JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 24, Number 1, Spring 2012

VOLTERRA INTEGRAL EQUATIONS AND FRACTIONAL CALCULUS: DO NEIGHBORING SOLUTIONS INTERSECT?

KAI DIETHELM AND NEVILLE J. FORD

Communicated by William McLean

ABSTRACT. In this paper we consider the solutions to two neighboring Hammerstein-type Volterra integral equations of the form

$$y(t) = \sigma + \int_0^t p(t,s) f(s, y(s))\, ds; \quad \sigma = y_0, z_0.$$

We give a theorem that guarantees that the solutions never intersect if $y_0 \neq z_0$, and we discuss several consequences of the main theorem that concern initial and boundary value problems for fractional calculus. Finally, we give an example that illustrates how one may calculate the history of the solution to a boundary value fractional differential equation.

1. Introduction. In this paper we consider the question of whether or not the solutions to two Volterra integral equations which have the same kernel but different forcing terms may intersect at some future time. Our discussions are motivated by the desire to set out a fairly general framework in which existing results about the intersection of solutions to ordinary differential equations can be extended to related problems such as solutions to equations of fractional order.

The Volterra second kind integral equations that we shall consider take the Hammerstein form

$$y(t) = \sigma + \int_0^t p(t,s) f(s, y(s))\, ds \tag{1}$$

with some constant $\sigma \in \mathbf{R}$ where f is assumed to be continuous whereas p may be singular. Equations of this type have been analyzed by many

The second author is the corresponding author.

Received by the editors on July 11, 2008, and in revised form on January 21, 2010.

DOI:10.1216/JIE-2012-24-1-25 Copyright ©2012 Rocky Mountain Mathematics Consortium

authors [**3, 11, 13, 14**], but as far as we are aware the problems addressed in the current paper provide new insights into questions not previously considered.

Note that (when $p(t,s) = 1$) equation (1) is equivalent to the first order ordinary differential equation

$$y'(t) = f(t, y(t)), \quad y(0) = \sigma. \tag{2}$$

When $p(t,s) = (t-s)$, equation (1) is equivalent to the second order ordinary differential equation

$$y''(t) = f(t, y(t)), \quad y(0) = \sigma,\ y'(0) = 0. \tag{3}$$

Caputo-type fractional differential equations arise when $p(t,s) = (t-s)^{\alpha-1}$; see [**5**]. For $0 < \alpha < 1$, this equation is equivalent to

$$D^{\alpha}_{*0} y(t) = \Gamma(\alpha) f(t, y(t)), \quad y(0) = \sigma, \tag{4}$$

and for $1 < \alpha < 2$, the equation becomes

$$D^{\alpha}_{*0} y(t) = \Gamma(\alpha) f(t, y(t)), \quad y(0) = \sigma,\ y'(0) = 0. \tag{5}$$

(See Section 2 for the precise definitions of these fractional differential operators.) Similar equivalences to fractional order equations can be derived for other values of α. We shall work in the general framework (1) and use the known theory for equations (2) and (3) as a point of reference for our results. We are able to deduce new results for fractional order equations.

2. Existing ordinary differential equation theory. One of the most fundamental and best known results in the theory of classical ordinary differential equations deals with the question whether the graphs of two different solutions to the same differential equation can meet or even cross each other. Under quite natural assumptions, the answer is negative, i.e., the graphs are strictly separated from each other. A mathematically precise formulation reads as follows (see, e.g., [**3**, Theorem 3.1]):

Theorem 2.1. *Let $f : [a,b] \times [c,d] \to \mathbf{R}$ be continuous and satisfy a Lipschitz condition with respect to the second variable. Consider two solutions y_1 and y_2 to the differential equation*

$$y_j'(t) = f(t, y_j(t)), \quad (j = 1, 2), \tag{6}$$

subject to the initial conditions $y_j(t_{j0}) = y_{j0} \in (c, d)$*, respectively. Then the functions* y_1 *and* y_2 *coincide either everywhere or nowhere.*

Proof. The proof is very simple: Assume that y_1 and y_2 coincide at some point t^*, i.e., $y_1(t^*) = y_2(t^*) =: y^*$, say. Then, both functions solve the initial value problem $y_j'(t) = f(t, y_j(t))$, $y_j(t^*) = y^*$. Since the assumptions imply that this problem has a unique solution, y_1 and y_2 must be identical, i.e., they coincide everywhere. □

The graphs of two different solutions to equation (6) thus never meet or cross each other. This result can be seen as the basis of graphical methods for solving first-order differential equations in the sense that it allows one to plot the graph of a solution on the basis of a direction field. If the graphs of two solutions would meet then the direction field would not give any useful information.

One of our goals is to generalize this result to the fractional setting, i.e., to differential equations of the form

$$D^{\alpha}_{*a} y(t) = f(t, y(t)) \tag{7}$$

where D^{α}_{*a} denotes the Caputo differential operator of order $\alpha \notin \mathbf{N}$ [**2**], defined by

$$D^{\alpha}_{*a} y(t) := D^{\alpha}_{a}(y - T[y])(t),$$

where $T[y]$ is the Taylor polynomial of degree $\lfloor \alpha \rfloor$ for y, centered at a, and D^{α}_{a} is the Riemann-Liouville derivative of order α [**15**]. The latter is defined by $D^{\alpha} := D^{\lceil \alpha \rceil} J_a^{\lceil \alpha \rceil - \alpha}$, with J_a^{β} being the Riemann-Liouville integral operator,

$$J_a^{\beta} y(t) := \frac{1}{\Gamma(\beta)} \int_a^t (t-s)^{\beta-1} y(s)\, ds$$

and $D^{\lceil \alpha \rceil}$ is the classical integer order derivative.

Our interest in the class of Volterra equations described by equation (1) is motivated by the well-known fact [**5**] that such initial value problems of fractional order are equivalent to a subclass of these Volterra equations.

We specifically draw the reader's attention to the fact that in Theorem 2.1 it does not matter whether the values t_1 and t_2, i.e., the

abscissae where the initial conditions are specified, coincide with each other or not. It has, however, been shown in [**4**] that in the fractional case the situation is much more involved. The nonlocal nature of the fractional derivative means that initial conditions can be imposed only at the starting point of the Caputo operator. In order to impose conditions at any other point we need to obtain the fuller understanding that is provided in this paper.

3. Theory for Volterra equations of the second kind. In this section we shall derive some theory for the Volterra equation (1). As we have already discussed, our aim is to find sufficient conditions (on p, f) for the solutions to two equations with different forcing terms not to meet after any finite time interval. It is important in this context to have a clear existence and uniqueness theory in place since we need to be sure that there is *precisely one* solution to each of the equations we are considering. It turns out that the theorem given by Linz [**13**, page 62] both meets this requirement and provides the fundamental building blocks for our new theorem:

Theorem 3.1. *For the equation*

$$y(t) = y_0 + \int_0^t p(t,s) f(s, y(s))\, ds \tag{8}$$

we make the following assumptions:

(1) *f is continuous on $[0,b] \times \mathbf{R}$ and is Lipschitz continuous with Lipschitz constant L with respect to its second argument.*

(2) *p satisfies the following conditions*:

(a) *for each continuous function x and for every $0 \le \tau_1 \le \tau_2 \le t$ the integrals*

$$\int_{\tau_1}^{\tau_2} p(t,s) f(s, x(s))\, ds \tag{9}$$

and

$$\int_0^t p(t,s) f(s, x(s))\, ds \tag{10}$$

are continuous functions of t,

(b) *p is absolutely integrable with respect to s for all $t \in [0, b]$,*

(c) *there exist points $0 = T_0 < T_1 < T_2 < \cdots < T_N = b, T_i \in \mathbf{R}$, such that with $t \geq T_i$,*

$$L \int_{T_i}^{\min(t, T_{i+1})} |p(t,s)|\, ds \leq \gamma < \frac{1}{2}, \tag{11}$$

(d) *for every $t \geq 0$,*

$$\lim_{\delta \to 0^+} \int_t^{t+\delta} |p(t+\delta, s)|\, ds = 0. \tag{12}$$

Then, (8) *has a unique continuous solution. Further, for every $c \in \mathbf{R}$, there exists precisely one value of $y_0 \in \mathbf{R}$ for which the solution y of* (8) *satisfies $y(b) = c$.*

The conditions on p are well known in the classical theory; see, e.g., [**13**, page 48].

Proof. The proof of existence and uniqueness of a continuous solution to (8) is given by [**13**, Theorem 4.8]. For the remainder we consider equation (8) in the form of a boundary value problem where we are given that $y(b) = c$.

In this case, the function y satisfies (8) if and only if it is also a solution of

$$y(t) = \int_0^t p(t,s) f(s, y(s)) ds - \int_0^b p(b,s) f(s, y(s))\, ds + c. \tag{13}$$

Recalling that $T_N = b$, it follows that y satisfies, on $[T_{N-1}, T_N]$, the equation

$$\begin{aligned} y(t) &= g(t) + \int_{T_{N-1}}^t p(t,s) f(s, y(s))\, ds \\ &\quad - \int_{T_{N-1}}^{T_N} p(T_N, s) f(s, y(s))\, ds \\ &= g(t) + \mathcal{F}(y)(t) \end{aligned} \tag{14}$$

where

$$g(t) = \int_0^{T_{N-1}} p(t,s) f(s, y(s))\, ds - \int_0^{T_{N-1}} p(b,s) f(s, y(s))\, ds \tag{15}$$

By (11), $\mathcal{F}$ (interpreted as a mapping from $C[T_{N-1}, T_N]$ to itself) is a contraction and it follows (by Picard iteration) that (14) has a unique solution on $[T_{N-1}, T_N]$.

We can repeat this process on successive intervals $[T_j, T_{j+1}]$, $j = N-2, N-3, \ldots, 0$ to establish that (13) has a unique solution on $[0, b]$. It follows that each value of the constant c corresponds to precisely one value of y_0. □

The following two corollaries indicate how this theorem can provide important practical results:

Corollary 3.1. *Let y, z satisfy* (1) *with, respectively, $\sigma = y_0$, $\sigma = z_0$. For every $t \in [0, b]$, $y(t) = z(t)$ if and only if $y_0 = z_0$.*

Corollary 3.2. *If $p(t,s) = (t-s)^{\alpha-1}/\Gamma(\alpha)$, for $0 < \alpha < 1$, then* (11) *is satisfied whenever each $[T_i, T_{i+1}]$ is an interval of length at most T satisfying $2LT^\alpha/\Gamma(\alpha+1) < 1$. It follows that the constant b can be chosen to be arbitrarily large.*

This corollary is related to a result of Agarwal et al. [**1**, Theorem 3.3]. Specifically, Agarwal et al. look at a slightly more general problem in the sense that, instead of the condition $y(b) = c$ that we use in our Theorem 3.1, and hence also in Corollary 3.2, they consider (in our notation) the two-point boundary condition $a_1 y(0) + a_2 y(T) = c$, and they show existence and uniqueness of solutions for $LT^\alpha(1 + |a_2|/|a_1 + a_2|)/\Gamma(\alpha+1) < 1$. Clearly, our setting corresponds to the case $a_1 = 0$ of theirs, in which case their condition reduces to our assumption $2LT^\alpha/\Gamma(\alpha+1) < 1$. However, it should be noted that, for this restricted class of problems, our result generalizes the above mentioned statement of [**1**] in the sense that the latter only deals with the case that the problem is considered on an interval of length T whereas we can now allow the length of the interval (in our notation, b) to be as large as we wish.

4. Discussion of the theorem. As we remarked earlier, if $p(t,s) = 1$ then equation (1) is equivalent to a first order ordinary differential equation. It is easy to check that in this case all the conditions of Theorem 3.1 are satisfied and so Theorem 2.1 could be deduced as a corollary to Theorem 3.1 and there would be no need to restrict the value of $b > 0$.

If $p(t,s) = (t-s)$, then we know that the equivalent second order ordinary differential equation does not have the *separation of solutions* property. It is instructive to explore how Theorem 3.1 breaks down in this case. It turns out that, for small enough values of b, the conditions of Theorem 3.1 are satisfied. But, for larger values of b,

$$L \int_{T_i}^{\min(t,T_{i+1})} |p(t,s)|\, ds \leq \gamma < \frac{1}{2}, \tag{16}$$

is not satisfied for t greater than some critical value. A moment's reflection shows why this is the case. Even for a second order equation, there will be an interval over which the two distinct solutions do not meet. If the two solutions meet for the first time at $t = B > 0$, then Theorem 3.1 will indicate that the two solutions do not meet on $[0,b]$ for any $b < B$. This is helpful because it shows us how to distinguish between the case where the solutions will never meet, and the case where they may meet after some finite time interval.

Now we can proceed with confidence to consider the fractional differential case where $p(t,s) = (t-s)^{\alpha-1}$.

Condition (11) can be used to show that for $0 < \alpha < 1$ there is no restriction on the choice of $b > 0$ and therefore the two solutions to the fractional differential equation will never meet. However, for $1 < \alpha < 2$, equation (11) is satisfied only for sufficiently small values of $b > 0$ and therefore we conclude that the solutions may meet at some value of $t > b$.

Theorem 4.1. *Let $0 < \alpha < 1$, and assume $f : [0,b] \times [c,d] \to \mathbf{R}$ to be continuous and satisfy a Lipschitz condition with respect to the second variable. Consider two solutions y_1 and y_2 to the differential equation*

$$D^{\alpha}_{*0} y_j(t) = f(t, y_j(t)) \quad (j = 1, 2), \tag{17}$$

subject to the initial conditions $y_j(0) = y_{j0}$*, respectively, where* $y_{10} \neq y_{20}$*. Then, for all* t *where both* $y_1(t)$ *and* $y_2(t)$ *exist, we have* $y_1(t) \neq y_2(t)$*.*

Proof. The result follows by applying Corollary 3.1 and Corollary 3.2. □

5. Insights for fractional differential equations. As we have seen in the previous section, for $0 < \alpha < 1$, for different initial conditions equation (17) has solutions that never intersect. However, for $1 < \alpha < 2$, this result no longer holds.

Insights from the linear case indicate why the result would not hold for $\alpha > 1$ as we can see in this example:

Example 5.1. Consider the fractional differential equation (17) for $1 < \alpha < 2$ with $f(t, y) = \lambda y$, i.e., a linear differential equation with constant coefficients, subject to the initial conditions $y_{10} = 1$, $y'_{10} = 0$, $y_{20} = 0$, $y'_{20} = 0$. We can write the exact solutions as

$$y_1(t) = E_\alpha(\lambda t^\alpha) = \sum_{j=0}^{\infty} \frac{(\lambda t^\alpha)^j}{\Gamma(1 + j\alpha)}, \tag{18}$$

$$y_2(t) = 0 \tag{19}$$

where E_α is a Mittag-Leffler function of the type discussed in [**10**]. As pointed out in that work, such functions possess at least one zero in $(0, \infty)$ if $\lambda < 0$. Thus, the solutions y_1 and y_2 would intersect at least once.

Turning our attention now to the conclusions of Theorem 4.1, we assess their significance in practice. We make the following points:

(1) We already knew that the dimension of the kernel of the fractional differential operator of order $\alpha \in (0, 1)$ is 1 at the starting point $t = 0$. By Theorem 4.1 we can now conclude that the dimension of the kernel of the operator remains unchanged for all $t > 0$. This is reassuring: the fact that the fractional differentiation operator is nonlocal and retains the full history of the function from the starting point might have

implied that the kernel of the operator would increase in dimension with the passage of time.

(2) Directly related to the previous point, we can now conclude that precisely one condition is needed to determine uniquely the solution to a given fractional differential equation of the form (17) and that the condition can be specified *at any value of* $t \geq 0$. This conclusion is of particular significance for those applied scientists who must collect data to provide the conditions for solving the equation. The implication is that it is perfectly acceptable to use data collected at some time point other than the starting point. In other words, the theorem makes it legitimate to attempt to solve a boundary value problem instead of an initial value problem.

(3) It has always been assumed that the history-dependence of the fractional differential operator implied that time could not be *reversed* (as it can be with an ordinary differential equation of order 1) so that the state of the system can be calculated at times before conditions are imposed. While there is still no constructive way to achieve this in the case of a fractional differential equation, Theorem 4.1 gives legitimacy to the desire to construct function values that predate the data condition. We consider in the next section how a simple shooting method can be employed to derive a good approximation to the initial condition based on knowledge of a boundary condition at a later time. As can be seen from this example, it becomes possible to provide a high accuracy approximation to the solution for all values of $t > 0$.

Finally, for this section, we remark on the use of a low order system of fractional equations to approximate a higher order problem. There are a number of possible ways to proceed here; see, e.g., [**6, 8, 9**]. It would seem from Theorem 4.1 that these approaches would guarantee (contrary to Example 5.1) that the solutions will not intersect. However this is not the case, because now we must consider the solution as a vector that contains y and its various fractional order derivatives. Theorem 4.1 guarantees that the two vector solutions do not intersect, but not that the two values of the function y do not coincide. In practice what this means is that at the point of intersection the two solution graphs will have different gradients and therefore the vectors representing the two solutions will be different.

6. Approximating the history of the solution. As we remarked in the previous section, it is now appropriate to consider how to evaluate the early part of the solution to a fractional differential equation of order $\alpha \in (0,1)$ based on a boundary condition given for some $t_1 > 0$. We know of no analytical approach for solving this problem although a construction could be based on the Picard iterations we have used here. We propose a very simple numerical approach for deriving an excellent approximation. It is based on the idea of a *shooting method* (see, e.g., [**12**]) which is commonly used in the solution of boundary value problems for ordinary differential equations.

The fundamental idea is that, by Theorem 4.1, we know that solution curves of

$$D^{\alpha}_{*0}y(t) = f(t, y(t)) \tag{20}$$

based on two different values for $y(0)$ never intersect.

We formulate the problem we are trying to solve:

For the equation (20) assume that the solution we require passes through the point (v, w) for some $v > 0$. Can we find the point $(0, y_0)$ that also lies on the same solution trajectory, and hence can we find the solution to (20) on $[0, v]$? Theorem 4.1 asserts that the required point is uniquely determined.

Therefore, we can begin by selecting a sequence of trial values of y_0 and evaluating (using a numerical scheme if necessary) the solution $y(v)$ in each case. The aim is to find values y_{01}, y_{02} satisfying $y(v)|_{y_0=y_{01}} < w < y(v)|_{y(0)=y_{02}}$. Now we follow a bisection method of search by trying $y(0) = (y_{01} + y_{02})/2$ and successively reducing the interval between the two approximants for $y(0)$ by a factor of 2 on each iteration. After several iterations one can reach any desired accuracy in the approximation of $y(0)$.

Example 6.1. As an example we consider the problem (20) with $\alpha = 1/2$ and $f(t, y) = \sin y$ subject to the boundary condition $y(1) = 2.5$. Table 1 shows the results of the bisection search based on a shooting method using the fractional Adams scheme of [**7**] with step size $1/200$. The continuous line in Figure 1 gives the graph of the solution based on the starting value $y_0 = 1.71875$ obtained from Table 1 and shows the trajectory passing close to the boundary value as expected. Moreover,

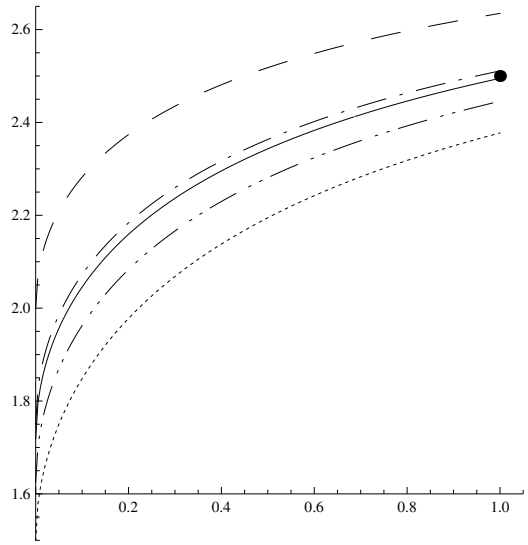

FIGURE 1. Graphs of neighboring approximate solutions to the boundary value problem of Example 6.1.

TABLE 1. Results of bisection search for Example 6.1.

y_0	1	2	1.5	1.75	1.625	1.6875	1.71875
$y(1)$	2.0556	2.63485	2.37728	2.51106	2.44567	2.47871	2.49496

Figure 1 gives the graphs of the neighboring solutions with $y_0 = 2$ (dashed line; top), $y_0 = 1.75$ (dash-dotted line; second from top), $y_0 = 1.5$ (dotted line, bottom) and $y_0 = 1.625$ (dashed and double dotted line; second from bottom), respectively, providing graphical evidence that two solutions belonging to different initial values indeed never meet.

7. Concluding remarks. The approach we have described, based on shooting and the bisection method, can be regarded as a prototype method that can be developed to be more sophisticated and computationally efficient. As a prototype, this works quite effectively, but efficiency would improve by using a better optimization method than bisection, and possibly by changing the numerical solver. In fact,

such decisions need to take account also of the accuracy and reliability of any experimental data in use. Obviously, the accuracy of the model depends critically on the accuracy of the initial or boundary conditions used in constructing the equation. It is reassuring to know (see, for example, [**5**]) that solutions depend continuously on these values so that any inaccuracy in the solution is likely to be reasonably small. In any event, there are fairly obvious ways to improve accuracy of the model at greater computational cost if the accuracy of the data or the critical nature of a particular application should warrant it.

Acknowledgments. Finally, the authors would like to acknowledge both the helpful comments from the referees and Editor, and the discussions we have had with several colleagues during the revision of this paper.

REFERENCES

1. R.P. Agarwal, M. Benchohra and S. Hamani, *A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions*, Acta Appl. Math. **109** (2010), 973–1033.

2. M. Caputo, *Linear models of dissipation whose Q is almost frequency independent*–II, Geophys. J. Royal Astronom. Soc. **13** (1967), 529–539; reprinted in Fract. Calc. Appl. Anal. **11** (2008), 4–14.

3. C. Corduneanu, *Principles of differential and integral equations*, 2nd edition, Chelsea Publishing Company, New York, 1977.

4. K. Diethelm, *On the separation of solutions of fractional differential equations*, Fract. Calc. Appl. Anal. **11** (2008), 259–268.

5. K. Diethelm and N.J. Ford, *Analysis of fractional differential equations*, J. Math. Anal. Appl. **265** (2002), 229–248.

6. ———, *Multi-order fractional differential equations and their numerical solution*, Appl. Math. Comput. **154** (2004), 621–640.

7. K. Diethelm, N.J. Ford and A.D. Freed, *Detailed error analysis for a fractional Adams method*, Numer. Algorithms **36** (2004), 31–52.

8. J.T. Edwards, N.J. Ford and A.C. Simpson, *The numerical solution of linear multi-term fractional differential equations: Systems of equations*, J. Comput. Appl. Math. **148** (2002), 401–418.

9. N.J. Ford and A.C. Simpson, *The approximate solution of fractional differential equations of order greater than* 1, in Proc. 16th IMACS World Congr. Sci. Comput., Appl. Math., and Simulation, M. Deville and R. Owens, eds., Lausanne, 2000.

10. R. Gorenflo and F. Mainardi, *Fractional oscillations and Mittag-Leffler functions*, in Proc. Inter. Workshop Recent Adv. Appl. Math. (RAAM '96), Kuwait University, Dept. of Mathematics and Computer Science, Kuwait, 1996.

11. G. Gripenberg, S.-O. Londen and O. Staffans, *Volterra integral and functional equations*, Cambridge University Press, Cambridge, 1990.

12. H.B. Keller, *Numerical methods for two-point boundary-value problems*, Blaisdell, Waltham, 1968.

13. P. Linz, *Analytical and numerical methods for Volterra equations*, SIAM, Philadelphia, 1985.

14. R.K. Miller, *Nonlinear Volterra integral equations*, Benjamin, Menlo Park, 1971.

15. S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional integrals and derivatives: Theory and applications*, Gordon and Breach, Yverdon, 1993.

Institut Computational Mathematics, Technische Universität Braunschweig, Pockelsstr. 14, 38106 Braunschweig, Germany; GNS Gesellschaft für Numerische Simulation mbH, Am Gaussberg 2, 38114 Braunschweig, Germany
Email address: k.diethelm@tu-bs.de, diethelm@gns-mbh.com

Department of Mathematics, University of Chester, Parkgate Road, Chester, CH1 4BJ, UK
Email address: njford@chester.ac.uk