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PROPERTIES OF THE SINGLE LAYER POTENTIAL FOR THE TIME FRACTIONAL DIFFUSION EQUATION

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ABSTRACT. This paper investigates the basic properties of the single layer potential for the time fractional diffusion equation (TFDE) in a bounded domain with Lyapunov boundary. We prove the continuity of the single layer potential across the boundary, that the normal derivative of the single layer potential satisfies the usual jump relation known for the heat equation and that the single layer potential is Hölder continuous. These properties are essential when investigating the boundary integral equation corresponding to TFDE.

Although the results are well known for the potentials of the heat equation corresponding to the case $\alpha = 1$, the proofs of the same properties seem not to be available in the case $0 < \alpha < 1$. Also, even if the proofs follow the same lines as in the case $\alpha = 1$, there are some additional difficulties to overcome. First of all, there is no explicit formula for the fundamental solution in terms of elementary functions unlike in the case of the heat potential. Secondly, the behavior of the Fox H-functions is different for small and large arguments. However, the known asymptotic behavior of the Fox H-functions allows us to prove the above-mentioned properties.

1. Introduction. We are interested in the boundary integral solution of the time fractional diffusion equation

$$
\begin{aligned}
\partial_t^\alpha \Phi - \Delta_x \Phi = 0 \qquad & \text{in } Q_T = \Omega \times (0,T),\\
\Phi = g, \qquad & \text{on } \Sigma_T = \Gamma \times (0,T),\\
\Phi(x,0) = 0, \quad & x \in \Omega,
\end{aligned} \tag{1}
$$

with the single layer potential. In problem (1) the domain $\Omega \subset \mathbf{R}^n$ is assumed to be bounded with the boundary $\Gamma \in \mathcal{C}^{1+\lambda}$, $0 < \lambda < 1$, and

$$
\partial_t^\alpha u(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} u'(\tau)\, \mathrm{d}\tau \tag{2}
$$

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is the fractional Caputo time derivative of order $0 < \alpha \le 1$. For $\alpha = 1$ the fractional derivative is interpreted as the limit $\lim_{\alpha \uparrow 1} \partial_t^\alpha u(t)$ which coincides with the usual time derivative $\frac{\mathrm{d}u(t)}{\mathrm{d}t}$ (see [**4**, page 68]).

The single layer potential is defined by

$$(S\varphi)(x,t) = \int_0^t \int_\Gamma G(x-y, t-\tau)\varphi(y,\tau)\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau, \quad (x,t) \in (\mathbf{R}^n \setminus \Gamma) \times \mathbf{R}_+, \tag{3}$$

where G is the fundamental solution of the fractional diffusion equation. It is known that

$$G(x,t) = \begin{cases} \pi^{-n/2} t^{\alpha-1} |x|^{-n} H_{12}^{20}\left[\frac{1}{4}|x|^2 t^{-\alpha}\Big|_{(n/2,1),(1,1)}^{(\alpha,\alpha)}\right] & x \in \mathbf{R}^n,\ t > 0, \\ 0 & x \in \mathbf{R}^n,\ t < 0, \end{cases} \tag{4}$$

where H is the Fox H-function (see [**6, 7, 9**]).

In order to simplify the notations we introduce the following functions defined for $z > 0$

$$H_{(1)}(z) := H_{12}^{20}\left[z \,\middle|\, \begin{matrix} (\alpha,\alpha) & \\ (n/2,1), & (1,1) \end{matrix} \right], \tag{5}$$

$$H_{(2)}(z) := H_{23}^{30}\left[z \,\middle|\, \begin{matrix} (\alpha,\alpha), & (0,1) & \\ (n/2,1), & (1,1), & (1,1) \end{matrix} \right] \tag{6}$$

$$H_{(3)}(z) := H_{34}^{40}\left[z \,\middle|\, \begin{matrix} (\alpha,\alpha), & (0,1), & (0,1) & \\ (n/2,1), & (1,1), & (1,1), & (1,1) \end{matrix} \right] \tag{7}$$

The paper is organized as follows. Since the proofs are rather lengthy, we prove the properties each in a different section. In Section 2 we prove the continuity of the single layer potential in $\mathbf{R}^n$ assuming that the density is in $L^\infty(\Sigma_\infty)$. In Section 3 we prove that the gradient of the single layer potential for density in $\mathcal{C}(\overline{\Sigma_T})$ has a nontangential limit when x tends to the boundary point. More precisely, we show that $\langle \nabla_x S\varphi(x,t), \nu(x_0)\rangle$ satisfy the jump relation when x tends to the boundary point x_0 along nontangential directions. Section 4 is devoted to the proof of Hölder continuity of the single layer potential for $L^\infty(\Sigma_T)$-density.

2. Continuity of the single layer potential. In our analysis we follow very closely the standard technique found, e.g., in ([**3**]). Now, because the behavior of the Fox H-function is different for small and large arguments, we have to treat them separately. Furthermore, the behavior near zero is very different in different dimensions, which gives us more subcases to analyze.

In our analysis the following properties of the functions (5)–(7) are needed.

Lemma 1. *For the functions $H_{(p)}$ there holds*:

(i) *Differentiation formula* $\frac{\mathrm{d}}{\mathrm{d}z} H_{(p)}(z) = -z^{-1} H_{(p+1)}(z)$ *for* $p = 1, 2$.

(ii) *The asymptotic behavior at infinity*:

(8)
$$|H_{(p)}(z)| \le C z^{(n+2p-2\alpha)/2(2-\alpha)} \exp(-\sigma z^{1/(2-\alpha)}),\ \sigma := \alpha^{\alpha/(2-\alpha)}(2-\alpha),$$

for $p = 1, 2, 3$ *and* $z \ge 1$.

(iii) *The asymptotic behavior near zero*:

$$|H_{(p)}(z)| \le C \begin{cases} z^{n/2} & \text{if } n = 2 \text{ or } n = 3, \\ z^2 |\log z| & \text{if } n = 4, \\ z^2 & \text{if } n > 4, \end{cases} \tag{9}$$

for $p = 1, 2, 3$ *and* $z \le 1$. *Moreover,*

$$|nH_{(p)}(z) + 2H_{(p+1)}(z)| \le C \begin{cases} z^2 |\log z| & \text{if } n = 2, \\ z^2 & \text{if } n \ge 3, \end{cases} \tag{10}$$

for $p = 1, 2$ *and* $z \le 1$.

The constants in (ii) *and* (iii) *can depend on* n, p *and* α.

Proof. The property (i) is an easy consequence of the Mellin-Barnes integral representation of the Fox H-functions and of the analyticity of the functions $H_{(p)}$. The asymptotic expansions for $H_{(p)}$ are stated in

([**2**, formulae (3.7), (3.14), (3.15) and (3.16)]). Note that the first terms in the series representation of $nH_{(1)} + 2H_{(2)}$ cancel out. For the proofs see ([**1, 6**]). □

In the sequel we use the notation $z := \frac{1}{4}|x|^2 t^{-\alpha}$. Because the Fox H-function depends only on z and the asymptotics are different for large and small values of z, we treat the cases $z \geq 1$ and $z \leq 1$ separately. We collect the asymptotic formulae in the following result.

Lemma 2. *For G the following asymptotic formulae hold*:

(i) *If $z \geq 1$, then*

$$|G(x,t)| \leq Ct^{-(\alpha n/2)-1+\alpha} \exp\Big(- \sigma t^{-\alpha/(2-\alpha)} |x|^{2/(2-\alpha)}\Big), \tag{11}$$

where $\sigma = 4^{1/(\alpha-2)} \alpha^{\alpha/(2-\alpha)}(2-\alpha)$.

(ii) *If $z \leq 1$, then*

$$|G(x,t)| \leq C \begin{cases} t^{-1} & \text{if } n = 2, \\ t^{-(\alpha/2)-1} & \text{if } n = 3, \\ t^{-\alpha-1}\big(\big|\log(|x|^2 t^{-\alpha})\big| + 1\big) & \text{if } n = 4, \\ t^{-\alpha-1}|x|^{-n+4} & \text{if } n > 4. \end{cases}$$

Proof. Because $0 < \alpha \leq 1$ and if $z \geq 1$, we may estimate $z^{(n+2p-2\alpha)/2(2-\alpha)} \leq z^{n/2}$ in Lemma 1. If we use this in (8) for $H_{(1)}$ and use the obtained bound in formula (4) for the fundamental solution, we get (i). Case (ii) follows from using the bounds (9) for $H_{(1)}$ in (4). □

Remark 1. Although we have an exact and optimal value for the constant σ in (i), in what follows σ may denote various positive constants. The only thing that matters is that $\sigma > 0$.

Theorem 1. *The single layer potential is continuous in $\mathbf{R}^n \times \mathbf{R}_+$ for any bounded measurable function φ. In particular, it is continuous across the lateral boundary Γ.*

Proof. The proof follows if we show that G is locally integrable (see [**3**, Lemma 1, page 7]). Let us first consider the case $z \geq 1$. Because

$z^\gamma \exp(-\sigma z^{1/(2-\alpha)})$ is uniformly bounded for any $\gamma > 0$, we have an estimate

$$|G(x,t)| \le Ct^{-(\alpha n/2)-1+\alpha+\alpha\gamma}|x|^{-2\gamma}. \tag{12}$$

From (12) we see that G is locally integrable if γ can be chosen such that $2\gamma < n-1$ and $-(\alpha n/2) + \alpha + \alpha\gamma - 1 > -1$. Both of these inequalities are satisfied for any

$$\frac{n-2}{2} < \gamma < \frac{n-1}{2}. \tag{13}$$

If $z \le 1$, then $1 \le 4^\gamma t^{\alpha\gamma}|x|^{-2\gamma}$ for any $\gamma > 0$, and from Lemma 2 we get the following estimate

$$|G(x,t)| \le C \begin{cases} t^{-1+\alpha\gamma}|x|^{-2\gamma} & \text{if } n = 2, \\ t^{-(\alpha/2)-1+\alpha\gamma}|x|^{-2\gamma} & \text{if } n = 3, \\ t^{-\alpha-1+\alpha\gamma}|x|^{-2\gamma} & \text{if } n = 4, \\ t^{-\alpha-1+\alpha\gamma}|x|^{-n+4-2\gamma} & \text{if } n > 4. \end{cases}$$

In the case $n = 4$ we have used the fact that $z^\gamma|\log z|$ is bounded in $(0,1]$ for $\gamma > 0$.

If we choose (i) γ as in (13) for $n = 2,3,4$ or (ii) $1 < \gamma < 3/2$ for $n > 4$, we see that G is locally integrable. □

3. The jump relation. In this section we study the boundary behavior of the gradient of the single layer potential. It follows from the asymptotic behavior of the fundamental solution that if $x \in \mathbf{R}^n \setminus \Gamma$,

$$(\nabla_x S\varphi)(x,t) = \int_0^t \int_\Gamma \nabla_x G(x-y, t-\tau)\varphi(y,\tau)\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau. \tag{14}$$

We are interested in what happens if $x \to x_0 \in \Gamma$ nontangentially. This means that if $K := K(x_0)$ is any finite closed cone in $\mathbf{R}^n$ with vertex x_0 such that $K(x_0) \subset \Omega \cup \{x_0\}$ the limit is taken in the sense $K \ni x \to x_0$. The nontangential approach from the exterior domain $\overline{\Omega}^c$ is defined analogously.

Here we need the partial derivative $\partial/\partial x_j$ of the fundamental solution. A straightforward calculation shows that

$$\frac{\partial}{\partial x_j}G(x,t) = -n\pi^{-n/2}\frac{x_j}{|x|^{n+2}}t^{\alpha-1}H_{(1)}\left(\frac{1}{4}t^{-\alpha}|x|^2\right) + \pi^{-n/2}|x|^{-n}t^{\alpha-1}\frac{\mathrm{d}}{\mathrm{dz}}H_{(1)}(z)|_{z=1/4t^{-\alpha}|x|^2}\cdot\frac{\partial z}{\partial x_j}.$$

Using the differentiation formula for $H_{(1)}$ in Lemma 1, we get

(15)
$$\frac{\partial}{\partial x_j}G(x,t)=-\pi^{-n/2}\frac{x_j}{|x|^{n+2}}t^{\alpha-1}\left\{nH_{(1)}\left(\frac{1}{4}t^{-\alpha}|x|^2\right)+2H_{(2)}\left(\frac{1}{4}t^{-\alpha}|x|^2\right)\right\}.$$

Before the proof of the jump relation, let us show that the integral

(16)
$$\int_0^t\int_\Gamma \partial_{\nu(x)}G(x-y,t-\tau)\varphi(y,\tau)\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau$$

is well-defined even if $x\in\Gamma$ when $\varphi\in L^\infty(\Sigma_T)$. We collect the necessary estimates for the kernel of (16) in the following lemma.

Lemma 3. *Let $z=(1/4)|x-y|^2t^{-\alpha}$ with $x,y\in\Gamma$. If $\Gamma\in\mathcal{C}^{1+\lambda}$, we have the following estimates for the normal derivative of G:*

(i) *If $z\geq 1$, then*

$$|\partial_{\nu(x)}G(x-y,t)|\leq Ct^{-\alpha n/2-1}|x-y|^{\lambda+1}\exp\left\{-\sigma t^{-\alpha/2-\alpha}|x-y|^{2/2-\alpha}\right\}.$$

(ii) *If $z\leq 1$, then*

$$|\partial_{\nu(x)}G(x-y,t)|\leq C\begin{cases} t^{-\alpha-1}|x-y|^{\lambda+1}|\log(|x-y|^2t^{-\alpha})| & \text{if } n=2,\\ t^{-\alpha-1}|x-y|^{\lambda+3-n} & \text{if } n\geq 3.\end{cases}$$

Proof. It follows from (15) that $\partial_{\nu(x)}G(x-y,t)$ can be written in a form

(17)
$$\partial_{\nu(x)}G(x-y,t)=-\pi^{-n/2}\frac{\langle x-y,\nu(x)\rangle}{|x-y|^{n+2}}t^{\alpha-1}\left\{nH_{(1)}(z)+2H_{(2)}(z)\right\},$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbf{R}^n$. Then case (i) follows by using the asymptotic estimate (8) in Lemma 1 for the Fox H-functions in (15) and from the fact that $|\langle x-y, \nu(x)\rangle| \leq C|x-y|^{\lambda+1}$ if $\Gamma \in \mathcal{C}^{1+\lambda}$ and $x, y \in \Gamma$.

The second case follows from the same observations as above and from estimate (10) in Lemma 1. □

By using the same arguments as in the proof of Lemma 2 we may conclude that the integral in (16) converges absolutely. As an example we consider case (i). Because $z^\gamma \exp(-\sigma z^{1/2-\alpha})$ is uniformly bounded for any $\gamma > 0$, we have an estimate

$$(18)\qquad |\partial_{\nu(x)} G(x-y, t-\tau)| \leq C(t-\tau)^{-\alpha n/2-1+\alpha\gamma}|x-y|^{-2\gamma+\lambda+1}.$$

From here we see that $\partial_{\nu(x)} G(x-y, t-\tau)$ is locally integrable if γ can be chosen such that $2\gamma - \lambda - 1 < n-1$ and $-(\alpha n/2) + \alpha\gamma - 1 > -1$. Both of these inequalities are satisfied for any

$$(19)\qquad \frac{n}{2} < \gamma < \frac{n+\lambda}{2}.$$

Choosing γ as above we find that the integral in (16) is finite if $\varphi \in L^\infty(\Sigma_T)$.

Now we are ready to state the jump relation of the single layer potential (see [**3**, Theorem 5.2.1, page 137, and its proof]).

Theorem 2. *Let $\Gamma \in \mathcal{C}^{1+\lambda}$ for some $0 < \lambda < 1$, and let $\varphi \in \mathcal{C}(\overline{\Sigma_T})$. Then, for any $x_0 \in \Gamma$, $0 < t \leq T$, the following jump relation holds*

$$(20)\qquad \lim_{\substack{x \to x_0 \\ x \in K}} \langle \nabla_x S\varphi(x,t), \nu(x_0)\rangle = \frac{1}{2}\varphi(x_0,t) + \int_0^t \int_\Gamma \partial_{\nu(x_0)} G(x_0-y, t-\tau)\varphi(y,\tau)\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbf{R}^n$.

Proof. We start with showing the limit along the outward normal $\nu(x_0)$. We have shown that the improper integral on the right-hand side of (20) exists and is absolutely convergent. Denote by $T(x_0)$ the tangent hyperplane to Γ at x_0 and $\Gamma_\delta := B(x_0, \delta) \cap \Gamma$. Since $\Gamma \in \mathcal{C}^{1+\lambda}$, it follows that the orthogonal projection $\pi : \Gamma_\delta \to T(x_0)$ is one-to-one if δ is small enough. We denote its image by $\Gamma'_\delta = \pi(\Gamma_\delta)$.

Write

$$\langle \nabla_x S\varphi(x,t), \nu(x_0)\rangle = I_\delta(x,t) + J_\delta(x,t), \tag{21}$$

where

$$I_\delta(x,t) = \int_0^t \int_{\Gamma_\delta} \langle \nabla_x G(x-y, t-\tau), \nu(x_0)\rangle \varphi(y,\tau)\, \mathrm{d}\sigma(y)\, \mathrm{d}\tau \tag{22}$$

and J_δ is its complementary part.

Also, denote

$$I'_\delta(x,t) = \int_0^t \int_{\Gamma'_\delta} \langle \nabla_x G(x-y', t-\tau), \nu(x_0)\rangle \varphi(x_0,\tau)\, \mathrm{d}\sigma'(y')\, \mathrm{d}\tau, \tag{23}$$

where $\mathrm{d}\sigma'(y')$ is the surface element on the hyperplane $T(x_0)$.

We prove (20) by showing that

$$\lim_{x\to x_0} I'_\delta(x,t) = \frac{1}{2}\varphi(x_0,t), \tag{24}$$

$$\lim_{x\to x_0} J_\delta(x,t) = J_\delta(x_0,t), \tag{25}$$

$$\lim_{x\to x_0} (I_\delta(x,t) - I'_\delta(x,t)) = I_\delta(x_0,t). \tag{26}$$

Proof of (24). With the change of variables $\tau \leftrightarrow \rho = |x-y'|^2/4(t-\tau)^\alpha$ in the definition of $I'_\delta(x,t)$ and integrating with respect to τ, we get

$$I'_\delta(x,t) = \int_{\Gamma'_\delta} \frac{\langle x-y', \nu(x_0)\rangle}{|x-y'|^n} H_{|x-y'|^2/4t^\alpha}(x,y',t)\, \mathrm{d}\sigma'(y'), \tag{27}$$

where

$$\begin{aligned} &H_\varepsilon(x,y',t) \\ &= -\frac{1}{4\alpha\pi^{n/2}} \int_\varepsilon^\infty \rho^{-2}\Big\{nH_{(1)}(\rho) + 2H_{(2)}(\rho)\Big\}\varphi\left(x_0, t - \frac{|x-y'|^{2/\alpha}}{(4\rho)^{1/\alpha}}\right) \mathrm{d}\rho. \end{aligned} \tag{28}$$

Since the integral $H_0(x, y', t)$ exists by Lemma 1 and the continuity of φ, we see that $H(x, y', t) := H_{|x-y'|^2/4t^\alpha}(x, y', t)$ is a continuous function of (x, y') for all $x = x_0 - h\nu(x_0)$ with $h > 0$ small enough and for all $y' \in \Gamma'_\delta$. In particular,

$$\begin{aligned}\lim_{\substack{x\to x_0\\ y'\to x_0}} H(x,y,t) &= H(x_0,x_0,t)\\ &= -\frac{\varphi(x_0,t)}{4\alpha\pi^{n/2}}\int_0^\infty \rho^{-2}\Big\{nH_{(1)}(\rho)+2H_{(2)}(\rho)\Big\}\,\mathrm{d}\rho.\end{aligned} \tag{29}$$

We divide Γ'_δ into two regions, $\Gamma'_{1\delta}$, which contains x_0 in its interior, and its complement $R_\delta = \Gamma'_\delta \setminus \Gamma'_{1\delta}$ exactly as in the proof of Theorem 1 of ([**3**, Section 5.2]). In $\Gamma'_{1\delta}$ we substitute $y'' = y' - x/|y' - x|$ and denote the resulting domain of the integration by $\Gamma''_{1\delta}$ and the corresponding area element by $\mathrm{d}\omega(y'')$ at y''. Then, because $\langle x - y', \nu(x_0)\rangle = -|x - y'| \cos(y' - x, \nu(x_0))$ and $\cos(y' - x, \nu(x_0))\,\mathrm{d}\sigma'(y') = |x - y'|^{n-1}\,\mathrm{d}\omega(y'')$, we have

$$\begin{aligned}I'_\delta(x,t) &= \frac{\varphi(x_0,t)}{4\alpha\pi^{n/2}} I_n \int_{\Gamma''_{1\delta}} \mathrm{d}\omega(y'')\\ &\quad - \int_{\Gamma''_{1\delta}} \Big(H(x,y',t) - H(x_0,x_0,t)\Big)\,\mathrm{d}\omega(y'') + R_\delta(x,t),\end{aligned} \tag{30}$$

where we have denoted the integral on the right-hand side of (29) by I_n.

Since $H(x, y', t)$ is a continuous function, the second integral in (30) can be made arbitrarily small. In $R_\delta(x, t)$ we note that $\langle x - y', \nu(x_0)\rangle \to 0$ as $x \to x_0$ and that $|x - y'|$ is bounded away from zero if $y' \in R_\delta$, which implies that this term tends to zero as well. For the first term, we observe that $\Gamma''_{1\delta}$ tends to a unit hemisphere, so the first term tends to $(\omega_n I_n/8\alpha\pi^{n/2})\varphi(x_0, t)$.

Now we have shown that $\lim_{x\to x_0} I'_\delta(x, t) = (\omega_n I_n/8\alpha\pi^{n/2})\varphi(x_0, t)$. Although we cannot calculate directly the exact value for the constant $\omega_n I_n/8\alpha\pi^{n/2}$, we prove that it must be $1/2$ in Remark 3 after the proof.

Proof of (25). It follows from the definition of J_δ in (21) that $|x - y| \geq \delta/2 > 0$ for $|x - x_0| < \delta/2$. Hence, the integral is a continuous function, which implies (25).

Proof of (26). The proof of this part is the most technical one. First of all, it should be noted that $I'_\delta(x_0,t)=0$ because $\nu(x_0)\perp(x_0-y')$. Now, take $\delta_1<\delta$ and write

$$\begin{aligned} I_\delta(x,t) &= I_{\delta_1}(x,t)+\overline{I}_{\delta_1}(x,t),\\ I'_\delta(x,t) &= I'_{\delta_1}(x,t)+\overline{I}'_{\delta_1}(x,t), \end{aligned} \tag{31}$$

where the integration in $\overline{I}_{\delta_1}$ ($\overline{I}'_{\delta_1}$) with respect to the spatial variable is taken over the set $\Gamma_\delta\setminus\Gamma_{\delta_1}$ ($\Gamma'_\delta\setminus\Gamma'_{\delta_1}$, respectively).

We prove that for any $\varepsilon>0$ there exists a $\delta_1>0$ such that

$$|I_{\delta_1}(x,t)-I'_{\delta_1}(x,t)|<\varepsilon, \tag{32}$$

$$|\overline{I}_{\delta_1}(x,t)-\overline{I}_{\delta_1}(x_0)|<\varepsilon, \tag{33}$$

$$|I_{\delta_1}(x_0,t)|, |\overline{I}_{\delta_1}(x,t)|<\varepsilon. \tag{34}$$

Proof of (32). We write $\langle\nabla_x G(x-y,t),\nu(x_0)\rangle$ as follows

$$\begin{aligned} &\langle\nabla_x G(x-y,t),\nu(x_0)\rangle\\ &\qquad=\langle y'-y,\nu(x_0)\rangle\widetilde{G}(x-y,t)+\langle x-y',\nu(x_0)\rangle\widetilde{G}(x-y,t). \end{aligned}$$

Because $\Gamma\in\mathcal{C}^{1+\lambda}$, we have $|y-y'|\le C|x_0-y|^{1+\lambda}\le C|x-y|^{1+\lambda}$ (see [**3**, formula 5.1.7, page 135]). Then we obtain the same bounds as before for the first term. When $z\ge 1$, for example, the first term is bounded by (see (18))

$$C(t-\tau)^{-(\alpha n/2)-1+\alpha\gamma}|x-y|^{-2\gamma+\lambda+1},$$

where $n/2<\gamma<(n+\lambda)/2$.

For the second term we proceed as follows. We denote $z=(1/4)|x-y|^2(t-\tau)^{-\alpha}$ and $z'=(1/4)|x-y'|^2(t-\tau)^{-\alpha}$. Let us decompose $\widetilde{G}$ as follows

$$\begin{aligned} \pi^{n/2}\widetilde{G}(x-y,t-\tau) &= \frac{(t-\tau)^{\alpha-1}}{|x-y|^{n+2}}\Big\{nH_{(1)}(z)+2H_{(2)}(z)\Big\}\\ &=\left(\frac{(t-\tau)^{\alpha-1}}{|x-y|^{n+2}}-\frac{(t-\tau)^{\alpha-1}}{|x-y'|^{n+2}}\right)\Big\{nH_{(1)}(z)+2H_{(2)}(z)\Big\} \end{aligned} \tag{35}$$

$$+\frac{(t-\tau)^{\alpha-1}}{|x-y'|^{n+2}}\Big\{nH_{(1)}(z)-nH_{(1)}(z')+2H_{(2)}(z)-2H_{(2)}(z')\Big\}$$
$$+\frac{(t-\tau)^{\alpha-1}}{|x-y'|^{n+2}}\Big\{nH_{(1)}(z')+2H_{(2)}(z')\Big\}=\widetilde{G}_1+\widetilde{G}_2+\widetilde{G}_3.$$

Therefore, we need estimates for the differences of the Fox H-functions and of the power functions. For that we use the Mean Value Theorem.

For the difference of the power functions we get

$$\left|\frac{1}{|x-y|^{n+2}}-\frac{1}{|x-y'|^{n+2}}\right|=(n+2)\frac{|\langle x-\widetilde{y},y'-y\rangle|}{|x-\widetilde{y}|^{n+4}} \tag{36}$$

for some $\widetilde{y}$ on the line between y and y'. Because $|x-y|\sim|x-\widetilde{y}|$ and Γ is of class $\mathcal{C}^{1+\lambda}$, we have

$$\left|\frac{1}{|x-y|^{n+2}}-\frac{1}{|x-y'|^{n+2}}\right|\le C|x-y|^{-n-2+\lambda}. \tag{37}$$

Then, using Lemma 1, we get the same bounds as before. If $n=3$ and $z\le 1$, for example, then by the estimates (10) and (37) we have a bound

$$|\widetilde{G}_1|\le C|x-y|^{\lambda-1}(t-\tau)^{-\alpha-1}, \tag{38}$$

which implies

$$|\langle x-y',\nu(x_0)\rangle\widetilde{G}_1|\le C|x-y|^{\lambda-2\gamma}(t-\tau)^{\alpha\gamma-\alpha-1}. \tag{39}$$

Choosing $1<\gamma<(2+\lambda/2)$ we see that this term is integrable.

For the estimation of $\langle x-y',\nu(x_0)\rangle\widetilde{G}_2$ we use the Mean Value Theorem for the function $nH_{(1)}+2H_{(2)}$ and differentiation formula in Lemma 1 to obtain

$$\begin{aligned} &nH_{(1)}(z)-nH_{(1)}(z')+2H_{(2)}(z)-2H_{(2)}(z') \\ &\qquad=-\frac{2\langle x-\widetilde{y},y'-y\rangle}{|x-\widetilde{y}|^2}\Big\{nH_{(2)}(\widetilde{z})+2H_{(3)}(\widetilde{z})\Big\} \end{aligned} \tag{40}$$

for some $\widetilde{z}=(1/4)|x-\widetilde{y}|^2(t-\tau)^{-\alpha}$ with $\widetilde{y}$ between y and y'.

Then we use the asymptotic formulae in Lemma 1. Finally, noting that $|y'-y| \le C|x-y|^{1+\lambda}$ and $|x-y| \sim |x-\widetilde{y}|$, we get an integrable upper bound. In the cases $n=3$ and $z \le 1$, for example, we have the following estimate

(41)
$$\begin{aligned}|\widetilde{G}_2| &= 2\pi^{-n/2}\frac{(t-\tau)^{\alpha-1}}{|x-y'|^{n+2}}\left|\frac{\langle x-\widetilde{y}, y'-y\rangle}{|x-\widetilde{y}|^2}\{nH_{(2)}(\widetilde{z})+2H_{(3)}(\widetilde{z})\}\right| \\ &\le C(t-\tau)^{\alpha-1}|x-y'|^{-5}|x-\widetilde{y}|^{-1}|y'-y|x-\widetilde{y}|^4(t-\tau)^{-2\alpha} \\ &\le C(t-\tau)^{-\alpha-1}|x-y|^{\lambda-1}.\end{aligned}$$

Multiplying the previous upper bound by $|\langle x-y', \nu(x_0)\rangle|$ we have a majorant

$$C(t-\tau)^{-\alpha-1}|x-y|^{\lambda} \le C(t-\tau)^{\alpha\gamma-\alpha-1}|x-y|^{\lambda-2\gamma},$$

which is integrable when we choose $1<\gamma<(2+\lambda/2)$. The other cases for n's are treated similarly.

The last difference

$$\pi^{-n/2}\int_0^t\int_{\Gamma_{\delta_1}}\langle x-y, \nu(x_0)\rangle\widetilde{G}_3(x-y', t-\tau)\varphi(y,\tau)\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau - I'_{\delta_1}(x,t)$$

can be estimated similarly as above.

Putting all of this together, we have proved that

(42)
$$\begin{aligned}|I_{\delta_1}(x,t)-I'_{\delta_1}(x,t)| \le C\int_0^t\int_{\Gamma_{\delta_1}}&\frac{1}{(t-\tau)^{\mu}}\frac{1}{|x-y|^{n-1-\mu'}}\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau \\ &+\sup\left|\frac{\varphi(y,\tau)}{\cos(\nu(x_0),\nu(y))}-\varphi(x_0,\tau)\right| \\ &\times\left|\int_0^t\int_{\Gamma'_{\delta_1}}\partial_{\nu(x_0)}G(x-y', t-\tau)\,\mathrm{d}\sigma'(y')\,\mathrm{d}\tau\right|\end{aligned}$$

for some $\mu<1$ and $\mu'>0$.

The integrand in the first term of (42) is integrable, so the corresponding integral can be made arbitrarily small by choosing δ_1 small enough. The second integral is bounded independently of δ_1 since it coincides with I'_δ as $\delta=\delta_1$ and $\varphi(x_0,\tau)\equiv 1$, and $\lim_{x\to x_0} I_\delta(x,t)$ was shown to exist. Also, the expression $\sup|\cdot|$ tends to zero as $\delta_1 \to 0$ because φ is continuous and $\cos(\nu(x_0),\nu(y)) \to 1$. Hence, for any ε we can choose δ_1 such that $|I_{\delta_1}(x,t) - I'_{\delta_1}(x,t)| < \varepsilon$ and the proof of (32) is finished.

Proof of (33). This term can be proved to be small by the same technique as (32). Indeed, using the Mean Value Theorem as above and noting that, for fixed δ_1, the terms $|x-y|, |x_0-y|$ in $\overline{I}_{\delta_1}(x,t)$ and $\overline{I}_{\delta_1}(x_0,t)$ are bounded away from zero. Then we can follow the same lines as in the proof of (32).

Proof of (34). Since the term $\langle x_0 - y, \nu(x_0)\rangle$ in the integrand of $I_{\delta_1}(x_0,t)$ is bounded by $|x_0-y|^{\lambda+1}$, we see, as before, that the kernel is locally integrable. This implies that

$$|I_{\delta_1}(x_0,t)| < \varepsilon \tag{43}$$

if δ_1 is small enough. Furthermore, since $|x-y'|$ in $\overline{I}'_{\delta_1}(x,t)$ is bounded away from zero, if δ_1 is now fixed, and since $\cos(\nu(x_0), y'-x) \to 0$ if $x \to x_0$,

$$|I'_{\delta_1}(x,t)| < \varepsilon \tag{44}$$

if x is sufficiently close to x_0.

Gathering everything together, we have proved (20) in the case where $x \to x_0$ along the normal $\nu(x_0)$. The general case where $K \ni x \to x_0$ can be proved as in ([**3**, pages 143–144]). □

Remark 2. Exactly the same proof applies also for the outward limit of the normal derivative. Note that in this case $\nu(x_0)$ is the inward normal and we have

$$\lim_{\substack{x\to x_0\\ x\in K'}} (\partial_{\nu(x_0)} S\varphi)(x,t) = -\frac{1}{2}\varphi(x_0,t) + \int_0^t \int_\Gamma \partial_{\nu(x_0)} G(x_0-y, t-\tau)\varphi(y,\tau)\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau, \tag{45}$$

where now $K' := K'(x_0) \subset \overline{\Omega}^c \cup \{x_0\}$.

Remark 3. Actually, we haven't proved that the constant $(\omega_n I_n)/8\alpha\pi^{n/2}$ is $1/2$. The same proof as in Theorem 2 gives the constant

$-(\omega_n I_n)/8\alpha\pi^{n/2}$ when the limit is taken outside Ω. However, it is proved in ([**5**, Theorem 2]), using a different technique, that the jump across the boundary is $-\varphi$. Then the proof of Theorem 2 and the previous result imply

$$\lim_{\substack{x\to x_0\\ x\in K'}} \langle \nabla_x S\varphi(x,t), \nu(x_0)\rangle - \lim_{\substack{x\to x_0\\ x\in K}} \langle \nabla_x S\varphi(x,t), \nu(x_0)\rangle = -2\frac{\omega_n I_n}{8\alpha\pi^{n/2}}\varphi(x_0,t) = -\varphi(x_0,t).$$

Hence the constant is $1/2$.

Remark 4. Although we consider in this paper the single layer potential, this remark concerns the double layer potential. The same technique as in the proof of Theorem 2 applies also for the proof of the boundary values for the double layer potential. We can prove that the double layer potential with continuous density φ can be continuously extended from Q_T to $\overline{\Omega}\times(0,T)$ with limiting values

$$u(x,t) = -\frac{1}{2}\varphi(x,t) + \int_0^t \int_\Gamma \partial_{\nu(y)} G(x-y,t-\tau)\varphi(y,\tau)\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau, \quad (x,t)\in\Sigma_T,$$

where the integral on the right hand side exists as an improper integral.

4. Hölder continuity. Finally, we concentrate on the proof of Hölder continuity of the single layer potential. We use the same technique as in Theorems 2 and 3 of [**8**].

Theorem 3. *Let $\varphi \in L^\infty(\Sigma_T)$. Then the single layer potential is Hölder continuous in $\overline{Q_T}$ for any $0<\kappa<1$, i.e.,*

$$|(S\varphi)(x,t) - (S\varphi)(x_1,t_1)| \le C \sup|\varphi|(|x-x_1|^\kappa + |t-t_1|^{\alpha\kappa/2})$$

for all $(x,t),(x_1,t_1)\in\overline{Q_T}$.

Proof. As in ([**8**, Theorems 2 and 3]), we divide the proof into two parts. At first we show that

$$|(S\varphi)(x,t) - (S\varphi)(x_1,t)| \le C\sup|\varphi|\cdot|x-x_1|^\kappa$$

and then that

$$|(S\varphi)(x,t) - (S\varphi)(x,t_1)| \leq C \sup|\varphi| \cdot |t-t_1|^{\alpha\kappa/2}$$

for any $0 < \kappa < 1$.

We denote the distance between x and x_1 by δ. Let x_0 be the closest point to x on Γ. It is enough to prove the result for x, x_1 in a neighborhood of Γ. Also, we may assume that δ is so small that the orthogonal projection $\pi : \Gamma_\delta = B(x, 2\delta) \cap \Gamma \to T(x_0)$ is one-to-one. Let I_δ and J_δ be as in the proof of Theorem 2. We write

$$(S\varphi)(x,t) = I_\delta(x,t) + J_\delta(x,t), \tag{46}$$

$$(S\varphi)(x_1,t) = I_\delta(x_1,t) + J_\delta(x_1,t). \tag{47}$$

Again, we have to prove the estimates for small z and for big z separately. If $z \geq 1$, we use the estimate (12) and get

$$|I_\delta(x,t)| \leq Ct^{-(\alpha n/2)+\alpha+\alpha\gamma} \sup|\varphi| \int_{\Gamma_\delta} \frac{\mathrm{d}\sigma(y)}{|x-y|^{2\gamma}}, \tag{48}$$

where we have chosen γ as in (13). We suppose that δ is so small that $\cos(\nu(y), \nu(x_0)) \geq 1/2$. Then, noting that $\mathrm{d}\sigma(y) = 1/[\cos(\nu(x_0), \nu(y))]\,\mathrm{d}\sigma'(y')$, we obtain

$$\begin{aligned} |I_\delta(x,t)| &\leq 2C't^{(\alpha/2)(1-\kappa)} \sup|\varphi| \int_{\Gamma'_\delta} \frac{\mathrm{d}\sigma'(y')}{|x_0-y'|^{n-1-\kappa}} \\ &\leq 2C't^{(\alpha/2)(1-\kappa)} \sup|\varphi| \int_0^{2\delta} \omega_{n-1} r^{\kappa-1}\,\mathrm{d}r \\ &\leq CT^{(\alpha/2)(1-\kappa)} \sup|\varphi| \cdot |x-x_1|^\kappa, \end{aligned} \tag{49}$$

where we have denoted $\kappa = -2\gamma + n - 1$. Note that (13) is equivalent with the condition $0 < \kappa < 1$.

When $z \leq 1$, we use the same majorants for the kernel as in the proof of Theorem 2 and get the same bound as above.

Similarly for $I_\delta(x_1,t)$ we have

$$\begin{aligned} |I_\delta(x_1,t)| &\leq CT^{(\alpha/2)(1-\kappa)} \sup|\varphi| \int_0^{3\delta} \omega_{n-1} r^{\kappa-1}\,\mathrm{d}r \\ &\leq CT^{(\alpha/2)(1-\kappa)} \sup|\varphi| \cdot |x-x_1|^\kappa. \end{aligned}$$

For the first part of the proof it remains to show the same bound for $|J_\delta(x,t) - J_\delta(x_1,t)|$. We divide this difference into two parts: I_1 and I_2 depending on whether $z \geq 1$ or $z \leq 1$. We start with the case $z \geq 1$. An application of the Mean Value Theorem gives the following majorant for the difference $|G(x-y,t-\tau) - G(x_1-y,t-\tau)|$,

$$C|\widetilde{x}-y|\cdot|x-x_1|(t-\tau)^{-(\alpha n/2)-1}\exp\{-\sigma t^{-(\alpha/2-\alpha)}|\widetilde{x}-y|^{2/(2-\alpha)}\}, \tag{50}$$

where $\widetilde{x}$ lies in the line between (x and x_1. If $y \in \Gamma \setminus \Gamma_\delta$, we have $|\widetilde{x}-y| \sim |x-y|$. This fact, integration and substitution $\tau \leftrightarrow \mu = |x-y|^{2/(2-\alpha)}(t-\tau)^{-\alpha/(2-\alpha)}$ give us

$$\begin{aligned} |I_1| &\leq C|x-x_1|\sup|\varphi|\int_0^\infty \mu^{(2-\alpha/2)n-1}\exp\{-\sigma_0\mu\}\,\mathrm{d}\mu\int_{\Gamma\setminus\Gamma_\delta}\frac{\mathrm{d}\sigma(y)}{|x-y|^{n-1}} \\ &= C_0\Gamma((2-\alpha)n/2)|x-x_1|\int_{\Gamma\setminus\Gamma_\delta}\frac{\mathrm{d}\sigma(y)}{|x-y|^{n-1}}. \end{aligned} \tag{51}$$

Then, proceeding as in the proof of Theorem 2 in [**8**], we get the desired result $|I_1| \leq C|x-x_1|^\kappa$.

When $z \leq 1$, we have to treat the cases for different n's separately. As an example we take $n = 3$ (the other cases can be handled by the same way). Note that in this case $\tau \leq t - |x-y|^{2/\alpha} \leq t - (2\delta)^{2/\alpha}$. As before, an application of the Mean Value Theorem, formula (15) and the asymptotic behavior of the Fox H-functions yield the estimate

$$|I_2| \leq C|x-x_1|\sup|\varphi|\int_0^{t-(2\delta)^{2/\alpha}}\int_{\Gamma\setminus\Gamma_\delta}(t-\tau)^{-\alpha-1}\,\mathrm{d}\tau\,\mathrm{d}\sigma(y),$$

where we have used the fact that $|\widetilde{x}-y| \sim |x-y|$ for $y \in \Gamma\setminus\Gamma_\delta$. Since $z \leq 1$, we have $(t-\tau)^{-\alpha} \leq |x-y|^{-2}$ and $t \geq (2\delta)^{2/\alpha}$, which implies

$$|I_2| \leq C\sup|\varphi|\delta\log(2\delta)\int_{\Gamma\setminus\Gamma_\delta}\frac{\mathrm{d}\sigma(y)}{|x-y|^2}.$$

Then, proceeding as in the proof of Theorem 2 in [**8**], we have

$$|I_2| \leq C_1\delta(|\log\delta|)^2 + C_2\delta|\log\delta|,$$

which finishes the first part of the proof since $\delta^{\varepsilon}|\log\delta|^{\beta} \to 0$ as $\delta \to 0$ for any $\beta, \varepsilon > 0$.

For the second part of the proof we take $x \in \overline{\Omega}$ and $0 \le t_1 < t \le T$. We write the difference of the potential at points (x,t) and (x,t_1) as follows

$$\begin{aligned}(S\varphi)(x,t) &- (S\varphi)(x,t_1)\\ &= \int_{t_1}^{t}\int_{\Gamma} G(x-y,t-\tau)\varphi(y,\tau)\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau\\ &\quad + \int_{0}^{t_1}\int_{\Gamma}(G(x-y,t-\tau)-G(x-y,t_1-\tau))\varphi(y,\tau)\,\mathrm{d}\sigma(y)\,\mathrm{d}\tau.\end{aligned}$$

Denote the integrals on the right hand side by I_1 and I_2. For the first integral an application of formula (12) gives an estimate

$$|I_1| \le C\sup|\varphi||t-t_1|^{\kappa'}, \quad \alpha < \kappa' < \frac{3\alpha}{2},$$

in the case $z \ge 1$. The case $z \le 1$ can be treated similarly. We fix $\delta > 0$ and divide the second integral into two parts as in the proof of the first part of the theorem, $I_2 = I_\delta + J_\delta$. Similarly as (49) we get

$$|I_\delta| \le C\sup|\varphi| t^{(\alpha/2)(1-\kappa)}\delta^{\kappa}. \tag{52}$$

For J_δ we use the Mean Value Theorem with respect to the time variable. Again, we have to treat the cases $z_1 := |x-y|^2/4(t_1-\tau)^{-\alpha} \le 1$ and $z_1 \ge 1$ separately. The time derivative of the fundamental solution is

$$\partial_t G(x,t) = \pi^{-n/2}t^{\alpha-2}|x|^{-n}\big\{(\alpha-1)H_{(1)}(z) + \alpha H_{(2)}(z)\big\}. \tag{53}$$

In the case $z_1 \ge 1$ Lemma 1 implies the estimate

$$|\partial_t G(x,t)| \le Ct^{\alpha-2}|x|^{-n}z^{((n/2)+1-\alpha)/(2-\alpha)}\exp\{-\sigma z^{1/(2-\alpha)}\}$$

with $\sigma < \alpha^{\alpha/(2-\alpha)}(2-\alpha)$. We estimate $((n/2)+1-\alpha)/(2-\alpha) \le (n/2)+(1/2)$ and use the fact that $z^{\gamma}\exp\{-\sigma z^{1/(2-\alpha)}\}$ is uniformly bounded for any $\gamma > 0$, which implies

$$|\partial_t G(x,t)| \le Ct^{\alpha\gamma+(\alpha/2)-(\alpha n/2)-2}|x|^{1-2\gamma}. \tag{54}$$

Now $z_1 \geq 1$, so $\tau \geq t_1 - |x-y|^{2/\alpha}$. We divide the time integration in J_δ into two parts $\int_0^{t_1} = \int_0^{t_1-|x-y|^{2/\alpha}} + \int_{t_1-|x-y|^{2/\alpha}}^{t_1}$. Denoting the corresponding parts by $J_\delta^{(1)}$ and $J_\delta^{(2)}$ and using the Mean Value Theorem we have

$$\begin{aligned} |J_\delta^{(2)}| &\leq C \sup|\varphi||t-t_1| \int_{\Gamma\setminus\Gamma_\delta} |x-y|^{2-n-(2/\alpha)} \mathrm{d}\sigma(y) \\ &\leq C \sup|\varphi||t-t_1|\delta^{1-(2/\alpha)}, \end{aligned} \tag{55}$$

where we have chosen γ as

$$\frac{1}{\alpha} + \frac{n-1}{2} < \gamma < \frac{2}{\alpha} + \frac{n-1}{2}$$

guaranteeing the convergence of the integral with respect to the time variable.

Next we choose $\delta = c|t-t_1|^{\varepsilon}$, where c is taken so small that the orthogonal projection π from the surface neighborhood to the tangent hypersurface is one-to-one. Moreover, we choose ε such that (compare (52) and (55))

$$\delta^\kappa = C|t-t_1|^{\varepsilon\kappa} = C|t-t_1|^{1-\varepsilon((2/\alpha)-1)}.$$

Then we have $\varepsilon = 1/(2/\alpha + \kappa - 1) > (\alpha/2)$, which implies the desired estimate for $J_\delta^{(2)}$.

In $J_\delta^{(1)}$ we have to treat the cases for different n's separately. As an example we take $n = 3$. Then (9) in Lemma 1 and (53) give us

$$|\partial_t G(x,t)| \leq C t^{-(\alpha/2)-2}.$$

Using this and the Mean Value Theorem we get

$$\begin{aligned} |J_\delta^{(1)}| &\leq C|t-t_1| \int_{\Gamma\setminus\Gamma_\delta} \int_0^{t_1-|x-y|^{2/\alpha}} (t_1-\tau)^{-(\alpha/2)-2} \mathrm{d}\tau \, \mathrm{d}\sigma(y) \\ &\leq C|t-t_1| \int_{\Gamma\setminus\Gamma_\delta} |x-y|^{-2/\alpha-1} \mathrm{d}\sigma(y). \end{aligned}$$

Similarly as (55), we obtain

$$|J_\delta^{(1)}| \leq C|t-t_1|\delta^{1-(2/\alpha)}. \tag{56}$$

Choosing $\delta = c|t - t_1|^\varepsilon$ and choosing ε such that $1 - \varepsilon(\frac{2}{\alpha} - 1) = \varepsilon\kappa$, i.e., $\varepsilon = 1/(2/\alpha + \kappa - 1) > (\alpha/2)$, which finishes the proof in the case $n = 3$. The proof for the other cases of n's are similar. $\square$

Remark 5. Note that the proof of Theorem 3 shows actually that the ratio between time and spatial regularity is $1/(2/\alpha + \kappa - 1)$ which is greater than $\alpha/2$. Compare this result with the fact that in the case of the single layer potential for the heat equation this ratio is always $1/2$. $\square$

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