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INTEGRO-DIFFERENTIAL EQUATIONS OF FIRST ORDER WITH AUTOCONVOLUTION INTEGRAL II

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ABSTRACT. In the paper two integro-differential equations of first order with autoconvolution integral and singular coefficients are investigated. For these equations the existence of a solitary solution is proved.

1. Introduction. Continuing our paper [**5**] (which we cite as part I of this paper in the following) we deal with two equations of the form

$$y'(x) + \frac{\gamma}{x} y(x) = a(x) \int_0^x y(\xi) y(x-\xi)\, d\xi, \quad x \in (0,T) \tag{1.1}$$

with given coefficient a and nonzero numbers $\gamma < 1$, $T \in (0,1)$. As reported in part I of the paper, equations of form (1.1) have applications in Burgers' turbulence theory. For functions a with singularity at $x = 0$ equation (1.1) is of type II in [**5**] and from Theorems 5 and 6 of [**5**] (cf. [**5**, 4.3]) we have the existence of a one-parametric family of solutions y_K, $K \in \mathbf{R}$, with $x^\gamma y_K \in C[0,T]$ and $y_K(x) \sim K x^{-\gamma}$ as $x \to 0$ if

$$x^{1-\gamma} a(x) \in L^1(0,T) \quad \text{for } 0 < \gamma < 1, \qquad x a(x) \in L^1(0,T) \quad \text{for } \gamma < 0, \tag{1.2}$$

respectively. (For $K = 0$ this is the trivial solution.) If (1.2) is not fulfilled, in general, we only know the trivial solution $y_0(x) \equiv 0$ for equation (1.1).

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In this paper we prove the existence of a further single solution ("solitary solution") y in two cases, where $a(x) = x^{\gamma-2}A_{1,2}(x)$, $\gamma < 1$ with (*first case*)

$$A_1(x) \sim C_1 \frac{1}{\ln^2 x}, \qquad C_1 \equiv C_{1,\gamma} = \frac{\Gamma(2-2\gamma)}{\Gamma^2(1-\gamma)} \tag{1.3}$$

and (*second case*)

$$A_2(x) \sim C_2 \left(\ln \frac{1}{x} \right)^{\beta}, \quad C_2 \equiv C_{2,\gamma,\beta} = (1+\beta)\frac{\Gamma(2-2\gamma)}{\Gamma(1-\gamma)} \ (\beta > -1) \tag{1.4}$$

as $x \to 0$. In the first case the solution y behaves like

$$y(x) \sim x^{-\gamma} \ln x \quad \text{as } x \to 0 \tag{1.5}$$

and in the second case like

$$y(x) \sim \frac{1}{\Gamma(1-\gamma)} x^{-\gamma} \left(\ln \frac{1}{x} \right)^{-1-\beta} \quad \text{as } x \to 0, \tag{1.6}$$

respectively. In the first case we have $x^{1-\gamma}a(x) \in L^1(0,T)$, whereas in the second case

$$x^{1-\gamma}a(x) \notin L^1(0,T) \quad \text{for } 0 < \gamma < 1, \qquad xa(x) \notin L^1(0,T) \quad \text{for } \gamma < 0,$$

so that (1.2) is fulfilled in the first case, but not fulfilled in the second case. The cases (1.3) and (1.4) are the simplest ones where we can show the existence of a further solution beyond the solutions from the general theory of such equations in part I.

As in part I of the paper, the proof of the existence theorem is based on the theorem by Janno for equations with bilinear operators together with a suitable ansatz for the function y according to its asymptotic behavior (1.5) and (1.6) as $x \to 0$, respectively, where (especially in the second case) the use of the generalized Volterra functions μ in the ansatz for y proves to be convenient.

The existence theorems for the two equations are dealt with in Sections 2 and 3 of the paper. In Appendix 1 we present two proofs of the basic Lemma 6 in part I of the paper; the first one is a completion of the proof in part I (which is incompletely given there) and we prove an additional lemma of this type which is basic for the existence proofs in Sections 2 and 3. In Appendix 2 some formulas for the generalized Volterra's functions μ are listed which are used in the proofs of the existence theorems.

Finally, in the references two papers by Fényes [**4**] and Seitkazieva [**6**] are added to the references in part I, where certain integral and integro-differential equations of auto-convolution type are solved by operational and Laplace transform methods, respectively.

2. First equation. Restricting ourselves to the special value $\gamma = 1/2$, for transparency, at first we deal with the equation

$$y'(x) + \frac{1}{2x} y(x) = \frac{A(x)}{\pi x^{3/2}} \int_0^x y(\xi) y(x-\xi)\, d\xi, \quad x \in (0, T), \tag{2.1}$$

with $T < 1$ where $A(x) \sim 1/(\ln^2 x)$ as $x \to 0$ and we are looking for a solution y with

$$y(x) = x^{-1/2} \ln x + \delta x^{-1/2} + o(x^{-1/2}), \quad \delta \in \mathbf{R} \text{ as } x \to 0.$$

Then

$$\frac{d}{dx}\left[x^{1/2} y(x) - \ln x\right] = \frac{A(x)}{\pi x} \int_0^x y(\xi) y(x-\xi)\, d\xi - \frac{1}{x} \tag{2.2}$$

and integrating (2.2), equation (2.1) is reduced to the *integral equation* for y

$$y(x) = \frac{1}{x^{1/2}} \left\{ \ln x + \delta + \int_0^x \left[\frac{A(\xi)}{\pi \xi} \int_0^\xi y(\eta) y(\xi - \eta)\, d\eta - \frac{1}{\xi} \right] d\xi \right\}. \tag{2.3}$$

We make the ansatz for y

$$y(x) = x^{-1/2}(\ln x + \delta) + x^{1/2} \ln x \cdot w(x), \quad w \in C[0, T] \tag{2.4}$$

and call a solution to integral equation (2.3) of form (2.4) the *generalized solution* of integro-differential equation (2.1). Inserting (2.4) into equation (2.3), we obtain the equation for w

$$w(x) = f(x) + G[w](x) + L[w, w](x) \tag{2.5}$$

where

(2.6) $f(x)$

$$= \frac{1}{x\ln x}\int_0^x \left[\frac{A(\xi)}{\pi\xi}\int_0^\xi \eta^{-1/2}\big(\ln\eta+\delta\big)(\xi-\eta)^{-1/2}\big(\ln(\xi-\eta)+\delta\big)\,d\eta - \frac{1}{\xi}\right]d\xi,$$

(2.7)

$$G[w](x) = \frac{2}{\pi x\ln x}\int_0^x \frac{A(\xi)}{\xi}\int_0^\xi (\xi-\eta)^{-1/2}\eta^{1/2}\ln\eta\big(\ln(\xi-\eta)+\delta\big)w(\eta)\,d\eta\,d\xi,$$

(2.8) $L[w_1, w_2](x)$

$$= \frac{1}{\pi x\ln x}\int_0^x \frac{A(\xi)}{\xi}\int_0^\xi \eta^{1/2}(\xi-\eta)^{1/2}\ln\eta\ln(\xi-\eta)\,w_1(\eta)w_2(\xi-\eta)\,d\eta\,d\xi.$$

From (2.6) and the integrals (A.10), (A.11) in the Appendix we have

$$f(x) = \frac{1}{x\ln x}\int_0^x \left[\frac{A(\xi)}{\xi}\big(\ln^2\xi + 2J_\delta\ln\xi + I_\delta\big) - \frac{1}{\xi}\right]d\xi \tag{2.9}$$

where

$$J_\delta = \delta - 2\ln 2, \qquad I_\delta = J_\delta^2 - \frac{\pi^2}{6}.$$

We now make the *assumption* that A has the form

$$A(x) = A_0(x)B(x), \qquad A_0(x) = \frac{1}{\ln^2 x + 2J_\delta\ln x + I_\delta} \tag{2.10}$$

where $B(x) = 1+D(x)$ with bounded measurable function D satisfying the asymptotic relation

$$D(x) = cx\ln x + o\left(x\ln\frac{1}{x}\right), \quad c\in\mathbf{R} \text{ as } x\to 0.$$

Thereby we eventually restrict the existence interval $(0,T)$ by $T < \min\big(1, x_\delta\big)$ with

$$x_\delta = \exp(-u_\delta), \quad \text{where } u_\delta = J_\delta + \frac{\pi}{\sqrt{6}}$$

is the larger root of the equation $u^2 - 2J_\delta u + I_\delta = 0$.

With (2.10) the expression (2.9) becomes

$$f(x) = \frac{1}{x \ln x} \int_0^x \frac{D(\xi)}{\xi}\, d\xi = c + o(1) \quad \text{as } x \to 0 \tag{2.11}$$

so that $f \in C[0,T]$ with $f(0) = c$.

To prove the existence of a solution $w \in C[0,T]$ to equation (2.5), as in part I of the paper, we use the iteration method with exponentially weighted norms

$$\|w\|_\sigma = \max_{0 \le x \le T} \left| e^{-\sigma x} w(x) \right|, \quad \sigma \ge 0$$

in $C[0,T]$. We show that for some $\sigma_0 > 0$ the estimations

$$\|G[w]\|_\sigma \le M(\sigma) \|w\|_\sigma, \quad \sigma \ge \sigma_0 \tag{2.12}$$

for any $w \in C[0,T]$ and

$$\|L[w_1, w_2]\|_\sigma \le N(\sigma) \|w_1\|_\sigma \|w_2\|_\sigma, \quad \sigma \ge \sigma_0 \tag{2.13}$$

for any pair $w_1, w_2 \in C[0,T]$ with continuous functions M, N, satisfying $M(\sigma) \to 0$ and $N(\sigma) \to 0$ as $\sigma \to \infty$ hold. From (2.12), (2.13) in view of a theorem by Janno the existence and uniqueness of the solution $w \in C[0,T]$ to equation (2.5) follow which depends continuously on the function f in the norm of $C[0,T]$.

By (2.7), (2.10) and (A.12), (A.13) for any $w \in C[0,T]$ we have

$$\begin{aligned} \left| e^{-\sigma x} G[w](x) \right| &\le \text{Const}\, \frac{\|w\|_\sigma}{x \ln(1/x)} \int_0^x \frac{e^{-\sigma(x-\xi)}}{\xi \ln^2 \xi} \\ &\quad \times \int_0^\xi (\xi - \eta)^{-1/2} \eta^{1/2} \ln \frac{1}{\eta} \left(\ln \frac{1}{\xi - \eta} + |\delta| \right) d\eta\, d\xi \\ &\le \text{Const}\, \frac{\|w\|_\sigma}{x \ln(1/x)} \int_0^x e^{-\sigma(x-\xi)} d\xi \le \text{Const}\, M_0(\sigma) \|w\|_\sigma \end{aligned}$$

with

$$M_0(\sigma) = \sup_{0 < x \le T} \left[\frac{1}{\ln(1/x)} \cdot \frac{1 - e^{-\sigma x}}{\sigma x} \right]$$

which tends to zero as $\sigma \to \infty$ in view of (A.2) in Lemma 2 in the Appendix. This proves (2.12).

Analogously by (2.8), (2.10) and (A.15) for any pair $w_1, w_2 \in C[0,T]$ we estimate

$$\begin{aligned}\left|e^{-\sigma x} L[w_1, w_2](x)\right| &\le \text{Const}\, \frac{\|w_1\|_\sigma \|w_2\|_\sigma}{x \ln(1/x)} \int_0^x \frac{e^{-\sigma(x-\xi)}}{\xi \ln^2 \xi} \\ &\quad \times \int_0^\xi \eta^{1/2}(\xi-\eta)^{1/2} \ln\frac{1}{\eta} \ln\frac{1}{\xi-\eta} d\eta\, d\xi \\ &\le \text{Const}\, \frac{\|w_1\|_\sigma \|w_2\|_\sigma}{x \ln(1/x)} \int_0^x e^{-\sigma(x-\xi)} \xi\, d\xi \\ &\le \text{Const}\, \frac{\|w_1\|_\sigma \|w_2\|_\sigma}{\ln(1/x)} \int_0^x e^{-\sigma(x-\xi)} d\xi \\ &\le \text{Const}\, \frac{1}{\sigma} \|w_1\|_\sigma \|w_2\|_\sigma,\end{aligned}$$

which proves (2.13).

Applying the theorem by Janno from part I of the paper, we get

Theorem 1. *Under the assumptions* (2.10) *there exists a unique generalized solution* y *of form* (2.4) *to equation* (2.1) *in some interval* $(0, T)$.

Corollary 1. *Taking* $x = 0$ *in the above relations of* $G[w]$ *and* $L[w_1, w_2]$, *we obtain* $G[w](0) = L[w,w](0) = 0$ *and from* (2.5) *we have* $w(0) = f(0) = c$ *in the solution* (2.4).

Corollary 2. *If the function* D *in* (2.10) *is continuous, from* (2.2) *it follows that the product* $x^{1/2}y$ *and therefore* y *itself possesses a continuous derivative for* $x > 0$. *That means,* y *is a classical solution of* (2.1).

Corollary 3. *If we make instead of* (2.4) *the ansatz*

$$y(x) = x^{-1/2} \ln x + \delta x^{-1/2} + x^{1/2} \cdot w(x), \quad w \in C[0,T],$$

we get the same existence and uniqueness result as for the ansatz (2.4) *if we assume* (2.10) *with*

$$D(x) = cx + o(x), \quad c \in \mathbf{R} \quad \text{as } x \to 0.$$

Then instead of (2.11), *it holds*

$$f(x) = \frac{1}{x}\int_0^x \frac{D(\xi)}{\xi}\,d\xi = c + o(1) \quad \text{as } x \to 0$$

and in the estimation of $G[w]$ *we use* (A.12) *and the monotonicity of the logarithmic function.*

We conclude this section with two remarks about the related approach using generalized Volterra's functions μ and a conjecture.

Remark 1. Instead of the ansatz (2.4) we can use the equivalent ansatz

$$y(x) = \pm\sqrt{\pi}\mu\left(x, -2, -\frac{1}{2}\right) + \lambda\mu\left(x, -1, -\frac{1}{2}\right) + \mu\left(x, -2, \frac{1}{2}\right)\cdot w(x), \quad \lambda \in \mathbf{R}$$

with the generalized Volterra's function $\mu(x, \beta, \alpha)$ (see [**2**, 18.3]). In particular, for $\lambda = 0$ we obtain

$$y(x) = \pm\sqrt{\pi}\mu\left(x, -2, -\frac{1}{2}\right) + \mu\left(x, -2, \frac{1}{2}\right)\cdot w(x), \quad w \in C[0, T], \tag{2.14}$$

where by (A.5),

$$\mu\left(x, -2, \frac{1}{2}\right) = \frac{2}{\sqrt{\pi}}\, x^{1/2}\left[\ln\frac{1}{x} + \Psi\left(\frac{3}{2}\right)\right]$$

with $\Psi(3/2) = 2(1 - \ln 2) - C \approx 0.0365 > 0$, C the Euler's constant (see [**1**, 1.7]) is a positive function and

$$\sqrt{\pi}\mu\left(x, -2, -\frac{1}{2}\right) = x^{-1/2}\left[\ln\frac{1}{x} + \Psi\left(\frac{1}{2}\right)\right]$$

with $\Psi(1/2) = -2\ln 2 - C \approx -1.9635 < 0$ is a positive function in $(0,T)$ if we require $T < \exp\Psi(1/2)$. In the calculation of f and the estimation of $G[w]$ and $L[w_1,w_2]$ we then use the integral formula (A.7) with the expression (A.6) for the functions $\mu(x,-3,\alpha)$, $\alpha = 1$ and $\alpha = 2$, respectively, in the Appendix. The function A is assumed to have the form (2.10) with $A_0(x) = \mp 1/(\mu(x,-3))$ where, by (A.6),

$$\mu(x,-3) = \ln^2 x + 2C\ln x + C^2 - \frac{\pi^2}{6}.$$

The ansatz (2.14) (with sign minus) corresponds to (2.4) with $\delta = -\Psi(1/2) = 2\ln 2 + C$ (and $u_\delta = C + (\pi/\sqrt{6}) \approx 1.8598$).

Remark 2. In the more general equation

$$y'(x) + \frac{\gamma}{x}y(x) = \frac{C_1 A(x)}{x^{2-\gamma}} \int_0^x y(\xi)y(x-\xi)\,d\xi, \quad x \in (0,T), \tag{2.15}$$

where $\gamma < 1$, $C_1 = \Gamma(2-2\gamma)/\Gamma^2(1-\gamma)$ and A has the form (2.10) with main part

$$A_0(x) = \sum_{k=2}^{\infty} \frac{a_k}{\ln^k x}, \quad a_2 = 1, \tag{2.16}$$

one can try to find a solution y with the ansatz

$$y(x) = y_0(x) + \mu(x,-2,1-\gamma)\cdot w(x), \quad w \in C[0,T] \tag{2.17}$$

where

$$y_0(x) = x^{-\gamma}[\ln x + \delta] + \sum_{n=0}^{N} c_n\,\mu(x,n,-\gamma) = \sum_{n=-2}^{N} c_n\,\mu(x,n,-\gamma)$$

with $\delta \in \mathbf{R}$ and

$$c_{-2} = -\Gamma(1-\gamma), \qquad c_{-1} = \Gamma(1-\gamma)[\delta + \Psi(1-\gamma)]$$

according to the formulas (A.5).

Using the formula for the derivative of the functions μ (A.17), the recurrence relation (A.16) and the integral formula (A.7), the ansatz (2.17) in (2.15) leads to the function

$$A_0(x) = \frac{x^{1-\gamma} \sum_{n=-2}^{N} (n+1) c_n \, \mu(x, n+1, -\gamma)}{C_1 \sum_{j=-2}^{N} \sum_{m=-2}^{N} c_j c_m \, \mu(x, j+m+1, 1-2\gamma)},$$

which has then to be compared with (2.16) by the asymptotic expansions for the functions μ as $x \to 0$ (see [**2**, 18.3, (13)]). The question remains open if for all functions A having a main part (2.16) with uniform convergent series in some interval $(0, T)$ there exists a solution to equation (2.15) of form (2.17) (in general, with $N = \infty$).

Conjecture. *The existence result of Theorem* 1 *for equation* (2.1) *is expected to hold for equation* (2.15) *with* $\gamma < 1$ *and* $A(x) \sim (\ln(1/x))^{\beta}$ *as* $x \to 0$ *with* $\beta < -1$*, too, where* y *behaves like (1.6).*

3. Second equation. Now the equation

$$y'(x) + \frac{\gamma}{x} y(x) = \frac{C_2 A(x)}{x^{2-\gamma}} \int_0^x y(\xi) y(x-\xi) \, d\xi, \quad x \in (0, T), \tag{3.1}$$

with $T < 1$ is considered where $\gamma < 1$, $C_2 = (1+\beta)\Gamma(2-2\gamma)/\Gamma(1-\gamma)$ and $A(x) \sim (\ln(1/x))^{\beta}$ as $x \to 0$ with $\beta > -1$. We are looking for a nontrivial solution y with $y(x) = o(x^{-\gamma})$ as $x \to 0$. From (3.1) we have

$$\frac{d}{dx} [x^{\gamma} y(x)] = \frac{C_2 A(x)}{x^{2-2\gamma}} \int_0^x y(\xi) y(x-\xi) \, d\xi, \tag{3.2}$$

and integrating (3.2) we obtain the *integral equation* for y

$$y(x) = \frac{C_2}{x^{\gamma}} \int_0^x \frac{A(\xi)}{\xi^{2-2\gamma}} \int_0^{\xi} y(\eta) y(\xi-\eta) \, d\eta \, d\xi. \tag{3.3}$$

Further, we make the ansatz

$$y(x) = \mu(x, \beta, -\gamma) + \mu(x, \beta_1, 1-\gamma) \cdot z(x), \quad z \in C[0, T] \tag{3.4}$$

with the generalized Volterra's function $\mu(x,\beta,\alpha)$, where $\beta_1 > -1$ is a further parameter. The functions $\mu(x,\beta,-\gamma)$ and $\mu(x,\beta_1,1-\gamma)$ are positive in $(0,T]$ and possess the asymptotic representations

$$\mu(x,\beta,-\gamma) \sim \frac{x^{-\gamma}}{\Gamma(1-\gamma)}\left(\ln\frac{1}{x}\right)^{-1-\beta},$$
$$\mu(x,\beta_1,1-\gamma) \sim \frac{x^{1-\gamma}}{\Gamma(2-\gamma)}\left(\ln\frac{1}{x}\right)^{-1-\beta_1}$$

as $x \to 0$. We call a solution y to integral equation (3.3) of form (3.4) the *generalized solution* of integro-differential equation (3.1).

Inserting the ansatz (3.4) into equation (3.3), we get the equation for z

$$z(x) = g(x) + G[z](x) + L[z,z](x) \tag{3.5}$$

where

$$g(x) = \frac{1}{\mu(x,\beta_1,1-\gamma)}\left[\frac{C_2}{x^\gamma}\int_0^x \frac{A(\xi)}{\xi^{2-2\gamma}}\int_0^\xi \mu(\eta,\beta,-\gamma) \times \mu(\xi-\eta,\beta,-\gamma)\,d\eta\,d\xi - \mu(x,\beta,-\gamma)\right], \tag{3.6}$$

$$G[z](x) = \frac{2C_2}{x^\gamma\mu(x,\beta_1,1-\gamma)}\int_0^x \frac{A(\xi)}{\xi^{2-2\gamma}} \times \int_0^\xi \mu(\xi-\eta,\beta,-\gamma)\mu(\eta,\beta_1,1-\gamma)z(\eta)\,d\eta\,d\xi, \tag{3.7}$$

$$L[z_1,z_2](x) = \frac{C_2}{x^\gamma\mu(x,\beta_1,1-\gamma)}\int_0^x \frac{A(\xi)}{\xi^{2-2\gamma}}\int_0^\xi \mu(\eta,\beta_1,1-\gamma) \times \mu(\xi-\eta,\beta_1,1-\gamma)z_1(\eta)z_2(\xi-\eta)\,d\eta\,d\xi. \tag{3.8}$$

From (3.6) and the integral (A.7) in the Appendix we have

$$g(x) = \frac{1}{\mu(x,\beta_1,1-\gamma)}\left[\frac{C_2}{x^\gamma}\int_0^x \frac{A(\xi)}{\xi^{2-2\gamma}}\,\mu(\xi,2\beta+1,1-2\gamma)\,d\xi - \mu(x,\beta,-\gamma)\right]. \tag{3.9}$$

We make the *assumption* that A has the form

$$A(x) = A_0(x) + C(x) \tag{3.10}$$

where

$$A_0(x) = \frac{\Gamma(1-\gamma)}{\Gamma(2-2\gamma)} x^{1-\gamma} \frac{\mu(x,\beta+1,-\gamma)}{\mu(x,2\beta+1,1-2\gamma)} \sim \left(\ln\frac{1}{x}\right)^{\beta} \quad \text{as } x \to 0,$$

and C is a bounded measurable function satisfying

$$C(x) = c_0 x\left(\ln\frac{1}{x}\right)^{1+2\beta-\beta_1} + o\left(x\left(\ln\frac{1}{x}\right)^{1+2\beta-\beta_1}\right), \quad c_0 \in \mathbf{R}, \tag{3.11}$$

as $x \to 0$. Then using the integral

$$(\beta+1)\int_0^x \xi^{\gamma-1}\mu(\xi,\beta+1,-\gamma)\,d\xi = x^{\gamma}\mu(x,\beta,-\gamma)$$

which follows from the recurrence formula (A.16) by integration, for example, from (3.9) we obtain

$$g(x) = \frac{1}{\mu(x,\beta_1,1-\gamma)}\,\frac{C_2}{x^{\gamma}}\int_0^x \frac{C(\xi)}{\xi^{2-2\gamma}}\,\mu(\xi,2\beta+1,1-2\gamma)\,d\xi. \tag{3.12}$$

Applying the asymptotic expansions for the functions $\mu(x,\beta_1,1-\gamma)$ and $\mu(x,2\beta+1,1-2\gamma)$ as $x \to 0$ in (3.12), (see [**2**, 18.3, (9)]) and the assumption (3.11) about the function C, we finally have that $g \in C[0,T]$ with $g(0) = c$, $c = c_0(1+\beta)\Gamma(2-\gamma)/\Gamma(1-\gamma)$.

For the existence proof to equation (3.5) in $C[0,T]$ we estimate in (3.7) using the integral (A.7)

$$\begin{aligned}
\left|e^{-\sigma x}G[z](x)\right| &\le \text{Const}\,\frac{\|z\|_\sigma}{x^{\gamma}\mu(x,\beta_1,1-\gamma)}\int_0^x \frac{e^{-\sigma(x-\xi)}}{\xi^{2-2\gamma}}\left(\ln\frac{1}{\xi}\right)^{\beta} \\
&\qquad\qquad \times \mu(\xi,\beta+\beta_1+1,2-2\gamma)\,d\xi \\
&\le \text{Const}\,\|z\|_\sigma \frac{1}{x}\left(\ln\frac{1}{x}\right)^{\beta_1+1}\int_0^x e^{-\sigma(x-\xi)}\left(\ln\frac{1}{\xi}\right)^{-\beta_1-2} d\xi \\
&\le \text{Const}\,\|z\|_\sigma \frac{1}{x}\left(\ln\frac{1}{x}\right)^{-1}\int_0^x e^{-\sigma(x-\xi)}\,d\xi \\
&\le \text{Const}\,M_0(\sigma)\|z\|_\sigma,
\end{aligned}$$

$$M_0(\sigma) = \sup_{0<x\le T}\left[\frac{1}{\ln(1/x)}\,\frac{1-e^{-\sigma x}}{\sigma x}\right]$$

where $M_0(\sigma) \to 0$ as $\sigma \to \infty$ by (A.2) in Lemma 2 in the Appendix.

Analogously in (3.8) we have

$$\begin{aligned}\left|e^{-\sigma x} L[z_1, z_2](x)\right| &\le \text{Const}\, \frac{\|z_1\|_\sigma \|z_2\|_\sigma}{x^\gamma \mu(x, \beta_1, 1-\gamma)} \int_0^x \frac{e^{-\sigma(x-\xi)}}{\xi^{2-2\gamma}} \left(\ln \frac{1}{\xi}\right)^\beta \\ &\qquad \times \mu(\xi, 2\beta_1 + 1, 3 - 2\gamma)\, d\xi \\ &\le \text{Const}\, \|z_1\|_\sigma \|z_2\|_\sigma\, \frac{1}{x} \left(\ln \frac{1}{x}\right)^{\beta_1 + 1} \\ &\quad \times \int_0^x e^{-\sigma(x-\xi)} \xi \left(\ln \frac{1}{\xi}\right)^{\beta - 2\beta_1 - 2} d\xi \\ &\le \text{Const}\, \|z_1\|_\sigma \|z_2\|_\sigma \int_0^x e^{-\sigma(x-\xi)} \left(\ln \frac{1}{\xi}\right)^{\beta - \beta_1 - 1} d\xi,\end{aligned}$$

where the integral tends to zero as $\sigma \to \infty$ in view of (A.1) in Lemma 1 in the Appendix. Therefore, the corresponding inequalities (2.12) and (2.13) are fulfilled. This proves

Theorem 2. *Under the assumptions* (3.10) *there exists a unique generalized solution y of form* (3.4) *to equation* (3.1) *in any interval* $(0, T)$ *with* $T < 1$.

Corollary 4. *Taking $x = 0$ in the formulas of $G[z]$ and $L[z_1, z_2]$, we get $G[z](0) = L[z_1, z_2](0) = 0$ implying from* (3.5) *that $z(0) = g(0) = c_0(1+\beta)\Gamma(2-\gamma)/\Gamma(1-\gamma)$ in the solution* (3.4).

Corollary 5. *For continuous function C in* (3.10) *the solution y possesses a continuous derivative for $x > 0$ and y is a classical solution of* (3.1).

We finish the paper with a remark concerning a more general equation (3.1) and an open problem.

Remark 3. In the equation (3.1) with

$$A_0(x) = \left(\ln \frac{1}{x}\right)^\beta \sum_{k=0}^\infty a_k \left(\ln \frac{1}{x}\right)^{-k}, \quad a_0 = 1 \tag{3.13}$$

we can look for a solution of the form

$$y(x) = y_0(x) + \mu(x, \beta, 1-\gamma) \cdot z(x), \quad z \in C[0,T],$$

where

$$y_0(x) = \sum_{n=0}^{N} c_n \, \mu(x, \beta+n, -\gamma), \quad c_0 = 1. \tag{3.14}$$

This leads to the expression

$$A_0(x) = \frac{x^{1-\gamma} \sum_{n=0}^{N} (\beta+n+1) c_n \, \mu(x, \beta+n+1, -\gamma)}{C_2 \sum_{j=0}^{N} \sum_{m=0}^{N} c_j c_m \, \mu(x, 2\beta+1+m+j, 1-2\gamma)}$$

for the function (3.13) from which the equations for determining the coefficients c_n in (3.14) can be obtained.

Open problem. The case $A_0(x) \sim (\ln(1/x))^{-1}$ in equation (3.1) is an open problem where a solution $y(x) \sim \text{Const}\, x^{-\gamma}/(\ln\ln(1/x))$ is to be expected.

APPENDIX

1. Two lemmas about limits. At first we give two proofs of the following basic technical lemma (cf. [**5**, Lemma 6 in part I]).

Lemma 1. *Let $l \in L^1(0,T)$, $l(x) \geq 0$ and $v_\sigma(x) = \int_0^x e^{-\sigma(x-\xi)} l(\xi)\, d\xi$. Then $v_\sigma \to 0$ in $C[0,T]$ as $\sigma \to \infty$.*

First proof. (This is a revised version of the proof presented in part I of the paper.) It is based on the following well-known result that is stated as Lemma 6a in part I.

Let v_σ, $\sigma \geq 0$, be an equicontinuous family of functions in $C[0,T]$ such that $v_\sigma(x) \to v(x)$ as $\sigma \to \infty$ for any $x \in [0,T]$ where $v \in C[0,T]$. Then $v_\sigma \to v$ in $C[0,T]$ as $\sigma \to \infty$.

We have to show the equicontinuity of the specific family v_σ. Let $0 < x_1 < x_2 < T$. It follows form the proof of Lemma 6 presented in part I of the paper that

$$v_\sigma(x_2) - v_\sigma(x_1) \le \omega(|x_1 - x_2|),$$

where ω is the modulus of continuity of the function $\int_0^x l(\xi)\, d\xi$. Further, using the non-negativity of l, we obtain

$$\begin{aligned}
v_\sigma(x_1) - v_\sigma(x_2) &= \int_0^{x_1} e^{-\sigma\xi} l(x_1 - \xi)\, d\xi - \int_0^{x_2} e^{-\sigma\xi} l(x_2 - \xi)\, d\xi \\
&= \int_0^{x_1} e^{-\sigma\xi} l(x_1 - \xi)\, d\xi - \int_0^{x_1} e^{-\sigma\xi} l(x_2 - \xi)\, d\xi \\
&\quad - \int_{x_1}^{x_2} e^{-\sigma\xi} l(x_2 - \xi)\, d\xi \\
&\le \int_0^{x_1} e^{-\sigma\xi} \left[l(x_1 - \xi) - l(x_2 - \xi) \right] d\xi \\
&\le \int_0^{x_1} |l(x_1 - \xi) - l(x_2 - \xi)|\, d\xi \\
&\le \int_0^{x_2} |g_{x_2}(\xi + \delta) - g_{x_2}(\xi)|\, d\xi,
\end{aligned}$$

where $g_x(\xi) = l(x - \xi)$ and $\delta = x_2 - x_1$. Thus,

$$|v_\sigma(x_1) - v_\sigma(x_2)| \le \max\{\omega(|x_1 - x_2|); q(|x_1 - x_2|)\}, \quad 0 < x_1 < x_2 < T$$

for any $\sigma > 0$ where $q(\delta) = \max_{0 \le x \le T} \int_0^x |g_x(\xi + \delta) - g_x(\xi)|\, d\xi$. This shows the equicontinuity of v_σ. Furthermore, since $e^{-\sigma(x-\xi)} l(\xi) \to 0$ as $\sigma \to \infty$ almost everywhere $\xi \in (0, x)$ for any $x \in [0, T]$ we have $v_\sigma(x) \to 0$ as $\sigma \to \infty$ for any $x \in [0, T]$. Consequently, by Lemma 6a from part I we obtain $v_\sigma \to 0$ as $\sigma \to \infty$ in $C[0, T]$.

Second proof. We define the following set

$$K_\sigma = \{x \in (0, T) : l(\xi) > \sqrt{\sigma}\}$$

and denote by χ_σ the characteristic function of K_σ. Then

$$\int_{K_\sigma} l(\xi)\, d\xi = \int_0^T l(\xi)\chi_\sigma(\xi)\, d\xi \longrightarrow 0 \quad \text{as } \sigma \to \infty$$

because $\chi_\sigma(\xi) \to 0$ as $\sigma \to \infty$ for any $\xi \in (0,T)$. Let us estimate for $x \in (0,T)$:

$$\begin{aligned} 0 \le v_\sigma(x) &= \int_{(0,x)\cap K_\sigma} e^{-\sigma(x-\xi)} l(\xi)\,d\xi + \int_{(0,x)\setminus K_\sigma} e^{-\sigma(x-\xi)} l(\xi)\,d\xi \\ &\le \int_{(0,x)\cap K_\sigma} l(\xi)\,d\xi + \sqrt{\sigma}\int_{(0,x)\setminus K_\sigma} e^{-\sigma(x-\xi)}\,d\xi \\ &\le \int_{K_\sigma} l(\xi)\,d\xi + \sqrt{\sigma}\int_0^\infty e^{-\sigma(x-\xi)}\,d\xi = \int_{K_\sigma} l(\xi)\,d\xi + \frac{1}{\sqrt{\sigma}}. \end{aligned}$$

Thus,

$$\|v_\sigma\|_{C[0,T]} \le \int_{K_\sigma} l(\xi)\,d\xi + \frac{1}{\sqrt{\sigma}} \longrightarrow 0 \quad \text{as } \sigma \to \infty.$$

This proves Lemma 1. □

From Lemma 1 it follows that

$$\lim_{\sigma\to\infty} \max_{0\le x\le T} \int_0^x e^{-\sigma(x-\xi)} l(\xi)\,d\xi = 0 \tag{A.1}$$

for any nonnegative summable function l. We further prove

Lemma 2. *It holds*

$$\lim_{\sigma\to\infty} \sup_{0<x\le T} \left[\left(\ln\frac{1}{x}\right)^{-\delta} \frac{1-e^{-\sigma x}}{\sigma x}\right] = 0 \tag{A.2}$$

for $\delta > 0$ *and* $T \in (0,1)$.

Proof. Putting $u = \sigma x$ we have to study the function

$$g(u) = \frac{1-e^{-u}}{u}(\ln\sigma - \ln u)^{-\delta}, \quad 0 \le u \le \sigma T$$

for sufficiently large $\sigma > 0$. Since

$$g'(u) = \frac{1}{u^2}(\ln\sigma - \ln u)^{-\delta}\left[\frac{\delta(1-e^{-u})}{\ln\sigma - \ln u} + (1+u)e^{-u} - 1\right]$$

the function g attains its maximum on $[0, \sigma T]$ either at the end point $u_1 = u_1(\sigma) = \sigma T$ with

$$g(u_1) = \frac{1 - e^{-\sigma T}}{\sigma T} \left(\ln \frac{1}{T} \right)^{-\delta}$$

or at the inner point $u_0 = u_0(\sigma)$ where

$$\text{(A.3)} \qquad \frac{\delta}{\ln \sigma - \ln u_0} = \frac{1 - (1 + u_0) e^{-u_0}}{1 - e^{-u_0}}$$

with the maximum

$$\text{(A.4)} \qquad g(u_0) = \frac{1 - e^{-u_0}}{u_0} (\ln \sigma - \ln u_0)^{-\delta}.$$

Obviously, we have $g(u_1) \to 0$ as $\sigma \to \infty$. Further, taking the limit $\sigma \to \infty$ in (A.3), we get $u_0 \to 0$ as $\sigma \to \infty$. Thus, performing $\sigma \to \infty$ and $u_0 \to 0$ in (A.4) and taking the inequality $\delta > 0$ into account we obtain $\lim_{\sigma \to \infty} g(u_0) = 0$. This proves (A.2). □

2. Some formulas for Volterra's functions. Generalized Volterra's function $\mu(x, \beta, \alpha)$ is defined (see [**2**, 18.3]) by the integral

$$\mu(x, \beta, \alpha) = \int_0^\infty \frac{x^{\alpha + t} \, t^\beta \, dt}{\Gamma(\beta + 1) \Gamma(\alpha + t + 1)}$$

for $\operatorname{Re} \beta > -1$ and analytic continuation for other values of β. In particular,

$$\mu(x, -m, \alpha) = (-1)^{m-1} \frac{d^{m-1}}{d\alpha^{m-1}} \left[\frac{x^\alpha}{\Gamma(\alpha + 1)} \right], \quad m = 1, 2, \dots ,$$

with the special cases
(A.5)

$$\mu(x, -1, \alpha) = \frac{x^\alpha}{\Gamma(\alpha + 1)}, \qquad \mu(x, -2, \alpha) = \frac{x^\alpha}{\Gamma(\alpha + 1)} [\Psi(\alpha + 1) - \ln x]$$

and
(A.6)

$$\mu(x, -3, \alpha) = \frac{x^\alpha}{\Gamma(\alpha + 1)} [\ln^2 x - 2\Psi(\alpha + 1) \ln x + \Psi^2(\alpha + 1) - \Psi'(\alpha + 1)]$$

for $\alpha > -1$ with the Gauss Ψ-function $\Psi(z) = \Gamma'(z)/\Gamma(z)$ (see [**1**, 1.7]). Further, one puts $\mu(x,\beta) = \mu(x,\beta,0)$, $\nu(x,\alpha) = \mu(x,0,\alpha)$ and $\nu(x) = \nu(x,0)$.

The function $\mu(x,\beta,\alpha)$ has the Laplace transform $p^{-\alpha-1}(\ln p)^{-\beta-1}$ for $\alpha > -1$ and $\alpha = -1$, $\beta > -1$. By the convolution theorem of Laplace transform from this the important integral formula arises

$$\int_0^x \mu(\xi,\beta,\alpha)\mu(x-\xi,\delta,\gamma)\,d\xi = \mu(x,\beta+\delta+1,\alpha+\gamma+1) \tag{A.7}$$

in cases $\alpha,\gamma > -1$ and $\alpha = -1, \gamma, \beta > -1$ and $\alpha = \gamma = -1, \beta, \delta > -1$. We further list the following integrals obtained in an analogous way using additionally corresponding Laplace transform relations in [**3**, Chap. IV and V]

$$\int_0^x \xi^\alpha \mu(x-\xi,\beta,\alpha)\,d\xi = \Gamma(\alpha+1)\mu(x,\beta,2\alpha+1), \tag{A.8}$$

$$\begin{aligned} &\int_0^x \xi^\alpha \ln\xi\, \mu(x-\xi,\beta,\alpha)\,d\xi \\ &\quad= \Gamma(\alpha+1)[\Psi(\alpha+1)\mu(x,\beta,2\alpha+1) - \mu(x,\beta-1,2\alpha+1)] \end{aligned} \tag{A.9}$$

for $\alpha > -1$ and

$$\int_0^x \xi^\alpha(x-\xi)^\alpha \ln\xi\, d\xi = \frac{\Gamma^2(\alpha+1)}{\Gamma(2\alpha+2)}\, x^{2\alpha+1}\,[\ln x + \Psi(\alpha+1) - \Psi(2\alpha+2)],$$

$$\begin{aligned} \int_0^x \xi^\alpha(x-\xi)^\alpha \ln\xi\, \ln(x-\xi)\, d\xi = \Gamma^2(\alpha+1)[\Psi^2(\alpha+1)\mu(x,-1,2\alpha+1) \\ - 2\Psi(\alpha+1)\mu(x,-2,2\alpha+1) + \mu(x,-3,2\alpha+1)] \end{aligned}$$

for $\alpha > -1$, in particular

$$\int_0^x \xi^{-1/2}(x-\xi)^{-1/2}\ln\xi\, d\xi = \pi\big(\ln x - 2\ln 2\big), \tag{A.10}$$

$$\int_0^x \xi^{-1/2}(x-\xi)^{-1/2}\ln\xi\, \ln(x-\xi)\, d\xi = \pi\left[\big(\ln x - 2\ln 2\big)^2 - \frac{\pi^2}{6}\right]. \tag{A.11}$$

Moreover, we have

$$\int_0^x \xi^{1/2}(x-\xi)^{-1/2}\ln\xi\,d\xi = \frac{\pi}{2}x\big(\ln x + 1 - 2\ln 2\big), \tag{A.12}$$

$$\int_0^x \xi^{1/2}(x-\xi)^{-1/2}\ln\xi\ \ln(x-\xi)\,d\xi = \frac{\pi}{2}x\left[\big(\ln x - 2\ln 2\big)^2 - \frac{\pi^2}{6}\right] \tag{A.13}$$

and

$$\int_0^x \xi^{1/2}(x-\xi)^{1/2}\ln\xi\,d\xi = \frac{\pi}{8}x^2\left(\ln x + \frac{1}{2} - 2\ln 2\right), \tag{A.14}$$

$$\begin{aligned}&\int_0^x \xi^{1/2}(x-\xi)^{1/2}\ln\xi\ \ln(x-\xi)\,d\xi \\ &= \pi\left[\left(1-\ln 2-\frac{C}{2}\right)^2\frac{x^2}{2} - 2\left(1-\ln 2-\frac{C}{2}\right)\mu(x,-2,2) + \mu(x,-3,2)\right]\end{aligned} \tag{A.15}$$

where $C \approx 0.5772$ is Euler's constant.

Finally, we state the recurrence formula

$$(\beta+1)\mu(x,\beta+1,\alpha) = x\mu(x,\beta,\alpha-1) - \alpha\mu(x,\beta,\alpha) \tag{A.16}$$

(cf. [**2**, 18.3, (11)] with missing factor $\beta+1$) and the formula for the derivative (cf. [**2**, 18.3, (13)])

$$\frac{d}{dx}\mu(x,\beta,\alpha) = \mu(x,\beta,\alpha-1). \tag{A.17}$$

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