JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 23, Number 3, Fall 2011

ADDITIONAL ORDER CONVERGENCE IN QUALOCATION FOR ELLIPTIC BOUNDARY INTEGRAL EQUATIONS

R.D. GRIGORIEFF

Communicated by Kendall Atkinson

Dedicated to I.H. Sloan

ABSTRACT. In this paper additional order of convergence is studied in the qualocation method for elliptic periodic pseudodifferential operators. Splines with multiple knots are used as trial and test spaces. Results are proved for both constant and variable coefficients.

1. Introduction. In this paper we study the qualocation method for pseudodifferential operators of the form

$$L = L_0 + L_1, \tag{1.1}$$

where

$$L_0 v(x) := \sum_{n=-\infty}^{\infty} \sigma_0(x,n)\widehat{v}(n)e^{i2\pi nx} \quad \text{for } x \in \mathbf{T}. \tag{1.2}$$

Here $\mathbf{T} := \mathbf{R}\backslash\mathbf{Z}$ is the one-dimensional torus of length 1 and

$$\widehat{v}(n) = \int_{\mathbf{T}} v(x)e^{-i2\pi nx}\,dx \quad \text{for } n \in \mathbf{Z}$$

are the complex Fourier coefficients of a 1-periodic distribution $v : \mathbf{T} \to \mathbf{R}$ so that

$$v(x) = \sum_{n=-\infty}^{\infty} \widehat{v}(n)e^{i2\pi nx} \quad \text{for } x \in \mathbf{T}.$$

Received by the editors on August 3, 2009.
DOI:10.1216/JIE-2011-23-3-383

The symbol σ_0 has the form

$$\sigma_0(x,\xi) := a^+(x)|\xi|^\beta + a^-(x)\text{sign}\,(\xi)|\xi|^\beta \quad \text{for } x \in \mathbf{T} \text{ and } 0 \neq \xi \in \mathbf{R} \tag{1.3}$$

with coefficients a^+ and a^- in $C^\infty(\mathbf{T})$, where $\beta \in \mathbf{R}$ is the order of L_0. We assume σ_0 to be normalized by $\sigma_0(x,0) = 1$ for $x \in \mathbf{T}$. L is assumed to be elliptic, i.e., $\sigma_0(x,\xi) \neq 0$ for $x \in \mathbf{T}$ and $|\xi| = 1$, and to have index $\kappa = 0$, where

$$\kappa := \frac{1}{2\pi}\left[\arg\frac{a^+(x)+a^-(x)}{a^+(x)-a^-(x)}\right]_0^1$$

is the winding number of the closed curve $(a^+ + a^-)/(a^+ - a^-)$ in the complex plane. It is known that then $L_0 : H^s \to H^{s-\beta}$ is a Fredholm operator with index 0 for all $s \in \mathbf{R}$, where $H^s = H^s(\mathbf{T})$ is the usual Sobolev space of periodic distributions v equipped with the norm

$$\|v\|_s := \left(\sum_{n=-\infty}^{\infty} \langle n\rangle^{2s}\,|\widehat{v}(n)|^2\right)^{1/2}, \quad \text{where } \langle n\rangle := \begin{cases} 1 & \text{if } n = 0, \\ |n| & \text{if } n \neq 0. \end{cases} \tag{1.4}$$

It is at least assumed that L_1 maps $H^s \to H^{s-\beta+\delta}$ for some $\delta > 0$ and all $s \in \mathbf{R}$, and hence L is also Fredholm with index 0.

We consider the discretization of (1.1) by qualocation using splines with multiple knots on equidistant meshes as test and trial spaces. Let r, M, N with $1 \leq M \leq r$ be positive integers. We define the set of knots

$$\pi_h := \{x_j = jh, j = 0, \dots, N-1\}, \quad h \in \mathcal{H} := \{1/N, N \in \mathbf{N}\},$$

and denote by $S_h^{r,M}$ the space of 1-periodic splines of order r with M-fold breakpoints in π_h. $S_h^{r,M}$ is a subspace of C^{r-M-1} of dimension MN, where $C^k = C^k(\mathbf{T})$ is the space of 1-periodic k times continuously differentiable functions (with C^{-1} meaning piecewise continuity with jumps only at the knots in π_h). By $\mathcal{H}_1$ we denote a final section of the null-sequence of stepsizes $\mathcal{H}$, not necessarily the same at different occurences.

Qualocation is based on a composite quadrature rule

$$Q_N f = h \sum_{k=0}^{N-1} \sum_{j=1}^{J} \omega_j f(x_{k,j}), \quad x_{k,j} := x_k + h\xi_j, \tag{1.5}$$

derived from the basic quadrature formula

$$Qf = \sum_{j=1}^{J} \omega_j f(\xi_j), \tag{1.6}$$

where the quadrature points $\{\xi_j\}$ and weights $\{\omega_j\}$ satisfy

$$0 \le \xi_1 < \xi_2 < \cdots < \xi_J < 1, \quad J \ge M, \ \sum_{j=1}^{J} \omega_j = 1, \ \omega_j > 0. \tag{1.7}$$

Associated with the quadrature rule we define an inner product

$$(v_h, w_h)_h := Q_N(v_h \overline{w}_h) \tag{1.8}$$

on the linear space W_h of 'grid' functions v_h and w_h, which are functions defined on the grid

$$\pi'_h := \{x_{k,j}, \ k = 0, \ldots, N-1, \ j = 1, \ldots, J\}. \tag{1.9}$$

The inner product in (1.8) can be thought of as an approximation to

$$(v, w)_0 := \int_0^1 v(x)\overline{w}(x)\, dx \quad \text{for } v, w \in L^2(\mathbf{T}). \tag{1.10}$$

In [**5**] we derived conditions that $(\cdot,\cdot)_h$ is an inner product on $S_h^{r,M}$ that are recalled in Section 2.

We choose now splines of order r as trial space and splines of a possibly different order r' as a test space. The qualocation method for solving the equation $Lu = f$ approximately is to find $u_h \in S_h^{r,M}$ such that

$$(Lu_h, z_h)_h = (f, z_h)_h \quad \text{for all } z_h \in S_h^{r',M}. \tag{1.11}$$

This method can be viewed as a discrete version of the Petrov-Galerkin method. Conditions to make (1.11) well-defined can be found in Section 5.

Based on approximation properties of periodic multiple knot splines obtained in [**5**], which are partly reviewed in Section 5, the authors proved basic convergence results for the qualocation method (1.11) in [**6**]. The main convergence result obtained there is reproduced as Theorem A.12.

It was first noted in [**2**] that, by specifically designed quadrature rules, it is possible to obtain higher convergence order than the standard one from (A.18). This feature was further developed in [**9, 10**] for more general pseudodifferential operators, also with variable coefficients, using smoothest splines. The aim of the present paper is to prove additional convergence order for the qualocation method using multiple knot splines. For operators with constant principal part which are discretized by collocation, such results can be found in [**7**]. Collocation is a special case of qualocation, obtained for $J = M$.

The principal results in this paper are for the constant coefficient case Theorem 2.4 and for the variable coefficient case Theorem 3.2. The additional convergence order results given here are, as in [**2, 7**], based on assumptions on the asymptotic behavior as $y \to 0$ of certain characteristic functions obtained when applying the qualocation method to L_0 (see Definitions 2.2 and 3.1). This differs from the analysis in [**10**] for smoothest splines, where the degree of exactness of certain quadrature formulas is directly used. In the last section we specialize the condition of additional order from Section 3 to the case of symmetric quadrature formulas thereby rediscovering the conditions in [**10**] if $M = 1$.

2. Additional order of convergence, constant coefficients. In this section we prove additional order convergence results for the case that L_0 has constant coefficients. We need some definitions from [**5, 6**].

Related to the operator L_0 are the following characteristic functions

$$\widetilde{\Omega}_k(\xi, y; x) := \sum_{\ell \neq 0} \sigma_0(x, y+\ell) \frac{\ell^{k-1}}{(y+\ell)^r} \Phi_\ell(\xi), \tag{2.1}$$

$$\Omega_1(\xi, y; x) := 1 + (\sigma_0(x, y))^{-1} y^r \widetilde{\Omega}_1(\xi, y; x) \quad \text{for } y \neq 0, \tag{2.2}$$

$$\Omega_1(\xi, 0; x) := 1, \tag{2.3}$$

$$\Omega_k(\xi, y; x) := \widetilde{\Omega}_k(\xi, y; x) \quad \text{for } k = 2, \dots, M. \tag{2.4}$$

We omit the variable x in the notation if σ_0 is independent of x. For Ω_k to be well defined we assume throughout the paper that

$$\beta + M < r. \tag{2.5}$$

Then it follows that the above Fourier series are absolutely convergent and $\Omega_1(\xi, \cdot; x)$ is continuous at $y = 0$ with value equal to 1.

Of special interest is the case that L_0 is the identity, i.e., $\beta = 0$ and $\sigma_0 \equiv 1$. With this setting the notation Δ in place of Ω is used. The condition (2.5) is in this case relaxed to $M \leq r$, where now the Fourier series are always understood to be the limit as $L \to \infty$ of the symmetric partial sums extended from $-L$ to L. With the aid of these functions the following basis of the spline space $S_h^{r,M}$ was found (see [**5**, Proposition 2.1]):

$$\psi_{k,\mu}(x) := \Phi_\mu(x)\Delta_k\left(Nx, \frac{\mu}{N}\right) \quad \text{for } k = 1, \dots, M \quad \text{and} \quad \mu \in \Lambda_h, \tag{2.6}$$

where

$$\Lambda_h := \left(-\frac{N}{2}, \frac{N}{2}\right] \cap \mathbf{Z} \quad \text{with } Nh = 1. \tag{2.7}$$

As test space the splines $S_h^{r',M}$ with r' possibly different from r are used, and we denote the corresponding quantities by $\psi'_{k,\mu}$, Δ'_k and $\widetilde{\Delta}'_k$.

A central role in the analysis is played by the so-called qualocation projection $R_h : W_h \to S_h^{r,M}$ defined by

$$(R_h f_h, \psi)_h = (f_h, \psi)_h \quad \text{for all } \psi \in S_h^{r,M}. \tag{2.8}$$

The following condition (R) is necessary and sufficient that the positive semidefinite sesquilinear form $(\cdot,\cdot)_h$ is an inner product on $S_h^{r,M}$ (see [**5**, Lemma 3.1]), thus guaranteeing that R_h is well-defined.

Definition 2.1. We say that the condition (R) or (R$'$) is satisfied if the functions $\{\Delta_k(\cdot, y), k = 1, \dots, M\}$ or $\{\Delta'_k(\cdot, y), k = 1, \dots, M\}$, respectively, restricted to the set (1.7) of quadrature points are linearly independent for $|y| \le 1/2$. □

The stability of the qualocation method is determined by the ellipticity of the numerical symbol defined as the $M \times M$-matrix $D = D(y; x)$ for $|y| \le 1/2$ and $x \in \mathbf{T}$ with elements

$$[D(y; x)]_{k,\ell} := Q\big(\Omega_\ell(\cdot, y; x), \Delta'_k(\cdot, y)\big) \quad \text{for } k, \ell = 1, \dots, M. \tag{2.9}$$

The numerical symbol is said to be elliptic if $D(y; x)$ is nonsingular for $x \in \mathbf{T}$ and $|y| \le 1/2$.

Throughout the paper we assume that conditions (R) and (R$'$) are satisfied and that the numerical symbol D is elliptic. We need the following definition.

Definition 2.2. Let $b > 0$. The qualocation method is said to have additional order b of convergence if

$$\Big| \sum_{k=1}^{M} D(y)^{-1}_{1,k} Q\big(\widetilde{\Omega}_1(\cdot, y), \Delta'_k(\cdot, y)\big) \Big| \le C|y|^b \quad \text{as } y \to 0, \tag{2.10}$$

where

$$Q(v, w) := \sum_{j=1}^{J} \omega_j v(\xi_j) \overline{w}(\xi_j). \tag{2.11}$$

Remark 2.3. If $0 < b \le r'$, Condition (2.10) is equivalent to

$$\begin{aligned} \Big| D(y)^{-1}_{1,1} Q\big(\widetilde{\Omega}_1(\cdot, y), 1\big) + \sum_{k=2}^{M} D(y)^{-1}_{1,k} Q\big(\widetilde{\Omega}_1(\cdot, y), \Delta'_k(\cdot, y)\big) \Big| \\ \le C|y|^b \text{ as } y \to 0. \end{aligned}$$

Proof. Since $\Delta'_1(\cdot, y) = 1 + y^{r'} \widetilde{\Delta}'_1(\cdot, y)$ it is sufficient to show that

$$\big| D(y)^{-1}_{1,1} Q(\widetilde{\Omega}_1(\xi, y), \widetilde{\Delta}'_1(\xi, y)) \big| \le C \quad \text{for } \xi \in \mathbf{R} \text{ and } |y| \le \frac{1}{2}.$$

Due to the ellipticity the factor $D(y)_{1,1}^{-1}$ is bounded for $|y| \leq 1/2$. Also $\widetilde{\Omega}_1(\xi, y)$ is bounded since the series in its definition is uniformly convergent with respect to (ξ, y). If $r' > 1$ also the series defining $\widetilde{\Delta}_1'(\xi, \eta)$ is uniformly convergent with respect to (ξ, y), and hence $\widetilde{\Delta}_1'(\cdot, \cdot)$ is bounded. We are left with the case $r' = 1$. Then $\widetilde{\Delta}_1'$ can be written in the form

$$\begin{aligned}\widetilde{\Delta}_1'(\xi, y) &= \sum_{\ell=1}^{\infty} \left(\frac{1}{y+\ell} \Phi_\ell(\xi) + \frac{1}{y-\ell} \Phi_{-\ell}(\xi) \right) \\ &= \sum_{\ell=1}^{\infty} \frac{2y}{y^2 - \ell^2} \cos 2\pi\ell\xi - \sum_{\ell=1}^{\infty} \frac{2i\ell}{y^2 - \ell^2} \sin 2\pi\ell\xi \\ &= \sum_{\ell=1}^{\infty} \frac{2y}{y^2 - \ell^2} \cos 2\pi\ell\xi - \sum_{\ell=1}^{\infty} \frac{2iy^2}{\ell(y^2 - \ell^2)} \sin 2\pi\ell\xi + 2\pi i \left(\frac{1}{2} - \xi \right)\end{aligned}$$

for $\xi \in (0, 1)$ and $|y| \leq 1/2$, from which the boundedness can be seen. In the last formula line it was used that

$$\sum_{\ell=1}^{\infty} \frac{2}{\ell} \sin 2\pi\ell\xi = 2\pi \left(\frac{1}{2} - \xi \right) \quad \text{for } \xi \in (0, 1). \qquad \square$$

In the case of collocation Definition 2.2 coincides with the one given in [**7**, Equation (2.12)]. If $M = 1$ Definition 2.2 is equivalent to

$$\left| Q\big(\widetilde{\Omega}_1(\cdot, y), \Delta_1'(\cdot, y)\big) \right| \leq C|y|^b \quad \text{as } y \to 0,$$

which can be identified with condition [**2**, Equation (3.9)].

The main result of this section is stated in the following theorem.

Theorem 2.4. *Assume that conditions* (R) *and* (R′) *are satisfied and that the numerical symbol D is elliptic. Let $L : H^\beta \to H^0$ be injective and*

$$s < r - M + \frac{1}{2}, \quad \beta - b \leq s \leq \beta < t - \frac{1}{2}, \quad s \leq t \leq r. \tag{2.12}$$

The qualocation equations (1.11) *are uniquely solvable for $h \in \mathcal{H}_1$. Let the qualocation method have additional order $0 < b \leq r - \beta$ of* *convergence, and let* L_1 *in* (1.1) *be a bounded map from* $H^q \to H^{q+b-\beta}$ *for all* $q \in \mathbf{R}$. *Then, if* $u \in H^{t-s+\beta}$, *the following error estimate with additional order of convergence holds*:

$$\|u - u_h\|_s \le Ch^{t-s}\|u\|_{t-s+\beta} \quad \text{for } h \in \mathcal{H}_1. \tag{2.13}$$

The corresponding additional order convergence results in [**2, 7**] are contained as special cases. The highest rate of convergence given by (2.13) occurs when $s = \beta - b$ and $t = r$, in which case

$$\|u - u_h\|_{\beta-b} \le Ch^{r-\beta+b}\|u\|_{r+b} \quad \text{for } u \in H^{r+b} \text{ and } h \in \mathcal{H}_1, \tag{2.14}$$

while the highest rate from the basic Convergence Theorem A.12 is only

$$\|u - u_h\|_\beta \le Ch^{r-\beta}\|u\|_r \quad \text{for } u \in H^r \text{ and } h \in \mathcal{H}_1. \tag{2.15}$$

The proof of the theorem is prepared by a couple of lemmas. In the next lemma we split the error $u - u_h$ in four parts which will be separately estimated afterwards. We always assume in this section that the symbol σ_0 has constant coefficients. A prime on a sum sign indicates that the $m = \mu$ term is to be omitted.

Lemma 2.5. *Assume* $s < r - M + (1/2)$, *let* $u \in H^s$, *and represent* $u_h \in S_h^{r,M}$ *in the form*

$$u_h = \sum_{k=1}^{M} \sum_{\mu \in \Lambda_h} c_{k,\mu}\psi_{k,\mu}. \tag{2.16}$$

Then

(2.17)
$$\begin{aligned}\|u - u_h\|_s^2 \le C\Big(&\sum_{\mu\in\Lambda_h} \langle\mu\rangle^{2s}\,|\widehat{u}(\mu) - c_{1,\mu}|^2 + N^{2s}\sum_{k=2}^{M}\sum_{\mu\in\Lambda_h}|c_{k,\mu}|^2\Big)\\ &+ \sum_{\mu\in\Lambda_h}\sum_{m\equiv\mu}{}' \langle m\rangle^{2s}\,|\widehat{u}(m)|^2\\ &+ \sum_{\mu\in\Lambda_h}\sum_{m\equiv\mu}{}' \langle m\rangle^{2s}\,|c_{1,\mu}\widehat{\psi}_{1,\mu}(m)|^2.\end{aligned}$$

Proof. Denote by $u_h^{(2)}$ the part of u_h in (2.16) in which the summation is only extended from $k = 2, \dots, M$. With the aid of Proposition A.3 applied to $u_h^{(2)}$ and with $\widehat{\psi}_{1,\mu}(\nu) = \delta_{\mu,\nu}$ for $\mu, \nu \in \Lambda_h$ (see (A.1)) we obtain

$$\begin{aligned}\|u - u_h\|_s^2 &\leq 2 \sum_{m\in\mathbf{Z}} \langle m\rangle^{2s} \left|\widehat{u}(m) - (\widehat{u}_h - \widehat{u}_h^{(2)})(m)\right|^2 + 2\|u_h^{(2)}\|_s^2 \\ &\leq C\Bigg(\sum_{\mu\in\Lambda_h} \sum_{m\equiv\mu}{}' \langle m\rangle^{2s} \left(|\widehat{u}(m)|^2 + \left|\sum_{\nu\in\Lambda_h} c_{1,\nu}\widehat{\psi}_{1,\nu}(m)\right|^2\right) \\ &\qquad + \sum_{\mu\in\Lambda_h} \langle\mu\rangle^{2s} |\widehat{u}(\mu) - c_{1,\mu}|^2 + N^{2s} \sum_{k=2}^{M} \sum_{\mu\in\Lambda_h} |c_{k,\mu}|^2 \Bigg).\end{aligned}$$

The assertion now follows by taking into account that $\widehat{\psi}_{1,\nu}(m) = 0$ if $\nu \not\equiv m$. ◻

Let us point out that it is important for upcoming estimates (see Lemmas 2.7 and 2.9) that the low order frequencies in the error $u - u_h$ are arranged such that the coefficient combination $\widehat{u}(\mu) - c_{1,\mu}$ appears.

Lemma 2.6. *Assume $s \leq \beta < t - (1/2)$ and $u \in H^{t-s+\beta}$. Then*

$$\sum_{\mu\in\Lambda_h} \left(\langle\mu\rangle^{s-\beta} \sum_{m\equiv\mu}{}' |\sigma_0(m)\widehat{u}(m)| \right)^2 \leq Ch^{2(t-s)}\|u\|_{t-s+\beta}^2. \tag{2.18}$$

Proof. Taking the form (1.3) of σ_0 into account we have the estimates

$$\begin{aligned}\sum_{m\equiv\mu}{}' |\sigma_0(m)\widehat{u}(m)| &\leq C \sum_{\ell\neq 0} |\mu + \ell N|^{\beta} |\widehat{u}(\mu + \ell N)| \\ &= CN^{\beta} \sum_{\ell\neq 0} \left|\frac{\mu}{N} + \ell\right|^{\beta} |\widehat{u}(\mu + \ell N)| \\ &\leq CN^{\beta} \left(\sum_{\ell\neq 0} \left|\frac{\mu}{N} + \ell\right|^{2(s-t)} \right)^{1/2}\end{aligned}$$

$$\times \left(\sum_{\ell \neq 0} \left| \frac{\mu}{N} + \ell \right|^{2(t-s+\beta)} |\widehat{u}(\mu + \ell N)|^2 \right)^{1/2}$$

$$\leq C N^{s-t} \left({\sum_{m \equiv \mu}}' |m|^{2(t-s+\beta)} |\widehat{u}(m)|^2 \right)^{1/2},$$

where the first sum in the third line converges uniformly for $\mu \in \Lambda_h$ since $2(s-t) < -1$ and $|\mu/N| \leq 1/2$. Note that $\langle \mu \rangle \geq 1$ and $s - \beta \leq 0$. The assertion is thus obtained after multiplying by $\langle \mu \rangle^{s-\beta}$, squaring and summing up with respect to μ. □

Lemma 2.7. *Assume $\beta + (1/2) < t$, and let $u \in H^t$. The qualocation solution u_h in the case $L = L_0$, written in the form* (2.16), *satisfies the linear system*

$$D\left(\frac{\mu}{N}\right) c_\mu^* = d_\mu^* \quad \textit{for } \mu \in \Lambda_h, \tag{2.19}$$

where

(2.20)
$$c_{k,\mu}^* = \begin{cases} \sigma_0(\mu)(c_{1,\mu} - \widehat{u}(\mu)) & \textit{for } k = 1, \\ N^\beta c_{k,\mu} & \textit{for } k = 2, \ldots, M, \end{cases}$$

(2.21)
$$d_\mu^* = d_\mu^{(1)} + d_\mu^{(2)},$$

(2.22)
$$d_{k,\mu}^{(1)} = -\sigma_0(\mu) \left(\frac{\mu}{N}\right)^r \sigma_0 \left(\frac{\mu}{N}\right)^{-1} Q\left(\widetilde{\Omega}_1 \left(\cdot, \frac{\mu}{N}\right), \Delta_k' \left(\cdot, \frac{\mu}{N}\right)\right) \widehat{u}(\mu),$$

(2.23)
$$d_{k,\mu}^{(2)} = {\sum_{m \equiv \mu}}' \sigma_0(m) Q\left(\Phi_{(m-\mu)/N}, \Delta_k' \left(\cdot, \frac{\mu}{N}\right)\right) \widehat{u}(m) \quad \textit{for } k = 1, \ldots, M.$$

Proof. The coefficients of the qualocation solution satisfy the linear system

$$D\left(\frac{\mu}{N}\right) \widetilde{c}_\mu = d_\mu \quad \text{for } \mu \in \Lambda_h, \tag{2.24}$$

where $\widetilde{c}_\mu$ and d_μ are from (A.16) and (A.17), respectively. Taking (A.4) (with $\psi_{\ell,\mu}$ replaced by $\psi'_{\ell,\mu}$) into account, the components of d_μ can be written as

$$
\begin{aligned}
d_{k,\mu} &= (L_0 u, \psi'_{k,\mu})_h = \sum_{m\in\mathbf{Z}} \sigma_0(m)\widehat{u}(m)(\Phi_m, \psi'_{k,\mu})_h \\
&= \sum_{m\equiv\mu} \sigma_0(m)\widehat{u}(m) Q\bigg(\Phi_{(m-\mu)/N}, \Delta'_k\Big(\cdot, \frac{\mu}{N}\Big)\bigg) \\
&= \sigma_0(\mu)\widehat{u}(\mu) Q\bigg(1, \Delta'_k\Big(\cdot, \frac{\mu}{N}\Big)\bigg) \\
&\quad + {\sum_{m\equiv\mu}}' \sigma_0(m)\widehat{u}(m) Q\bigg(\Phi_{(m-\mu)/N}, \Delta'_k\Big(\cdot, \frac{\mu}{N}\Big)\bigg).
\end{aligned}
$$

Subtracting $\sigma_0(\mu)\widehat{u}(\mu)$ from the variable $\widetilde{c}_{1,\mu}$ in the system (2.24) the equations (2.19)–(2.23) are now readily verified upon recalling the definition (2.9) and (2.1) of $D(y)$ and $\widetilde{\Omega}_1(\cdot, y)$, respectively. □

Lemma 2.8. *Assume $\beta + (1/2) < t \leq r$. Let $u \in H^t$ and $L = L_0$. If the qualocation solution u_h is written in the form* (2.16), *then*

$$
N^{2s} \sum_{k=2}^{M} \sum_{\mu\in\Lambda_h} |c_{k,\mu}|^2 \leq C h^{t-s} \|u\|_t. \tag{2.25}
$$

Proof. Consider the linear system (2.19). $D(y)$ is continuous for $|y| \leq 1/2$, and the ellipticity implies that the matrix $D(y)$ has a bounded inverse uniformly in y. Since $c_{k,\mu} = N^{-\beta} c^*_{k,\mu}$ for $k \geq 2$, we can derive the estimate (2.25) by bounding $N^{s-\beta} d^*_\mu$ correspondingly. Note that the quantities of the form $Q(\cdot,\cdot)$ in (2.22) and (2.23) are uniformly bounded. Thus, for $k = 1, \ldots, M$,

$$
\begin{aligned}
\sum_{\mu\in\Lambda_h} \left|N^{s-\beta} d^{(1)}_{k,\mu}\right|^2 &\leq C N^{2(s-\beta)} \sum_{\mu\in\Lambda_h} \left|\sigma_0(\mu)\Big(\frac{\mu}{N}\Big)^r \sigma_0\Big(\frac{\mu}{N}\Big)^{-1} \widehat{u}(\mu)\right|^2 \\
&= C N^{2(s-t)} \sum_{0\neq\mu\in\Lambda_h} \left|\Big(\frac{\mu}{N}\Big)^{r-t} \mu^t \widehat{u}(\mu)\right|^2 \\
&\leq C h^{2(t-s)} \|u\|_t^2,
\end{aligned}
$$

where we took $|\mu/N| \le 1/2$ and $r - t \ge 0$ into account and that, due to the ellipticity of L_0, $|\sigma_0(\mu)| \ge \gamma \langle\mu\rangle^\beta$ for some $\gamma > 0$. Also

$$\sum_{\mu\in\Lambda_h} \left|N^{s-\beta} d^{(2)}_{k,\mu}\right|^2 \le CN^{2(s-\beta)} \sum_{\mu\in\Lambda_h} \Big({\sum_{m\equiv\mu}}' |\sigma_0(m)\widehat{u}(m)|\Big)^2 \le Ch^{2(t-s)}\|u\|_t^2,$$

where we applied (2.18) with $s = \beta$. The two estimates together furnish the result. □

Lemma 2.9. *Assume $\beta - b \le s \le \beta < t - (1/2)$ and $t \le r$. Let $u \in H^{t-s+\beta}$ and $L = L_0$. Assume that the qualocation method has additional order b of convergence. If the qualocation solution u_h is written in the form* (2.16) *then*

$$\sum_{\mu\in\Lambda_h} \langle\mu\rangle^{2s} |\hat{u}(\mu) - c_{1,\mu}|^2 \le Ch^{2(t-s)}\|u\|^2_{t-s+\beta}. \tag{2.26}$$

Proof. We start from the system (2.24). Since D^{-1} is bounded we obtain, by virtue of the additional order of convergence, the bound

$$\Big|\sum_{k=1}^{M} D\Big(\frac{\mu}{N}\Big)^{-1}_{1,k} d^*_{k,\mu}\Big| \le C\bigg(\Big|\frac{\mu}{N}\Big|^{r+b}\Big|\widehat{u}(\mu)\sigma_0(\mu)\sigma_0\Big(\frac{\mu}{N}\Big)^{-1}\Big| + {\sum_{m\equiv\mu}}' |\sigma_0(m)\widehat{u}(m)|\bigg),$$

where d^*_μ is from (2.21). Recall that $|\sigma_0(\mu)| \ge \gamma \langle\mu\rangle^\beta$ holds for some $\gamma > 0$. Thus, inverting the left-hand side of (2.19) and multiplying the resulting first equation by $\langle\mu\rangle^s / |\sigma_0(\mu)|$ $(\le C\langle\mu\rangle^{s-\beta})$ we obtain

$$\langle\mu\rangle^s |c_{1,\mu} - \widehat{u}(\mu)| \le C\bigg(\langle\mu\rangle^s \Big|\frac{\mu}{N}\Big|^{r+b} \Big|\sigma_0\Big(\frac{\mu}{N}\Big)\Big|^{-1} |\widehat{u}(\mu)| + \langle\mu\rangle^{s-\beta} {\sum_{m\equiv\mu}}' |\sigma_0(m)\widehat{u}(m)|\bigg). \tag{2.27}$$

Now square both sides of (2.27), estimate the right-hand side using the sum of squares and sum up with respect to $\mu \in \Lambda_h$. The last quantity in the resulting right-hand side of (2.27) can be estimated with the aid of (2.18) in the form we need. The first one vanishes for $\mu = 0$ and can be bounded by

$$CN^{s-t}\left|\frac{\mu}{N}\right|^{s+b-\beta+r-t} |\mu^t \widehat{u}(\mu)|.$$

Taking $|\mu/N| \le (1/2)$ and $s+b-\beta+r-t \ge 0$ into account the desired bound is obtained. □

Lemma 2.10. *Assume $s \le t$ and $u \in H^t$. Then*

$$\sum_{\mu\in\Lambda_h} {\sum_{m\equiv\mu}}' \langle m\rangle^{2s} |\widehat{u}(m)|^2 \le Ch^{2(t-s)}\|u\|_t^2. \tag{2.28}$$

Proof. The assertion follows from the chain of relations

$$\begin{aligned}\sum_{\mu\in\Lambda_h} {\sum_{m\equiv\mu}}' \langle m\rangle^{2s} |\widehat{u}(m)|^2 &= \sum_{\mu\in\Lambda_h} {\sum_{m\equiv\mu}}' |m|^{2(s-t)}|m|^{2t}|\widehat{u}(m)|^2 \\ &= N^{2(s-t)} \sum_{\mu\in\Lambda_h} \sum_{\ell\neq 0} \left|\frac{\mu}{N} + \ell\right|^{2(s-t)} \\ &\qquad \times |\mu + \ell N|^{2t} |\widehat{u}(\mu+\ell N)|^2 \\ &\le Ch^{2(t-s)}\|u\|_t^2,\end{aligned}$$

where in the final step we used $|\mu/N + \ell|^{2(s-t)} \le C$ since $|\mu/N| \le (1/2)$ and $s - t \le 0$. □

Lemma 2.11. *Assume $s < r - M + (1/2)$ and $\beta \le t \le r$. If u_h is written in the form* (2.16), *then*

$$\sum_{\mu\in\Lambda_h} {\sum_{m\equiv\mu}}' \langle m\rangle^{2s} |c_{1,\mu}\widehat{\psi}_{1,\mu}(m)|^2 \le Ch^{2(t-s)}\|u\|_t^2. \tag{2.29}$$

Proof. From (A.1) we see that $\widehat{\psi}_{1,\mu}(m) = (\mu/m)^r$ for $m \equiv \mu$, $m \neq \mu$, and thus

$$\begin{aligned}\sum_{\mu\in\Lambda_h}\sum_{m\equiv\mu}{}' \langle m\rangle^{2s}\,|c_{1,\mu}\widehat{\psi}_{1,\mu}(m)|^2 &\\ &\le \sum_{\mu\in\Lambda_h}\sum_{m\equiv\mu}{}'|m|^{2(s-r)}|\mu|^{2r}|c_{1,\mu}|^2\\ &\le N^{2(s-t)}\sum_{0\neq\mu\in\Lambda_h}\Bigg(\sum_{\ell\neq 0}\left|\frac{\mu}{N}+\ell\right|^{2(s-r)}\left|\frac{\mu}{N}\right|^{2(r-t)}\Bigg)|\mu|^{2t}|c_{1,\mu}|^2.\end{aligned}$$

Since $2(s-r) < -1$, $|\mu/N| \le 1/2$ and $r-t \ge 0$ the inner sum converges uniformly for $\mu \in \Lambda_h$ and we obtain

$$\sum_{\mu\in\Lambda_h}\sum_{m\equiv\mu}{}' \langle m\rangle^{2s}\,|c_{1,\mu}\widehat{\psi}_{1,\mu}(m)|^2 \le Ch^{2(t-s)}\|u_h\|_{t,h}^2 \le Ch^{2(t-s)}\|u\|_t^2.$$

The last inequality is implied by Lemma 2.12 with $v_h := u_h$, where it is known from the basic Convergence Theorem A.12 that u_h satisfies the condition in the lemma with $v := u$. □

Lemma 2.12. *Assume condition* (R) *holds. Let* $v_h \in S_h^{r,M}$ *satisfy* $\|v - v_h\|_\beta \le Ch^{t-\beta}\|v\|_t$ *for some* $v \in H^t$ *with* $t \ge \beta$*. Then* $\|v_h\|_{t,h} \le C\|v\|_t$.

Proof. Recall the definition (A.11) of $P_h^{r,1}$. With the aid of the inverse inequality (A.8), the approximation power (A.12) of $P_h^{r,1}$, Lemma A.8 and Proposition A.3, we obtain

$$\begin{aligned}\|v_h\|_{t,h} &\le \|v_h - P_h^{r,1}v\|_{t,h} + \|P_h^{r,1}v\|_{t,h}\\ &\le h^{\beta-t}\|v_h - P_h^{r,1}v\|_{\beta,h} + \|v\|_t\\ &\le Ch^{\beta-t}\|v_h - P_h^{r,1}v\|_\beta + \|v\|_t\\ &\le Ch^{\beta-t}\big(\|v_h - v\|_\beta + \|v - P_h^{r,1}v\|_\beta\big) + \|v\|_t\\ &\le Ch^{\beta-t}\big(h^{t-\beta}\|v\|_t + h^{t-\beta}\|v\|_t\big) + \|v\|_t\\ &\le C\|v\|_t. \qquad \square\end{aligned}$$

Proof of Theorem 2.4. We prove the theorem for the case $L = L_0$, a proof of the extension to the general case is given at the end of the proof of Theorem 3.2. The four quantities on the right-hand side of (2.17) are estimated in Lemmas 2.8–2.11 in accordance with the asserted estimate. □

3. Additional order of convergence, variable coefficients. In the variable coefficient case a direct Fourier analysis as in the section before cannot be carried through. The proof of additional powers of convergence relies instead, as in [**10**], on a dual argument and a strengthened condition of additional order of convergence.

We denote by L_β^+ and L_β^- the special case of pseudodifferential operators L_0 that are defined with the symbol coefficients $a^+ = 1, a^- = 0$ and $a^+ = 0, a^- = 1$, respectively, such that $L_0 = a^+ L_\beta^+ + a^- L_\beta^-$. Recall our principal assumption

$$\beta + M < r \quad \text{and} \quad M \le r'. \tag{3.1}$$

Definition 3.1. Let $b \in \mathbf{N}$. The qualocation method is said to have strong additional order b of convergence if

$$\left| Q\big(\widetilde{\Omega}_k(\cdot, y), 1\big) \right| \le C|y|^b \quad \text{as } y \to 0 \text{ for } k = 1, \dots, M \tag{3.2}$$

with $\widetilde{\Omega}_k$ to be taken for both L_β^+ and L_β^- and

$$\left| Q\big(1, \widetilde{\Delta}_1'(\cdot, y)\big) \right| \le C|y|^{r-\beta+b-r'} \quad \text{as } y \to 0, \tag{3.3}$$

$$\left| Q\big(1, \widetilde{\Delta}_k'(\cdot, y)\big) \right| \le C|y|^{r-\beta} \quad \text{as } y \to 0 \text{ for } k = 2, \dots, M. \tag{3.4}$$

It may be helpful for interpreting the two conditions in Definition 3.1 to hint to the fact that $Q\big(\widetilde{\Omega}_k(\cdot, y), 1\big)$ can be considered as the result of the quadrature rule Q applied to the integral $(\widetilde{\Omega}_k(\cdot, y), 1)_0$, which vanishes as is immediately seen from the definition of $\widetilde{\Omega}_k$. If one thinks of an expansion of $\widetilde{\Omega}_k(\cdot, y)$ with respect to y, where the coefficients are functions of ξ (see (3.33)), this means that some of these functions at the beginning of the series are integrated exactly, i.e., to zero. These observations apply also to $Q\big(1, \widetilde{\Delta}_k'(\cdot, y)\big)$.

In the case $M = 1$ and an x-independent elliptic numerical symbol D Condition (3.2) coincides with the condition in Remark 2.3. The relation to the conditions in [**10**] (for the case $M = 1$) is elaborated in Section 4.

We denote by $L^{\pm}_{\beta-i}$, $i \in \mathbf{N}$, the operators corresponding to $L^{\pm}_{\beta}$ but of order $\beta - i$ in place of β. The main theorem in this section is

Theorem 3.2. *Assume that Conditions* (R) *and* (R′) *hold and that the numerical symbol* D *is elliptic. Let* (3.1) *and*

$$s < r - M + \frac{1}{2},\ \beta - s < r' - M + \frac{1}{2},\ s \le \beta < t - \frac{1}{2},\ s \le t \le r \tag{3.5}$$

be satisfied. Let the qualocation method have strong additional order b *of convergence with* $b \in \mathbf{N}$ *satisfying*

$$\beta - s \le b \le \min(r', r - \beta). \tag{3.6}$$

Let L *have the form* $L = L_0 + L_1 + K$, *where*

$$L_0 = a_0^+(x)L_\beta^+ + a_0^-(x)L_\beta^-,$$
$$L_1 = \sum_{i=1}^{b-1}\left(a_i^+(x)L_{\beta-i}^+ + a_i^-(x)L_{\beta-i}^-\right)$$

and K *is a bounded map from* $H^q \to H^{q+b-\beta}$ *for* $q \in \mathbf{R}$. *Let* L *map* $H^\beta \to H^0$ *injectively. Then the qualocation equations* (1.11) *are uniquely solvable for* $h \in \mathcal{H}_1$. *If* $u \in H^{t-s+\beta}$ *the following error estimate holds*:

$$\|u - u_h\|_s \le Ch^{t-s}\|u\|_{t-s+\beta} \quad \text{for } h \in \mathcal{H}_1. \tag{3.7}$$

Remark 3.3. If $a_i^- = 0$ or $a_i^+ = 0$ for $i = 0, \dots, b$, then it is sufficient that the qualocation method has strong additional order b of convergence to require (3.2) for L_β^+ or L_β^- only, respectively.

We prepare the proof of Theorem 3.2 by a couple of lemmas. In all proofs the norm equivalence from Proposition A.3 and the boundedness of the quantities

$$Q\Big(\widetilde{\Omega}_k\Big(\cdot, \frac{\mu}{N}\Big), \Delta'_\ell\Big(\cdot, \frac{\mu}{N}\Big)\Big) \quad \text{and} \quad \Big(\Phi_k, \Delta'_\ell\Big(\cdot, \frac{\mu}{N}\Big)\Big)_0$$

for $\mu \in \Lambda_h$ will be used without further notice. In the following lemmas $u_h \in S_h^{r,M}$ and $v_h \in S_h^{r',M}$ will be represented in the form

$$u_h = \sum_{k=1}^{M} \sum_{\mu \in \Lambda_h} c_{k,\mu} \psi_{k,\mu} \quad \text{and} \quad v_h = \sum_{\ell=1}^{M} \sum_{\mu \in \Lambda_h} d_{\ell,\mu} \psi'_{\ell,\mu}, \tag{3.8}$$

respectively. We recall our general assumption that (R) and (R′) hold and D is elliptic.

The following duality argument is an essential ingredient in the proof of the next lemma.

Lemma 3.4. *Assume that* s *and* σ *satisfy*

$$0 \le \beta - s < r' - M + \frac{1}{2}, \quad s < r - M + \frac{1}{2}$$

and

$$\sigma \in \Big(\frac{1}{2}, r - M + \frac{1}{2} - \beta\Big) \cap \Big(\frac{1}{2}, 1\Big].$$

Let $u \in H^{\beta+\sigma}$, *and let* $u_h \in S_h^{r,M}$ *satisfy the qualocation equations*

$$(L_0(u - u_h), v_h)_h = 0 \quad \text{for } v_h \in S_h^{r',M}.$$

Then the estimate

$$\begin{aligned} \|u - u_h\|_s \le C\Big[& h^{\beta-s}\|u - u_h\|_\beta + h^{\beta-s+\sigma}\|u - u_h\|_{\beta+\sigma} \\ & + \max_{\pm} \sup_{\|v_h\|_{\beta-s} \le 1} \Big\{ \delta^{\pm} |(L_\beta^{\pm}(u - u_h), v_h)_0 - (L_\beta^{\pm}(u - u_h), v_h)_h|, \\ & \qquad v_h \in S_h^{r',M} \Big\}\Big] \end{aligned} \tag{3.9}$$

holds, where

$$\delta^+ := \begin{cases} 1 & \text{if } a^+ \neq 0 \\ 0 & \text{if } a^+ = 0, \end{cases} \qquad \delta^- := \begin{cases} 1 & \text{if } a^- \neq 0 \\ 0 & \text{if } a^- = 0. \end{cases}$$

Proof. Since L_0 is injective and a Fredholm operator with index zero, it follows that L_0 maps $H^s \to H^{s-\beta}$ bijectively and hence the adjoint L_0^* of L_0 with respect to the $L^2(\mathbf{T})$ inner product $(\cdot,\cdot)_0$ maps the dual spaces $H^{\beta-s} \to H^{-s}$ bijectively. For a given $w \in H^{-s}$ let $v \in H^{\beta-s}$ be the solution of $L_0^* v = w$. Invoking the approximation power (A.12) of the projection P_h' on $S_h^{r',M}$ from (A.11) with $r = r'$ we obtain

(3.10)
$$\begin{aligned} \left|(u-u_h, w)_0\right| &= \left|(u-u_h, L_0^* v)_0\right| = \left|(L_0(u-u_h), v)_0\right| \\ &\leq \left|(L_0(u-u_h), v - P_h' v)_0\right| + \left|(L_0(u-u_h), P_h' v)_0\right| \\ &\leq C h^{\beta-s} \|u-u_h\|_\beta \|v\|_{\beta-s} + \left|(L_0(u-u_h), P_h' v)_0\right|. \end{aligned}$$

The last quantity is bounded by

$$\begin{aligned} &\left|(L_0(u-u_h), P_h' v)_0\right| \\ &\quad = \left|(L_0(u-u_h), P_h' v)_0 - (L_0(u-u_h), P_h' v)_h\right| \\ &\quad \leq \left|(L_\beta^+(u-u_h), \overline{a^+} P_h' v)_0 - (L_\beta^+(u-u_h), \overline{a^+} P_h' v)_h\right| \\ &\qquad + \left|(L_\beta^-(u-u_h), \overline{a^-} P_h' v)_0 - (L_\beta^-(u-u_h), \overline{a^-} P_h' v)_h\right| \\ &\quad \leq \max_\pm \Big[\left|(L_\beta^\pm(u-u_h), (1-P_h')(\overline{a^\pm} P_h' v))_0\right| \\ &\qquad\qquad + \left|(L_\beta^\pm(u-u_h), (1-P_h')(\overline{a^\pm} P_h' v))_h\right| \\ &\qquad\qquad + \delta^\pm \left|(L_\beta^\pm(u-u_h), v_h^\pm)_0 - (L_\beta^\pm(u-u_h), v_h^\pm)_h\right|\Big], \end{aligned}$$

where $v_h^\pm := P_h'(\overline{a^\pm} P_h' v) \in S_h^{r',M}$. With the aid of Theorem A.10, the inverse inequality (A.8) on $S_h^{r',M}$ (in the case $\beta - s > r' - M$ use the hypothesis $r' - M + 1/2 > \beta - s$ instead) and $\|v\|_{\beta-s} \leq C\|w\|_{-s}$ we see that

$$\begin{aligned}|(L^{\pm}_{\beta}(u-u_h),(1-P'_h)(\overline{a^{\pm}}P'_h v))_0| &\le Ch^{r'-M+1}\|u-u_h\|_{\beta}\|P'_h v\|_{r'-M}\\ &\le Ch^{\beta-s}\|u-u_h\|_{\beta}\|P'_h v\|_{\min(\beta-s,r'-M)}\\ &\le Ch^{\beta-s}\|u-u_h\|_{\beta}\|v\|_{\min(\beta-s,r'-M)}\\ &\le Ch^{\beta-s}\|u-u_h\|_{\beta}\|w\|_{-s}.\end{aligned}$$

Similarly, with (A.10) for P'_h in place of P_h, Proposition A.5 and Theorem A.10 we obtain

$$\begin{aligned}&|(L^{\pm}_{\beta}(u-u_h),(1-P'_h)(\overline{a^{\pm}}P'_h v))_h|\\ &\le \left(\|(1-P'_h)L^{\pm}_{\beta}(u-u_h)\|_h+\|P'_hL^{\pm}_{\beta}(u-u_h)\|_h\right)\|(1-P'_h)(\overline{a^{\pm}}P'_h v)\|_h\\ &\le C\left(h^{\sigma}\|L^{\pm}_{\beta}(u-u_h)\|_{\sigma}+\|P'_hL^{\pm}_{\beta}(u-u_h)\|_0\right)h^{\sigma}\|(1-P'_h)(\overline{a^{\pm}}P'_h v)\|_{\sigma}\\ &\le C\left(h^{\sigma}\|u-u_h\|_{\beta+\sigma}+\|u-u_h\|_{\beta}\right)h^{r'-M+1}\|P'_h v\|_{r'-M}\\ &\le Ch^{\beta-s}\left(h^{\sigma}\|u-u_h\|_{\beta+\sigma}+\|u-u_h\|_{\beta}\right)\|w\|_{-s}.\end{aligned}$$

We combine the above estimates and end up with

$$\begin{aligned}|(u-u_h,w)_0| \le C&\left[h^{\beta-s}\|u-u_h\|_{\beta}+h^{\beta-s+\sigma}\|u-u_h\|_{\beta+\sigma}\right]\|w\|_{-s}\\ &+\max_{\pm}\delta^{\pm}\left|(L^{\pm}_{\beta}(u-u_h),v^{\pm}_h)_0-(L^{\pm}_{\beta}(u-u_h),v^{\pm}_h)_h\right|.\end{aligned}$$

After taking the supremum with respect to $\|w\|_{-s}\le 1$ and since $\|v^{\pm}_h\|_{\beta-s}\le C\|w\|_{-s}$, the assertion follows. □

In the next lemma we use the elementary operator

(3.11) $$J: H^s \longrightarrow H^t,\quad Jv:=(v,1)_0 \quad \text{for } v\in H^s \text{ and } s,t\in\mathbf{R}.$$

Lemma 3.5. *Assume condition* (3.4) *holds. Let* $v_h\in S_h^{r',M}$. *Then*

(3.12) $$(v_h,1)_h=\widehat{v}_h(0)=(v_h,1)_0$$

and

$$(3.13) \qquad (Jv, v_h)_h = (Jv, v_h)_0 \quad \textit{for } v \in H^s \textit{ and } s \in \mathbf{R}.$$

Proof. Note that $Q(1, \Delta'_k(\cdot, 0)) = 0$ for $k = 2, \dots, M$ as a consequence of (3.4) since $\Delta'_k(\xi, y)$ is for each $\xi \in [0, 1)$ continuous in $y = 0$. The continuity of $\Delta'_k(\xi, y)$ for $k = r'$ can be seen by a similar argument as in the proof of Remark 2.3. We then calculate, taking (A.4), (3.4) for $y = 0$, the definition of Δ'_1 and (A.5) into account,

$$\begin{aligned}(v_h, 1)_h &= \sum_{k=1}^{M} \sum_{\mu \in \Lambda_h} d_{k,\mu} (\psi'_{k,\mu}, \Phi_0)_h \\ &= \sum_{k=1}^{M} d_{k,0} Q(\Delta'_k(\cdot, 0), 1) \\ &= d_{1,0} Q(1, 1) = d_{1,0} = (v_h, \Phi_0)_0 = \widehat{v}_h(0) = (v_h, 1)_0\end{aligned}$$

proving (3.12). Then also (3.13) is obtained from

$$(Jv, v_h)_h = (v, 1)_0 (1, v_h)_h = (v, 1)_0 (1, v_h)_0 = (Jv, v_h)_0. \qquad \square$$

Lemma 3.6. *Assume* $\beta + (1/2) < t$, $0 \le \beta - s < r' - M + (1/2)$ *and* $u \in H^{t-s+\beta}$. *Then, for* $\ell = 1, \dots, M$

$$(3.14) \qquad \sum_{\mu \in \Lambda_h} |d_{\ell,\mu}| \sum_{m \equiv \mu}{}' |\sigma_0(m) \widehat{u}(m)| \le C h^{t-s} \|v_h\|_{\beta - s} \|u\|_{t-s+\beta}.$$

Proof. With Schwarz's inequality the square of the left-hand side of (3.14) can be bounded by

$$\sum_{\mu \in \Lambda_h} \langle \mu \rangle^{2(\beta - s)} |d_{\ell,\mu}|^2 \sum_{\mu \in \Lambda_h} \Big(\langle \mu \rangle^{s-\beta} \sum_{m \equiv \mu}{}' |\sigma_0(m) \widehat{u}(m)| \Big)^2.$$

The assertion follows after an application of Proposition A.3 and Lemma 2.6, where in the case $\ell = 2, \ldots, M$ the estimate $\langle \mu \rangle \leq N$ was used in the first sum. □

Lemma 3.7. *Assume $\beta - s < r' - M + (1/2)$ and $\beta + (1/2) < t \leq r$. Then, for $\ell = 2, \ldots, M$*

(3.15)
$$\sum_{\mu \in \Lambda_h} \left| \sigma_0(\mu) c_{1,\mu} \left(\frac{\mu}{N}\right)^r \sigma_0\left(\frac{\mu}{N}\right)^{-1} d_{\ell,\mu} Q\left(\widetilde{\Omega}_1\left(\cdot, \frac{\mu}{N}\right), \Delta'_\ell\left(\cdot, \frac{\mu}{N}\right)\right)\right| \leq C h^{t-s} \|v_h\|_{\beta-s} \|u\|_t.$$

Proof. The term with $\mu = 0$ in the sum vanishes. Taking $|\mu/N| \leq 1/2$ and $r - t \geq 0$ into account, the left-hand side of (3.15) can be bounded by

$$\begin{aligned}
&\sum_{0 \neq \mu \in \Lambda_h} N^{s-t} \left|\frac{\mu}{N}\right|^{r-t} |\mu|^t |c_{1,\mu}| N^{\beta - s} |\, d_{\ell,\mu}| \\
&\quad \leq C h^{t-s} \left(\sum_{0 \neq \mu \in \Lambda_h} |\mu|^{2t} |c_{1,\mu}|^2 \right)^{1/2} \left(\sum_{0 \neq \mu \in \Lambda_h} |N|^{2(\beta - s)} |\, d_{\ell,\mu}|^2 \right)^{1/2} \\
&\quad \leq C h^{t-s} \|v_h\|_{\beta-s} \|u_h\|_{t,h} \leq C h^{t-s} \|v_h\|_{\beta-s} \|u\|_t,
\end{aligned}$$

where in the last step we used Lemma 2.12 with $v_h := u_h$, where it is known from the basic Convergence Theorem A.12 that u_h satisfies the condition in Lemma 2.12 with $v := u$. □

The proof of the next lemma follows along the same lines.

Lemma 3.8. *Assume $\beta - s < r' - M + (1/2)$ and $\beta + (1/2) < t \leq r$. For $k, \ell = 2, \ldots, M$*
(3.16)
$$N^\beta \sum_{\mu \in \Lambda_h} \left| c_{k,\mu} d_{\ell,\mu} Q\left(\widetilde{\Omega}_k\left(\cdot, \frac{\mu}{N}\right), \Delta'_\ell\left(\cdot, \frac{\mu}{N}\right)\right)\right| \leq C h^{t-s} \|u\|_t \|v_h\|_{\beta-s}.$$

Lemma 3.9. *Let the qualocation method have strong additional order $b \le r'$ of convergence. Assume $\beta - s \le b$, $\beta - s < r' - M + (1/2)$ and $\beta + (1/2) < t \le r$. Then, for $k = 2, \dots, M$*

(3.17)
$$N^\beta \sum_{\mu \in \Lambda_h} \left| c_{k,\mu} d_{1,\mu} Q\Big(\widetilde{\Omega}_k\Big(\cdot, \frac{\mu}{N}\Big), \Delta_1'\Big(\cdot, \frac{\mu}{N}\Big)\Big)\right| \le C h^{t-s} \|u\|_t \|v_h\|_{\beta - s}.$$

Proof. As in Remark 2.3 it can be seen that conditions (3.2) are equivalent to

$$\big|Q\big(\widetilde{\Omega}_k(\cdot, y), \Delta_1'(\cdot, y)\big)\big| \le C|y|^b \quad \text{as } y \to 0 \text{ for } k = 1, \dots, M. \tag{3.18}$$

Thus the left-hand side of (3.17) can be estimated by

$$C \sum_{0 \ne \mu \in \Lambda_h} \left|\frac{\mu}{N}\right|^{s-\beta+b} N^{s-t} N^t |c_{k,\mu}| |\mu|^{\beta - s} |d_{1,\mu}|.$$

Taking $|\mu/N| \le 1/2$ and $s - \beta + b \ge 0$ into account, the assertion follows similarly as in the proof of Lemma 3.7. □

Lemma 3.10. *Assume $\beta - s < r' - M + (1/2)$ and $\beta + (1/2) < t \le r$. Then, for $k = 2, \dots, M$*

(3.19)
$$N^\beta \sum_{\mu \in \Lambda_h} \left| c_{k,\mu} d_{1,\mu} \Big(\widetilde{\Omega}_k\Big(\cdot, \frac{\mu}{N}\Big), \Delta_1'\Big(\cdot, \frac{\mu}{N}\Big)\Big)_0\right| \le C h^{t-s} \|u\|_t \|v_h\|_{\beta - s}.$$

Proof. The proof is as in the lemma before, this time using

$$\big|(\widetilde{\Omega}_k(\cdot, y), \Delta_1'(\cdot, y))_0\big| \le C|y|^{r'} \quad \text{for } k = 1, \dots, M \tag{3.20}$$

in place of (3.18). Inequality (3.20) follows with the aid of

$$\Delta_1'(\cdot, y) = 1 + y^{r'} \widetilde{\Delta}_1'(\cdot, y) \quad \text{and} \quad (\widetilde{\Omega}_k(\cdot, y), 1)_0 = 0. \qquad □$$

Lemma 3.11. *Let the qualocation method have strong additional order $b \le r'$ of convergence. Assume $\beta - s \le b$, $\beta - s < r' - M + (1/2)$ and $\beta + (1/2) < t \le r$. Then*

(3.21)
$$\sum_{\mu \in \Lambda_h} \left| \sigma_0(\mu) c_{1,\mu} \left(\frac{\mu}{N}\right)^r \sigma_0\left(\frac{\mu}{N}\right)^{-1} d_{1,\mu} Q\left(\widetilde{\Omega}_1\left(\cdot, \frac{\mu}{N}\right), \Delta_1'\left(\cdot, \frac{\mu}{N}\right)\right)\right| \le C h^{t-s} \|u\|_t \|v_h\|_{\beta - s}.$$

Proof. By virtue of (3.18) for $k = 1$ the left-hand side of (3.21) can be estimated by

$$C \sum_{0 \ne \mu \in \Lambda_h} N^{s-t} |\mu|^t |c_{1,\mu}| \left|\frac{\mu}{N}\right|^{r-t+s+b-\beta} |\mu|^{\beta - s} |d_{1,\mu}|.$$

Hence, taking $|\mu/N| \le 1/2, r - t \ge 0$ and $s + b - \beta \ge 0$ into account, the assertion follows similarly as in the proof of Lemma 3.7. □

Lemma 3.12. *Assume $\beta - s < r' - M + (1/2)$ and $\beta + (1/2) < t \le r$. Then*

(3.22)
$$\sum_{\mu \in \Lambda_h} \left| \sigma_0(\mu) c_{1,\mu} \left(\frac{\mu}{N}\right)^r \sigma_0\left(\frac{\mu}{N}\right)^{-1} d_{1,\mu} \left(\widetilde{\Omega}_1\left(\cdot, \frac{\mu}{N}\right), \Delta_1'\left(\cdot, \frac{\mu}{N}\right)\right)_0\right| \le C h^{t-s} \|u\|_t \|v_h\|_{\beta - s}.$$

Proof. The proof is as in the lemma before, this time using (3.20) with $k = 1$ in place of (3.4). □

Lemma 3.13. *Assume $\beta - s < r' - M + (1/2)$, $\beta + (1/2) < t \le r$, and let the qualocation method have strong additional order $b \le r'$ of* *convergence. Then for* $k = 2, \dots, M$,

(3.23)
$$\sum_{\mu \in \Lambda_h} \left| \sigma_0(\mu) \widehat{u}(\mu)\, d_{k,\mu} Q\left(1, \Delta_k'\left(\cdot, \frac{\mu}{N}\right)\right)\right| \le C h^{t-s} \|u\|_t \|v_h\|_{\beta-s}$$

and

(3.24)
$$\sum_{\mu \in \Lambda_h} \left| \sigma_0(\mu) c_{1,\mu} d_{k,\mu} Q\left(1, \Delta_k'\left(\cdot, \frac{\mu}{N}\right)\right)\right| \le C h^{t-s} \|u\|_t \|v_h\|_{\beta-s}.$$

Proof. We prove (3.24) only, the proof of (3.23) is similar. In the proof of Lemma 3.5 we have shown that $Q(1, \widetilde{\Delta}_k'(\cdot, 0)) = 0$ by virtue of (3.4) and, consequently, the $\mu = 0$ term vanishes in the sum which thus can be estimated by

$$C \sum_{\mu \in \Lambda_h} N^{s-t} |\mu|^t |c_{1,\mu}| \left| \frac{\mu}{N} \right|^{r-t} N^{\beta-s} |d_{k,\mu}|.$$

Taking $|\mu/N| \le 1/2$ and $r - t \ge 0$ into account, the assertion follows similarly as in the proof of Lemma 3.7. □

Lemma 3.14. *Assume* $\beta - s < r' - M + (1/2), \beta + (1/2) < t \le r$, *and let the qualocation method have strong additional order* $b \le r'$ *of convergence. Then*

(3.25)
$$\sum_{\mu \in \Lambda_h} \left| \sigma_0(\mu) \widehat{u}(\mu)\, d_{1,\mu} \left(\frac{\mu}{N}\right)^{r'} Q\left(1, \widetilde{\Delta}_1'\left(\cdot, \frac{\mu}{N}\right)\right)\right| \le C h^{t-s} \|u\|_t \|v_h\|_{\beta-s}$$

and

(3.26)
$$\sum_{\mu \in \Lambda_h} \left| \sigma_0(\mu) c_{1,\mu}\, d_{1,\mu} \left(\frac{\mu}{N}\right)^{r'} Q\left(1, \widetilde{\Delta}_1'\left(\cdot, \frac{\mu}{N}\right)\right)\right| \le C h^{t-s} \|u_h\|_t \|v_h\|_{\beta-s}.$$

Proof. We prove the second inequality, the proof of the first one is similar. The $\mu = 0$ term vanishes. With the aid of (3.3) the left-hand side of (3.26) can then be estimated by

$$C \sum_{\mu \in \Lambda_h} N^{s-t} |\mu|^t |c_{1,\mu}| \left| \frac{\mu}{N} \right|^{r-t+s+b-\beta} |\mu|^{\beta - s} |d_{1,\mu}|.$$

Taking $|\mu/N| \leq 1/2$, $s + b - \beta > 0$ and $r - t \geq 0$ into account the assertion follows similarly as in the proof of Lemma 3.7. □

In the proof of the next theorem we need the following representation of some quantities, which can be verified with the aid of Lemmas A.2 and A.11. If L_0 has constant coefficients then

(3.27)
$$\begin{aligned}(L_0 u, \psi'_{\ell,\mu})_h &= \sigma_0(\mu)\widehat{u}(\mu) Q\Big(1, \Delta'_\ell\Big(\cdot, \frac{\mu}{N}\Big)\Big) \\ &\quad + \sum_{m \equiv \mu}{}' \sigma_0(m)\widehat{u}(m) Q\Big(\Phi_{(m-\mu)/N}, \Delta'_\ell\Big(\cdot, \frac{\mu}{N}\Big)\Big),\end{aligned}$$

(3.28)
$$\begin{aligned}(L_0 u, \psi'_{\ell,\mu})_0 &= \sigma_0(\mu)\widehat{u}(\mu)\delta_{\ell,1} \\ &\quad + \sum_{m \equiv \mu}{}' \sigma_0(m)\widehat{u}(m) \Big(\Phi_{(m-\mu)/N}, \Delta'_\ell\Big(\cdot, \frac{\mu}{N}\Big)\Big)_0,\end{aligned}$$

(3.29)
$$\begin{aligned}(L_0 u_h, \psi'_{\ell,\mu})_h &= \sigma_0(\mu) c_{1,\mu} \Big[Q\Big(1, \Delta'_\ell\Big(\cdot, \frac{\mu}{N}\Big)\Big) \\ &\quad + \Big(\frac{\mu}{N}\Big)^r \sigma_0\Big(\frac{\mu}{N}\Big)^{-1} Q\Big(\widetilde{\Omega}_1\Big(\cdot, \frac{\mu}{N}\Big), \Delta'_\ell\Big(\cdot, \frac{\mu}{N}\Big)\Big)\Big] \\ &\quad + N^\beta \sum_{k=2}^{M} c_{k,\mu} Q\Big(\widetilde{\Omega}_k\Big(\cdot, \frac{\mu}{N}\Big), \Delta'_\ell\Big(\cdot, \frac{\mu}{N}\Big)\Big),\end{aligned}$$

(3.30)
$$(L_0 u_h, \psi'_{\ell,\mu})_0 = \sigma_0(\mu) c_{1,\mu} \Big[\delta_{\ell,1} + \Big(\frac{\mu}{N}\Big)^r \sigma_0\Big(\frac{\mu}{N}\Big)^{-1}$$

$$\times \left(\widetilde{\Omega}_1\left(\cdot, \frac{\mu}{N}\right), \Delta'_\ell\left(\cdot, \frac{\mu}{N}\right)\right)_0\Bigg]$$
$$+ N^\beta \sum_{k=2}^{M} c_{k,\mu}\left(\widetilde{\Omega}_k\left(\cdot, \frac{\mu}{N}\right), \Delta'_\ell\left(\cdot, \frac{\mu}{N}\right)\right)_0.$$

We are now in a position to prove our theorem on additional order of convergence. The basic convergence result is stated in Theorem A.12.

Proof of Theorem 3.2. The proof will be obtained by estimating the right-hand side of (3.9). We start with observing that from (A.18) with $s = \beta$, the approximation power (A.12) of P_h and the inverse inequality on $S_h^{r,M}$ follows

$$\begin{aligned} \|u - u_h\|_{\beta+\sigma} &\le \|u - P_h u\|_{\beta+\sigma} + \|P_h u - u_h\|_{\beta+\sigma} \\ &\le \|u - P_h u\|_{\beta+\sigma} + h^{-\sigma}\|P_h u - u_h\|_\beta \\ &\le C h^{t-\beta-\sigma}\|u\|_t \end{aligned} \tag{3.31}$$

for $(1/2) < \sigma \le t-\beta$ and $u \in H^t$. Consequently, the first two quantities on the right-hand side of (3.9) are seen to have the asserted order of convergence.

We are left to estimate the third quantity on the right-hand side of (3.9) and consider

$$\left(L_\beta^\pm(u - u_h), v_h\right)_0 - \left(L_\beta^\pm(u - u_h), v_h\right)_h \tag{3.32}$$

in more detail. For each basis function $v_h = \psi'_{\ell,\mu}$ we have calculated (3.32) in (3.27)–(3.30) with $\sigma_0(\xi) = |\xi|^\beta$ or $\sigma_0(\xi) = \operatorname{sign}\xi\,|\xi|^\beta$ in which case L_0 equals L_β^+ or L_β^-, respectively. An essential property is that for $\ell = 1$, where $\Delta'_\ell(\xi, y) = 1 + y^{r'}\widetilde{\Delta}'_\ell(\xi, y)$, the terms $\sigma_0(\mu)\widehat{u}(\mu)(Q(1,1) - \delta_{\ell,1})$ and $\sigma_0(\mu)c_{1,\mu}(Q(1,1) - \delta_{\ell,1})$ in (3.32) vanish since $Q(1,1) = 1$. By suitably collecting terms it follows from the Lemmas 3.6–3.14 that all the remaining contributions in (3.32) are bounded as needed.

We turn to consider the additional terms in the definition of L but still with $K = 0$. The line of reasoning for this case is given in [**10**], and we adapt it to our situation. Firstly, Lemma 3.4 has to be proved for L replacing L_0. We can assume that $L : H^\beta \to H^0$ is injective.

Otherwise, we change the definition of L to $L + aJ$ (see (3.11)) and of K to $K - aJ$, where $a \in \mathbf{R}$ is chosen such that $L + aJ$ is injective, which is possible since J is a smoothing operator. An inspection of the proof of Lemma 3.4 shows that also in the presence of the additional term aJ the estimate (3.9) still holds true due to the relation (3.13). Now the application of Lemma 3.4 leads in (3.9) to additional contributions

$$\big(L^{\pm}_{\beta-i}(u-u_h), v_h\big)_0 - \big(L^{\pm}_{\beta-i}(u-u_h), v_h\big)_h \quad \text{for } i = 1, \dots, b-1.$$

They can be bounded in the same way as before with one major difference concerning the condition (3.4) of strong additional order of convergence that is needed for the functions $\widetilde{\Omega}^{\pm}_k$ belonging to $L^{\pm}_{\beta-i}$. By definition (2.1) for L^{+}_{β} and L^{-}_{β} these functions are given by

$$\widetilde{\Omega}^{+}_k(\xi, y) = \sum_{\ell \neq 0} |y+\ell|^{\beta} \frac{\ell^{k-1}}{(y+\ell)^r} \Phi_\ell(\xi)$$

and

$$\widetilde{\Omega}^{-}_k(\xi, y) = \sum_{\ell \neq 0} \operatorname{sign}(y+\ell)|y+\ell|^{\beta} \frac{\ell^{k-1}}{(y+\ell)^r} \Phi_\ell(\xi),$$

respectively. Expanding in a Taylor series with respect to $y = 0$ we obtain

$$\widetilde{\Omega}^{\pm}_k(\xi, y) = \sum_{n=0}^{b-1} \varphi^{\pm}_{k,n}(\xi) y^n + O(|y|^b) \quad \text{as } y \to 0 \tag{3.33}$$

with certain explicitly calculable functions $\varphi^{\pm}_{k,n}$. Condition (3.2) for strong additional order b of convergence for $L^{\pm}_{\beta}$ is equivalent to

$$Q(\varphi^{\pm}_{k,n}, 1) = 0 \quad \text{for } n = 0, \dots, b-1. \tag{3.34}$$

It is easy to check that conditions (3.34) for $n = 1, \dots, b-1$ are identical with the corresponding conditions for the functions $\widetilde{\Omega}^{\pm}_k$ belonging to $L^{\pm}_{\beta-1}$ with strong additional order $b-1$ of convergence in place of b. This conclusion can be repeated to show that $L^{\pm}_{\beta-i}$ has strong additional order $b-i$ of convergence for $i = 1, \dots, b-1$. Taking into account that the order of the pseudodifferential operators $L^{\pm}_{\beta-i}$ is only $\beta - i$ it is not difficult to verify that we also get the correct order of convergence for all the additional contributions in (3.9) coming from L_1.

For the ease of the reader we provide the proof for the case $K \neq 0$ which is similar to the one in [**7**, page 439]. Both $L_2 := L_0 + L_1$ and $L = L_2 + K$ are bounded and invertible operators from H^s on $H^{s-\beta}$, so the operator $L_2^{-1}L = I + L_2^{-1}K$ has a bounded inverse on H^s for all $s \in \mathbf{R}$. Thus,

$$\|u - u_h\|_s \leq C\|u - u_h + w\|_s, \text{ where } w := L_2^{-1}K(u - u_h). \tag{3.35}$$

The qualocation equations (1.11) in operator form are $R_h L u_h = R_h L u$ and are equivalent to

$$R_h L_2 u_h = R_h L_2 (u + w).$$

Consequently, if $\widetilde{u}_h, w_h \in S_h^{r,M}$ satisfy the qualocation equations

$$R_h L_2 \widetilde{u}_h = R_h L_2 u \quad \text{and} \quad R_h L_2 w_h = R_h L_2 w$$

then $u_h = \widetilde{u}_h - w_h$. Now assume that (3.5) and (3.6) are satisfied. The error bound (3.7) for L_2 gives

$$\|u - \widetilde{u}_h\|_s \leq C h^{s-t} \|u\|_{t-s+\beta}.$$

Since $u, u_h \in H^s$ and $L_2^{-1}K : H^s \to H^{s+b}$ is bounded for $s \in \mathbf{R}$, we have $w \in H^{\beta+b}$. Applying the basic error estimate (A.18) yields

$$\begin{aligned} \|w - w_h\|_\beta &\leq C h^{\beta+b-\beta} \|w\|_{\beta+b} \leq C h^b \|u - u_h\|_\beta \\ &\leq C h^{b+t-\beta} \|u\|_t \leq C h^{t-s} \|u\|_{t-s+\beta}, \end{aligned}$$

where we used $\beta + b \leq r$, and in the second line $b - \beta \geq -s$ and $\beta - s \geq 0$. Using the triangle inequality in (3.35) these estimates imply (3.7).

The remark at the end of the theorem is clear from the proof given so far. □

Remark 3.15. Under the conditions of Theorem 3.2 the map $L : H^\beta \to H^0$ is injective if and only if $L : H^q \to H^{q-\beta}$ is injective for

$q \in \mathbf{R}$ since a solution $u \in H^\beta$ of $Lu = 0$ lies in all spaces $H^q, q \in \mathbf{R}$, due to the bijectivity of L_0 and the order less than β of $L_1 + K$.

4. Symmetric quadrature formulas. For symmetric quadrature rules the conditions in Definition 3.1 can be further elaborated. A basic quadrature rule Q is said to be symmetric if it satisfies the condition that if $\xi \in (0, (1/2))$ is a quadrature point then so is $(1 - \xi)$ with the same weight ω. In the case of smoothest splines in [**9, 10**] exactness conditions for symmetric quadrature rules are given to provide additional order. In this section we sharpen these conditions and extend them to multiple knot splines.

We need the following functions G_α for $\alpha > 0$ and $\xi \in (0, 1)$ which have been studied in [**1**]:

$$G_\alpha(\xi) := 2\sum_{\ell=1}^{\infty} \frac{1}{\ell^\alpha} \cos 2\pi\ell\xi.$$

In [**4**] conditions (3.2)–(3.4) were given another form. The symbol σ_0 is said to be even or odd if the coefficient a^- or a^+ vanishes, respectively.

Lemma 4.1. *If σ_0 is even (odd), then condition (3.2) is equivalent to*

$$\sum_{j=1}^{J} \omega_j G_{r-\beta+\ell}(\xi_j) = 0 \quad \text{for even (odd) } \ell \in [-M + 1, b - 1] \tag{4.1}$$

if σ_0 and r have like (opposite) parity. If σ_0 is neither even nor odd, then (3.2) is equivalent to the equation in (4.1) for all even and odd $\ell \in [-M + 1, b - 1]$.

Let $\lfloor x \rfloor$ denote truncation of $x \in \mathbf{R}$ to the next integer not larger than x.

Lemma 4.2. *Condition (3.3) is equivalent to*

$$\sum_{j=1}^{J} \omega_j G_\ell(\xi_j) = 0 \quad \textit{for even } \ell \in [r', r - \lfloor\beta\rfloor + b - 1] \tag{4.2}$$

and Condition (3.4) *is void if* $M = 1$ *and if* $M > 1$ *is equivalent to*

$$(4.3)\quad \sum_{j=1}^{J} \omega_j G_\ell(\xi_j) = 0 \quad \textit{for even } \ell \in [-M+1+r', r-\lfloor\beta\rfloor+r'-2].$$

Sufficient conditions for (4.1)–(4.3) can be derived by noting that G_α is for even α a multiple of the Bernoulli polynomial B_α (see [**2**]). From this observation the next corollary follows easily from Lemmas 4.1 and 4.2. In its formulation we use the notation of an extended symmetric quadrature formula Q. By this we mean a modification of Q, which is symmetric for periodic functions only, into a general symmetric formula $\widetilde{Q}$. The modification is necessary only in the case $\xi_1 = 0$. To obtain $\widetilde{Q}$ the additional quadrature point $\xi_{J+1} := 1$ is introduced with weight $\omega_{J+1} := \omega_1/2$ and the weight for $\xi_1 = 0$ is changed to also be equal to $\omega_1/2$.

Corollary 4.3. *Let* $\beta \in \mathbf{Z}$. *Let* σ_0 *and* r *have like parity, and let* $r-\beta$ *be even or let* σ_0 *and* r *have opposite parity and* $r-\beta$ *odd. Then the conditions* (4.1)–(4.3) *are satisfied if the extended symmetric quadrature rule* Q *has at least order* $2\lfloor q \rfloor$ *of exactness, where* $q = (r-\beta+b-1)/2$ *unless* $M > 1$ *and* $b < r'-1$, *where* $q = (r-\beta+r'-2)/2$.

In the case of general operator L, observe that by our index assumptions we have $r' \geq 1$, $-M+1+r' \geq 1$ and $r-\beta > 0$ and the following corollary can be derived from Lemmas 4.1 and 4.2.

Corollary 4.4. *Let* $\beta \in \mathbf{Z}$. *If the symmetric quadrature formula* Q *satisfies*

$$(4.4)\quad \sum_{j=1}^{J} \omega_j B_\ell(\xi_j) = 0 \quad \textit{for even } \ell \in [2, r-\beta+b-1],$$

$$(4.5)\quad \sum_{j=1}^{J} \omega_j G_\ell(\xi_j) = 0 \quad \textit{for odd } \ell \in [r-\beta-M+1, r-\beta+b-1]$$

and, additionally, if $M > 1$ *and* $b < r' - 1$

$$\sum_{j=1}^{J} \omega_j B_\ell(\xi_j) = 0 \quad \text{for even } \ell \in [r - \beta + b, r - \beta + r' - 2] \tag{4.6}$$

then Conditions (4.1)–(4.3) *hold true for general variable coefficient operators* L.

In the case $M = 1$ these conditions coincide with (1.15) and (1.20) in [**10**].

TABLE 1. Quadrature formulas from [**9**] and Table 2 providing additional order b of convergence for general variable coefficient operators L.

M	$r - \beta$	b	r'	formula
2	3	1	2	$G_{3,2,2}, L_{3,2,2}$
2	4	1	2	$G_{4,3,2}, L_{4,3,2}$
2	3	2	2,3	$G_{4,3,2}, L_{4,3,2}$
3	4	1	3	$G_{4,3,2}, L_{4,3,2}$
3	4	2	3	$G_{5,3,3}, L_{5,3,3}$

TABLE 2. Quadrature formulas providing for $M = 3$ additional order $b = 2$ of convergence.

J	ξ_j	ω_j	Rule name
5	0.03675444410510	0.09796641612174	$G_{5,3,3}$
	0.20980173750308	0.24512752237399	
	0.5	0.31381212300853	
	0.79019826249692	0.24512752237399	
	0.96324555589490	0.09796641612174	
5	0.0	0.04767138349495	$L_{5,3,3}$
	0.09758560632523	0.17451387385978	
	0.34287284360121	0.30165043439274	
	0.65712715639879	0.30165043439274	
	0.90241439367477	0.17451387385978	

In [**9**] a list of symmetric quadrature formulas with various exactness properties is provided. In Table 1 we collect those formulas which satisfy the conditions of Corollary 4.4 for certain choices of the parameters and, additionally, include the two new formulas from Table 2. We keep the notation in [**9**]. Useful information for us is the first index indicating the number J. The parameters of the quadrature formula in Table 2 have been calculated in [**4**].

Remark 4.5. The stability of the formulas from [**9**] has been numerically checked there for strongly and oddly elliptic operators with integer $\beta \in [-1, 1]$. For some of the rules stability was proved analytically in [**8**].

APPENDIX

For the convenience of the reader we provide here some results for the spline spaces $S_h^{r,M}$ and the convergence of the qualocation method obtained in [**5, 6**]. Additionally, some easy consequences which are needed are given.

In the next two lemmas some relations for the spline basis are given.

Lemma A.1 [**5**, Lemma A.5]. *The Fourier series of the spline functions $\psi_{k,\mu}$ are*

$$\psi_{1,\mu}(x) = \Phi_\mu(x) + \sum_{m\equiv\mu}{}' \left(\frac{\mu}{m}\right)^r \Phi_m(x), \tag{A.1}$$

$$\psi_{k,\mu}(x) = \sum_{m\equiv\mu}{}' \left(\frac{N}{m}\right)^r \left(\frac{m-\mu}{N}\right)^{k-1} \Phi_m(x) \quad \textit{for } k = 2, \dots, M. \tag{A.2}$$

Recall that a prime on a sum sign indicates that the $m = \mu$ term is to be omitted.

Lemma A.2 [**5**, Corollaries A.3 and A.4]. *For $\mu, \nu \in \Lambda_h$, $k, \ell = 1, \dots, M$ and $m \in \mathbf{Z}$,*

$$(\psi_{k,\mu}, \psi_{\ell,\nu})_h = \delta_{\mu,\nu} Q\left(\Delta_k\left(\cdot, \frac{\mu}{N}\right), \Delta_\ell\left(\cdot, \frac{\nu}{N}\right)\right), \tag{A.3}$$

$$(\Phi_m, \psi_{\ell,\mu})_h = \begin{cases} 0 & \text{if } m \not\equiv \mu, \\ Q\left(\Phi_{(m-\mu)/N}, \Delta_\ell(\cdot, \frac{\mu}{N})\right) & \text{if } m \equiv \mu, \end{cases} \tag{A.4}$$

$$(\Phi_m, \psi_{\ell,\mu})_0 = \begin{cases} 0 & \text{if } m \not\equiv \mu, \\ \left(\Phi_{(m-\mu)/N}, \Delta_\ell(\cdot, \tfrac{\mu}{N})\right)_0 & \text{if } m \equiv \mu. \end{cases} \tag{A.5}$$

A useful relation for the subsequent analysis is the norm equivalence given in the next lemma in which we use the definition

$$\|v_h\|_{s,h} := \left(\sum_{\mu \in \Lambda_h} \left[\langle \mu \rangle^{2s} |c_{1,\mu}|^2 + N^{2s} \sum_{k=2}^{M} |c_{k,\mu}|^2 \right] \right)^{1/2} \quad \text{for } s \in \mathbf{R} \tag{A.6}$$

for a function $v_h \in S_h^{r,M}$ written in the form

$$v_h = \sum_{k=1}^{M} \sum_{\mu \in \Lambda_h} c_{k,\mu} \psi_{k,\mu}. \tag{A.7}$$

Proposition A.3 [**5**, Proposition 3.3]. *Let $s < r - M + (1/2)$. On $S_h^{r,M}$ the norms $\|\cdot\|_s$ and $\|\cdot\|_{s,h}$ are uniform for $h \in \mathcal{H}$ equivalent.*

We also need an inverse inequality for the norms $\|v_h\|_{s,h}$ that is proved in the next lemma.

Lemma A.4. *If $s, t \in \mathbf{R}$ with $s \le t$, then*

$$\|v_h\|_{t,h} \le h^{s-t} \|v_h\|_{s,h} \quad \textit{for } v_h \in S_h^{r,M}. \tag{A.8}$$

Proof. Since $\langle \mu \rangle / N \le 1$ the assertion is obtained from the following relations:

$$\begin{aligned} \|v_h\|_{t,h}^2 &= \sum_{\mu \in \Lambda_h} \langle \mu \rangle^{2t} |c_{1,\mu}|^2 + N^{2t} \sum_{\mu \in \Lambda_h} \sum_{k=2}^{M} |c_{k,\mu}|^2 \\ &\le N^{2(t-s)} \left(\sum_{\mu \in \Lambda_h} \langle \mu \rangle^{2s} |c_{1,\mu}|^2 + N^{2s} \sum_{\mu \in \Lambda_h} \sum_{k=2}^{M} |c_{k,\mu}|^2 \right) \\ &= h^{2(s-t)} \|v_h\|_{t,h}^2. \qquad \square \end{aligned}$$

A further norm equivalence is given in the next proposition.

Proposition A.5 [**5**, Proposition 3.5]. *The norms* $\|\cdot\|_0$ *and* $\|\cdot\|_h$ *are equivalent on* $S_h^{r,M}$*, uniformly for* $h \in \mathcal{H}$*, if and only if condition* (R) *is satisfied.*

Next we cite results on the stability and approximation power of R_h.

Proposition A.6 [**5**, Proposition 3.7]. *Let condition* (R) *be satisfied and assume* $0 \le s < r - M + (1/2)$*,* $s \le t \le r$ *and* $(1/2) < t$*. Then*

$$\|f - R_h f\|_s \le C h^{t-s} \|f\|_t \quad \text{for } f \in H^t. \tag{A.9}$$

With respect to the norm $\|\cdot\|_h$ the qualocation projection R_h has the same approximation power as with respect to the norm $\|\cdot\|_0$.

Proposition A.7 [**5**, Proposition 3.11]. *Let condition* (R) *be satisfied and assume* $(1/2) < t \le r$*. Then*

$$\|f - R_h f\|_h + \|f - P_h f\|_h \le C h^t \|f\|_t \quad \text{for } f \in H^t. \tag{A.10}$$

Here $P_h := P_h^{r,M} : H^t \to S_h^{r,M}$ is the projection that was introduced in [**7**, page 428] through the definition

$$P_h^{r,M} f \in S_h^{r,M}, \quad (P_h^{r,M} f, \Phi)_0 = (f, \Phi)_0 \quad \text{for } \Phi \in S_h^{\infty,M}, \tag{A.11}$$

where

$$S_h^{\infty,M} := \operatorname{span}\{\Phi_{\mu+\ell N}, \mu \in \Lambda_h, \ell \in [-M/2, M/2)\}.$$

It is shown in [**7**, Theorem 3.4] that for $s < r - M + (1/2)$ and $s \le t \le r$,

$$\|f - P_h^{r,M} f\|_s \le C h^{t-s} \|f\|_t \quad \text{for } f \in H^t. \tag{A.12}$$

Lemma A.8. *If* $t \in \mathbf{R}$ *and* $f \in H^t$*, then* $\|P_h^{r,1} f\|_{t,h} \le \|f\|_t$.

Proof. We have $M = 1$ and obtain with the aid of (A.1) the representation

$$P_h^{r,1} f = \sum_{\mu \in \Lambda_h} c_{1,\mu} \psi_{1,\mu} = \sum_{\mu \in \Lambda_h} c_{1,\mu} \Phi_\mu + \sum_{\mu \in \Lambda_h} \sum_{m \equiv \mu}{}' \left(\frac{\mu}{m}\right)^r \Phi_m.$$

The double sum is orthogonal to Φ_ν for $\nu \in \Lambda_h$ and we see from the defining equations (A.11) of $P_h^{r,1}$ that $c_{1,\mu} = \widehat{f}(\mu)$. Thus

$$\|P_h^{r,1} f\|_{t,h}^2 = \sum_{\mu\in\Lambda_h} \langle\mu\rangle^{2t} |c_{1,\mu}|^2 = \sum_{\mu\in\Lambda_h} \langle\mu\rangle^{2t} |\widehat{f}(\mu)|^2 \le \|f\|_t^2. \qquad \square$$

On the space W_h of grid functions on the mesh π_h' the qualocation projection is bounded with respect to the norm $\|\cdot\|_h$.

Proposition A.9 [**5**, Proposition 3.14]. *Let condition* (R) *be satisfied. Then*

$$\|R_h f_h\|_0 \le C\|f_h\|_h \quad \textit{for } f_h \in W_h \textit{ and } h \in \mathcal{H}. \tag{A.13}$$

Finally, we state the superapproximation property for P_h.

Theorem A.10 [**5**, Theorem 4.1], [**3**, Theorem 1.1]. *Let $g \in C^r(\mathbf{T})$ and $M < r$, $0 \le s < r - M + (1/2)$, $t \le r - M$. Then*

$$\|(I - P_h)(gv_h)\|_s \le Ch^{1+t-s}\|g'\|_{r-1,\infty}\|v_h\|_t \quad \textit{for } v_h \in S_h^{r,M}. \tag{A.14}$$

The following lemma provides formulas for certain inner products, in which the spline space basis is involved.

Lemma A.11 [**6**, Lemma 4.1]. *Let σ_0 be independent of x. For $\mu, \nu \in \Lambda_h$ and $\ell = 1, \dots, M$,*

$$(L_0\psi_{1,\mu}, \psi'_{\ell,\nu})_h = \delta_{\mu,\nu}\sigma_0(\mu) Q\left(\Omega_1\left(\cdot, \frac{\mu}{N}\right), \Delta'_\ell\left(\cdot, \frac{\nu}{N}\right)\right),$$

$$(L_0\psi_{k,\mu}, \psi'_{\ell,\nu})_h = \delta_{\mu,\nu} N^\beta Q\left(\Omega_k\left(\cdot, \frac{\mu}{N}\right), \Delta'_\ell\left(\cdot, \frac{\nu}{N}\right)\right) \quad \textit{for } k = 2, \dots, M.$$

With the aid of this lemma the qualocation equations (1.11) can be written (see [**6**, Equation 4.4]) as a block diagonal system

$$D\left(\frac{\mu}{N}\right)\widetilde{c}_\mu = d_\mu \quad \text{for } \mu \in \Lambda_h, \tag{A.15}$$

where D is from (2.9) and $\widetilde{c}_\mu, d_\mu$ for $\mu \in \Lambda_h$ are vectors in $\mathbf{C}^M$ with components

$$\widetilde{c}_{1,\mu} := \sigma_0(\mu)c_{1,\mu}, \qquad \widetilde{c}_{k,\mu} := N^\beta c_{k,\mu} \quad \text{for } k = 2, \dots, M, \tag{A.16}$$

$$d_{k,\mu} := (f, \psi'_{k,\mu})_h \quad \text{for } k = 1, \dots, M. \tag{A.17}$$

The basic convergence result reads as follows.

Theorem A.12 [**6**, Theorem 6.3]. *Let the numerical symbol be elliptic, and let $L : H^\beta \to H^0$ be injective. Assume $\beta + M < r$, and let s and t be real numbers satisfying*

$$s < r - M + \frac{1}{2}, \quad \beta + \frac{1}{2} < t, \quad \beta \le s \le t \le r.$$

Then the qualocation equations (1.11) *are uniquely solvable for $h \in \mathcal{H}_1$. Moreover, if $u \in H^t$*

$$\|u - u_h\|_s \le Ch^{t-s}\|u\|_t \quad \textit{for } h \in \mathcal{H}_1. \tag{A.18}$$

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Technische Universität Berlin, Strasse des 17. Juni 135, 10623 Berlin, Germany
Email address: grigo@math.tu-berlin.de