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SPECTRUM OF A VISCOELASTIC BOUNDARY DAMPING MODEL

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ABSTRACT. The undamped linear wave equation with convolution boundary conditions is considered. Physically, the boundary condition models the interaction of a viscoelastic boundary material with memory and the incident waves. We consider three geometric configurations: an interval, a disc and a cylinder. The main result is that the rate of exponential decay can be arbitrarily low. This is demonstrated by proving the existence of eigenvalues that are arbitrarily close to the imaginary axis. The assumptions on the convolution kernel a are: it is of positive type and its Laplace transform $\widehat{a}(\lambda)$ tends to zero when $\Im\lambda$ goes to ∞ while $\Re\lambda$ stays bounded.

1. Introduction. Consider a model for the evolution of sound in a compressible fluid with viscoelastic surface (cf. [**5**]):

$$\begin{aligned} &p_{tt}(t,x) - \Delta p(t,x) = 0, \quad x \in \Omega, \\ &\frac{\partial p}{\partial n}(t,x) + a \star p_t(t,x) = 0, \quad x \in \partial\Omega, \end{aligned} \tag{1}$$

where $p(t,x) \in \mathbf{R}$ denotes acoustic pressure, $\Omega \subset \mathbf{R}^3$ is a domain with smooth boundary and $n(x)$ the outer normal to $\partial\Omega$ at x. The convolution is $a \star v(t,.) := \int_{-\infty}^{t} a(t-s)v(s,.)\,ds$, a is a given real-valued function on $[0,\infty)$.

One can consider either solutions on the line ($t \in \mathbf{R}$), or solutions on the halfline ($t \in [0,\infty)$) with initial conditions $p(0,.) = p_0(.)$,

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$p_t(0,.) = p_1(.)$. The latter case corresponds to solutions on the line vanishing on $(-\infty, 0)$, and in this case $a \star v(t,.) = \int_0^t a(t-s)v(s,.)\,ds$.

Let p be a solution on the halfline. Taking formally the Laplace transform $\widehat{p}(.,x)$ of $p(.,x)$, we obtain the equation

$$\begin{aligned} &\lambda^2 \widehat{p}(\lambda,x) - \Delta\widehat{p}(\lambda,x) = \lambda p(0,x) + p_t(0,x), \quad x \in \Omega, \\ &\frac{\partial \widehat{p}}{\partial n}(\lambda,x) + \lambda\widehat{a}(\lambda)\widehat{p}(\lambda,x) = \widehat{a}(\lambda)p(0,x), \quad x \in \partial\Omega. \end{aligned} \tag{2}$$

We say that $\lambda \in \mathbf{C}\setminus(-\infty,0]$ is in the spectrum of equation (1), $\lambda \in \mathfrak{S}$, if the following elliptic boundary value problem admits a nontrivial solution $v \in H^1(\Omega)$,

$$\begin{aligned} &\lambda^2 v - \Delta v = 0, \quad x \in \Omega, \\ &\frac{\partial v}{\partial n} + \lambda\widehat{a}(\lambda)v = 0, \quad x \in \partial\Omega. \end{aligned} \tag{3}$$

Under the hypotheses below, the Laplace transform $\widehat{a}(\lambda)$ need not exist for $\lambda \in (-\infty, 0]$. If $\lambda \in \mathfrak{S}$, then also $\overline{\lambda} \in \mathfrak{S}$, since $\overline{\widehat{a}(\lambda)} = \widehat{a}(\overline{\lambda})$ and $\overline{v}$ is the solution for $\overline{\lambda}$.

Note that if $\lambda \in \mathfrak{S}$, then there is a nontrivial solution of (1) on the line of the form $p(t,x) = e^{\lambda t}v(x)$. If $\Re\lambda > 0$, this gives an exponentially growing solution. However, under our hypotheses, there are no such points in the spectrum. Also note, that one could solve equation (1) by Laplace transform, provided there is an inverse Laplace transform formula outside $\mathfrak{S}$. This makes important the location of $\mathfrak{S}$.

Our goal in this paper is to express the closeness of the spectrum to the imaginary axis. In particular, we prove that there exist points of the spectrum with non-positive real part and arbitrarily close to the imaginary axis. This is in some sense (concerning Laplace inversion method and concerning stability of the system) a negative result. We do this in three examples of Ω: the interval, a cylindrical domain, the disc.

As one of the referees pointed out, an argument to exclude exponentially decreasing solutions on the halfline might be the following: if a solution of (1) would decay exponentially with $\widehat{p_t}|_{\partial\Omega} \neq 0$ in a neighborhood of the origin, then from the boundary condition in (1) one would obtain that $\widehat{a}$ is extendible meromorphically to a neighborhood of the origin, which is not the case for $a(t) = t^{-\alpha}$, $\alpha \in (0,1)$. We would like to emphasize that in this article we prove actual existence of solutions on the line with either an arbitrary slow exponential decay rate as $t \to \infty$ (case $\lambda \in \mathfrak{S}, \Re\lambda < 0$) or no decay (case $\lambda \in \mathfrak{S}, \Re\lambda = 0$). In fact, if $\lambda \in \mathfrak{S}$, then $p(t,x) = e^{\lambda t} v(x)$ is a nontrivial solution of (1) on the line. In the parallel article[**4**] we consider solutions on the halfline and prove that they converge to 0 as $t \to \infty$, but without specifying any decay rate. In [**4**] a different method is employed, and other technical assumptions on the kernel a, namely that a is completely monotonic and integrable on $(0,\infty)$, are used. In the recent article [**2**] the same model, including $a(t) = t^{-\alpha} e^{-\varepsilon t}$, $\alpha \in (0,1)$, $\varepsilon > 0$, is treated and polynomial decay rate of solutions on the halfline is proved by yet a different method. The general assumptions on a in [**2**] are structurally different from ours and also from the ones in [**4**].

We will use the following hypotheses:

(H1) The kernel a is a function of positive type; equivalently, $\Re\widehat{a}(\lambda) \geq 0$ for all $\Re\lambda > 0$.

(H2) The Laplace transform of a, $\widehat{a}(\lambda)$, has a holomorphic extension to $\mathbf{C} \setminus (-\infty, 0]$ which satisfies $\lim_{|\lambda|\to\infty,\, \lambda\notin(-\infty,0]} \widehat{a}(\lambda) = 0$.

A weaker version of (H2) is

(H2′) There exists an $\varepsilon_2 > 0$ such that $\lim_{|\Im\lambda|\to\infty, |\Re\lambda|<\varepsilon_2} \widehat{a}(\lambda) = 0$.

(H3) There exist constants $\varepsilon_1 > 0$ and $M_1 > 0$ such that

$$\begin{aligned} &\arg\widehat{a}(\lambda) < -\varepsilon_1 \quad \text{provided } \Im\lambda > 0,\ \Re\lambda \leq 0 \text{ and } |\lambda| \geq M_1, \\ &\arg\widehat{a}(\lambda) > \varepsilon_1 \quad \text{provided } \Im\lambda < 0,\ \Re\lambda \leq 0 \text{ and } |\lambda| \geq M_1. \end{aligned}$$

Actually, we assume only (H1) and the weaker version (H2′) for the results about existence of points of $\mathfrak{S}$ near to the imaginary axis (first part of Theorem 3.1, Theorem 4.3 and Theorem 5.2); whereas (H1), the stronger version (H2) and (H3) are needed for statements about the location of $\mathfrak{S}$ (second part of Theorem 3.1, Theorem 4.2.)

An example of a which satisfies all the hypotheses is $a(t) = t^{-\alpha}$ with $\alpha \in (0,1)$, since $\widehat{a}(\lambda) = \Gamma(1-\alpha)\lambda^{\alpha-1}$. (N.B. $\arg\widehat{a}(\lambda) = (\alpha - 1)\arg\lambda$, $\Re\widehat{a}(\lambda) = \Gamma(1-\alpha)|\lambda|^{\alpha-1}\cos((\alpha-1)\arg\lambda)$.)

In [**4**] it is assumed that a is completely monotonic and $a \in L^1(0,\infty)$. As Lemma 2.3 below shows, these assumptions imply our hypotheses

(H1) and (H2′) and thus there is no exponential decay of solutions of (1), neither under the assumptions (H1), (H2′) nor under the assumptions of [**4**]. In neither case do we get any decay rates.

The approach of the present paper is based on Rouché's theorem ([**6**]): Let $\Omega \subset \mathbf{C}$ be a domain, $\gamma \subset \Omega$ a closed path such that $\mathrm{Ind}_\gamma(z) = 0$ for all $z \notin \Omega$, and $\mathrm{Ind}_\gamma(z) = 0$ or 1 for all $z \in \Omega \setminus \gamma$. Denote $\Omega_1 := \{z \in \mathbf{C}; \mathrm{Ind}_\gamma(z) = 1\}$. If the functions F and G are holomorphic in Ω and

$$|F(z) - G(z)| < |F(z)|, \quad z \in \gamma,$$

then F and G have the same number of zeros, counted with multiplicity, in Ω_1.

For the sake of definiteness, in this paper we consider the branch of the square root with negative real part, and the branch of argument $\arg z \in (-\pi, \pi)$.

2. Preliminaries. The following two lemmas are general results about the location of $\mathfrak{S}$. The third stands to compare the hypotheses of [**4**] to (H2′).

Lemma 2.1. *If a satisfies* (H1), *then* $\mathfrak{S} \subset \{\Re\lambda \le 0\}$.

Proof. Let $\Re\lambda > 0$. We multiply equation (3) by $\overline{\lambda}\overline{v}$, integrate over Ω, use the divergence theorem and the boundary condition to obtain

$$\lambda \int_\Omega |\lambda v|^2 + \overline{\lambda} \int_\Omega |\nabla v|^2 + \widehat{a}(\lambda) \int_{\partial\Omega} |\lambda v|^2 = 0.$$

By taking the real part and using that a is of positive type, we obtain that necessarily $v = 0$. □

Lemma 2.2. *If $\Re\widehat{a}(i\varrho) \ne 0$ for some $\varrho \in \mathbf{R} \setminus \{0\}$, then $i\varrho \notin \mathfrak{S}$.*

Proof. Suppose that $\lambda = i\varrho \in \mathfrak{S} \cap i\mathbf{R}$. Then there exists a $v \in H^1(\Omega)$, $v \not\equiv 0$, such that $-\varrho^2 v - \triangle v = 0$ in Ω, $\partial v/\partial n + i\varrho\widehat{a}(i\varrho)v = 0$ on $\partial\Omega$. Multiplying the first equation by $\overline{v}$, using the divergence theorem and the second equation, we obtain

$$-\varrho^2 \int_\Omega |v|^2 + \int_\Omega |\nabla v|^2 + i\varrho\widehat{a}(i\varrho)\int_{\partial\Omega} |v|^2 = 0.$$

By taking the imaginary part, we have

$$\varrho\Re\widehat{a}(i\varrho)\int_{\partial\Omega} |v|^2 = 0.$$

Hence, if $\varrho \neq 0$ and $\Re\widehat{a}(i\varrho) \neq 0$ we obtain necessarily that $v = 0$ on $\partial\Omega$, and from the boundary condition also that $\partial v/\partial n = 0$ on $\partial\Omega$. Therefore, v can be extended (by 0 outside Ω) to a function $v_1 \in H^1(\mathbf{R}^3)$ which satisfies $\triangle v_1 = -\varrho^2 v_1$ on $\mathbf{R}^3$. Since the Laplacian on $\mathbf{R}^n$ has an empty point spectrum, this is a contradiction. □

Now let us turn to the assumptions used in [**4**]. The assumption that a is completely monotonic is equivalent to the existence of some nondecreasing function $\nu : [0,\infty) \to [0,\infty)$ such that $a(t) = \int_0^\infty e^{-ts} d\nu(s)$ ([**7**]). This implies that $\widehat{a}(\lambda) = \int_0^\infty \frac{1}{\lambda+s}\, d\nu(s)$; therefore, any completely monotonic a is of positive type, and $\widehat{a}$ admits a holomorphic extension to $\mathbf{C}\backslash(-\infty, 0]$. Moreover, for a completely monotonic a, the assumption $a \in L^1(0,\infty)$ is equivalent to $d\nu(\{0\}) = 0$, $\int_0^\infty \frac{1}{s}\, d\nu(s) < \infty$.

Lemma 2.3. *Suppose that* $a(t) = \int_0^\infty e^{-st}\, d\nu(s)$, $t \geq 0$, *with* $\nu : [0,\infty) \to [0,\infty)$ *nondecreasing,* $\int_0^\infty \frac{1}{s}\, d\nu(s) < \infty$. *Then for any* $\varepsilon > 0$ *one has*

$$\lim_{\substack{|\Im\lambda|\to\infty \\ |\Re\lambda|<\varepsilon}} \widehat{a}(\lambda) = 0.$$

Proof. In order to estimate $|\widehat{a}(\lambda)|$, split the integral into one over $[0,1)$ plus one over $[1,\infty)$. The first one converges to zero as $|\Im\lambda| \to \infty$. The second one can be estimated by showing that $\sup_{s\in[1,\infty)} \frac{s}{|\lambda+s|}$ is finite, with a bound independent on λ, if $|\Im\lambda|$ is large and $|\Re\lambda| < \varepsilon$. □

3. Example on the interval. We consider the one-dimensional form of (1), i.e., $\Omega = (0,1) \subset \mathbf{R}$.

Theorem 3.1. *If a satisfies* (H1) *and* (H2′)*, then there exists a sequence $\lambda_n \in \mathfrak{S}$ such that $\Re\lambda_n \le 0$, $\Re\lambda_n \to 0$, $|\Im\lambda_n| \to \infty$.*

If a also satisfies (H2)*, then for all $\varrho > 0$ (small) there exists an $M > 0$ such that*

$$\mathfrak{S} \subset \{\lambda \in \mathbf{C},\ |\lambda| \le M\} \cup \{\Re\lambda \ge -\varrho\} \cup (-\infty, 0].$$

Proof. The equation to be solved for v reads

$$\begin{aligned} \lambda^2 v - v_{xx} &= 0, \quad x \in (0,1),\\ -v_x(0) + \lambda\widehat{a}(\lambda)v(0) &= 0,\\ v_x(1) + \lambda\widehat{a}(\lambda)v(1) &= 0. \end{aligned}$$

A general solution of the differential equation is $v(x) = c_1 e^{\lambda x} + c_2 e^{-\lambda x}$; the boundary conditions have the form

$$\begin{aligned} (-\lambda + \lambda\widehat{a}(\lambda))c_1 + (\lambda + \lambda\widehat{a}(\lambda))c_2 &= 0\\ (\lambda e^{\lambda} + \lambda\widehat{a}(\lambda)e^{\lambda})c_1 + (-\lambda e^{-\lambda} + \lambda\widehat{a}(\lambda)e^{-\lambda})c_2 &= 0. \end{aligned}$$

A nontrivial solution $(c_1, c_2) \ne (0,0)$ exists if and only if the determinant is zero; this gives

$$\left(\frac{\lambda\widehat{a}(\lambda) - \lambda}{\lambda\widehat{a}(\lambda) + \lambda}\right)^2 = e^{2\lambda}.$$

We restrict us to solving

$$\frac{\widehat{a}(\lambda) - 1}{\widehat{a}(\lambda) + 1} = e^{\lambda}, \quad \text{i.e.} \quad \widehat{a}(\lambda) + \frac{e^{\lambda} + 1}{e^{\lambda} - 1} = 0. \tag{4}$$

We will use Rouché's theorem to compare the number of zeros of $G(\lambda) = \widehat{a}(\lambda) + \frac{e^{\lambda}+1}{e^{\lambda}-1}$ and of $F(\lambda) = \frac{e^{\lambda}+1}{e^{\lambda}-1}$. The function $\frac{e^{\lambda}+1}{e^{\lambda}-1}$ is $2\pi i$-periodic, has zeros at $\widetilde{\lambda}_m = -\pi i + 2m\pi i$, poles at $2m\pi i$, $m \in \mathbf{Z}$. On the circle $\partial B(\widetilde{\lambda}_m, \varepsilon), \varepsilon < \pi$, we have $|\frac{e^{\lambda}+1}{e^{\lambda}-1}| > \delta(\varepsilon) > 0$, independently of m. Choosing $\varepsilon = \varepsilon_m \to 0$ we have

$$|G(\lambda) - F(\lambda)| = |\widehat{a}(\lambda)| < \delta(\varepsilon_{m_n}) < |F(\lambda)| \quad \text{for} \quad \lambda \in \partial B(\widetilde{\lambda}_{m_n}, \varepsilon_{m_n}),$$

where, using (H2′), the subsequence m_n is chosen to satisfy $|\widehat{a}(\lambda)| < \delta(\varepsilon_{m_n})$ for $|\lambda| > 2\pi m_n$. Rouché's theorem says that G has exactly one zero λ_n in $B(\widetilde{\lambda}_{m_n}, \varepsilon_{m_n})$ which, in turn, is a solution of (4).

To prove the second part of the statement, let $\varrho > 0$ be small. Then $|\frac{e^\lambda+1}{e^\lambda-1}| > \frac{1-e^{-\varrho}}{1+e^{-\varrho}}$ for $\Re\lambda < -\varrho$. By (H2), choose M such that $|\widehat{a}(\lambda)| < \frac{1-e^{-\varrho}}{1+e^{-\varrho}}$ for $|\lambda| > M$, so that $\Re\lambda < -\varrho$ along with $|\lambda| > M$ implies $\widehat{a}(\lambda) + \frac{e^\lambda+1}{e^\lambda-1} \neq 0$. □

4. Example on a cylindrical domain. We consider $\Omega = (0, L)\times\Omega_0$, $\Omega_0 \subset \mathbf{R}^2$ such that the operator Δ on $L^2(\Omega_0)$ with Neumann boundary condition has a spectral resolution, i.e., that there exists an orthonormal basis of $L^2(\Omega_0)$ consisting of eigenfunctions. We will consider the case when the viscoelastic damping takes place only on the faces of the body, whereas, on the side, homogeneous Neumann conditions hold:

$$\begin{aligned} p_{tt} - \Delta p = 0 &\quad \text{in } (0, L) \times \Omega_0, \\ \frac{\partial p}{\partial n} + a \star p_t = 0 &\quad \text{on } \{0\} \times \Omega_0, \{L\} \times \Omega_0, \\ \frac{\partial p}{\partial n} = 0 &\quad \text{on } (0, L) \times \partial\Omega_0. \end{aligned}$$

Let ϕ_k, $-\eta_k^2 \leq 0$, $k \in \mathbf{N}$, be the eigenfunctions and eigenvalues of the operator Δ on Ω_0 with Neumann boundary condition, that is, $\Delta\phi_k = -\eta_k^2\phi_k$ in Ω_0, and $\frac{\partial \phi_k}{\partial n} = 0$ on $\partial\Omega_0$.

Lemma 4.1. *Let $\lambda \in \mathbf{C} \setminus (-\infty, 0]$. One has $\lambda \in \mathfrak{S}$ if and only if there exists a $k \in \mathbf{N}$ such that λ is a solution of one of the following two equations*:

$$\lambda\widehat{a}(\lambda) = \frac{1 \pm e^{L\sqrt{\lambda^2+\eta_k^2}}}{1 \mp e^{L\sqrt{\lambda^2+\eta_k^2}}}\sqrt{\lambda^2 + \eta_k^2}. \tag{5}$$

Proof. We have to solve the following equation for v:

$$\begin{aligned} \lambda^2 v - \Delta v = 0, &\quad x \in \Omega, \\ \nabla v \cdot n + \lambda\widehat{a}(\lambda)v = 0 &\quad \text{on } \{0\} \times \Omega_0, \{L\} \times \Omega_0 \\ \nabla v \cdot n = 0 &\quad \text{on } (0, L) \times \partial\Omega_0. \end{aligned}$$

Decompose $v(x_1,x_2,x_3)=\sum_{k=1}^{\infty}\widetilde{v}_k(x_1)\phi_k(x_2,x_3)$. Then the equation for v is equivalent to

$$\begin{aligned} \lambda^2\widetilde{v}_k(x)-\widetilde{v}_k''(x)+\widetilde{v}_k(x)\eta_k^2&=0, \quad x\in(0,L),\\ -\widetilde{v}_k'(0)+\lambda\widehat{a}(\lambda)\widetilde{v}_k(0)&=0,\\ \widetilde{v}_k'(L)+\lambda\widehat{a}(\lambda)\widetilde{v}_k(L)&=0, \quad k\in\mathbf{N}. \end{aligned}$$

A general solution is

$$\widetilde{v}_k(x)=c_{k,1}e^{\sqrt{\lambda^2+\eta_k^2}x}+c_{k,2}e^{-\sqrt{\lambda^2+\eta_k^2}x}.$$

The boundary conditions say

$$\begin{aligned} &\left(-\sqrt{\lambda^2+\eta_k^2}\,e^0+\lambda\widehat{a}(\lambda)e^0\right)c_{k,1}\\ &\qquad\qquad+\left(\sqrt{\lambda^2+\eta_k^2}\,e^0+\lambda\widehat{a}(\lambda)e^0\right)c_{k,2}=0\\ &\left(\sqrt{\lambda^2+\eta_k^2}\,e^{\sqrt{\lambda^2+\eta_k^2}L}+\lambda\widehat{a}(\lambda)e^{\sqrt{\lambda^2+\eta_k^2}L}\right)c_{k,1}\\ &\quad+\left(-\sqrt{\lambda^2+\eta_k^2}\,e^{-\sqrt{\lambda^2+\eta_k^2}L}+\lambda\widehat{a}(\lambda)e^{-\sqrt{\lambda^2+\eta_k^2}L}\right)c_{k,2}=0. \end{aligned}$$

A nontrivial solution exists if and only if the determinant of the system for $(c_{k,1},c_{k,2})$ is zero, i.e.,

$$\left(\frac{\lambda\widehat{a}(\lambda)-\sqrt{\lambda^2+\eta_k^2}}{\lambda\widehat{a}(\lambda)+\sqrt{\lambda^2+\eta_k^2}}\right)^2=e^{2L\sqrt{\lambda^2+\eta_k^2}}.$$

This is equivalent to (5). □

Theorem 4.2. *Assume that a satisfies* (H1), (H2) *and* (H3). *For all $\varrho>0$ (small) there exists $M>0$ such that*

$$\mathfrak{S}\subset\{\lambda\in\mathbf{C},\ |\lambda|\le M\}\cup\{\Re\lambda\ge-\varrho\}\cup(-\infty,0].$$

Proof. As $z\mapsto z\frac{1\pm e^{Lz}}{1\mp e^{Lz}}$ is an even function, both branches of the square root can be used in (5) (with either combination of $+$ and $-$).

For the sake of definiteness we consider the branch with negative real part of the square roots, and $\arg z \in (-\pi, \pi)$.

In the proof we will make use of the following properties of complex numbers, which can be checked by elementary calculations: if $\Re\lambda < 0$ and $\eta > 0$, then $\Re\sqrt{\lambda^2+\eta^2} \le \Re\lambda$ and $\arg\sqrt{\lambda^2+\eta^2} \ge \arg\lambda$ provided $\Im\lambda > 0$, and $\arg\sqrt{\lambda^2+\eta^2} \le \arg\lambda$ provided $\Im\lambda < 0$.

We divide the proof into several steps.

Step 1. Let ε_1 and M_1 be given in (H3). We prove that there exists an $r > 0$ (possibly large) such that if $\Re\sqrt{\lambda^2+\eta_k^2} < -r$ and $|\lambda| > M_1$ then λ cannot be a solution of (5). (N.B. From this follows, since $\Re\sqrt{\lambda^2+\eta_k^2} \le \Re\lambda$, in particular that

$$\Re\lambda < -r, |\lambda| > M_1 \quad \text{implies} \quad \lambda \notin \mathfrak{S}.)$$

In fact, since $\lim_{\Re z\to-\infty} \frac{1\pm e^{Lz}}{1-e^{Lz}} = 1$, we can find an $r > 0$ such that

$$|\arg \frac{1\pm e^{L\sqrt{\lambda^2+\eta_k^2}}}{1\mp e^{L\sqrt{\lambda^2+\eta_k^2}}}| < \frac{\varepsilon_1}{2} \quad \text{provided} \quad \Re\sqrt{\lambda^2+\eta_k^2} < -r.$$

Suppose, by contradiction, that (5) holds. We will compare the arguments of the left-hand side, respectively right-hand side of (5). For $\Im\lambda > 0$ we have $\arg\sqrt{\lambda^2+\eta_k^2} > \arg\lambda$, and (5) yields:

$$\begin{aligned}\arg\lambda - \varepsilon_1 &> \arg\lambda + \arg\widehat{a}(\lambda) = \arg(\lambda\widehat{a}(\lambda))\\ &= \arg\frac{1\pm e^{L\sqrt{\lambda^2+\eta_k^2}}}{1\mp e^{L\sqrt{\lambda^2+\eta_k^2}}}\sqrt{\lambda^2+\eta_k^2}\\ &= \arg\frac{1\pm e^{L\sqrt{\lambda^2+\eta_k^2}}}{1\mp e^{L\sqrt{\lambda^2+\eta_k^2}}} + \arg\sqrt{\lambda^2+\eta_k^2} > -\frac{\varepsilon_1}{2} + \arg\lambda,\end{aligned}$$

which is a contradiction. Similarly, for $\Im\lambda < 0$ we have $\arg\sqrt{\lambda^2+\eta^2} < \arg\lambda$, and therefore (5) implies:

$$\begin{aligned}\arg\lambda + \varepsilon_1 &< \arg\lambda + \arg\widehat{a}(\lambda)\\ &= \arg\frac{1\pm e^{L\sqrt{\lambda^2+\eta_k^2}}}{1\mp e^{L\sqrt{\lambda^2+\eta_k^2}}} + \arg\sqrt{\lambda^2+\eta_k^2} < \frac{\varepsilon_1}{2} + \arg\lambda,\end{aligned}$$

which is a contradiction, too. This proves Step 1.

Step 2. We prove that for any $\varrho \in (0, r)$ fixed there exists a constant $M_2 \in (0,1)$ such that if λ and $\eta > 0$ satisfy

$$\Re\lambda \le -\varrho \quad \text{and} \quad \Re(\sqrt{\lambda^2+\eta^2}) \in [-r, 0],$$

then

$$|\frac{\sqrt{\lambda^2+\eta^2}}{\lambda}| \ge M_2 \quad \text{and} \quad \Re(\sqrt{\lambda^2+\eta^2}) \le -\varrho M_2.$$

We proceed by contradiction and assume that no such M_2 exists. Then there exist sequences λ_n, η_n such that

$$\lim_{n\to\infty} \frac{|\lambda_n^2+\eta_n^2|}{|\lambda_n^2|} = 0.$$

Since $\Re\sqrt{\lambda_n^2+\eta_n^2} \le \Re\lambda_n$, we have actually $\Re\lambda_n \in [-r, -\varrho]$.

If $\lambda_n^2+\eta_n^2$ would go to 0, then its real and imaginary parts would go to 0 too,

$$\begin{aligned} \varrho^2 - \Im^2\lambda_n &\le \Re^2\lambda_n - \Im^2\lambda_n + \eta_n^2 \\ &= \Re(\lambda_n^2+\eta_n^2) \to 0, \\ 2\Im\lambda_n\Re\lambda_n &= \Im(\lambda_n^2+\eta_n^2) \to 0, \end{aligned}$$

but the first line above implies that $|\Im\lambda_n|$ is bounded away from 0, while the second one implies that in contrary $\Im\lambda_n \to 0$, so $\lambda_n^2+\eta_n^2$ cannot go to 0.

Therefore we must have $\lim|\lambda_n^2| = \infty$. Since $|\lambda_n^2| = \Re^2\lambda_n + \Im^2\lambda_n$ and since $\Re\lambda_n$ is bounded, we actually have $\lim|\Im\lambda_n| = \infty$. This yields that $\Re\lambda_n^2 = \Re^2\lambda_n - \Im^2\lambda_n \to -\infty$ and $\frac{|\lambda_n^2|}{-\Re\lambda_n^2} \to 1$.

On the other hand, using that $\Re z^2 = \Re^2 z - \Im^2 z = \Re^2 z - \frac{\Im^2 z^2}{4\Re^2 z}$ first for $z = \sqrt{\lambda_n^2+\eta_n^2}$, then for $z = \lambda_n$ we have

$$\begin{aligned} \Re(\lambda_n^2+\eta_n^2) &= \Re^2\sqrt{\lambda_n^2+\eta_n^2} - \frac{\Im^2(\lambda_n^2+\eta_n^2)}{4\Re^2\sqrt{\lambda_n^2+\eta_n^2}} \\ &\le r^2 - \frac{\Im^2\lambda_n^2}{4r^2} \\ &= r^2 - \frac{1}{4r^2}(\Re^2\lambda_n - \Re\lambda_n^2)4\Re^2\lambda_n \\ &\le r^2 - \frac{\varrho^4}{r^2} + \Re\lambda_n^2. \end{aligned}$$

Therefore,

$$\liminf \frac{|\lambda_n^2+\eta_n^2|}{|\lambda_n^2|} \geq \liminf \frac{-\Re(\lambda_n^2+\eta_n^2)}{|\lambda_n^2|} \geq \liminf \frac{-(r^2-(\varrho^4/r^2))-\Re\lambda_n^2}{|\lambda_n^2|} = 1.$$

This contradiction proves the first part of Step 2.

The second part of Step 2 is a direct consequence of the first part and of $|\arg\sqrt{\lambda^2+\eta^2}| \geq |\arg\lambda|$:

$$\frac{-\Re\sqrt{\lambda^2+\eta^2}}{|\sqrt{\lambda^2+\eta^2}|} = \cos(\pi - |\arg\sqrt{\lambda^2+\eta^2}|) \geq \cos(\pi - |\arg\lambda|) = \frac{-\Re\lambda}{|\lambda|} \geq \frac{\varrho}{|\lambda|} \geq \frac{\varrho M_2}{|\sqrt{\lambda^2+\eta^2}|}.$$

Step 3. Now we prove the theorem by contradiction. Suppose that there exist a $\varrho > 0$ and sequences $\{\lambda_n\}_{n\in\mathbf{N}} \subset \mathfrak{S}$ and $\{\eta_{k_n}^2\}_{n\in\mathbf{N}}$, such that $\Re\lambda_n < -\varrho$, $|\lambda_n| \to \infty$ and

$$\lambda_n\widehat{a}(\lambda_n) = \begin{cases} \frac{1+e^{L\sqrt{\lambda_n^2+\eta_{k_n}^2}}}{1-e^{L\sqrt{\lambda_n^2+\eta_{k_n}^2}}}\sqrt{\lambda_n^2+\eta_{k_n}^2} & \text{or} \\ \frac{1-e^{L\sqrt{\lambda_n^2+\eta_{k_n}^2}}}{1+e^{L\sqrt{\lambda_n^2+\eta_{k_n}^2}}}\sqrt{\lambda_n^2+\eta_{k_n}^2}, & \end{cases} \tag{6}$$

$n \in \mathbf{N}$. Then, by Step 1, we must have $\Re\sqrt{\lambda_n^2+\eta_{k_n}^2} \geq -r$ for large n's. By Step 2 we obtain

$$\left|\frac{\sqrt{\lambda_n^2+\eta_{k_n}^2}}{\lambda_n}\right| \geq M_2 \quad \text{and} \quad \Re\Big(\sqrt{\lambda_n^2+\eta_{k_n}^2}\Big) \leq -\varrho M_2 < 0.$$

The latter inequality implies that there exists an $M_3 > 0$ small, depending only on $M_2\varrho$, but not on n, such that

$$\left|\frac{1+e^{L\sqrt{\lambda_n^2+\eta_{k_n}^2}}}{1-e^{L\sqrt{\lambda_n^2+\eta_{k_n}^2}}}\right| \geq M_3 \quad \text{and} \quad \left|\frac{1-e^{L\sqrt{\lambda_n^2+\eta_{k_n}^2}}}{1+e^{L\sqrt{\lambda_n^2+\eta_{k_n}^2}}}\right| \geq M_3.$$

So we arrive at

$$|\lambda_n \widehat{a}(\lambda_n)| = \left| \frac{1 \pm e^{L\sqrt{\lambda_n^2+\eta_{k_n}^2}}}{1 \mp e^{L\sqrt{\lambda_n^2+\eta_{k_n}^2}}} \right| \cdot \left| \sqrt{\lambda_n^2 + \eta_{k_n}^2} \right| \geq M_3 M_2 |\lambda_n|,$$

which is a contradiction with $\widehat{a}(\lambda_n) \to 0$ (hypothesis (H2)). The theorem is proved. □

Theorem 4.3. *If a satisfies* (H1) *and* (H2′) *then there exists a sequence $\lambda_n \in \mathfrak{S}$ such that $\Re\lambda_n \leq 0$, $\Re\lambda_n \to 0$, $|\Im\lambda_n| \to \infty$.*

Proof. As before, we use the branch of the square root with negative real part.

Let $\eta > 0$ be fixed, and denote

$$F(\lambda) := \frac{e^{L\lambda}+1}{e^{L\lambda}-1}, \quad G_\eta(\lambda) := \widehat{a}(\lambda) + \frac{e^{L\sqrt{\lambda^2+\eta^2}}+1}{e^{L\sqrt{\lambda^2+\eta^2}}-1} \frac{\sqrt{\lambda^2+\eta^2}}{\lambda}.$$

We are looking for a sequence of zeros of G_η. The function F is $\frac{2\pi i}{L}$-periodic, has zeros at $\widetilde{\lambda}_m := \frac{i\pi + i2m\pi}{L}$ and poles at $\widetilde{\lambda}_m - \frac{i\pi}{L}$, $m \in \mathbf{Z}$. Using Rouché's theorem we will find zeros of G_η in small neighborhoods of zeros of F.

To this end let us fix $0 < \varepsilon < \frac{\pi}{L}$. Denote $\gamma_m := \partial B(\widetilde{\lambda}_m, \varepsilon)$, $m \in \mathbf{Z}$. The function F is holomorphic on a neighborhood of the closed ball $\overline{B(\widetilde{\lambda}_m, \varepsilon)}$ having exactly one zero inside the ball. Because of the periodicity, the number

$$\delta := \min\{|F(\lambda)| : \lambda \in \gamma_m\} > 0$$

is independent of $m \in \mathbf{Z}$ (it depends on ε). Note that

$$\frac{\sqrt{\lambda^2+\eta^2}}{\lambda} = \begin{cases} \sqrt{1+\frac{\eta^2}{\lambda^2}} \to -1 & \text{as } |\lambda| \to \infty \text{ with } \Re\lambda \leq 0, \\ -\sqrt{1+\frac{\eta^2}{\lambda^2}} \to 1 & \text{as } |\lambda| \to \infty \text{ with } \Re\lambda > 0. \end{cases}$$

We distinguish these two cases in the calculation of the difference $G_\eta(\lambda) - F(\lambda)$:

$$(7)\quad G_\eta(\lambda) - F(\lambda) = \widehat{a}(\lambda) + \frac{e^{L\sqrt{\lambda^2+\eta^2}}+1}{e^{L\sqrt{\lambda^2+\eta^2}}-1}\frac{\sqrt{\lambda^2+\eta^2}}{\lambda} - \frac{e^{L\lambda}+1}{e^{L\lambda}-1}$$

$$= \begin{cases} \widehat{a}(\lambda) - \frac{e^{L\lambda}+1}{e^{L\lambda}-1}\left(\frac{\sqrt{\lambda^2+\eta^2}}{\lambda}+1\right) \\ \qquad + \left(\frac{e^{L\sqrt{\lambda^2+\eta^2}}+1}{e^{L\sqrt{\lambda^2+\eta^2}}-1} + \frac{e^{L\lambda}+1}{e^{L\lambda}-1}\right)\frac{\sqrt{\lambda^2+\eta^2}}{\lambda} & \text{if } \Re\lambda \le 0, \\ \widehat{a}(\lambda) + \frac{e^{L\lambda}+1}{e^{L\lambda}-1}\left(\frac{\sqrt{\lambda^2+\eta^2}}{\lambda}-1\right) \\ \qquad + \left(\frac{e^{L\sqrt{\lambda^2+\eta^2}}+1}{e^{L\sqrt{\lambda^2+\eta^2}}-1} - \frac{e^{L\lambda}+1}{e^{L\lambda}-1}\right)\frac{\sqrt{\lambda^2+\eta^2}}{\lambda} & \text{if } \Re\lambda > 0. \end{cases}$$

By (H2$'$), there exists an $m_1 \in \mathbf{N}$ such that if $m \ge m_1$ and $\lambda \in \gamma_m$ then $|\widehat{a}(\lambda)| < \frac{\delta}{2}$. By periodicity, there exists a $K > 0$ such that

$$\left|\frac{e^{L\lambda}+1}{e^{L\lambda}-1}\right| \le K, \quad \lambda \in \gamma_m,\ m \in \mathbf{Z}.$$

There exists an $m_2 \in \mathbf{N}$ such that, for $m \ge m_2$, $\lambda \in \gamma_m$, one has

$$\left|\frac{\sqrt{\lambda^2+\eta^2}}{\lambda}+1\right| < \frac{\delta}{4K}, \quad \text{for } \Re\lambda \le 0,$$
$$\left|\frac{\sqrt{\lambda^2+\eta^2}}{\lambda}-1\right| < \frac{\delta}{4K}, \quad \text{for } \Re\lambda > 0.$$

For the third term in the difference $G_\eta(\lambda) - F(\lambda)$ in (7) we will work with the following expressions:

$$\frac{e^{L\sqrt{\lambda^2+\eta^2}}+1}{e^{L\sqrt{\lambda^2+\eta^2}}-1} + \frac{e^{L\lambda}+1}{e^{L\lambda}-1} = 2\frac{e^{L(\sqrt{\lambda^2+\eta^2}+\lambda)}-1}{(e^{L\sqrt{\lambda^2+\eta^2}}-1)(e^{L\lambda}-1)},$$
$$\frac{e^{L\sqrt{\lambda^2+\eta^2}}+1}{e^{L\sqrt{\lambda^2+\eta^2}}-1} - \frac{e^{L\lambda}+1}{e^{L\lambda}-1} = 2e^{L\lambda}\frac{1-e^{L(\sqrt{\lambda^2+\eta^2}-\lambda)}}{(e^{L\sqrt{\lambda^2+\eta^2}}-1)(e^{L\lambda}-1)}.$$

Note that one has

$$\sqrt{\lambda^2+\eta^2}+\lambda = \frac{\eta^2}{\lambda\left(\frac{\sqrt{\lambda^2+\eta^2}}{\lambda}-1\right)} \to 0 \quad \text{as } |\lambda|\to\infty \text{ with } \Re\lambda \le 0,$$
$$\sqrt{\lambda^2+\eta^2}-\lambda = \frac{\eta^2}{\lambda\left(\frac{\sqrt{\lambda^2+\eta^2}}{\lambda}+1\right)} \to 0 \quad \text{as } |\lambda|\to\infty \text{ with } \Re\lambda > 0.$$

Further, we express

$$\text{for } \Re\lambda \le 0: e^{L\sqrt{\lambda^2+\eta^2}}-1 = e^{-L\lambda}\left(e^{L(\sqrt{\lambda^2+\eta^2}+\lambda)}-1+1-e^{L\lambda}\right),$$
$$\text{for } \Re\lambda > 0: e^{L\sqrt{\lambda^2+\eta^2}}-1 = e^{L\lambda}\left(e^{L(\sqrt{\lambda^2+\eta^2}-\lambda)}-1\right)+\left(e^{L\lambda}-1\right).$$

Therefore, the third term in the difference of $G_\eta(\lambda)-F(\lambda)$ can be written in the following form:

$$\begin{cases} 2\left(e^{L(\sqrt{\lambda^2+\eta^2}+\lambda)}-1\right) \\ \qquad \frac{e^{L\lambda}}{\left(e^{L(\sqrt{\lambda^2+\eta^2}+\lambda)}-1\right)+(1-e^{L\lambda})}\frac{1}{e^{L\lambda}-1}\frac{\sqrt{\lambda^2+\eta^2}}{\lambda}, & \Re\lambda \le 0, \\ 2\left(1-e^{L(\sqrt{\lambda^2+\eta^2}-\lambda)}\right) \\ \qquad \frac{e^{L\lambda}}{e^{L\lambda}\left(e^{L(\sqrt{\lambda^2+\eta^2}-\lambda)}-1\right)+(e^{L\lambda}-1)}\frac{1}{e^{L\lambda}-1}\frac{\sqrt{\lambda^2+\eta^2}}{\lambda}, & \Re\lambda > 0. \end{cases}$$

Let $\delta_1 := \min\{|e^{L\lambda}-1|,\ \lambda\in\gamma_m\}$, which is positive and independent of $m \in \mathbf{Z}$. There exists an $m_3 \in \mathbf{N}$ such that, for $m \ge m_3$, $\lambda \in \gamma_m$, one has

$$\frac{\sqrt{\lambda^2+\eta^2}}{\lambda}| < 2,$$
$$\text{for } \Re\lambda \le 0: \quad |e^{L(\sqrt{\lambda^2+\eta^2}+\lambda)}-1| < \min\left\{\frac{\delta_1}{2}, \frac{\delta_1^2\delta}{32e^{L\varepsilon}}\right\},$$
$$\text{for } \Re\lambda > 0: \quad |1-e^{L(\sqrt{\lambda^2+\eta^2}-\lambda)}| < \min\left\{e^{-L\varepsilon}\frac{\delta_1}{2}, \frac{\delta_1^2\delta}{32e^{L\varepsilon}}\right\}.$$

Then, for $m \geq m_0 := \max\{m_1, m_2, m_3\}$ (which depends on $K, \varepsilon, \delta, \delta_1$), and $\lambda \in \gamma_m$ with $\Re\lambda > 0$, we can estimate

$$|G_\eta(\lambda) - F(\lambda)| < \frac{\delta}{2} + K\frac{\delta}{4K} + 2\frac{\delta_1^2\delta}{32e^{L\varepsilon}}\frac{e^{L\varepsilon}}{-\frac{\delta_1}{2} + \delta_1}\frac{1}{\delta_1}2 = \delta.$$

We get the same result when considering $\Re\lambda \leq 0$. So

$$|G_\eta(\lambda) - F(\lambda)| < \delta \leq |F(\lambda)|, \quad \lambda \in \gamma_m,$$

and Rouché's theorem says that G_η has exactly one zero in $B(\widetilde{\lambda}_m, \varepsilon)$, for any $m \geq m_0$. Repeating this procedure with $\varepsilon = \varepsilon_n \to 0$, we obtain a desired sequence $\{\lambda_n\}$ of zeros of G_η. $\square$

Remark. It follows from the above proof that the points $\lambda \in \mathfrak{S}$ which correspond to a particular η_k approach the imaginary axis from the left as $\Im\lambda \to \infty$. But, since the estimates in the proof are not uniform in η, the convergence is not shown to be uniform in $\eta_k, k \in \mathbf{N}$; in fact, non-uniformity is more probable.

5. Example on the disc. We consider the two-dimensional version of (1) with $\Omega = B(0,1) = \{(x,y) \in \mathbf{R}^2;\ x^2 + y^2 < 1\}$.

Lemma 5.1. *Let $\lambda \in \mathbf{C} \setminus (-\infty, 0]$. One has $\lambda \in \mathfrak{S}$ if and only if there exists a $k \in \mathbf{N}_0$ for which the problem*

$$u''(r) + \frac{1}{r}u'(r) - \left(\frac{k^2}{r^2} + \lambda^2\right)u(r) = 0, \quad r \in (0,1), \tag{8}$$

$$u'(1) + \lambda\widehat{a}(\lambda)u(1) = 0, \tag{9}$$

$$|\lim_{r\to 0+} u(r)| < \infty \tag{10}$$

has a nontrivial solution u.

Proof. The equation to be solved reads

$$\lambda^2 v - \Delta v = 0 \quad \text{in } B(0,1),$$
$$\frac{\partial v}{\partial n} + \lambda\widehat{a}(\lambda)v = 0 \quad \text{on } \partial B(0,1).$$

To solve the equation for v we introduce polar coordinates: $x = r\cos\varphi$, $y = r\sin\varphi$, $\Delta = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\varphi^2}$, $\frac{\partial}{\partial n} = \frac{\partial}{\partial r}$, and arrive at

$$\lambda^2 v - v_{rr} - \frac{1}{r}v_r - \frac{1}{r^2}v_{\varphi\varphi} = 0, \quad r \in (0,1),\ \varphi \in (0, 2\pi),$$
$$v_r + \lambda\widehat{a}(\lambda)v = 0, \quad r = 1,\ \varphi \in (0, 2\pi).$$

We decompose $v(r,.)$ in Fourier series with respect to $1, \cos k\varphi, \sin k\varphi$, $k \in \mathbf{N}$: $v(r,\varphi) = \sum_{k=0}^{\infty} \widetilde{v}_k(r)\Phi_k(\varphi)$ with $\Phi_0 = 1$, $\Phi_{2k-1}(\varphi) = \cos k\varphi$, $\Phi_{2k}(\varphi) = \sin k\varphi$, $k \in \mathbf{N}$, $\frac{\partial^2}{\partial\varphi^2}\Phi_k = \eta_k\Phi_k$ with $\eta_0 = 0$, $\eta_{2k} = \eta_{2k-1} = -k^2$, $k \in \mathbf{N}$. Then the equation for v is equivalent to the system

$$\begin{aligned} \lambda^2\widetilde{v}_k - \widetilde{v}_k'' - \frac{1}{r}\widetilde{v}_k' - \frac{1}{r^2}\widetilde{v}_k\eta_k = 0, &\quad r \in (0,1), \\ \widetilde{v}_k' + \lambda\widehat{a}(\lambda)\widetilde{v}_k = 0, &\quad r = 1,\ k \in \mathbf{N}_0. \end{aligned}$$

Having a nontrivial v is equivalent to having a nontrivial $\widetilde{v}_k$ for some $k \in \mathbf{N}_0$. □

Theorem 5.2. *If a satisfies* (H1) *and* (H2′), *then there exists a sequence $\lambda_n \in \mathfrak{S}$ such that $\Re\lambda_n \leq 0$, $\Re\lambda_n \to 0$, $|\Im\lambda_n| \to \infty$.*

Proof. The theory of second order linear ordinary differential equations (e.g., [**3**, Chapter 9, Section 7]) gives us the general solution to (8) in the form of the linear combination $C_1u_k + C_2v_k$ with coefficients $C_1, C_2 \in \mathbf{C}$, where

(11)
$$u_k(r) = \left(\frac{2}{i\lambda}\right)^k J_k(i\lambda r) = r^k \sum_{n=0}^{\infty} \frac{1}{(k+n)!}\frac{1}{n!}\left(\frac{\lambda r}{2}\right)^{2n},$$

(12)
$$v_k(r) = u_k(r)\log(r) + r^{-k}\sum_{n=0}^{\infty} c_{2n}r^{2n}, \quad c_n \in \mathbf{C},\ (c_0 = 1),$$

and J_k are the Bessel functions

$$J_k(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+k)!\,n!}\left(\frac{x}{2}\right)^{2n+k},$$

$k \in \mathbf{N}_0$.

Note that $u_0(0) = 1$, and all other u_k's are 0 at the origin. Therefore, for all $k \in \mathbf{N}_0$, v_k is unbounded as $r \downarrow 0$. This is ruled out by the boundary condition (10), and therefore $C_2 = 0$.

The boundary condition (9) reads

$$C_1\left(\frac{2}{i\lambda}\right)^k (i\lambda J_k'(i\lambda) + \lambda\widehat{a}(\lambda)J_k(i\lambda)) = 0.$$

We are interested in $\Re\lambda < 0$. Then $\Im(i\lambda) < 0$. Since J_k has only real zeros ([**1**, page 372]), we can equivalently write

$$-i\widehat{a}(\lambda) + \frac{J_k'(i\lambda)}{J_k(i\lambda)} = 0. \tag{13}$$

We arrived at the following conclusion: $\lambda \in \mathfrak{S}$ if and only if there exists a $k \in \mathbf{N}_0$ such that λ solves (13).

Using the recurrence formula $2J_k' = J_{k-1} - J_{k+1}$ and asymptotic expansions of $J_k(z)$ for large $|z|$ ([**1**, 9.2.1]) we have, for fixed $k \in \mathbf{N}_0$,

$$\frac{J_k'(i\lambda)}{J_k(i\lambda)} = \frac{\sin(i\lambda - (2k+1)\pi/4) + e^{|\Im(i\lambda)|}\,O(|i\lambda|^{-1})}{\cos(i\lambda - (2k+1)\pi/4) + e^{|\Im(i\lambda)|}\,O(|i\lambda|^{-1})} \quad \text{as } |\lambda| \to \infty. \tag{14}$$

Note at this point, that we do not claim uniformity of the terms $O(|i\lambda|^{-1})$ with respect to k.

As $\widehat{a}(\lambda) \to 0$ for $|\Im\lambda| \to \infty$ (actually hypothesis (H2′)), Rouché's theorem implies that G_k has zeros in small neighborhoods of zeros of F_k, with

$$F_k(\lambda) := \tan(i\lambda - (2k+1)\pi/4), \quad G_k(\lambda) := -i\widehat{a}(\lambda) + \frac{J_k'(i\lambda)}{J_k(i\lambda)}.$$

More specifically, let $\widetilde{\lambda}_m$ be the zeros of $F_k(\lambda)$, i.e.,

$$\widetilde{\lambda}_m := -i(m\pi + (2k+1)\pi/4), \quad m \in \mathbf{Z}.$$

The poles of F_k are $\widetilde{\lambda}_m + i\pi/2$. Let $0 < \varepsilon < \pi/2$. The function $F(\lambda)$ is holomorphic in a neighborhood of the closed ball $\overline{B(\widetilde{\lambda}_m, \varepsilon)}$ and has exactly one zero in its interior. Let $\gamma_m := \partial B(\widetilde{\lambda}_m, \varepsilon)$, $m \in \mathbf{Z}$. Because of the π-periodicity of tan, the positive number δ,

$$\delta := \min\{|F_k(\lambda)| : \lambda \in \gamma_m\}$$

is independent of m.

Choose m_1 such that for all $m \geq m_1$ and $\lambda \in \gamma_m$ the estimate $|\widehat{a}(\lambda)| < \delta/2$ holds. We can write

$$G_k(\lambda) - F_k(\lambda) = -i\widehat{a}(\lambda) + \frac{c_k(\lambda)\, e^{|\Im(i\lambda)|}\, O(|i\lambda|^{-1}) \;-\; s_k(\lambda)\, e^{|\Im(i\lambda)|}\, O(|i\lambda|^{-1})}{c_k(\lambda)\,(c_k(\lambda) + e^{|\Im(i\lambda)|}\, O(|i\lambda|^{-1}))}$$

as $|\lambda| \to \infty$, where $s_k(\lambda) := \sin(i\lambda - (2k+1)\pi/4)$ and $c_k(\lambda) := \cos(i\lambda - (2k+1)\pi/4)$. The functions s_k and c_k are $2\pi i$-periodic and bounded on $|\Re\lambda| \leq \varepsilon$. Thus, there exists a $K > 0$ such that $|c_k(\lambda)|, |s_k(\lambda)| < K/2$ for all $\lambda \in \gamma_m$, $m \in \mathbf{Z}$. Since $|c_k(\lambda)|$, $\lambda \in \gamma_0, \gamma_1$, is bounded away from zero, $e^{|\Im(i\lambda)|} \leq e^{\varepsilon}$ for $|\Re\lambda| \leq \varepsilon$, as well as $|i\lambda|^{-1} \to 0$ for $\lambda \in \gamma_m$ and $m \to \infty$, we can choose an $m_2 > 0$ with the following property: There exists an $L > 0$ such that for all $m \geq m_2$ and $\lambda \in \gamma_m$

$$|c_k(\lambda)\,(c_k(\lambda) + e^{|\Im(i\lambda|}\, O(|i\lambda|^{-1}))| > \frac{1}{L}$$

as well as

$$e^{\varepsilon}\, |O(|i\lambda|^{-1})| < \frac{\delta}{2KL}.$$

Thus, for $m \geq m_0 := \max\{m_1, m_2\}$ and all $\lambda \in \gamma_m$,

$$|G_k(\lambda) - F_k(\lambda)| < \frac{\delta}{2} + L\,\frac{K}{2}\,\frac{\delta}{KL} = \delta \leq |F_k(\lambda)|.$$

Therefore, by Rouché's theorem, for $m \geq m_0$, G_k has the same number of zeros in $B(\widetilde{\lambda}_m, \varepsilon)$ as has F_k, namely one, say λ_m.

Since $\varepsilon \in (0, \pi)$ can be chosen arbitrarily small, we can obtain the desired sequence λ_{m_n} of zeros of G_k. $\square$

REFERENCES

1. M. Abramovic and I.A. Stegun, *Handbook of mathematical functions*, Dover, New York, 1965.

2. F. Alabau-Boussouira, J. Prüss and R. Zacher, *Exponential and polynomial stability of a wave equation for boundary memory damping with singular kernels*, C.R. Acad. Sci. Paris **347** (2009), 277–282.

3. G. Birkhoff and G.-C. Rota, *Ordinary differential equations*, 2nd edition, Blaisdell, Waltham, Massachusetts, 1969.

4. W. Desch, E. Fašangová, J. Milota and G. Propst, *Stabilization through viscoelastic boundary damping: A semigroup approach*, Semigroup Forum **80** (2010), 405–415.

5. G. Propst and J. Prüss, *On wave equations with boundary dissipation of memory type*, J. Integral Equations and Applications **8** (1996), 99–123.

6. W. Rudin, *Real and complex analysis*, McGraw-Hill, Inc., New York, 1974.

7. D.V. Widder, *The Laplace transform*, Princeton University Press, Princeton, 1941.

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