JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 23, Number 4, Winter 2011

BOUNDARY BEHAVIOR OF THE LAYER POTENTIALS FOR THE TIME FRACTIONAL DIFFUSION EQUATION IN LIPSCHITZ DOMAINS

JUKKA KEMPPAINEN

Communicated by William McLean

ABSTRACT. This paper investigates the boundary behavior of layer potentials for the time fractional diffusion equation (TFDE) in Lipschitz domains. The paper is a continuation of [**9**]. Since now the boundary of the spatial domain Ω admits only Lipschitz smoothness, we have to replace the classical technique used in [**9**] with a more delicate harmonic analysis technique.

We prove that certain nontangential maximal functions related to the layer potentials are bounded in $L^p(\Sigma_T)$, which in particular implies the usual jump relations known for the heat equation. Although the results are well known in the case of the heat potential corresponding to the case $\alpha = 1$, the proofs of the same properties seem not to be available in the case $0 < \alpha < 1$.

1. Introduction. We study the boundary behavior of layer potentials for the time fractional diffusion equation

$$\begin{aligned} \partial_t^\alpha \Phi - \Delta\Phi &= 0, \quad \text{in } Q_T = \Omega \times (0,T), \\ \Phi &= g, \quad \text{on } \Sigma_T = \Gamma \times (0,T) \\ \Phi(x,0) &= 0, \ x \in \Omega, \end{aligned} \tag{1}$$

where $\Omega \subset \mathbf{R}^n$ is a bounded domain with Lipschitz boundary Γ and

$$\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} u'(\tau)\, \mathrm{d}\tau \tag{2}$$

is the fractional Caputo time derivative of order $0 < \alpha \le 1$. For $\alpha = 1$ the fractional derivative is interpreted as the limit $\lim_{\alpha\uparrow 1} \partial_t^\alpha u(t)$, which coincides with the usual time derivative $\mathrm{d}u(t)/\mathrm{d}t$ [**8**, page 68].

Received by the editors on June 6, 2009, and in revised form on November 1, 2009.

DOI:10.1216/JIE-2011-23-4-541

In the case of the Dirichlet and Neumann problems for the heat equation the nontangential maximal functions associated with the solution have been shown to be bounded in $L^p(\Sigma_T)$ [**2, 5, 6**]. In this paper we extend these well-known results for TFDE.

In [**9**] we have studied the boundary behavior of the single layer potential in $\mathcal{C}^{1+\lambda}$ domains, $0 < \lambda < 1$. The technique for showing similar results for the double layer potential is analogous. In this paper we study both the single layer and the double layer potentials. Detailed proofs are given only for the double layer potential since the arguments for the single layer potential are similar. The technique used in our considerations can be found in [**6, 14**]. The mapping properties of the layer potentials in L^p-spaces follow from the well-known technique found, e.g., from [**16**].

2. Notation and main results. We define the single layer potential by

$$(S\varphi)(x,t) = \int_0^t \int_\Gamma G(x-y,t-\tau)\varphi(y,\tau)\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau, \tag{3}$$

where G is the fundamental solution of the fractional diffusion equation. It is known that

(4)
$$G(x,t) = \begin{cases} \pi^{-n/2}t^{\alpha-1}|x|^{-n}H^{20}_{12}\left[\frac{1}{4}|x|^2t^{-\alpha}\,\middle|\,{}^{(\alpha,\alpha)}_{(n/2,1),\ (1,1)}\right] & x\in\mathbf{R}^n,\ t>0,\\ 0 & x\in\mathbf{R}^n,\ t<0,\end{cases}$$

where H is the Fox H-function [**11–13**]. The double layer potential is defined by

$$(D\varphi)(x,t) = \int_0^t \int_\Gamma \partial_{n(y)}G(x-y,t-\tau)\varphi(y,\tau)\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau. \tag{5}$$

The boundary behavior of the double layer potential is based on the properties of the operator

$$J_\varepsilon\varphi(x,t) = \int_0^{t-\varepsilon} \int_\Gamma \partial_{n(y)}G(x-y,t-\tau)\varphi(y,\tau)\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau \tag{6}$$

for given $0<\varepsilon<t$ and $\varphi \in L^1_{\mathrm{loc}}(\Sigma_\infty)$.

Our main result states that the nontangential function for the double layer potential defines a bounded linear operator in $L^p(\Sigma_T)$. As a consequence, the double layer potential has a nontangential limit pointwise almost everywhere in Σ_T. The nontangential maximal function of u is defined by

$$N(u)(x,t) = \sup\{|u(y,t)| : \; y \in \Omega \cap B(x,\delta) \\ \text{such that } \langle x-y, n(x)\rangle > \beta|x-y|\}$$

for some $0<\beta<1$ and a positive constant δ depending on Ω and β.

In the paper we prove the following results.

Theorem 1. *Let* $1<p<\infty$ *and* $\varphi \in L^p(\Sigma_\infty)$. *We have*

(i) *The operator* $J_\varepsilon : L^p(\Sigma_\infty) \to L^p(\Sigma_\infty)$ *is bounded independently of* ε.

(ii) *The operator* $\widetilde{J}(\varphi) = \sup_{\varepsilon>0} |J_\varepsilon \varphi|$ *is bounded in* $L^p(\Sigma_\infty)$.

(iii) *The limit* $(J\varphi)(x,t) = \lim\limits_{\varepsilon\downarrow 0}(J_\varepsilon\varphi)(x,t)$ *exists pointwise almost everywhere in* Σ_∞ *and in* $L^p(\Sigma_\infty)$. *In particular,* J *is bounded in* $L^p(\Sigma_\infty)$.

Theorem 2. *There exists a constant* $0<\beta<1$ *depending on the Lipschitz constant for* Ω *and a constant* $\delta=\delta(\beta,\Omega)$ *such that the nontangential maximal function* $N(D\psi)$ *is bounded in* $L^p(\Sigma_T)$. *Moreover,*

$$(D\psi)(x,t) \longrightarrow -\frac{1}{2}\psi(x_0,t) + (J\psi)(x_0,t)$$

pointwise for almost every $(x_0,t) \in \Sigma_T$ *as* $\Omega \ni x \to x_0$ *such that* $\langle x_0 - x, n(x_0)\rangle > \beta|x-x_0|$.

The same technique as in the proof of Theorem 2 may be used to prove a similar result for the single layer potential. We only state the corresponding result without proof.

Theorem 3. *There exists a constant* $0<\beta<1$ *depending on the Lipschitz constant for* Ω *and a constant* $\delta=\delta(\beta,\Omega)$ *such that* *the nontangential maximal function* $N(\nabla_x S\psi)$ *is bounded in* $L^p(\Sigma_T)$. *Moreover,*

$$\langle \nabla_x S\psi(x,t), n(x_0)\rangle \longrightarrow \frac{1}{2}\psi(x_0,t) + (J'\psi)(x_0,t)$$

with

$$(J'\psi)(x,t) = \lim_{\varepsilon\downarrow 0}\int_0^{t-\varepsilon}\int_\Gamma \partial_{n(x)}G(x-y,t-\tau)\psi(y,\tau)\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau,$$

pointwise for almost every $(x_0,t) \in \Sigma_T$ *as* $\Omega \ni x \to x_0$ *such that* $\langle x_0 - x, n(x_0)\rangle > \beta|x-x_0|$.

3. Estimates for the fundamental solution. In our analysis we need the following Fox H-functions defined for $z > 0$

(7) $$H_{(1)}(z) := H^{20}_{12}\left[z\,\middle|\,\begin{smallmatrix}(\alpha,\alpha)\\ (n/2,1)\ (1,1)\end{smallmatrix}\right],$$

(8) $$H_{(2)}(z) := H^{30}_{23}\left[z\,\middle|\,\begin{smallmatrix}(\alpha,\alpha)\quad(0,1)\\ (n/2,1),\ (1,1),\ (1,1)\end{smallmatrix}\right]$$

(9) $$H_{(3)}(z) := H^{40}_{34}\left[z\,\middle|\,\begin{smallmatrix}(\alpha,\alpha)\quad(0,1)\ (0,1)\\ (n/2,1)\ (1,1)\ (1,1)\ (1,1)\end{smallmatrix}\right].$$

The following properties of the functions (7)–(9) are in a central role.

Lemma 1. *For the functions* $H_{(p)}$ *the following holds*:

(i) *Differentiation formula* $(\mathrm{d}/\mathrm{d}z)H_{(p)}(z) = -z^{-1}H_{(p+1)}(z)$ *for* $p = 1,2$.

(ii) *The asymptotic behavior at infinity*:

(10) $$|H_{(p)}(z)| \le Cz^{(n+2p-2\alpha)/2(2-\alpha)}\exp(-\sigma z^{1/(2-\alpha)}),\quad \sigma := \alpha^{\alpha/(2-\alpha)}(2-\alpha),$$

for $p = 1,2,3$ *and* $z \ge 1$.

(iii) *The asymptotic behavior near zero*:

(11) $$|nH_{(p)}(z) + 2H_{(p+1)}(z)| \le C\begin{cases} z^2|\log z| & \text{if } n = 2,\\ z^2 & \text{if } n \ge 3,\end{cases}$$

for $p = 1,2$ *and* $z \le 1$. *The constants in* (ii) *and* (iii) *can depend on* n, p *and* α.

Proof. Property (i) is an easy consequence of the Mellin-Barnes integral representation of Fox H-functions and of the analyticity of the functions $H_{(p)}$. The asymptotic expansions for $H_{(p)}$ are stated in [**5**, formulae (3.7), (3.14), (3.15) and (3.16)]. Note that the first terms in the series representation of $nH_{(1)} + 2H_{(2)}$ cancel out. For the proofs of the asymptotics we refer to [**1, 11**]. □

Remark 1. Although we have an exact and optimal value for the constant σ in (ii), in what follows σ may denote various positive constants. The only thing that matters is that $\sigma > 0$.

The next result gives estimates for the partial derivatives of the fundamental solution.

Lemma 2. *Let β be any multi-index with $|\beta| = 1$ or $|\beta| = 2$. The following asymptotic formulae hold for the partial derivatives of the fundamental solution.*

(i) *The asymptotic behavior at infinity*:

$$(12)\quad |\partial_x^\beta \partial_t^{2-|\beta|} G(x,t)| \\ \leq Ct^{-(\alpha n/2)-1+[(\alpha/2)-1](2-|\beta|)} \exp(-\sigma t^{-\alpha/(2-\alpha)}|x|^{2/(2-\alpha)})$$

for a positive constant σ as $t^{-\alpha}|x|^2 \to \infty$.

(ii) *The asymptotic behavior near zero*:

$$(13)\quad |\partial_x^\beta \partial_t^{2-|\beta|} G(x,t)| \leq Ct^{|\beta|-\alpha-3} \begin{cases} |x|^{4-|\beta|-n} & \text{if } n \geq 3, \\ |x|^{2-|\beta|}|\log(t^{-\alpha}|x|^2)| & \text{if } n = 2, \end{cases}$$

as $t^{-\alpha}|x|^2 \to 0$.

Proof. From Lemma 1 and (4) we get the formula

$$(14)\quad \partial_{x_j} G(x,t) \\ = -\pi^{-n/2}\frac{x_j}{|x|^{n+2}}t^{\alpha-1}\left\{nH_{(1)}\left(\frac{1}{4}t^{-\alpha}|x|^2\right) + 2H_{(2)}\left(\frac{1}{4}t^{-\alpha}|x|^2\right)\right\}$$

for the partial derivative of the fundamental solution.

For the case $|\beta| = 2$ we differentiate in x once more to get

$$
\begin{aligned}
\partial^2_{x_i x_j} G(x,t) = \frac{1}{\pi^{n/2}} \frac{t^{\alpha-1}}{|x|^{n+4}} \Big\{ & 2x_i 2x_j \big(nH_{(2)}(z) + 2H_{(3)}(z)\big) \\
& + \big((n+2)x_i x_j - \delta_{ij}|x|^2\big)\big(nH_{(1)}(z) + 2H_{(2)}(z)\big)\Big\},
\end{aligned} \tag{15}
$$

where we have denoted $z = (1/4)t^{-\alpha}|x|^2$. Then Lemma 1 gives

$$
\begin{aligned}
|\partial^2_{x_i x_j} G(x,t)| &\le C \frac{t^{\alpha-1}}{|x|^{n+2}} \sum_{p=1}^{3} |H_{(p)}(z)| \\
&\le C t^{-(\alpha n/2)-1} z^{(n(\alpha-1)+2)/(2(2-\alpha))} \exp(-\sigma z^{1/(2-\alpha)}).
\end{aligned}
$$

Since $z^\gamma \exp(-cz^\beta)$ is uniformly bounded in $[1,\infty)$ for any $c, \beta > 0$ and $\gamma \in \mathbf{R}$, we may estimate

$$
|\partial^2_{x_i x_j} G(x,t)| \le C t^{-(\alpha n/2)-1} \exp(-\sigma_0 z^{1/(2-\alpha)})
$$

for any $0 < \sigma_0 < \sigma$, from which (12) follows in the case $|\beta| = 2$.

If z is small, we get from (15) and Lemma 1 an estimate

$$
|\partial^2_{x_i x_j} G(x,t)| \le C \begin{cases} t^{-\alpha-1}|x|^{-n+2} & \text{if } n \ge 3, \\ t^{-\alpha-1}|\log(t^{-\alpha}|x|^2)| & \text{if } n = 2. \end{cases} \tag{16}
$$

Then (13) follows from (16) in the case $|\beta| = 2$).

In the case $|\beta| = 1$ differentiation in time in 14) gives us

$$
\begin{aligned}
\partial_t \partial_{x_j} G(x,t) = -\pi^{-n/2}(\alpha - 1) & \frac{x_j t^{\alpha-2}}{|x|^{n+2}} \Big\{ nH_{(1)}(z) + 2H_{(2)}(z) \Big\} \\
& - \frac{\alpha \pi^{-n/2} x_j t^{\alpha-2}}{|x|^{n+2}} \Big\{ nH_{(2)}(z) + 2H_{(3)}(z) \Big\}.
\end{aligned} \tag{17}
$$

Now Lemma 1 gives the following estimate

$$
|\partial_t \partial_{x_j} G(x,t)| \le C t^{\alpha-2} |x|^{-n-1} z^{(n+6-2\alpha)/(2(2-\alpha))} \exp(-\sigma z^{1/(2-\alpha)}), \qquad z \to \infty.
$$

Then we may choose $0 < \sigma_0 < \sigma$ and argue as before to obtain

$$|\partial_t \partial_{x_j} G(x,t)| \le C t^{\alpha-2} |x|^{-n-1} \exp(-\sigma_0 z^{1/(2-\alpha)}) \\ \le C t^{[(\alpha(n-1))/2]-2} \exp(-\sigma_0 z^{1/(2-\alpha)}),$$

which completes the proof of (12) in the case $|\beta| = 1$.

If z is small, Lemma 1 together with (17) gives us an estimate

$$|\partial_t \partial_{x_j} G(x,t)| \le C \begin{cases} t^{-\alpha-2}|x|^{-n+3} & \text{if } n \ge 3, \\ t^{-\alpha-2}|x \log(t^{-\alpha}|x|^2)| & \text{if } n = 2, \end{cases} \tag{18}$$

which completes the proof of (13) in the case $|\beta| = 1$. □

Now we can prove important properties for the gradient of the fundamental solution, which is crucial for the weak type estimate.

Lemma 3. *Assume that* $|x-y| + |t-\tau|^{\alpha/2} > 2(|x-x_1| + |t-t_1|^{\alpha/2})$, *and let* $z = (1/4)(t-\tau)^{-\alpha}|x-y|^2$. *For the gradient of the fundamental solution we have*

(i) *If* $z \ge 1$, *then*

$$|\nabla G(x-y, t-\tau) - \nabla G(x_1 - y, t_1 - \tau)| \\ \le C \frac{|x-x_1| + |t-t_1|^{\alpha/2}}{\big(|x-y| + |t-\tau|^{\alpha/2}\big)^{n+(2/\alpha)}}. \tag{19}$$

(ii) *If* $z \le 1$ *and* $\gamma > 0$, *then*

$$\frac{|\nabla G(x-y, t-\tau) - \nabla G(x_1 - y, t_1 - \tau)|}{|x-x_1| + |t-t_1|^{\alpha/2}} \\ \le C \begin{cases} (t-\tau)^{-\alpha-1}|x-y|^{-n+2} & \text{if } n \ge 3, \\ (t-\tau)^{\alpha\gamma-\alpha-1}|x-y|^{-2\gamma} & \text{if } n = 2. \end{cases} \tag{20}$$

Proof. We consider the change in x and the change in t separately.

Let us first consider the change in the x-variable. If $z \geq 1$, then using the Mean Value Theorem and estimate (12) in Lemma 2 we obtain

$$\begin{aligned}|\nabla G(x-y,t-\tau)-\nabla G(x_1-y,t-\tau)|&\\ &\leq C|x-x_1|(t-\tau)^{-(\alpha(n+2))/2-1+\alpha}\\ &\times \exp\{-\sigma(t-\tau)^{-\alpha/(2-\alpha)}|\widetilde{x}-y|^{2/(2-\alpha)}\}.\end{aligned}$$

Now $z^\gamma \exp(-\sigma z^{1/(2-\alpha)})$ is uniformly bounded in $\mathbf{R}_+$ for any γ. Choosing $\gamma = (n/2) + (1/\alpha)$ we get

$$|\nabla G(x-y,t-\tau)-\nabla G(x_1-y,t-\tau)| \leq C|x-x_1||\widetilde{x}-y|^{-n-2/\alpha}.$$

Since $z \geq 1$, we have $2|x-x_1| < |t-\tau|^{\alpha/2} + |x-y| < 3|x-y|/2$. Then by the triangle inequality we have $|\widetilde{x}-y| \geq |x-y|/4$. Using the previous two inequalities we may conclude the claim in this case.

If $z \leq 1$, then using the Mean Value Theorem and estimate (16) in Lemma 2 we get

$$|\nabla G(x-y,t)-\nabla G(x_1-y,t)| \leq C|x-x_1|t^{-\alpha-1}\begin{cases}|x-y|^{-n+2} & \text{if } n \geq 3,\\ t^{\alpha\gamma}|x-y|^{-2\gamma} & \text{if } n = 2.\end{cases}$$

We used the fact that $z^\gamma \log z$ is bounded near zero for any $\gamma > 0$ above in the case $n = 2$.

For the change in t Lemma 2 gives the following estimate

$$|\partial_t \partial_{x_j} G(x,t)| \leq Ct^{-(\alpha(n-1))/2-2+\alpha\gamma}|x|^{-2\gamma}.$$

for any $\gamma > 0$ in the case $z \geq 1$. If we choose $\gamma = (n-1/2) + (2/\alpha)$, we get

$$|\partial_t \partial_{x_j} G(x,t)| \leq C|x|^{-n-(4/\alpha)+1}. \tag{21}$$

Using (21) and the Mean Value Theorem we have

$$|\nabla G(x-y,t-\tau)-\nabla G(x-y,t_1-\tau)| \leq C|t-t_1||x-y|^{-n-(4/\alpha)+1}. \tag{22}$$

Notice that in this case $|t-t_1|^{\alpha/2} < |x-y|$, which gives $|t-t_1| \leq |t-t_1|^{\alpha/2}|x-y|^{(2/\alpha)-1}$. Also, $|t-\tau|^{\alpha/2} + |x-y| < 2|x-y|$ and the desired inequality follows from these observations and (22) in the case $z \geq 1$.

If $z \leq 1$, then the Mean Value Theorem together with Lemma 2 gives

$$|\nabla G(x, t-\tau) - \nabla G(x, t_1 - \tau)| \\ \leq C|t-t_1|(t-\tau)^{-\alpha-2} \begin{cases} |x|^{3-n} & \text{if } n \geq 3, \\ |x \log((t-\tau)^{-\alpha}|x|^2)| & \text{if } n = 2. \end{cases}$$

In the case $n = 2$ we use the fact that $z^{1/2} \log z$ is bounded near zero and obtain

$$(23) \quad |\nabla G(x-y, t-\tau) - \nabla G(x-y, t_1-\tau)| \leq C|t-t_1|(t-\tau)^{-(\alpha/2)-2}.$$

Note now that $|t-t_1| \leq C|t-t_1|^{\alpha/2}(t-\tau)^{1-(\alpha/2)}$, which together with (23) and $z^{-\gamma} \geq 1$ implies the inequality (19).

We easily see that the inequality holds also in the case $n \geq 3$. Indeed, the estimate (13) in Lemma 2 gives an estimate

$$(24) \quad |\nabla G(x-y, t-\tau) - \nabla G(x-y, t_1-\tau)| \\ \leq C|t-t_1|(t-\tau)^{-\alpha-2}|x-y|^{-n+3} \\ \leq C|t-t_1|^{\alpha/2}(t-\tau)^{-(3\alpha/2)-1}|x-y|^{-n+3}$$

and the claim follows, since $(t-\tau)^{-\alpha/2} \leq 2|x-y|^{-1}$. □

4. Properties of the operator J_ε. We need a proper condition to guarantee the weak type $(1,1)$ estimate [**14**, Theorem 4.1]. The condition is given by the next result [**14**, Theorems 4.1 and 5.1].

Proposition 1. *Define $U_r := U_r(x,t) = \{(y,\tau) \in \Sigma_T \mid |x-y| + |t-\tau|^{\alpha/2} < r\}$. If $(x_1, t_1) \in U_r$, then the integral*

$$\int_{\Sigma_\infty \setminus U_{2r}} |\nabla G(x-y, t-\tau) - \nabla G(x_1 - y, t_1 - \tau)| \, \mathrm{d}\sigma(y) \, \mathrm{d}\tau$$

is uniformly bounded independently of r.

Proof. It is enough to prove the theorem in the case $\Gamma = \mathbf{R}^{n-1} \times \{0\}$ since for all $x, y \in \Gamma$, $x = (x', \varphi(x'))$ and $y = (y', \varphi(y'))$ we have

$$|x' - y'| \le |x - y| \le \sqrt{1 + \sup_{x' \in \mathbf{R}^{n-1}} |\nabla \varphi(x')|}\, |x' - y'|.$$

Denote $M = \sup_{x \in \mathbf{R}^{n-1}} |\nabla \varphi(x)|$. Then

$$\mathrm{d}x' \le \mathrm{d}\sigma(x) = \sqrt{1 + |\nabla \varphi(x')|^2}\, r dx' \le \sqrt{1 + M^2}\, \mathrm{d}x'.$$

Since the technique is the same in any dimension, we consider only the case $n = 2$. Denote the integral by I. We divide the integral into two parts I_1 and I_2, where the domains of the integrations are

$$(\Sigma_\infty \setminus U_{2r}) \cap \{(y, \tau) \in \Sigma_\infty \mid |x - y|(t - \tau)^{-\alpha/2} \ge 1\}$$

for I_1 and

$$(\Sigma_\infty \setminus U_{2r}) \cap \{(y, \tau) \in \Sigma_\infty \mid |x - y|(t - \tau)^{-\alpha/2} \le 1\}$$

for I_2. Then substituting $\tau \leftrightarrow t - |x-y|^{2/\alpha}\mu$, we have $0 < \mu = z^{-2/\alpha} < 1$, and from Lemma 3 we get

$$\begin{aligned} |I_1| &\le Cr \int_0^1 \int_{|x-y|>(2r/(1+\mu^{\alpha/2}))} (1 + \mu^{\alpha/2})^{-2-(2/\alpha)} |x - y|^{-2}\, \mathrm{d}\sigma(y)\, \mathrm{d}\mu \\ &\le Cr \int_0^1 \int_{(2r/(1+\mu^{\alpha/2}))}^{\infty} (1 + \mu^{\alpha/2})^{-2-(2/\alpha)} R^{-2}\, \mathrm{d}R\, \mathrm{d}\mu \\ &\le C \int_0^1 (1 + \mu)^{-1-(2/\alpha)}\, \mathrm{d}\mu \le C. \end{aligned}$$

For I_2 we use the inequality $|x - y| + (t - \tau)^{\alpha/2} < 2(t - \tau)^{\alpha/2}$ and then make the same substitution as above, which again by Lemma 3 yields an estimate

$$\begin{aligned} |I_2| &\le Cr \int_1^\infty \int_{|x-y|>(2r/(1+\mu^{\alpha/2}))} (1 + \mu^{\alpha/2})^{2\gamma-2-(2/\alpha)} |x - y|^{-2}\, \mathrm{d}\sigma(y)\, \mathrm{d}\mu \\ &\le Cr \int_1^\infty \int_{(2r/(1+\mu^{\alpha/2}))}^{\infty} (1 + \mu^{\alpha/2})^{2\gamma-2-(2/\alpha)} R^{-2}\, \mathrm{d}R\, \mathrm{d}\mu \\ &\le C \int_1^\infty \mu^{\alpha\gamma-(\alpha/2)-1}\, \mathrm{d}\mu. \end{aligned}$$

Since this is true for any $\gamma > 0$, we can choose $\gamma < (1/2)$ and the proof is complete. $\square$

Now we are ready to prove Theorem 1.

Proof of Theorem 1. The technique we have used can be found in [**6**, Theorem 1.1] and [**14**, Theorems 4.1 and 5.1].

(i) Taking the Laplace transform in the time variable and using a convolution property, we obtain

$$\mathcal{L}(J_\varepsilon\psi)(x,i\eta) = \int_\Gamma \int_\varepsilon^\infty \partial_{n(y)} G(x-y,t)\exp(-\mathrm{i}\eta t)\,\mathrm{d}t(\mathcal{L}\psi)(y,\mathrm{i}\eta)\,\mathrm{d}\sigma(y). \tag{25}$$

Making the change of variables $t \leftrightarrow |x-y|^{2/\alpha}\tau$, we have

$$\mathcal{L}(J_\varepsilon\psi)(x,i\eta) = \int_\Gamma \frac{\langle x-y, n_y\rangle}{|x-y|^n} H(\varepsilon/|x-y|^{2/\alpha}, |x-y|^{2/\alpha}\eta)(\mathcal{L}\psi)(y,\mathrm{i}\eta)\,\mathrm{d}\sigma(y), \tag{26}$$

where

$$H(\varepsilon,\eta) = \frac{1}{\pi^{n/2}}\int_\varepsilon^\infty \exp(-\mathrm{i}\eta\tau)\tau^{\alpha-1}\left\{ nH_{(1)}\left(\frac{1}{4}\tau^{-\alpha}\right) + 2H_{(2)}\left(\frac{1}{4}\tau^{-\alpha}\right)\right\}\mathrm{d}\tau. \tag{27}$$

We split the integral over Γ into two parts depending on whether $|x-y| > \varepsilon^{\alpha/2}$ or $|x-y| < \varepsilon^{\alpha/2}$. Denote the resulting integrals by $I_1(x,\eta)$ and $I_2(x,\eta)$. Let us first study the integral I_2. Now, because $\varepsilon/|x-y|^{2/\alpha} > 1$, the lower bound in the integral of H in (26) is greater than 1, so in the case $n=3$ we have the following bound for $\varepsilon > 1$

$$|H(\varepsilon,\eta)| \le C\varepsilon^{-\alpha} \tag{28}$$

because the behavior of $nH_{(1)}((1/4)\tau^{-\alpha}) + 2H_{(2)}((1/4)\tau^{-\alpha})$ is of order $\tau^{-2\alpha}$. The other cases for n's are treated similarly. From (28) we have

$$\begin{aligned}|I_2(x,\eta)| &\le C\varepsilon^{-\alpha}\int_{|x-y|<\varepsilon^{\alpha/2}} |(\mathcal{L}\psi)(y,\mathrm{i}\eta)\,\mathrm{d}\sigma(y) \\ &\le CM_\Gamma(\mathcal{L}\psi)(\cdot,\eta)(x),\end{aligned} \tag{29}$$

where M_Γ denotes the Hardy-Littlewood maximal operator on Γ, i.e.,

$$M_\Gamma(g)(x) = \sup_{r>0} r^{-(n-1)} \int_{|x-y|<r} |g(y)| \, \mathrm{d}\sigma(y). \tag{30}$$

For the integral I_1 we use a cut-off function $\varphi \in \mathcal{C}_0^\infty(\mathbf{R})$ such that $\varphi(t) \equiv 1$ for $|t| \leq 1$ and $\varphi(t) \equiv 0$ for $|t| > 2$. Set $H(\eta) := H(0,\eta)$ and, to shorten the formulas below, we denote $\widetilde{\varepsilon} = \varepsilon/|x-y|^{2/\alpha}$ and $\widetilde{\eta} = |x-y|^{2/\alpha}\eta$. It is clear from the asymptotics of the Fox H-function that $H(\eta)$ is finite. We divide the integral I_1 into four parts as in [**5**]:

(31)

$$\begin{aligned} I_1(x,\eta) = & \int\limits_{|x-y|>\varepsilon^{\alpha/2}} \frac{\langle x-y, n_y\rangle}{|x-y|^n} \Big\{ H(\widetilde{\varepsilon},\widetilde{\eta}) - H(\widetilde{\eta}) \Big\} (\mathcal{L}\psi)(x,\mathrm{i}\eta)\,\mathrm{d}\sigma(y) \\ & + \int\limits_{|x-y|>\varepsilon^{\alpha/2}} \frac{\langle x-y, n_y\rangle}{|x-y|^n} \big(H(\widetilde{\eta}) - H(0)\varphi(\widetilde{\eta}) \big) \mathcal{L}\psi)(x,\mathrm{i}\eta)\,\mathrm{d}\sigma(y) \\ & + H(0)\cdot \int\limits_{\substack{|x-y|>\varepsilon^{\alpha/2} \\ |x-y|>1/|\eta|^{\alpha/2}}} \frac{\langle x-y, n_y\rangle}{|x-y|^n} \varphi(\widetilde{\eta})(\mathcal{L}\psi)(y,\eta)\,\mathrm{d}\sigma(y) \\ & + H(0)\cdot \int\limits_{1/|\eta|^{\alpha/2}\geq|x-y|>\varepsilon^{\alpha/2}} \frac{\langle x-y, n_y\rangle}{|x-y|^n} (\mathcal{L}\psi)(y,\eta)\,\mathrm{d}\sigma(y) \\ = {} & I_{11} + \cdots . \end{aligned}$$

In the first integral I_{11} we proceed as in the proof of Lemma 2 and choose $0 < \delta_0 < \delta := 4^{1/(\alpha-2)}\alpha^{2/(2-\alpha)}(2-\alpha)$. Then we have an estimate

$$\begin{aligned} I_{12}(x,\eta) \leq {} & C \int_{|x-y|>\varepsilon^{\alpha/2}} |x-y|^{-n+1} \\ & \times \int_0^{\varepsilon/|x-y|^{2/\alpha}} \tau^{-(\alpha n/2)-1} \exp\{-\delta\tau^{-\alpha}/(2-\alpha)\}\,\mathrm{d}\tau \\ & \qquad \times |(\mathcal{L}\psi)(y,\eta)|\,\mathrm{d}\sigma(y) \\ \leq {} & C\varepsilon^{-(\alpha n/2)} \int_{|x-y|>\varepsilon^{\alpha/2}} |x-y| \exp\{-\delta_0|x-y|^2/\varepsilon^\alpha\} \\ & \times |(\mathcal{L}\psi)(y,\eta)|\,\mathrm{d}\sigma(y) \int_0^1 \tau^{-1} \exp\{-(\delta-\delta_0)\tau^{-2/(2-\alpha)}\}\,\mathrm{d}\tau \end{aligned}$$

$$\leq C\varepsilon^{-(\alpha(n-1))/2}\int_\Gamma \frac{|x-y|}{\varepsilon^{\alpha/2}}\exp\{-\delta_0|x-y|^2/\varepsilon^\alpha\} \times |(\mathcal{L}\psi)(y,\eta)|\,\mathrm{d}\sigma(y).$$

The last integral is of convolution type $(\varphi_{\varepsilon^{\alpha/2}} * \mathcal{L}\psi)(x)$, where φ is integrable and $\varphi_\varepsilon(x) = \varepsilon^{n-1}\varphi(x/\varepsilon)$. Therefore we can use the same argument as in [**6**] to say that I_{11} is bounded by $CM_\Gamma((\mathcal{L}\psi)(\cdot,\eta))(x)$ [**16**, Theorem 2 (a), pages 62–63]. Now, it follows from the asymptotic behavior of the Fox H-functions that H is the Fourier-transform of an integrable function and $\eta^\mu(\mathcal{F}^{-1}H)(\eta)$ is integrable for any $0 < \mu < \alpha$. Therefore H is Hölder-continuous with exponent μ. This implies that the second integral is bounded by

$$C\mathrm{diam}\,(\Gamma)^{2\mu((1/\alpha)-1)}|\eta|^{(n-1)/2} \times \int_\Gamma \left(|\eta||x-y|^2\right)^{\mu-(n-1/2)}|(\mathcal{L}\psi)(y,\eta)|\,\mathrm{d}\sigma(y), \tag{32}$$

where $0 < \mu < \alpha$. Since this is valid for any η, we can choose $\eta = 1/\varepsilon$ and use the same argument as before [**16**, Theorem 2 (a)] to have a bound $CM_\Gamma((\mathcal{L}\psi)(\cdot,\eta))(x)$. The same argument applies also for the third integral. Finally, the last integral is bounded by $C\widetilde{K}((\mathcal{L}\psi)(\cdot,\tau))(x)$, where

$$\widetilde{K}(g)(x) = \sup_{\varepsilon>0}\left|\int_{|x-y|>\varepsilon}\frac{\langle x-y,n_y\rangle}{|x-y|^n}g(y)\,\mathrm{d}\sigma(y)\right|. \tag{33}$$

The Hardy-Littlewood maximal function is bounded in $L^p(\Gamma)$ and boundedness of $\widetilde{K}$ in $L^2(\Gamma)$ follows from [**3**, Theorem 9, page 382] and [**5**]. These results, together with Parseval's theorem, imply that J_ε is bounded on $L^2(\Sigma_\infty)$ with a norm independent on ε. The fact that J_ε is of weak type $(1,1)$ follows from Proposition 1 [**14**, Theorem 4.1]. Then the Marcinkiewitz interpolation theorem and the duality argument finishes the proof of (i).

(ii) Next, we are going to show that $\sup_{\varepsilon>0}|J_\varepsilon\psi|$ is bounded in $L^p(\Sigma_\infty)$. To show that, we take ψ to be a simple function with compact support and start with the following term

$$\int_0^t\int_\Gamma\Big|\partial_{n(y)}G(x-y,t-\tau)\big\{\chi_{t-\tau>\varepsilon}(\tau) - \chi_{|x-y|^{2/\alpha}+(t-\tau)>\varepsilon}(\tau)\big\}\psi(y,\tau)\Big|\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau. \tag{34}$$

Now, the difference of characteristic functions is nonzero if and only if $t-\tau \le \varepsilon$ and $|x-y|^{2/\alpha}+(t-\tau) > \varepsilon$. We consider two cases: (a) $t-\tau \le (\varepsilon/2)$ and (b) $(\varepsilon/2) < t-\tau \le \varepsilon$. In the first case $|x-y|^{2/\alpha} > \varepsilon-(t-\tau) \ge (\varepsilon/2) \ge t-\tau$. It follows that $|x-y|^2(t-\tau)^{-\alpha} \ge 1$, and we can use Lemma 1. We choose $0<\sigma_0<\sigma$ and estimate the $\exp(-(\sigma-\sigma_0)\cdot)$-term in Lemma 1 from above by $C(|x-y|^2(t-\tau)^{-\alpha})^{-(\alpha n+2)/(2\alpha)}$, which implies the upper bound

$$C\frac{2}{\varepsilon}\int_{t-\varepsilon/2}^{t}\int_{|x-y|>(\varepsilon/2)^{\alpha/2}}|x-y|^{1-n} \times \exp\{-\sigma_0 z^{1/(2-\alpha)}\}|\psi(y,\tau)|\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau. \tag{35}$$

Further, this can be bounded by

$$\frac{C}{\varepsilon}\int_{t-\varepsilon/2}^{t}\left(\frac{\varepsilon}{2}\right)^{(\alpha/2)(1-n)} \times\int_{|x-y|>(\varepsilon/2)^{\alpha/2}}\exp\left\{-\sigma_0\left(\frac{2^{\alpha/2}|x-y|}{\varepsilon^{\alpha/2}}\right)^{2/(2-\alpha)}\right\} |\psi(y,\tau)|\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau, \tag{36}$$

which, by the same argument as before, is bounded by $C(M_1(M_\Gamma\psi))(x,t)$, where M_1 is a one-dimensional Hardy-Littlewood maximal function and M_Γ is as before. This shows that this term is bounded in $L^p(\Sigma_\infty)$. In case (b) we have to consider two more subcases depending on whether $z=|x-y|^2(t-\tau)^{-\alpha}\le 1$ or $z\ge 1$. If $z\ge 1$, we have from Lemma 1 and formula (14) a bound

$$C\frac{2\sqrt{2}^{\alpha}}{\varepsilon}\int_{t-\varepsilon}^{t-\varepsilon/2}\varepsilon^{-(\alpha(n-1))/2} \times\int_{\Gamma}\frac{|x-y|}{\varepsilon^{\alpha/2}}\exp\left\{-\sigma 2^{\alpha}\frac{|x-y|^2}{\varepsilon^{\alpha}}\right\}|\psi(y,\tau)|\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau, \tag{37}$$

which is bounded by $C(M_1(M_\Gamma\psi))(x,t)$ and therefore bounded in $L^p(\Sigma_\infty)$ [**16**, Theorem 2 (a), pages 62–63]. If $z\le 1$, then $|x-y|\le (t-\tau)^{\alpha/2}\le\varepsilon^{\alpha/2}$ and from formula (11) in Lemma 1 in the case $n=3$ we have a majorant

$$C\frac{2}{\varepsilon}\int_{t-\varepsilon}^{t-\varepsilon/2}\varepsilon^{-\alpha}\int_{|x-y|<\varepsilon^{\alpha/2}}|\psi(y,\tau)|\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau, \tag{38}$$

which is bounded by $C(M_1(M_\Gamma\psi))(x,t)$. The estimates (35), (37) and (38) imply that (34) is bounded in $L^p(\Sigma_\infty)$.

Finally, we are left to show the boundedness of

$$
\begin{aligned}
(39)\quad &\sup_{\varepsilon>0}|\widetilde{J}_\varepsilon(\psi)(x,t)| \\
&= \sup_{\varepsilon>0}\left|\int_0^t\int_\Gamma \partial_{n(y)}G(x-y,t-\tau)\chi_{|x-y|^{2/\alpha}+(t-\tau)>\varepsilon}(y,\tau)\psi(y,\tau)\mathrm{d}\sigma(y)\mathrm{d}\tau\right|.
\end{aligned}
$$

We define the neighborhood U_ε of 0 in $\mathbf{R}^n\times\mathbf{R}$ by $U_\varepsilon=\{(x,t)\in\mathbf{R}^n\times\mathbf{R}\mid |x|^{2/\alpha}+|t|<\varepsilon\}$ and denote $U_\varepsilon(x,t)=\{(y,\tau)|\,(y-x,\tau-t)\in U_\varepsilon\}$. Then $\{U_\varepsilon,\varepsilon>0\}$ is a Vitali family [**14**, Definition 2.1]. In particular, we note that if $(x,t),(y,\tau)\in U_\varepsilon$ then

$$
\begin{aligned}
|x-y|^{2/\alpha}+|t-\tau| &\le 2^{2/\alpha-1}(|x|^{2/\alpha}+|y|^{2/\alpha})+|t|+|\tau| \\
&\le 2^{2/\alpha-1}(|x|^{2/\alpha}+|t|+|y|^{2/\alpha}+|\tau|)\le 2^{2/\alpha}\varepsilon,
\end{aligned}
$$

so we have $U_\varepsilon-U_\varepsilon\subset U_{\Phi(\varepsilon)}$ for $\Phi(t)=2^{2/\alpha}t$. On the boundary Γ we have (on local coordinates) $y=(y',\varphi(y'))$, where φ is a compactly supported Lipschitz-function with $\varphi(0)=0$ and $y'\in\mathbf{R}^{n-1}$. Therefore $|y|\sim|y'|$ and we have to integrate in the domain $\mathbf{R}^{n-1}\times\mathbf{R}_+$.

We split the integral in (39) into three parts. For that, recall the definition of the operator

$$
(J\psi)(x,t)=\lim_{\varepsilon\downarrow 0}(J_\varepsilon\psi)(x,t).
$$

The fact that the operator J is well-defined is proved later when we prove pointwise convergence. Choose $(x_1,t_1)\in U_{\Phi^{-1}(\varepsilon)}(x,t)\cap\Sigma_\infty$. Then we write $\widetilde{J}_\varepsilon$ as follows

$$
\begin{aligned}
(\widetilde{J}_\varepsilon\psi)(x,t) &= (J\psi)(x_1,t_1)-(J\psi\chi_{U_\varepsilon(x,t)})(x_1,t_1) \\
(40)\qquad &\quad+\int_0^t\int_\Gamma \partial_{n(y)}(G(x-y,t-\tau) \\
&\quad -G(x_1-y,t_1-\tau))\psi(y,\tau)\chi_{U_\varepsilon^c}\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau
\end{aligned}
$$

and average the expression for $\widetilde{J}_\varepsilon\psi$ in the variables (x_1,t_1) over the set $U_{\Phi^{-1}(\varepsilon)}(x,t)\cap\Sigma_\infty$, and take the supremum over all $\beta=\Phi^{-1}(\varepsilon)$. Then we have

$$
(41)\qquad |(\widetilde{J}_\varepsilon\psi)(x,t)|\le(I_1\psi)(x,t)+(I_2\psi)(x,t),
$$

where

(42)

$$(I_1\psi)(x,t) = \sup_\beta \frac{1}{|U_\beta|}\int_{U_\beta(x,t)\cap\Sigma_\infty} |(J\psi)(x_1,t_1)|\,\mathrm{d}\sigma(x_1)\,\mathrm{d}t_1 + \sup_\beta \frac{1}{|U_\beta|}\int_{U_\beta(x,t)\cap\Sigma_\infty} |(J\psi\chi_{U_\epsilon(x,t)})(x_1,t_1)|\,\mathrm{d}\sigma(x_1)\,\mathrm{d}t_1$$

and

$$(I_2\psi)(x,t) = \sup_\beta \frac{1}{|U_\beta|}\int_{U_\beta(x,t)\cap\Sigma_\infty} \bigg\{ \int_{\Sigma_\infty\setminus U_\varepsilon(x,t)} |\partial_{n(y)}(G(x-y,t-\tau) - G(x_1-y,t_1-\tau))\psi(y,\tau)|\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau \bigg\}\,\mathrm{d}\sigma(x_1)\,\mathrm{d}t_1. \tag{43}$$

By Proposition 1 we have $\|I_2\psi\|_\infty \le C\|\psi\|_\infty$, and I_1 can be estimated as in [**6**, page 182] and [**14**, proof of Theorem 5.1] to get

$$(I_1\psi)(x,t) \le C_1 M_1(M_\Gamma(J\psi))(x,t) + C_2 M_1(M_\Gamma(|\psi|^q))(x,t)^{1/q}, \tag{44}$$

where $1<q<p$. The boundedness of $\widetilde{J}$ follows from the proof of [**14**, Theorem 5.1].

(iii) For the pointwise limit we first show that the limit exists for simple functions with compact support. It is enough to prove the existence of a limit

$$\lim_{\varepsilon\downarrow 0}\int_0^{t-\varepsilon}\int_\Gamma \partial_{n(y)}G(x-y,t-\tau)\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau. \tag{45}$$

For this we use the Gauss divergence theorem and the fact that $\partial_{n(y)}G(x-y,t-\tau)$ solves fractional diffusion equation, when $x\neq y$ and $t\neq\tau$, to write it in a form

$$\int_0^{t-\varepsilon}\int_\Omega \Delta_y G(x-y,t-\tau)\,\mathrm{d}y\,\mathrm{d}\tau = \int_0^{t-\varepsilon}\int_\Omega \Delta_x G(x-y,t-\tau)\,\mathrm{d}y\,\mathrm{d}\tau = \int_0^{t-\varepsilon}\int_\Omega (\partial_t^\alpha G(x-y,\cdot))(t-\tau)\,\mathrm{d}y\,\mathrm{d}\tau.$$

Using the asymptotic behavior of $\Delta_y G(x-y,t-\tau)$ we see that it is absolutely integrable over the domain $\Omega\times(0,t-\varepsilon)$ by Lemma 2, so we may change the order of the integration.

Because $G(x,0+)=0$ for any $x\neq 0$, the Caputo and Riemann-Liouville derivatives coincide and the last integral can be written in a form

$$\begin{aligned}(46)\quad \int_\Omega\int_0^{t-\varepsilon} &(D^\alpha_{0+}G(x-y,\cdot))(t-\tau)\,\mathrm{d}y\,\mathrm{d}\tau\\ &=\int_\Omega\int_\varepsilon^t (D^1(J^{1-\alpha}_{0+}G))(\tau)\,\mathrm{d}y\,\mathrm{d}\tau\\ &=\int_\Omega\Big(J^{1-\alpha}_{0+}G(t)-J^{1-\alpha}_{0+}G(\varepsilon)\Big)\,\mathrm{d}y,\end{aligned}$$

where D^α_{0+} is the Riemann-Liouville fractional derivative and $J^{1-\alpha}_{0+}$ is the Riemann-Liouville fractional integral. Denote by E the parametrix of the fractional diffusion. Then $D^{1-\alpha}_{0+}E(t)=G(t)$ [**4**, formula (3.6)]. Moreover,

$$(J^{1-\alpha}_{0+}D^{1-\alpha}_{0+}f)(t)=f(t)-(J^\alpha_{0+}f)(0)\frac{t^{-\alpha}}{\Gamma(1-\alpha)}$$

[**12**, formula (2.108)]. An easy calculation shows that

$$J^\alpha_{0+}H_{(1)}(\omega t^{-\alpha})=\Gamma(\alpha)t^\alpha H^{20}_{12}\left[\omega t^{-\alpha}\,\middle|\,\begin{smallmatrix}(1+\alpha,\alpha)\\ (\frac{n}{2},1)\quad(1,1)\end{smallmatrix}\right].$$

The asymptotic behavior of this function shows that the initial value is 0 and that is why $J^{1-\alpha}_{0+}$ and $D^{1-\alpha}_{0+}$ commute for E. This observation implies that the last integral in formula (46) can be written in a form

$$(47)\qquad \int_\Omega (E(x-y,t)-E(x-y,\varepsilon))\,\mathrm{d}y,$$

which can be dominated by

$$(48)\qquad \int_{\mathbf{R}^n}(E(x-y,t)+E(x-y,\varepsilon))\,\mathrm{d}y.$$

But this integral converges and the value is independent of ε [**4**, formula (4.13)]. This finally shows that the pointwise limit exists for simple functions. In particular, this implies that J is well-defined and, further, Proposition 1 implies that J can be extended by continuity to the whole class $L^p(\Sigma_\infty)$.

The existence of a pointwise limit follows now from [**14**, Theorem 5.2]. Also, the limit in $L^p(\Sigma_\infty)$ exists by Lebesgue's dominated convergence theorem. Hence, in particular, J is a bounded linear operator in $L^p(\Sigma_\infty)$. $\square$

5. Proof of Theorem 2. The technique used for the proof is identical to the one used in [**6**]. By using a partition of unity we may assume that $\operatorname{supp}\psi \subset B(y_0,\delta)\cap\Gamma$, where $\delta > 0$ and $y_0 \in \Gamma$. To simplify the notation, we also assume that $\psi \geq 0$. Then, if $x_0 \notin \overline{B(y_0,2\delta)} \cap \Gamma$ and $|x - x_0| < \delta$, we have $|x-y| \geq \delta$. From (14) we get the following bound for the kernel of the double layer potential

$$
\begin{aligned}
(49)\quad & |\partial_{n(y)} G(x-y,t-\tau)| \\
&\qquad \leq C\frac{(t-\tau)^{\alpha-1}}{|x-y|^{n+1}}\{nH_{(1)}(z) + 2H_{(2)}(z)\} \\
&\qquad = C|x-y|^{n-1+(2/\alpha)} z^{-1+(1/\alpha)}\{nH_{(1)}(z) + 2H_{(2)}(z)\}.
\end{aligned}
$$

The asymptotic behavior of Fox H-functions guarantees that the function

$$z \mapsto z^{-1+(1/\alpha)}\{nH_{(1)}(z) + 2H_{(2)}(z)\}$$

is uniformly bounded in $\mathbf{R}_+$ which implies an estimate

$$
(50)\qquad |(D\psi)(x,t)| \leq C\delta^{1-n-(2/\alpha)}\|\psi\|_{L^1(\Sigma_T)} \leq C_\delta\|\psi\|_{L^p(\Sigma_T)},
$$

where we used Hölder's inequality. Hence $\widetilde{D\psi}(x_0,t) \leq C_\delta\|\psi\|_{L^p(\Sigma_T)}$. Now, since time is finite and Γ is bounded, the desired result follows after integration.

We are left with the case $x_0 \in \overline{B(y_0,2\delta)}\cap\Gamma$. We write the kernel of the double layer potential in a form $\partial_{n(y)}G(x-y,t) = \langle x-y, n(y)\rangle \widetilde{G}(x-y,t)$

and split D as follows

$$\begin{aligned}(D\psi)(x,t) &= J_{|x-x_0|^{2/\alpha}}(x_0,t) \\ &\quad + \int_0^t \int_\Gamma \langle x - x_0, n(y)\rangle \widetilde{G}(x-y,t-\tau)\psi(y,\tau)\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau \\ &\quad + \int_{t-|x-x_0|^{2/\alpha}}^t \int_\Gamma \langle x_0 - y, n(y)\rangle \\ &\qquad\qquad \times \widetilde{G}(x-y,t-\tau)\psi(y,\tau)\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau \\ &\quad + \int_0^{t-|x-x_0|^{2/\alpha}} \int_\Gamma \langle x_0 - y, n(y)\rangle \big\{\widetilde{G}(x-y,t-\tau) \\ &\qquad\qquad - \widetilde{G}(x_0-y,t-\tau)\big\}\psi(y,\tau)\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau \\ &= J_{|x-x_0|^{2/\alpha}}(x_0,t) + I_1 + I_2 + I_3.\end{aligned}$$

We split the integral I_1 into two parts and have an estimate

$$\begin{aligned}I_1 &\le |x-x_0| \int_0^t \int_{|x_0-y|\le 2|x-x_0|} |\widetilde{G}(x-y,t-\tau)\psi(y,\tau)|\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau \\ &\quad + |x-x_0| \int_0^t \int_{|x_0-y|> 2|x-x_0|} |\widetilde{G}(x-y,t-\tau)\psi(y,\tau)|\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau.\end{aligned} \tag{51}$$

As usual, we have to treat the cases $z \le 1$ and $z \ge 1$ separately. If $z \ge 1$, we have a bound

$$|\widetilde{G}(x-y,t-\tau)| \le C(t-\tau)^{-(\alpha n/2)-1}\exp\{-\sigma z^{1/(2-\alpha)}\}. \tag{52}$$

For the small arguments we have to treat different n's separately. As an example, we consider $n = 2$. In this case we have a bound

$$|\widetilde{G}(x-y,t-\tau)| \le C(t-\tau)^{-\alpha-1}|\log z|. \tag{53}$$

It is enough to consider the case $\Gamma = \mathbf{R}^{n-1} \times \{0\}$. Also, noting that $|x-y| > (1/2)|x_0 - y|$, we have the following bound for the second integral in (51),

$$\int_0^t (t-\tau)^{-(\alpha/2)-1}\varphi_{(t-\tau)^{\alpha/2}} * \psi(\cdot,\tau)(x_0)\,\mathrm{d}\tau, \tag{54}$$

where $\varphi_\varepsilon(x) = 1/(\varepsilon^{n-1})\varphi(x/\varepsilon)$, and a decreasing integrable majorant is

$$\varphi(x) = \begin{cases} C_1(1+|\log|x||) & \text{if } |x| \le 1 \\ C_2 \exp\{-\sigma_0 |x|^{2/(2-\alpha)}\} & \text{if } |x| \ge 1 \end{cases} \tag{55}$$

for some positive constants C_1, C_2 and $\sigma_0 < \sigma$. Then, using the result [**16**, Theorem 2 (a), pages 62–63], we have

$$\begin{aligned} r\int_0^{t-r^{2/\alpha}} \int_{|x_0-y|>r} &|\widetilde{G}(x-y,t-\tau)\psi(y,\tau)|\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau \\ &\le Cr\int_0^{t-r^{2/\alpha}} (t-\tau)^{-(\alpha/2)-1}(M_\Gamma\psi(\cdot,\tau))(x_0)\,\mathrm{d}\tau \\ &\le CM_1(M_\Gamma\psi)(x_0,t). \end{aligned} \tag{56}$$

When $t-r^{2/\alpha} < \tau < t$ and $|x_0-y| > r$, we have $z = |x-y|^2/(t-\tau)^\alpha \ge 1/4$. Using the asymptotic behavior of the fox H-function for large arguments, we have in this case an estimate

$$\begin{aligned} r\int_{t-r^{2/\alpha}}^{t} &\int_{|x_0-y|>r} |\widetilde{G}(x-y,t-\tau)\psi(y,\tau)|\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau \\ &\le \frac{C}{r^{2/\alpha}}\int_{t-r^{2/\alpha}}^{t} r^{1+2/\alpha}\int_{|x_0-y|>r} \frac{\psi(y,\tau)}{|x_0-y|^{n+(2/\alpha)}}\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau \\ &\le CM_1(M_\Gamma\psi)(x_0,t). \end{aligned} \tag{57}$$

For the first term on the right hand side of (51) we use the inequality $|x-y| > (\beta/2)|x-x_0|$, which holds for δ sufficiently small, and get an upper bound (in the case $n = 2$)

$$\varphi_{|x-x_0|^{2/\alpha}} * (M_\Gamma\psi)(x_0,\cdot)(t) \le CM_1(M_\Gamma\psi)(x_0,t), \tag{58}$$

where we have used [**16**, Theorem 2, pages 62–63] and $\varphi_\varepsilon(t) = (1/\varepsilon)\varphi(t/\varepsilon)$ with an integrable majorant

$$\varphi(t) = Ct^{-\alpha-1}\begin{cases} 1+|\log t| & \text{if } t > 1, \\ \exp\{-\sigma t^{-\alpha/(2-\alpha)}\} & \text{if } 0 < t \le 1, \\ 0 & \text{if } t < 0 \end{cases}$$

for some positive constants C, σ.

Similarly, as above, the spatial integral in I_2 is divided into two parts depending upon whether $|x_0 - y| > 2|x - x_0|$ or $|x_0 - y| \leq 2|x - x_0|$. If $|x_0 - y| \leq 2|x - x_0|$, we obtain

$$\begin{aligned}\int_{t-|x-x_0|^{2/\alpha}}^{t} \int_{|x_0-y|<2|x-x_0|} &|\langle x_0 - y, n(y)\rangle \\ &\times \widetilde{G}(x-y, t-\tau)\psi(y,\tau)|\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau \\ \leq 2|x-x_0| \int_{t-|x-x_0|^{2/\alpha}}^{t} &|\widetilde{G}(x-y,t-\tau)\psi(y,\tau)|\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau,\end{aligned}$$

which can be estimated similarly as for I_1. On the other hand, if $|x_0 - y| > 2|x - x_0|$, then $z \geq 1$, and we get an upper bound

$$\begin{aligned}&\int_{t-|x-x_0|^{2/\alpha}}^{t} \int_{|x-y|>|x-x_0|} |x-y| \\ &\qquad\qquad \times |\widetilde{G}(x-y,t-\tau)\psi(y,\tau)|\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau \\ &\quad \leq \frac{C}{r^{2/\alpha}} \int_{t-r^{2/\alpha}}^{t} r^{2/\alpha} \int_{|x_0-y|>r} \frac{\psi(y,\tau)}{|x_0-y|^{n+(2/\alpha)-1}}\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau \\ &\quad \leq CM_1(M_\Gamma\psi)(x_0,t).\end{aligned}$$

Finally, we use the Mean Value Theorem for the difference in I_3 to get

$$I_3 \leq |x - x_0| \int_0^{t-|x-x_0|^{2/\alpha}} \int_\Gamma |x_0 - y||\nabla\widetilde{G}(\tilde{x} - y, t-\tau)|\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau,$$

where $\widetilde{x}$ is some point between x and x_0. When δ is small enough, then $|\widetilde{x} - y|$ is comparable with $|x_0 - y|$, and we get an upper bound (in the two-dimensional case)

$$\begin{aligned}C|x-x_0| \int_{t-\tau>|x-x_0|^{2/\alpha}} &(t-\tau)^{-(\alpha/2)-1} \\ &\times \int_\Gamma \varphi_{(t-\tau)^{\alpha/2}}(x_0-y)\psi(y,\tau)\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau,\end{aligned}$$

where the least decreasing majorant for φ is integrable. Hence [**16**, Theorem 2 (a), pages 62–63] and the same argument as before leads to the majorant $C(M_1(M_\Gamma\psi))(x_0,t)$.

Putting all this together we have proved that

$$N(D\psi)(x_0,t) \leq C\{\sup_{\varepsilon>0} |(J_\varepsilon\psi)(x_0,t)| + (M_1(M_\Gamma\psi))(x_0,t)\}.$$

Hence $\|N(D\psi)\|_{L^p(\Sigma_T)} \leq C\|\psi\|_{L^p(\Sigma_T)}$.

The pointwise limit may be proved first for smooth functions and then using the first part of the proof and [**5**, Theorem 1.3]. See also [**9**, Theorem 2]. □

Acknowledgments. The author would like to thank the referee's valuable comments for the improvement of this paper.

REFERENCES

1. B.L.J. Braaksma, *Asymptotic expansions and analytic continuations for a class of Barnes integrals*, Compos. Math. **15** (1964), 239–341.

2. R.M. Brown, *The method of layer potentials for the heat equation in Lipschitz cylinders*, Amer. J. Math. **111** (1989), 339–379.

3. R.R. Coifman, A. MacIntosh and Y. Meyer, *L'Intgrale de Cauchy définit un opérateur borné sur L^2 pour les courbes lipschitziennes*, Annals Math. **116** (1982), 361–387.

4. S.D. Eidelman and A.N. Kochubei, *Cauchy problem for fractional diffusion equations*, J. Differential Equations **199** (2004), 211–255.

5. E.B. Fabes, M. Jodeit, Jr. and N.M. Riviére, *Potential techniques for boundary value problems on C^1-domains*, Acta Math. **141** (1978), 165–186.

6. E.B. Fabes and N.M. Riviére, *Dirichlet and Neumann problems for the heat equation in C^1-cylinders*, in Proc. Symposia Pure Math. **35** (1979), 179–196.

7. A. Friedman, *Partial differential equations of parabolic type*, Robert E. Krieger Publishing Company, Inc., Malabar, USA, 1983.

8. L. Hörmander, *The analysis of linear partial differential operators* I, Distribution Theory Fourier Anal., Springer-Verlag, Heidelberg, 2003.

9. J. Kemppainen, *Properties of the single layer potential for the time fractional diffusion equation*, J. Integral Equations Applications, to appear.

10. J. Kemppainen and K. Ruotsalainen, *Boundary integral solution of the time fractional diffusion equation*, Integral Equation Operator Theory **64** (2009), 239–249.

11. A.A. Kilbas and M. Saigo, H-*transforms*: *Theory and applications*, CRC Press, Boca Raton, FL, 2004.

12. I. Podlubny, *Fractional differential equations*, Academic Press, San Diego, California, 1999.

13. A.P. Prudnikov, Y.A. Brychkov and O.I Marichev, *Integrals and series*, vol. 3. *More special functions*, Overseas Publishers Association, Amsterdam, 1990.

14. N.M. Riviére, *Singular integrals and multiplier operators*, Arkiv Matematik **9** (1971), 243–278.

15. S.G. Samko, A.A. Kilbas and O.I. Marichev, *Fractional integrals and derivatives*: *Theory and applications*, Gordon and Breach Science Publishers, Amsterdam, 1993.

16. E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, Princeton, NJ, 1970.

University of Oulu, Finland
Email address: jukemppa@paju.oulu.fi