\times \left(f^{(\nu-1)}(s_i+) - f^{(\nu-1)}(s_i-) \right) + O\left(h^{m+2}\right).$$

If f is $m+2$ times differentiable on $[0,1]$, then

(4.8)
$$S_n(f) = \int_0^1 f(t)\,dt + \sum_{\substack{\nu=4 \\ \nu \text{ even}}}^{m} \frac{h^\nu}{\nu!} \left(\frac{B_\nu(0) + 4B_\nu(1/2) + B_\nu(1)}{6} \right) \times \left(f^{(\nu-1)}(1) - f^{(\nu-1)}(0) \right) + O\left(h^{m+2}\right).$$

Proof. Note that

$$\mathcal{S}_n(f) = \frac{h}{6} \sum_{j=1}^{n} f((j-1)h) + \frac{4h}{6} \sum_{j=1}^{n} f\left(\left(j - \frac{1}{2} \right) h \right) + \frac{h}{6} \sum_{j=1}^{n} f(jh).$$

Hence, using (3.7) and (3.8) we obtain

(4.9)
$$\begin{aligned} S_n(f) = \int_0^1 f(t)\,dt &+ \sum_{\nu=1}^{m} \frac{h^\nu}{\nu!} \left(\frac{B_\nu(0) + 4B_\nu(1/2) + B_\nu(1)}{6} \right) \times \left(f^{(\nu-1)}(1) - f^{(\nu-1)}(0) \right) \\ &- \sum_{\nu=2}^{m} \frac{h^\nu}{\nu!} \left(\frac{\overline{B}_\nu(-v/h) + 4\overline{B}_\nu((1/2) - (v/h)) + \overline{B}_\nu(1 - (v/h))}{6} \right) \times \left(f^{(\nu-1)}(v+) - f^{(\nu-1)}(v-) \right) \\ &+ \widetilde{R}_m, \end{aligned}$$

where

$$\widetilde{R}_m = -\frac{h^m}{m!} \int_0^1 \frac{\overline{B}_m(-t/h) + 4\overline{B}_m((1/2) - (t/h))}{6} + \frac{\overline{B}_m(1 - (t/h))}{6} f^{(m)}(t)\,dt. \tag{4.10}$$

For ν odd, from (2.5) we have

$$B_\nu(1) = -B_\nu(0) \text{ and } B_\nu(\frac{1}{2}) = 0,$$

and hence

$$\frac{B_\nu(0) + 4B_\nu(1/2) + B_\nu(1)}{6} = 0.$$

Also, since the Simpson rule is exact for cubic polynomials,

$$\frac{B_2(0) + 4B_2(1/2) + B_2(1)}{6} = \int_0^1 B_2(t)\, dt = \int_0^1 \left(t^2 - t + \frac{1}{6}\right) dt = 0.$$

If $v = t_i = (i-1)h$, $i = 2, 3, \dots, n$, then since $\overline{B}_\nu$ is periodic function with period 1,

$$\frac{\overline{B}_\nu(-v/h) + 4\overline{B}_\nu((1/2) - (v/h)) + \overline{B}_\nu(1 - (v/h))}{6} = \frac{B_\nu(0) + 4B_\nu(1/2) + B_\nu(1)}{6}$$

which vanishes for $\nu = 2$ and for ν odd, $\nu \geq 3$.

Also, from (3.13), it follows that

$$\widetilde{R}_m = O\left(h^{m+2}\right).$$

Using the above results, we deduce (4.6) from (4.9).

If $v = s_i$, $i = 1, 2, \dots, n$, then

$$\frac{\overline{B}_\nu(-v/h) + 4\overline{B}_\nu((1/2) - (v/h)) + \overline{B}_\nu(1 - (v/h))}{6} = \frac{B_\nu(1/2) + 4B_\nu(0) + B_\nu(1/2)}{6}$$

which vanishes for ν odd, $\nu \geq 3$. The expansion (4.7) then follows from (4.9).

The expansion (4.8) is deduced from (3.9) in a similar fashion. □

5. Nyström operator. Let

$$\sum_{j=1}^{N} w_j\, x(r_j) \approx \int_0^1 x(t)\, dt$$

be a convergent composite quadrature rule with respect to the uniform partition (2.4) of $[0,1]$. Here N depends on n. For example, in the composite Midpoint rule $N = n$, and in the composite Simpson rule, $N = 2n+1$. The Nyström approximation of T is defined as

$$T_n x(s) = \sum_{j=1}^{N} w_j k(s, r_j) x(r_j). \tag{5.1}$$

Then, since the kernel is continuous and the quadrature rule is convergent for all continuous functions, it follows that T_n converges to T pointwise and the set $\{T_n : n \geq 1\}$ is collectively compact. (See Atkinson [**1**, Section 4.1.1].) In order to obtain an asymptotic series expansions for $(T_n x - Tx)(r_i)$, we use Theorems 4.1 and 4.2 with $f(t) = k(s,t)\, x(t)$, where $s \in [0,1]$ is fixed. Note that the function f fails to be differentiable at s.

Composite midpoint rule: Let

$$(T_n x)(s) = \mathcal{M}_n(k(s,\cdot)x(\cdot)) = h \sum_{j=1}^{n} k\left(s, \left(j - \frac{1}{2}\right)h\right) x\left(\left(j - \frac{1}{2}\right)h\right).$$

Theorem 5.1. *Let $x \in C^{m+2}[0,1]$. Then for $i = 2, 3, \ldots, n$,*

$$\begin{aligned} (T_n x)(t_i) &= (Tx)(t_i) + \sum_{p=1}^{m/2} (A_{2p} x)(t_i) h^{2p} \\ &\quad + \sum_{p=1}^{m/2} (C_{2p} x)(t_i) h^{2p} + O\left(h^{m+2}\right), \end{aligned} \tag{5.2}$$

whereas, for $i = 1$ or $i = n+1$,

$$(T_n x)(t_i) = (Tx)(t_i) + \sum_{p=1}^{m/2} (A_{2p} x)(t_i) h^{2p} + O\left(h^{m+2}\right), \tag{5.3}$$

where

$$(A_{2p} x)(t_i) = \frac{B_{2p}(1/2)}{(2p)!} \left[\left(\frac{\partial}{\partial t} \right)^{2p-1} (k(t_i, t) x(t)) \right]_{t=0}^{t=1},$$

$$(C_{2p} x)(t_i) = -\frac{B_{2p}(1/2)}{(2p)!} \left[\left(\frac{\partial}{\partial t} \right)^{2p-1} (k(t_i, t) x(t)) \right]_{t=t_i-}^{t=t_i+}.$$

Also, for $i = 1, 2, \ldots, n$,

$$(T_n x)(s_i) = (Tx)(s_i) + \sum_{p=1}^{m/2} (A_{2p}x)(s_i) h^{2p} + \sum_{p=1}^{m/2} (\widetilde{C}_{2p}x)(s_i) h^{2p} + O\left(h^{m+2}\right), \tag{5.4}$$

where

$$(\widetilde{C}_{2p}x)(s_i) = -\frac{B_{2p}(0)}{(2p)!} \left[\left(\frac{\partial}{\partial t} \right)^{2p-1} (k(s_i, t)x(t)) \right]_{t=s_i-}^{t=s_i+}.$$

Proof. The expansions (5.2), (5.3) and (5.4) follow respectively from (4.1), (4.3) and (4.2). □

Composite Simpson rule: Let

$$\begin{aligned}(T_n x)(s) &= \mathcal{S}_n(k(s,\cdot)x(\cdot)) \\ &= h \sum_{j=1}^{n} \frac{k(s,(j-1)h)x((j-1)h)}{6} \\ &\quad + \frac{4k\,(s,(j-(1/2))h)\,x\,((j-(1/2))h) + k\,(s,jh)\,x(jh)}{6}.\end{aligned}$$

Theorem 5.2. *Let* $x \in C^{m+2}[0,1]$. *Then for* $i = 2, 3, \ldots, n$,

$$(T_n x)(t_i) = (Tx)(t_i) + \sum_{p=2}^{m/2} (A_{2p}x)(t_i) h^{2p} + \sum_{p=2}^{m/2} (C_{2p}x)(t_i) h^{2p} + O\left(h^{m+2}\right), \tag{5.5}$$

whereas, for $i = 1$ *or* $i = n + 1$,

$$(T_n x)(t_i) = (Tx)(t_i) + \sum_{p=2}^{m/2} (A_{2p}x)(t_i) h^{2p} + O\left(h^{m+2}\right), \tag{5.6}$$

where

$$
\begin{aligned}
(A_{2p}x)(t_i) &= \frac{1}{(2p)!}\left(\frac{B_{2p}(0)+4B_{2p}(1/2)+B_{2p}(1)}{6}\right)\\
&\quad\times\left[\left(\frac{\partial}{\partial t}\right)^{2p-1}(k(t_i,t)x(t))\right]_{t=0}^{t=1}\\
(C_{2p}x)(t_i) &= -\frac{1}{(2p)!}\left(\frac{B_{2p}(0)+4B_{2p}(1/2)+B_{2p}(1)}{6}\right)\\
&\quad\times\left[\left(\frac{\partial}{\partial t}\right)^{2p-1}(k(t_i,t)x(t))\right]_{t=t_i-}^{t=t_i+}.
\end{aligned}
$$

Also, for $i = 1, 2, \ldots, n$,

$$
\begin{aligned}
(T_n x)(s_i) &= (Tx)(s_i) + \sum_{p=2}^{m/2}(A_{2p}x)(s_i)h^{2p}\\
&\quad + \sum_{p=1}^{m/2}(\widetilde{C}_{2p}x)(s_i)h^{2p} + O\left(h^{m+2}\right),
\end{aligned}
\tag{5.7}
$$

where

$$
\begin{aligned}
(\widetilde{C}_{2p}x)(s_i) &= -\frac{1}{(2p)!}\left(\frac{B_{2p}(1/2)+4B_{2p}(0)+B_{2p}(1/2)}{6}\right)\\
&\quad\times\left[\left(\frac{\partial}{\partial t}\right)^{2p-1}(k(s_i,t)x(t))\right]_{t=s_i-}^{t=s_i+}.
\end{aligned}
$$

Proof. The expansions (5.5), (5.6) and (5.7) follow respectively from (4.6), (4.8) and (4.7). □

Modified Simpson rule: As can be seen from (5.5), (5.6) and (5.7) that, in the case of the composite Simpson rule, we get

$$|(Tx)(t_i) - (T_n x)(t_i)| = O(h^4),$$

and

$$|(Tx)(s_i) - (T_n x)(s_i)| = O(h^2).$$

In order to restore the order of convergence of h^4, we consider the following modified Simpson method. (See Atkinson-Shampine [**2**], Cubillos [**5**].)

For $i = 1, 2, \ldots, n$, let $\widetilde{x}\,|_{[t_i, t_{i+1}]}$ be a quadratic polynomial such that

$$\widetilde{x}(t_i) = x(t_i), \qquad \widetilde{x}(s_i) = x(s_i), \qquad \widetilde{x}(t_{i+1}) = x(t_{i+1}).$$

Then $\widetilde{x} : [0,1] \to \mathbf{R}$ is a continuous piecewise quadratic polynomial with respect to the uniform partition (2.4).

We introduce the following notation:

$$S\{f, a, b\} = \frac{b-a}{6}\left(f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right).$$

Thus,

$$\mathcal{S}_n(f) = \sum_{j=1}^{n} S\{f, (j-1)h, jh\}.$$

We define
(5.8)

$$\begin{aligned}(T_{2n}\widetilde{x})(s) &= \sum_{i=1}^{n} \left(S\{k(s,\cdot)\,\widetilde{x}(\cdot), t_i, s_i\} + S\{k(s,\cdot)\,\widetilde{x}(\cdot), s_i, t_{i+1}\}\right) \\ &= \frac{h}{12}\sum_{i=1}^{n}\Bigg[k(s,t_i)\,\widetilde{x}(t_i) + 4k\left(s, \frac{t_i + s_i}{2}\right)\widetilde{x}\left(\frac{t_i+s_i}{2}\right) \\ &\qquad\qquad + k(s,s_i)\,\widetilde{x}(s_i)\Bigg] \\ &\quad + \frac{h}{12}\sum_{i=1}^{n}\Bigg[k(s,s_i)\,\widetilde{x}(s_i) + 4k\left(s, \frac{s_i + t_{i+1}}{2}\right)\widetilde{x}\left(\frac{s_i+t_{i+1}}{2}\right) \\ &\qquad\qquad + k(s,t_{i+1})\,\widetilde{x}(t_{i+1})\Bigg].\end{aligned}$$

Note that the above formula is completely determined by the values of x at t_i and at s_i.

We would like to use the results of Theorem 5.2 for obtaining asymptotic expansions for $(T_{2n}\widetilde{x})(t_i)$ and $(T_{2n}\widetilde{x})(s_i)$. However, since $\widetilde{x}$ is only continuous on $[0,1]$, we cannot use Theorem 5.2 directly. Hence, we define $y \in C^{m+2}[0,1]$ as follows.

For $i = 1, 2, \ldots, n$, let $y\,|_{[t_i, s_i]}$ be a polynomial of degree $\leq 2m + 6$ such that

$$y^{(\nu)}(t_i) = x^{(\nu)}(t_i), \qquad y^{(\nu)}(s_i) = x^{(\nu)}(s_i), \quad \nu = 0, 1, 2, \ldots, m + 2,$$

and

$$y\left(\frac{t_i + s_i}{2}\right) = \widetilde{x}\left(\frac{t_i + s_i}{2}\right).$$

Similarly, for $i = 1, 2, \ldots, n$, let $y\,|_{[s_i, t_{i+1}]}$ be a polynomial of degree $\leq 2m + 6$ such that

$$y^{(\nu)}(s_i) = x^{(\nu)}(s_i), \qquad y^{(\nu)}(t_{i+1}) = x^{(\nu)}(t_{i+1}), \quad \nu = 0, 1, 2, \ldots, m + 2,$$

and

$$y\left(\frac{s_i + t_{i+1}}{2}\right) = \widetilde{x}\left(\frac{s_i + t_{i+1}}{2}\right).$$

Then $y \in C^{m+2}[0, 1]$ and

$$(T_{2n}y)(s) = (T_{2n}\widetilde{x})(s), \quad s \in [0, 1]. \tag{5.9}$$

Also, for $p = 1, 2, \ldots, m/2$,

$$(A_{2p}x)(t_i) = (A_{2p}y)(t_i), \qquad (C_{2p}x)(t_i) = (C_{2p}y)(t_i), \quad i = 1, 2, \ldots, n + 1, \tag{5.10}$$

and

$$(A_{2p}x)(s_i) = (A_{2p}y)(s_i), \qquad (C_{2p}x)(s_i) = (C_{2p}y)(s_i), \quad i = 1, 2, \ldots, n, \tag{5.11}$$

where A_{2p} and C_{2p} are defined in Theorem 5.2. Note that, since $x \in C^{m+2}[0, 1]$,

$$\|x - y\|_\infty \leq C h^{m+2},$$

and hence

$$\|Tx - Ty\|_\infty \leq \|k\|_\infty \|x - y\|_\infty \leq C h^{m+2}. \tag{5.12}$$

We obtain the following asymptotic expansion for the modified Simpson rule.

Theorem 5.3. *Let $x \in C^{m+2}[0,1]$. Then, for $i = 2, 3, \dots, n$,*

$$(5.13)\quad \begin{aligned}(T_{2n}\widetilde{x})(t_i) = (Tx)(t_i) + \sum_{p=2}^{m/2}(A_{2p}x)(t_i)\left(\frac{h}{2}\right)^{2p} \\ + \sum_{p=2}^{m/2}(C_{2p}x)(t_i)\left(\frac{h}{2}\right)^{2p} + O\left(h^{m+2}\right),\end{aligned}$$

whereas, for $i = 1$ or $i = n+1$,

$$(5.14)\quad (T_{2n}\widetilde{x})(t_i) = (Tx)(t_i) + \sum_{p=2}^{m/2}(A_{2p}x)(t_i)\left(\frac{h}{2}\right)^{2p} + O\left(h^{m+2}\right).$$

For $i = 1, 2, \dots, n$,

$$(5.15)\quad \begin{aligned}(T_{2n}\widetilde{x})(s_i) = (Tx)(s_i) + \sum_{p=2}^{m/2}(A_{2p}x)(s_i)\left(\frac{h}{2}\right)^{2p} \\ + \sum_{p=2}^{m/2}(C_{2p}x)(s_i)\left(\frac{h}{2}\right)^{2p} + O\Big(h^{m+2}\Big),\end{aligned}$$

where A_{2p} and C_{2p} are as defined in Theorem 5.2.

Proof. Since $y \in C^{m+2}[0,1]$, from equation (5.5) of Theorem 5.2, we obtain

$$(5.16)\quad \begin{aligned}(T_{2n}y)(t_i) = (Ty)(t_i) + \sum_{p=2}^{m/2}(A_{2p}y)(t_i)\left(\frac{h}{2}\right)^{2p} \\ + \sum_{p=2}^{m/2}(C_{2p}y)(t_i)\left(\frac{h}{2}\right)^{2p} + O\Big(h^{m+2}\Big).\end{aligned}$$

$$(5.17)\quad \begin{aligned}(T_{2n}y)(s_i) = (Ty)(s_i) + \sum_{p=2}^{m/2}(A_{2p}y)(s_i)\left(\frac{h}{2}\right)^{2p} \\ + \sum_{p=2}^{m/2}(C_{2p}y)(s_i)\left(\frac{h}{2}\right)^{2p} + O\left(h^{m+2}\right).\end{aligned}$$

Using (5.9), (5.10), (5.11) and (5.12), we deduce (5.13) and (5.15).

In a similar fashion, (5.14) follows from (5.6). □

It follows from (5.13) and (5.15) that

$$|(Tx)(t_i) - (T_{2n}\widetilde{x})(t_i)| = O(h^4), \qquad |(Tx)(s_i) - (T_{2n}\widetilde{x})(s_i)| = O(h^4).$$

Thus, we have restored the order of convergence h^4 and, in addition, have obtained an asymptotic expansion at the node points in the modified Simpson method.

6. Iterated collocation method. Choose $r \geq 1$. Let X_n be the space of all piecewise polynomials of order r (i.e., of degree $\leq r-1$) with breakpoints at $t_2, \dots, t_n$. Let $\tau_1, \dots, \tau_r$ be r distinct points in [0,1]. Define

$$t_{ij} = (i - 1 + \tau_j)h, \quad j = 1, \dots, r, \ i = 1, \dots, n.$$

The interpolation operator $\pi_n : C[0,1] \to X_n$ is defined by

$$(\pi_n x)(t_{ij}) = x(t_{ij}), \quad j = 1, \dots, r, \ i = 1, \dots, n.$$

Since $\pi_n \to I$ pointwise, the iterated collocation operator $T\pi_n$ converges to T pointwise and $\{T\pi_n\}$ is a collectively compact family. (See [**1**].) Let $[r_1, r_2, \dots, r_q]$ denote the divided difference operator associated with the points $r_1, \dots, r_q$. Define

$$\omega_r(\tau) = (\tau - \tau_1) \cdots (\tau - \tau_r).$$

The proof of the following theorem is very much similar to that of Theorem 4.1 in McLean [**11**].

Theorem 6.1. *Let $x \in C^{m+2}[0,1]$ and $s \in (0,1)$. Then*

(6.1)

$$(T\pi_n x)(s) = (Tx)(s) + \sum_{p=r}^{m+1} (A_p x)(s)h^p + \sum_{p=r+2}^{m+1} (C_p x)(s)h^p + O(h^{m+2}),$$

where

(6.2)
$$(A_p x)(s) = b_{pp}(Tx^{(p)})(s) + \sum_{j=r}^{p-1} b_{pj} \left[\left(\frac{\partial}{\partial t}\right)^{p-j-1} \Big(k(s,t)x^{(j)}(t)\Big) \right]_{t=0}^{t=1},$$

(6.3)
$$(C_p x)(s) = \sum_{j=r}^{p-2} c_{pj} \left[\left(\frac{\partial}{\partial t}\right)^{p-j-1} \Big(k(s,t)x^{(j)}(t)\Big) \right]_{t=s-}^{t=s+},$$

(6.4)
$$b_{pj} = -\int_0^1 \Phi_j(\tau) \frac{B_{p-j}(\tau)}{(p-j)!} w_r(\tau)\, d\tau,$$
$$c_{pj} = \int_0^1 \Phi_j(\tau) \frac{\overline{B}_{p-j}(\tau - (s/h))}{(p-j)!} w_r(\tau)\, d\tau,$$

and

(6.5)
$$\Phi_j(\tau) = \int_0^1 \frac{(\sigma-\tau)^{j-r}}{(j-r)!} \frac{[\tau_1, \dots, \tau_r, \tau](\bullet - \sigma)_+^{r-1}}{(r-1)!}\, d\sigma.$$

If $s = 0$ *or* $s = 1$*, then*

(6.6)
$$(T\pi_n x)(s) = (Tx)(s) + \sum_{p=r}^{m+1} (A_p x)(s) h^p + O(h^{m+2}).$$

Proof. For $t \in [(i-1)h, ih]$, we have

$$(I - \pi_n)x(t) = [t_{i1}, \dots, t_{ir}, t]x\,(t - t_{i1}) \cdots (t - t_{ir}),$$

and, by the Peano representation for the divided difference, we get

$$[t_{i1}, \dots, t_{ir}, t]x = \int_{(i-1)h}^{ih} \frac{[t_{i1}, \dots, t_{ir}, t](\bullet - z)_+^{r-1}}{(r-1)!} x^{(r)}(z)\, dz.$$

Put $t = (i-1+\tau)h$ and $z = (i-1+\sigma)h$ to obtain

$$[t_{i1}, \ldots, t_{ir}, (i-1+\tau)h]x = \int_0^1 \frac{[\tau_1, \ldots, \tau_r, \tau](\bullet - \sigma)_+^{r-1}}{(r-1)!} x^{(r)}[(i-1+\sigma)h]\, d\sigma.$$

Taylor's theorem implies

$$x^{(r)}[(i-1+\sigma)h] = \sum_{j=r}^{m+1} \frac{x^{(j)}[(i-1+\tau)h]}{(j-r)!}(\sigma-\tau)^{j-r}h^{j-r} + O(h^{m-r+2}).$$

Thus,

$$[t_{i1}, \ldots, t_{ir}, (i-1+\tau)h]x = \sum_{j=r}^{m+1} \Phi_j(\tau) x^{(j)}[(i-1+\tau)h]h^{j-r} + O(h^{m-r+2}).$$

Note that

$$(t - t_{i1}) \cdots (t - t_{ir}) = h^r \omega_r(\tau).$$

Hence,

$$(I-\pi_n)x((i-1+\tau)h) = \sum_{j=r}^{m+1} h^j \Phi_j(\tau) x^{(j)}[(i-1+\tau)h]\, \omega_r(\tau) + O(h^{m+2}).$$

It follows that

(6.7)

$$\begin{aligned} T(I-\pi_n)x(s) &= \sum_{i=1}^{n} \int_{(i-1)h}^{ih} k(s,t)(I-\pi_n)x(t)\, dt \\ &= \sum_{i=1}^{n} \int_0^1 h\, k(s,(i-1+\tau)h)(I-\pi_n)x[(i-1+\tau)h]\, d\tau \\ &= \sum_{j=r}^{m+1} h^j \int_0^1 \Phi_j(\tau) \\ &\quad \left\{ h \sum_{i=1}^{n} k(s,(i-1+\tau)h) x^{(j)}[(i-1+\tau)h] \right\} \omega_r(\tau)\, d\tau \\ &\quad + O(h^{m+2}) \end{aligned}$$

If $s \in (0,1)$, then using (3.12) and the summation formula (3.7), with m replaced by $m+1$, we get

$$
\begin{aligned}
(6.8)\quad & h\sum_{i=1}^{n} k(s,(i-1+\tau)h)x^{(j)}[(i-1+\tau)h] \\
&= \int_0^1 k(s,t)x^{(j)}(t)\,dt \\
&+ \sum_{p=1}^{m+1-j} \frac{B_p(\tau)}{p!}\left[\left(\frac{\partial}{\partial t}\right)^{p-1}\Big(k(s,t)x^{(j)}(t)\Big)\right]_{t=0}^{t=1} h^p \\
&- \sum_{p=2}^{m+1-j} \frac{\overline{B}_p(\tau-\frac{s}{h})}{p!}\left[\left(\frac{\partial}{\partial t}\right)^{p-1}\Big(k(s,t)x^{(j)}(t)\Big)\right]_{t=s-}^{t=s+} h^p \\
&+ O(h^{m+2-j}).
\end{aligned}
$$

Substituting the above formula in (6.7) and rearranging terms, we obtain

$$
(6.9)\quad (T\pi_n x)(s) = (Tx)(s) + \sum_{p=r}^{m+1}(A_p x)(s)h^p + \sum_{p=r+2}^{m+1}(C_p x)(s)h^p + O(h^{m+2}).
$$

If $s=0$ or $s=1$, then using the summation formula (3.9), we get

$$
\begin{aligned}
& h\sum_{i=1}^{n} k(s,(i-1+\tau)h)x^{(j)}[(i-1+\tau)h] \\
&\qquad = \int_0^1 k(s,t)x^{(j)}(t)\,dt \\
&\qquad + \sum_{p=1}^{m+1-j} \frac{B_p(\tau)}{p!}\left[\left(\frac{\partial}{\partial t}\right)^{p-1} k(s,t)x^{(j)}(t)\right]_{t=0}^{t=1} h^p \\
&\qquad + O(h^{m+2-j}).
\end{aligned}
$$

Substituting the above formula in (6.7), we obtain the expansion (6.6). □

Remark 6.2. Note that b_{pj} which is defined by (6.4) does not depend upon s.

Suppose that the points $\tau_1, \tau_2, \dots, \tau_r$ are placed symmetrically in the interval $[0,1]$, that is, $\tau_{r-p+1} = 1 - \tau_p$ for all p. Then (see [**11**])
(6.10)
$$\Phi_j(1-\tau) = (-1)^{j-r}\Phi_j(\tau), \qquad \omega_r(1-\tau) = (-1)^r \omega_r(\tau), \quad \tau \in [0,1].$$
Recall from (2.5) that
$$B_{p-j}(1-\tau) = (-1)^{p-j} B_{p-j}(\tau).$$
It follows that
$$b_{pj} = -\int_0^1 \Phi_j(\tau)\frac{B_{p-j}(\tau)}{(p-j)!}\, w_r(\tau)\, d\tau = (-1)^p b_{pj},$$
and hence for p odd,
$$b_{pj} = 0. \tag{6.11}$$
As a consequence, we deduce the following expansion from (6.1).
(6.12)
$$(T\pi_n x)(s) = (Tx)(s) + \sum_{\substack{p=r \\ p \text{ even}}}^{m} (A_p x)(s) h^p + \sum_{p=r+2}^{m+1} (C_p x)(s) h^p + O(h^{m+2}),$$
where $(A_p x)(s)$ and $(C_p x)(s)$ are defined respectively by (6.2) and (6.3).

We now consider some cases where the points $\tau_1, \tau_2, \dots, \tau_r$ are placed symmetrically in the interval $[0,1]$.

Special cases: 1. Let X_n be the space of piecewise constant functions with $r = 1$ and $\tau_1 = 1/2$. The interpolation operator $\pi_n : C[0,1] \to X_n$ is defined as
$$(\pi_n x)(s_i) = x(s_i), \quad i = 1, 2, \dots, n,$$
where $s_i = (i - (1/2))h$.

Corollary 6.3. *Let $x \in C^{m+2}[0,1]$. Then, for $i = 2, \dots, n$,*
$$\begin{aligned} (T\pi_n x)(t_i) &= (Tx)(t_i) + \sum_{p=1}^{m/2} (A_{2p}x)(t_i) h^{2p} \\ &\quad + \sum_{p=2}^{m/2} (C_{2p}x)(t_i) h^{2p} + O\left(h^{m+2}\right), \end{aligned} \tag{6.13}$$

whereas, for $i = 1$ or $i = n+1$,

$$(6.14) \qquad (T\pi_n x)(t_i) = (Tx)(t_i) + \sum_{p=1}^{m/2} (A_{2p}x)(t_i)h^{2p} + O\left(h^{m+2}\right),$$

where $c_{p,j} = -b_{p,j}$ and A_{2p} and C_{2p} are respectively given by (6.2) *and* (6.3).

For $i = 1, 2, \ldots, n$,

$$(6.15) \qquad \begin{aligned} (T\pi_n x)(s_i) = (Tx)(s_i) &+ \sum_{p=1}^{m/2} (A_{2p}x)(s_i)h^{2p} \\ &+ \sum_{p=2}^{m/2} (C_{2p}x)(s_i)h^{2p} + O\left(h^{m+2}\right), \end{aligned}$$

where

$$c_{pj} = \int_0^1 \Phi_j(\tau) \frac{\overline{B}_{p-j}(\tau - (1/2))}{(p-j)!} \left(\tau - \frac{1}{2}\right) d\tau.$$

Proof. Let $s = t_i = (i-1)h$. Then

$$c_{pj} = -b_{pj} = \int_0^1 \Phi_j(\tau) \frac{B_{p-j}(\tau)}{(p-j)!} \left(\tau - \frac{1}{2}\right) d\tau$$

and, using (6.11) and (6.12), we obtain (6.13). Expansion (6.14) follows from (6.6).

Let

$$s = s_i = \left(i - \frac{1}{2}\right)h, \quad i = 1, 2, \ldots n.$$

Then

$$\overline{B}_{p-j}\left(\tau - \frac{s}{h}\right) = \overline{B}_{p-j}\left(\tau - \frac{1}{2}\right).$$

Note that

$$\overline{B}_j\left(\tau - \frac{1}{2}\right) = \begin{cases} B_j(\tau + (1/2)) & \text{if } \tau \in [0, (1/2)], \\ B_j(\tau - (1/2)) & \text{if } \tau \in [(1/2), 1]. \end{cases}$$

Since

$$B_j(1-\tau) = (-1)^j B_j(\tau), \quad \tau \in [0,1],$$

it can be verified that

$$\overline{B}_j\left((1-\tau) - \frac{1}{2}\right) = (-1)^j \overline{B}_j\left(\tau - \frac{1}{2}\right), \quad \tau \in [0,1]. \tag{6.16}$$

Since $r = 1$, from (6.10),

$$\Phi_j(1-\tau) = (-1)^{j-1}\Phi_j(\tau) \quad \text{and} \quad \omega_1(1-\tau) = -\omega_1(\tau).$$

Thus,

$$\begin{aligned} c_{pj} &= \int_0^1 \Phi_j(\tau) \frac{\overline{B}_{p-j}(\tau - (1/2))}{(p-j)!}\left(\tau - \frac{1}{2}\right) d\tau \\ &= (-1)^p \int_0^1 \Phi_j(\tau) \frac{\overline{B}_{p-j}(\tau - (1/2))}{(p-j)!}\left(\tau - \frac{1}{2}\right) d\tau = (-1)^p c_{pj}. \end{aligned}$$

Hence, for p odd, $c_{pj} = 0$. The expansion (6.15) then follows from (6.12). □

2. Let X_n be the space of piecewise linear continuous functions with $r = 2$ and $\tau_1 = 0$, $\tau_2 = 1$. The interpolation operator $\pi_n : C[0,1] \to X_n$ is defined as

$$(\pi_n x)(t_i) = x(t_i), \quad i = 1, 2, \ldots, n+1.$$

Corollary 6.4. *Let $x \in C^{m+2}[0,1]$. Then, for $i = 2, \ldots, n$,*

$$\begin{aligned} (T\pi_n x)(t_i) = (Tx)(t_i) &+ \sum_{p=1}^{m/2} (A_{2p}x)(t_i) h^{2p} \\ &+ \sum_{p=2}^{m/2} (C_{2p}x)(t_i) h^{2p} + O\left(h^{m+2}\right), \end{aligned} \tag{6.17}$$

whereas, for $i = 1$ or $i = n+1$,

$$(T\pi_n x)(t_i) = (Tx)(t_i) + \sum_{p=1}^{m/2} (A_{2p}x)(t_i) h^{2p} + O\left(h^{m+2}\right), \tag{6.18}$$

where $c_{pj} = -b_{pj}$ *and* A_{2p} *and* C_{2p} *are respectively given by* (6.2) *and* (6.3).

Proof. Let

$$s = t_i = (i-1)h, \quad i = 1, 2, \dots, n+1.$$

Then

$$\overline{B}_{p-j}\left(\tau - \frac{s}{h}\right) = B_{p-j}(\tau).$$

As a consequence,

$$c_{pj} = -b_{pj} = \int_0^1 \Phi_j(\tau) \frac{B_{p-j}(\tau)}{(p-j)!}(\tau)(\tau - 1)\, d\tau.$$

Since $\tau_2 = 1 - \tau_1$, by (6.11), for p odd,

$$b_{pj} = 0.$$

Thus, expansions (6.17) and (6.18) follow respectively from (6.12) and (6.6). □

3. Let X_n be the space of piecewise quadratic continuous functions with $r = 3$ and $\tau_1 = 0$, $\tau_2 = 1/2$ and $\tau_3 = 1$. The interpolation operator $\pi_n : C[0,1] \to X_n$ is defined as

$$(\pi_n x)(s_i) = x(s_i), \qquad (\pi_n x)(t_i) = x(t_i), \quad i = 1, 2, \dots, n,$$

and

$$(\pi_n x)(t_{n+1}) = x(t_{n+1}).$$

Corollary 6.5. *Let* $x \in C^{m+2}[0,1]$. *For* $i = 2, \dots, n$,

$$\begin{aligned}(T\pi_n x)(t_i) &= (Tx)(t_i) + \sum_{p=2}^{m/2} (A_{2p}x)(t_i) h^{2p} \\ &\quad + \sum_{p=3}^{m/2} (C_{2p}x)(t_i) h^{2p} + O\left(h^{m+2}\right),\end{aligned} \tag{6.19}$$

whereas, for $i = 1$ *and* $i = n + 1$,

$$(6.20) \qquad (T\pi_n x)(t_i) = (Tx)(t_i) + \sum_{p=2}^{m/2} (A_{2p}x)(t_i)h^{2p} + O\left(h^{m+2}\right),$$

where

$$c_{pj} = -b_{pj} = \int_0^1 \Phi_j(\tau) \frac{B_{p-j}(\tau)}{(p-j)!}(\tau)\left(\tau - \frac{1}{2}\right)(\tau - 1)\, d\tau$$

and A_{2p} *and* C_{2p} *are respectively given by* (6.2) *and* (6.3).

For $i = 1, 2, \ldots, n$,

$$(6.21) \qquad \begin{aligned}(T\pi_n x)(s_i) = (Tx)(s_i) &+ \sum_{p=2}^{m/2} (A_{2p}x)(s_i)h^{2p} \\ &+ \sum_{p=3}^{m/2} (C_{2p}x)(s_i)h^{2p} + O\left(h^{m+2}\right),\end{aligned}$$

where

$$c_{pj} = \int_0^1 \Phi_j(\tau) \frac{\overline{B}_{p-j}(\tau - (1/2))}{(p-j)!}(\tau)\left(\tau - \frac{1}{2}\right)(\tau - 1)\, d\tau.$$

As the proof is similar to the earlier proofs, we skip it.

7. Operator equations. Consider the second kind Fredholm integral equation

$$(7.1) \qquad u(s) - (Tu)(s) = f(s), \quad s \in [0, 1],$$

where T is an integral operator with a kernel of the type of Green's function defined by (2.1). We assume that (7.1) has a unique solution and that the exact solution is in $C^{m+2}[0, 1]$. The above equation is approximated by

$$(7.2) \qquad u_n(s) - (T_n u_n)(s) = f(s), \quad s \in [0, 1],$$

where T_n is one of the approximating operator defined in Sections 5 or 6. Note that T_n is a collectively compact family of operators which converge to T pointwise and hence, for n large enough, (7.2) has a unique solution (see [**1**]).

In Sections 5 and 6, we obtained asymptotic series expansions for various choices of T_n of the following form:
(7.3)

$$(T_n x)(r_i) = (Tx)(r_i) + \sum_{p=\alpha}^{m/2} (R_{2p}x)(r_i)h^{2p} + \sum_{p=\alpha+1}^{m/2} (S_{2p}x)(r_i)h^{2p} + O(h^{m+2}), \quad i = 1, 2, \ldots, N.$$

The points r_i are the node points in the Nyström method and are the collocation points in the iterated collocation methods. In the Nyström method associated with the midpoint rule and the iterated collocation method based on piecewise constant functions, the above expansion is valid, in addition, at the partition points $t_i = (i-1)/n = (i-1)h$, $i = 1, \ldots, n+1$. This expansion at the partition points is needed for extrapolation.

Using the technique from Ford et al. [**6**], we can write asymptotic series expansions for $u_n(r_i) - u(r_i)$ as described below.

Theorem 7.1. *Let T be an integral operator, defined on $C[0,1]$, with a kernel of the type of Green's function defined in Section 2. We assume that, for m even and for each fixed $s \in [0,1]$, the kernel $k(s,t)$ of T is $m+2$ times differentiable with respect to the second variable t in $(0,s) \cup (s,1)$. Let the exact solution u of (7.1) be in $C^{m+2}[0,1]$. Let*

$$s_i = \frac{t_i + t_{i+1}}{2}, \quad i = 1, \ldots, n.$$

Then the following expansions are valid.

1. **Nyström method: Composite midpoint rule, composite Simpson rule, iterated collocation method: Piecewise con-**

stant functions.

$$u_n(t_i) = u(t_i) + \sum_{p=1}^{m/2} \eta_p(t_i) h^{2p} + O(h^{m+2}), \tag{7.4}$$

$$u_n(s_i) = u(s_i) + \sum_{p=1}^{m/2} \delta_p(s_i) h^{2p} + O(h^{m+2}). \tag{7.5}$$

2. **Nyström method: Composite modified Simpson rule, iterated collocation method: Piecewise quadratic functions.**

$$u_n(t_i) = u(t_i) + \sum_{p=2}^{m/2} \eta_p(t_i) h^{2p} + O(h^{m+2}), \tag{7.6}$$

$$u_n(s_i) = u(s_i) + \sum_{p=2}^{m/2} \delta_p(s_i) h^{2p} + O(h^{m+2}). \tag{7.7}$$

3. **Iterated collocation method: Piecewise linear functions.**

$$u_n(t_i) = u(t_i) + \sum_{p=1}^{m/2} \eta_p(t_i) h^{2p} + O(h^{m+2}). \tag{7.8}$$

Note that the functions $\eta_p \in C^{m+2-2p}[0,1]$ and $\delta_p \in C^{m+2-2p}[0,1]$ and are independent of h.

For each $t_i = (i-1)/n$, $i = 1, \dots, n+1$, and for $u_n(t_i)$ satisfying (7.4) or (7.8), define

$$u_{n,0}(t_i) = u_n(t_i)$$

and

$$u_{n,l}(t_i) = \frac{2^{2l} u_{2n,l-1}(t_i) - u_{n,l-1}(t_i)}{2^{2l}-1}, \quad l = 1, 2, \dots, m/2 - 1. \tag{7.9}$$

Then we have the following result.

Corollary 7.2. *Suppose that the conditions of Theorem* 7.1 *hold. Then*

$$u_{n,l}(t_i) = u(t_i) + \sum_{p=l+1}^{m/2} e_{l,p}(t_i)h^{2p} + O(h^{m+2}), \qquad i = 1, 2, \dots, n+1, \tag{7.10}$$

where the functions $e_{l,p}$ *are independent of* h.

In a similar fashion, for each $t_i = (i-1)/n$, $i = 1, \dots, n+1$, and for $u_n(t_i)$ satisfying (7.6), define

$$u_{n,0}(t_i) = u_n(t_i)$$

and

$$u_{n,l}(t_i) = \frac{2^{2l+2}u_{2n,l-1}(t_i) - u_{n,l-1}(t_i)}{2^{2l+2} - 1}, \qquad l = 1, 2, \dots, m/2 - 1. \tag{7.11}$$

Then we have the following result.

Corollary 7.3. *Suppose that the conditions of Theorem* 7.1 *hold. Then*

$$u_{n,l}(t_i) = u(t_i) + \sum_{p=l+2}^{m/2} e_{l,p}(t_i)h^{2p} + O(h^{m+2}), \qquad i = 1, 2, \dots, n+1, \tag{7.12}$$

where the functions $e_{l,p}$ *are independent of* h.

In the next section, we validate the above results by numerical examples.

Remark 7.4. We now show that the result of Lin and Liu [**9**] is a special case of our results. Let X_n be the space of piecewise linear continuous functions with $r = 2$ and $\tau_1 = 0$, $\tau_2 = 1$. The interpolation operator $\pi_n : C[0,1] \to X_n$ is defined as

$$(\pi_n x)(t_i) = x(t_i), \quad i = 1, 2, \ldots, n+1.$$

Let $x \in C^4[0,1]$. Then from (6.12) we obtain the following asymptotic expansion which is valid for each $s \in [0,1]$.

$$(T\pi_n x)(s) = (Tx)(s) + (A_2 x)(s) h^2 + O(h^4), \quad s \in [0,1], \tag{7.13}$$

where

$$(A_2 x)(s) = b_{22} T x^{(2)}(s) = -\left(\int_0^1 \Phi_2(\tau)\tau(\tau - 1)\, d\tau \right) T x^{(2)}(s).$$

Let u be the exact solution of (7.1), and let u_n be the approximate solution satisfying

$$u_n(s) - T\pi_n u_n(s) = f(s), \quad s \in [0,1].$$

Using the resolvent identity, we obtain

$$u_n(s) = u(s) + h^2 \eta(s) + O(h^4), \quad s \in [0,1], \tag{7.14}$$

where the function η is independent of h. Let u_{2n} denote the approximate solution associated with the uniform partition of $[0,1]$ consisting of $2n$ subintervals. Then

$$u_{2n}(s) = u(s) + \frac{h^2}{4}\eta(s) + O(h^4), \quad s \in [0,1]. \tag{7.15}$$

Define

$$u_{n,1}(s) = \frac{4u_{2n}(s) - u_n(s)}{3}, \quad s \in [0,1]. \tag{7.16}$$

Then, for $s \in [0,1]$,

$$u_{n,1}(s) - u(s) = O(h^4), \tag{7.17}$$

which is the result proved by Lin and Liu [**9**].

8. Numerical results. We consider the following integral equation from Atkinson and Shampine [**2**].

$$u(s) - \int_0^1 k(s,t)u(t)\,dt = \left(1 - \frac{1}{\pi^2}\right)\sin(\pi s), \quad 0 \le s \le 1 \tag{8.1}$$

with

$$k(s,t) = \begin{cases} s(1-t) & \text{if } s \le t, \\ t(1-s) & \text{if } t < s. \end{cases}$$

The exact solution is $\sin(\pi s)$.

The above integral equation is approximated by

$$u_n - T_n u_n = f,$$

for various choices of approximating operators T_n.

We use the following notation. Let $s = 1/2$ or $s = 1/4$ and

$$\begin{aligned} E_1^n &= |u(s) - u_n(s)|, & E_2^n &= |u(s) - u_{n,1}(s)|, \\ E_3^n &= |u(s) - u_{n,2}(s)|, & E_4^n &= |u(s) - u_{n,3}(s)|, \end{aligned}$$

where $u_{n.l}$, $l = 1, 2, 3$, are defined by (7.9) or by (7.11). The orders of convergence are computed by using the following formula:

$$\alpha_{l-1} = \frac{\log\left(E_l^n / E_l^{2n}\right)}{\log(2)}, \quad l = 1, 2, 3, 4.$$

8.1. Nyström method. The expected orders of convergence are as follows.

Composite midpoint rule, Composite Simpson rule: $\alpha_0 = 2$, $\alpha_1 = 4$, $\alpha_2 = 6$, $\alpha_3 = 8$. (See Theorem 7.1 and Corollary 7.2.)

Composite modified Simpson rule: $\alpha_0 = 4$, $\alpha_1 = 6$, $\alpha_2 = 8$. (See Theorem 7.1 and Corollary 7.3.)

It is seen in Tables 1–6 that the computed orders match well with the expected orders.

TABLE 1. Composite midpoint rule: $s = 1/2$.

n	E_1^n	α_0	E_2^n	α_1	E_3^n	α_2	E_4^n	α_3
8	5.13×10^{-4}							
16	1.27×10^{-4}	2.02	2.12×10^{-6}					
32	3.15×10^{-5}	2.00	1.31×10^{-7}	4.01	1.11×10^{-9}			
64	7.88×10^{-6}	2.00	8.19×10^{-9}	4.00	1.72×10^{-11}	6.01	1.28×10^{-13}	
128	1.97×10^{-6}	2.00	5.12×10^{-10}	4.00	2.68×10^{-13}	6.00	8.88×10^{-16}	8.05

TABLE 2. Composite Simpson rule: $s = 1/2$.

n	E_1^n	α_0	E_2^n	α_1	E_3^n	α_2	E_4^n	α_3
8	1.53×10^{-4}							
16	4.63×10^{-5}	1.72	1.08×10^{-5}					
32	1.21×10^{-5}	1.94	6.50×10^{-7}	4.05	2.60×10^{-8}			
64	3.05×10^{-6}	1.99	4.03×10^{-8}	4.01	3.94×10^{-10}	6.04	1.21×10^{-11}	
128	7.64×10^{-7}	1.99	2.51×10^{-9}	4.00	6.11×10^{-12}	6.01	4.60×10^{-14}	8.05

TABLE 3. Composite modified Simpson rule: $s = 1/2$.

n	E_1^n	α_0	E_2^n	α_1	E_3^n	α_2
8	1.53×10^{-5}					
16	4.63×10^{-6}	4.11	5.59×10^{-8}			
32	1.21×10^{-7}	4.03	8.49×10^{-10}	6.04	2.50×10^{-10}	
64	3.05×10^{-9}	4.01	1.32×10^{-11}	6.01	9.46×10^{-14}	8.04
128	7.64×10^{-10}	4.00	2.06×10^{-13}	6.00	0	—

TABLE 4. Composite midpoint rule: $s = 1/4$.

n	E_1^n	α_0	E_2^n	α_1	E_3^n	α_2	E_4^n	α_3
8	3.63×10^{-3}							
16	8.95×10^{-4}	2.02	1.50×10^{-6}					
32	2.23×10^{-5}	2.00	9.29×10^{-8}	4.01	7.85×10^{-10}			
64	5.57×10^{-6}	2.00	5.79×10^{-9}	4.00	1.22×10^{-11}	6.01	9.14×10^{-14}	
128	1.39×10^{-6}	2.00	3.62×10^{-10}	4.00	1.90×10^{-13}	6.00	1.11×10^{-16}	8.04

TABLE 5. Composite Simpson rule: $s = 1/4$.

n	E_1^n	α_0	E_2^n	α_1	E_3^n	α_2	E_4^n	α_3
8	1.08×10^{-4}							
16	3.27×10^{-5}	1.72	7.63×10^{-5}					
32	8.54×10^{-5}	1.94	4.60×10^{-7}	4.05	1.84×10^{-7}			
64	2.16×10^{-6}	1.99	2.85×10^{-8}	4.01	2.78×10^{-10}	6.04	8.58×10^{-12}	
128	5.40×10^{-7}	2.00	1.78×10^{-9}	4.00	4.32×10^{-12}	6.01	3.33×10^{-14}	8.01

TABLE 6. Composite modified Simpson rule: $s = 1/4$.

n	E_1^n	α_0	E_2^n	α_1	E_3^n	α_2
8	3.24×10^{-5}					
16	1.99×10^{-6}	4.03	3.95×10^{-8}			
32	1.24×10^{-7}	4.01	6.00×10^{-10}	6.04	1.77×10^{-10}	
64	7.72×10^{-9}	4.00	9.31×10^{-12}	6.01	6.68×10^{-14}	8.04
128	4.82×10^{-10}	4.00	1.45×10^{-13}	6.00	1.11×10^{-16}	9.23

TABLE 7. Piecewise constant collocation at midpoints: $s = 1/2$.

n	E_1^n	α_0	E_2^n	α_1	E_3^n	α_2	E_4^n	α_3
8	7.32×10^{-4}							
16	1.82×10^{-4}	2.01	1.80×10^{-6}					
32	4.53×10^{-5}	2.00	1.11×10^{-7}	4.01	8.75×10^{-10}			
64	1.13×10^{-6}	2.00	6.97×10^{-9}	4.00	1.36×10^{-11}	6.01	9.37×10^{-14}	
128	2.83×10^{-7}	2.00	4.35×10^{-10}	4.00	2.12×10^{-13}	6.00	4.44×10^{-16}	8.04

TABLE 8. Piecewise linear collocation at endpoints: $s = 1/2$.

n	E_1^n	α_0	E_2^n	α_1	E_3^n	α_2	E_4^n	α_3
8	3.61×10^{-4}							
16	9.05×10^{-5}	2.00	2.07×10^{-7}					
32	2.26×10^{-5}	2.00	1.30×10^{-8}	4.00	8.08×10^{-12}			
64	5.66×10^{-6}	2.00	8.10×10^{-10}	4.00	1.26×10^{-13}	6.02	2.22×10^{-16}	
128	1.41×10^{-6}	2.00	5.06×10^{-11}	4.00	2.00×10^{-15}	6.00	0	—

TABLE 9. Piecewise constant collocation at midpoints: $s = 1/4$.

n	E_1^n	α_0	E_2^n	α_1	E_3^n	α_2	E_4^n	α_3
8	5.17×10^{-4}							
16	1.28×10^{-4}	2.01	1.27×10^{-6}					
32	3.20×10^{-5}	2.00	7.89×10^{-8}	4.01	6.19×10^{-10}			
64	8.01×10^{-6}	2.00	4.92×10^{-9}	4.00	9.61×10^{-12}	6.01	6.61×10^{-14}	
128	2.00×10^{-6}	2.00	3.08×10^{-10}	4.00	1.49×10^{-13}	6.01	7.77×10^{-16}	–

TABLE 10. Piecewise linear collocation at endpoints: $s = 1/4$.

n	E_1^n	α_0	E_2^n	α_1	E_3^n	α_2	E_4^n	α_3
8	2.56×10^{-4}							
16	6.40×10^{-5}	2.00	1.47×10^{-7}					
32	1.60×10^{-5}	2.00	9.17×10^{-9}	4.00	5.71×10^{-12}			
64	4.00×10^{-6}	2.00	5.73×10^{-10}	4.00	8.86×10^{-14}	6.01	6.66×10^{-16}	
128	1.00×10^{-6}	2.00	3.58×10^{-11}	4.00	1.78×10^{-15}	5.64	3.33×10^{-16}	–

8.2. Iterated collocation method. We consider the following two cases.

1. Let X_n be the space of piecewise constant functions with respect to a uniform partition of $[0, 1]$ with n subintervals, and let $\pi_n : C[0, 1] \to X_n$ be defined as

$$(\pi_n x)(s_i) = x(s_i), \quad i = 1, 2, \ldots, n.$$

2. Let X_n be the space of piecewise linear continuous functions with respect to a uniform partition of $[0, 1]$ with n subintervals, and let $\pi_n : C[0, 1] \to X_n$ be defined as

$$(\pi_n x)(t_i) = x(t_i), \quad i = 1, 2, \ldots, n + 1.$$

Let $T_n = T\pi_n$.

The expected orders of convergence in both cases are $\alpha_0 = 2$, $\alpha_1 = 4$, $\alpha_2 = 6$, $\alpha_3 = 8$ (Theorem 7.1, Corollary 7.2), and Tables 7–10 show that the computed orders match well with the expected orders.

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