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A CLASS OF STOCHASTIC FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MARKOVIAN SWITCHING

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ABSTRACT. This paper investigates a class of stochastic functional differential equations with Markovian switching. Under the local Lipschitz condition but not the linear growth condition, this paper establishes existence-and-uniqueness theorems for the global solutions of these equations. This paper also examines asymptotic boundedness of the global solution, including boundedness in moment, stochastically ultimate boundedness and the moment average boundedness in time. To illustrate our idea more clearly, we consider a scalar stochastic polynomial equation and a special n-dimensional equation in detail.

1. Introduction. Stochastic differential equations with Markovian switching appear in many branches of science and have therefore received a great deal of attention since they may experience abrupt changes in their structure and parameters caused by phenomena such as component failures or repairs, changing subsystem interconnections and abrupt environmental disturbances. There is an extensive literature concerned with these equations (cf. [**5, 6, 8, 9, 12–14, 16, 18, 19, 22, 24**]), in which the coefficients are required to satisfies the local Lipschitz condition and the linear growth condition to guarantee existence and uniqueness of the global solution.

However, coefficients of many well-known systems such as the stochastic Lotka-Volterra model under regime switching (cf. [**9**]) do not satisfy the linear growth condition, so it is necessary to investigate existence

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and uniqueness of global solutions for stochastic differential equations with Markovian switching without the linear growth condition. By the recent technique (cf. [**23**]), this paper examines a class of nonlinear stochastic functional differential equations with Markovian switching whose coefficients do not satisfy the linear growth condition and establishes the existence-and-uniqueness theorems of global solutions. This paper also examines asymptotic boundedness of this global solution, including boundedness in moment, stochastically ultimate boundedness and moment average boundedness in time.

Consider the n-dimensional stochastic functional differential equation with Markovian switching

$$dx(t) = f(x_t, r(t))\,dt + g(x_t, r(t))\,dw(t), \quad x_0 = \xi(\theta) \in C([-\tau, 0]; \mathbf{R}^n), \tag{1.1}$$

where $x_t \in C([-\tau, 0]; \mathbf{R}^n)$ is defined by $x_t(\theta) = x(t+\theta), \theta \in [-\tau, 0]$, $r(t)$ is a Markov chain taking values $\mathbf{S} = \{1, 2, \ldots, N\}$, $f, g : C([-\tau, 0]; \mathbf{R}^n) \times \mathbf{S} \to \mathbf{R}^n$ and $w(t)$ is a scalar Brownian motion. This equation can be regarded as the following N equations

$$dx(t) = f(x_t, i)\,dt + g(x_t, i)\,dw(t), \quad i \in \mathbf{S}$$

switching from one to the other according to the movement of the Markov chain.

As direct applications, this paper also examines the following two special cases of equation (1.1):

$$dx(t) = f(x_t, r(t))\,dt + g(x(t), r(t))\,dw(t), \; x_0 = \xi(\theta) \in C([-\tau, 0]; \mathbf{R}^n), \tag{1.2}$$

$$dx(t) = f(x(t), r(t))\,dt + g(x_t, r(t))\,dw(t), \; x_0 = \xi(\theta) \in C([-\tau, 0]; \mathbf{R}^n). \tag{1.3}$$

In this paper, we will show that equation (1.1), as well as (1.2) and (1.3), has the following properties:

- This equation almost surely admits a global solution.

- The solution of this equation is bounded in the pth moment and stochastically ultimately bounded in the sense that for any $\varepsilon > 0$ and $p > 0$, there exist positive constants $H = H(\varepsilon)$ and K_p such that the solution of this equation has the properties that

$$\limsup_{t\to\infty} \mathbf{E}|x(t)|^p \le K_p, \tag{1.4}$$

$$\lim_{t\to\infty} \mathbf{P}\{|x(t)| \le H\} > 1-\varepsilon. \tag{1.5}$$

• The average in time of the $(\alpha+p)$th moment of the solution is bounded, namely there is a K_p^* such that the solution of this equation obeys

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{E}|x(s)|^{\alpha+p} ds \le K_p^*, \tag{1.6}$$

where α is a parameter defined later.

In the next section, we give some necessary notations, definitions and lemmas. In order to illustrate our idea clearly, Section 3 gives a general result, which includes the existence-and-uniqueness theorem and asymptotic boundedness for the solution of equation (1.1). In Section 4, we examine two classes of conditions on coefficients f and g, under which there exists a unique global solution for equation (1.1). This global solution is asymptotic bounded by the previous general result. Section 5 examines a scalar system whose coefficients are polynomial in detail. In the last section, a special n-dimensional example is discussed.

2. Preliminaries. Throughout this paper, unless otherwise specified, we use the following notations. Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions, that is, it is right continuous and increasing while $\mathcal{F}_0$ contains all $\mathbf{P}$-null sets. Let $w(t)$ be a scalar Brownian motion defined on this probability space. Let $|\cdot|$ be the Euclidean norm in $\mathbf{R}^n$. If A is a vector or matrix, its transpose is denoted by A^T. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\mathrm{trace}\,(A^TA)}$ and operator norm is denoted by $\|A\|$. Let $\tau > 0$, and denote by $C([-\tau,0];\mathbf{R}^n)$ the family of continuous functions from $[-\tau, 0]$ to $\mathbf{R}^n$ with the norm $\|\varphi\| = \sup_{-\tau\le\theta\le 0}|\varphi(\theta)|$ (without any confusion with the operator norm), which forms a Banach space. For any $a = (a_1, \dots, a_n)^T \in \mathbf{R}^n$, let $\widehat{a} = \min_{1\le i\le n}\{a_i\}$ and $\check{a} = \max_{1\le i\le n}\{a_i\}$.

Let $r(t)$ $(t \geq 0)$ be a right-continuous Markov chain (see [**1**]) on this probability space taking values in a finite state space $\mathbf{S} = \{1, 2, \dots, N\}$ with generator $\Gamma = [\gamma_{ij}]_{N\times N}$ satisfying

$$\gamma_{ij} \geq 0 \ (i \neq j), \quad \sum_{j=1}^{N} \gamma_{ij} = 0. \tag{2.1}$$

We assume that Markov chain $r(t)$ is independent of Brownian motion $w(t)$. For any symmetric matrix $Q_i \in \mathbf{R}^{n\times n}(i \in \mathbf{S})$, let $q_i = \lambda_{\min}(Q_i)$ represent its smallest eigenvalue, and let $\widehat{q} = \min_{i\in\mathbf{S}}\{q_i\}$.

The following lemma establishes the boundedness of polynomial functions.

Lemma 2.1. *For any $h(x) \in C(\mathbf{R}^n; \mathbf{R})$, $\alpha, b > 0$, if $h(x) = o(|x|^\alpha)$ as $|x| \to \infty$, then*

$$\sup_{x\in\mathbf{R}^n} [h(x) - b|x|^\alpha] < \infty. \tag{2.2}$$

Proof. Define $\varphi(x) = h(x) - b|x|^\alpha$. Choose $r > 0$ such that $|h(x)| < b|x|^\alpha$ when $x \in \mathbf{R}^n$ and $|x| > r$, which implies $\varphi(x) < 0$. We therefore have

$$\sup_{x\in\mathbf{R}^n} \varphi(x) = \sup_{\substack{x\in\mathbf{R}^n \\ |x|\leq r}} \varphi(x) < \infty,$$

which implies the desired assertion. □

The following simple lemma is the link between the ultimate boundedness and boundedness in moment.

Lemma 2.2. *For any $p > 0$, if stochastic process $x(t)$ is bounded in the pth moment, i.e., $\limsup_{t\to\infty} \mathbf{E}|x(t)|^p \leq K_p$, where K_p is a constant dependent on p, then $x(t)$ is ultimately bounded, namely, for any $\epsilon \in (0, 1)$, there exists a constant $M = M(\epsilon)$ such that*

$$\limsup_{t\to\infty} \mathbf{P}\{|x(t)| \leq M\} \geq 1 - \epsilon. \tag{2.3}$$

Proof. For any $\epsilon \in (0,1)$, let $M = K_p^{1/p}/\epsilon^{1/p}$. Then by the Markov inequality,

$$\mathbf{P}\{|x(t)| > M\} \le \frac{\mathbf{E}|x(t)|^p}{M^p}.$$

Hence

$$\limsup_{t\to\infty} \mathbf{P}\{|x(t)| \le M\} \ge 1-\epsilon,$$

as desired. □

Throughout this paper, *const* represents a positive constant, whose precise value is not important. For $x \in \mathbf{R}^n$, $I(x) \le const$ implies that $I(x)$ satisfies the super boundedness property. Hence Lemma 2.1 can be rewritten as

$$-b|x|^\alpha + o(|x|^\alpha) \le const. \tag{2.4}$$

In this paper, the notation $o(|x|^\alpha)$ implies that $h(x) = o(|x|^\alpha)$ is continuous.

Let $C^2(\mathbf{R}^n \times \mathbf{S}; \mathbf{R}_+)$ denote the family of all nonnegative functions on $\mathbf{R}^n \times \mathbf{S}$ which are continuously twice differentiable in x. If $V \in C^2(\mathbf{R}^n \times \mathbf{S}; \mathbf{R}_+)$, define an operator $\mathcal{L}V : C([-\tau,0];\mathbf{R}^n) \times \mathbf{S} \to \mathbf{R}$ by

$$\begin{aligned}\mathcal{L}V(\varphi,i) &= V_x(\varphi(0),i)f(\varphi,i) + \frac{1}{2}g^T(\varphi,i)V_{xx}(\varphi(0),i)g(\varphi,i)\\ &\quad + \sum_{j=1}^{N} r_{ij}V(\varphi(0),j)\end{aligned} \tag{2.5}$$

for any $\varphi \in C([-\tau,0];\mathbf{R}^n)$, where

$$\begin{aligned}V_x(x,i) &= \left(\frac{\partial V(x,i)}{\partial x_1}, \frac{\partial V(x,i)}{\partial x_2}, \dots, \frac{\partial V(x,i)}{\partial x_n}\right),\\ V_{xx}(x,i) &= \left[\frac{\partial^2 V(x,i)}{\partial x_k \partial x_l}\right]_{n\times n}.\end{aligned}$$

3. The elementary lemma. In this paper the following assumption is imposed as a standing hypothesis.

Assumption 3.1. *Both f and g are locally Lipschitz continuous, namely, for any $k > 0$, there exists a constant c_k such that*

$$|f(\varphi, i) - f(\phi, i)| \vee |g(\varphi, i) - g(\phi, i)| \le c_k \|\varphi - \phi\|$$

for all $\varphi, \phi \in C([-\tau, 0]; \mathbf{R}^n)$ with $\|\varphi\| \vee \|\phi\| \le k$ and any $i \in \mathbf{S}$.

In order for a stochastic differential equation to have a unique global solution for any given initial value, the coefficients of this equation are generally required to satisfy the linear growth condition and the local Lipschitz condition (see [**2, 11**]). These two standard conditions exclude many well-known equations such as the stochastic Lotka-Volterra model under regime switching (cf. [**9**]). The aim of this paper is to establish existence-and-unique theorems and asymptotic results for the global solution of equation (1.1) under the local Lipschitz condition but not the linear growth condition.

It is well known for stochastic differential equations that the linear growth condition for global solutions may be replaced by use of the Lyapunov functions (see [**7, 17, 21**]). This paper investigates this method with Markovian switching to discuss the solution of equation (1.1). To show this idea, we need the concept of local solutions (see [**10**]).

Definition 3.1. Set $\mathcal{F}_t = \mathcal{F}_0$ for $-\tau \le t \le 0$ and let $x(t)$, $-\tau \le t < \rho_e$ be a continuous $\mathbf{R}^n$-valued $\mathcal{F}_t$-adapted process. It is called a local strong solution of equation (1.1) with initial data $\xi \in C([-\tau, 0]; \mathbf{R}^n)$ if $x(t) = \xi(t)$ on $-\tau \le t \le 0$ and

$$x(t \wedge \rho_k) = \xi(0) + \int_0^{t \wedge \rho_k} f(x_s, r(s))\, ds + \int_0^{t \wedge \rho_k} g(x_s, r(s))\, dw(s), \quad \forall t \ge 0$$

for each $k \ge 1$, $\{\rho_k\}_{k \ge 1}$ is a nondecreasing sequence of finite stopping times such that $\rho_k \to \rho_e$ almost surely as $k \to \infty$. If, moreover, $\limsup_{t \to \rho_e} |x(t)| = \infty$ is satisfied almost everywhere when $\rho_e < \infty$, it is called a maximal local strong solution and ρ_e is called the explosion time. A maximal local strong solution $x(t)$, $-\tau \le t < \rho_e$ is said to be unique if for any other maximal local strong solution $\overline{x}(t)$, $-\tau \le t < \overline{\rho}_e$, we have $\rho_e = \overline{\rho}_e$ and $x(t) = \overline{x}(t)$ for $-\tau \le t < \rho_e$ almost surely.

Applying the standing truncation technique (see [**10**, Theorem 3.2.2, page 95] and [**19**, Theorem 8.3, page 303]) to equation (1.1) yields the following result.

Theorem 3.1. *Under Assumption* 3.1, *equation* (1.1) *almost surely has a unique maximal local strong solution for any initial data* $\xi \in C([-\tau, 0]; \mathbf{R}^n)$.

Then the following general result follows.

Lemma 3.2. *Under Assumption* 3.1, *if there exist constants* $\alpha \geq 0$, b_i, ε, p, K_{i0}, K_{ij}, $\alpha_j > 0$ $(i \in \mathbf{S}, 1 \leq j \leq m)$ *for any given integer* m, *probability measures* μ_{ij} *on* $[-\tau, 0]$ *and function* $V \in C^2(\mathbf{R}^n \times \mathbf{S}; \mathbf{R})$ *such that*

$$\lim_{|x|\to\infty} V(x,i) = \infty, \quad (x,i) \in \mathbf{R}^n \times \mathbf{S}, \tag{3.1}$$

$$\begin{aligned} \mathcal{L}V(\varphi, i) + \varepsilon V(\varphi(0), i) &\leq -b_i|\varphi(0)|^{\alpha+p} + K_{i0} \\ &\quad + \sum_{j=1}^m K_{ij}\left[\int_{-\tau}^0 |\varphi(\theta)|^{\alpha_j} d\mu_{ij}(\theta) - e^{\varepsilon\tau}|x|^{\alpha_j}\right], \\ &\qquad \varphi \in C([-\tau, 0]; \mathbf{R}^n), \end{aligned} \tag{3.2}$$

then for any initial data $\xi \in C([-\tau, 0]; \mathbf{R}^n)$, *there exists a unique global solution* $x(t, \xi)$ *to equation* (1.1) *and this solution has properties* (1.6) *and*

$$\limsup_{t\to\infty} \mathbf{E}V(x(t), r(t)) \leq \overline{K}, \tag{3.3}$$

where $\overline{K}$ *is a constant independent of initial data* ξ.

Proof. Clearly, condition (3.2) includes the following three inequalities:

$$\mathcal{L}V(\varphi, i) \leq K_{i0} + \sum_{j=1}^m K_{ij}\left[\int_{-\tau}^0 |\varphi(\theta)|^{\alpha_j} d\mu_{ij}(\theta) - |\varphi(0)|^{\alpha_j}\right], \tag{3.4}$$

$$\mathcal{L}V(\varphi,i)+\varepsilon V(\varphi(0),i)\le K_{i0}+\sum_{j=1}^{m}K_{ij}\left[\int_{-\tau}^{0}|\varphi(\theta)|^{\alpha_j}d\mu_{ij}(\theta)-e^{\varepsilon\tau}|\varphi(0)|^{\alpha_j}\right], \tag{3.5}$$

$$b_i|\varphi(0)|^{\alpha+p}\le -\mathcal{L}V(\varphi,i)+K_{i0}+\sum_{j=1}^{m}K_{ij}\left[\int_{-\tau}^{0}|\varphi(\theta)|^{\alpha_j}d\mu_{ij}(\theta)-|\varphi(0)|^{\alpha_j}\right]. \tag{3.6}$$

For any given initial data $\xi\in C([-\tau,0];\mathbf{R}^n)$, we divide the proof into three steps.

Step 1. *Existence and uniqueness of the global solution.* For any initial data $x_0=\xi\in C([-\tau,0];\mathbf{R}^n)$, under Assumption 3.1, Theorem 3.1 shows that there exists a unique maximal local solution $x(t)$ on $t\in[-\tau,\sigma)$, where σ is the explosive time. To show that $x(t)$ is actually global, we need to show $\sigma=\infty$, almost surely. To prove this statement, for sufficiently large integer k (namely, $k>\max_{-\tau\le\theta\le0}\{V(\xi(\theta),i)\}$ almost surely), define stopping time

$$\sigma_k=\inf\{-\tau\le t<\sigma: V(x(t),r(t))>k\}, \tag{3.7}$$

where, as usual, $\inf\varnothing=\infty$ (as usual $\varnothing$ denotes the empty set). Clearly, σ_k is nondecreasing and $\lim_{k\to\infty}\sigma_k=\sigma_\infty\le\sigma$. This proof can be completed if $\sigma_\infty=\infty$, almost surely. This is equivalent to proving that for any $t>0$, $\mathbf{P}(\sigma_k\le t)\to0$ as $k\to\infty$. To prove this result, noting the right continuity of $r(t)$ and $V(x(\sigma_k),r(\sigma_k))=k$, by condition (3.2), applying the generalized Itô formula (see [**19**]) to $V(x)$ yields

$$\begin{aligned}k\mathbf{P}(\sigma_k\le t)&=V(x(\sigma_k),r(\sigma_k))\mathbf{P}(\sigma_k\le t)\\&\le \mathbf{E}V(x(t\wedge\sigma_k),r(t\wedge\sigma_k))\\&\le \mathbf{E}V(\xi(0),r(0))+\mathbf{E}\int_0^{t\wedge\sigma_k}\mathcal{L}V(x_s,r(s))\,ds\\&\le \mathbf{E}V(\xi(0),r(0))\\&\quad+\mathbf{E}\int_0^{t\wedge\sigma_k}\left[K_{r0}+\sum_{j=1}^{m}K_{rj}\left(\int_{-\tau}^{0}|x(t+\theta)|^{\alpha_j}d\mu_{rj}(\theta)\right.\right.\\&\qquad\left.\left.-|x(s)|^{\alpha_j}\right)\right]ds,\end{aligned}$$

where $r \in \mathbf{S}$ represents the initial value of the process $r(t)$. By the Fubini theorem, it follows that

$$\begin{aligned}\int_0^{t\wedge\tau_k}\int_{-\tau}^0 |x(s+\theta)|^{\alpha_j}\,d\mu_{rj}(\theta)\,ds - \int_0^{t\wedge\tau_k}|x(s)|^{\alpha_j}\,ds \\ &= \int_{-\tau}^0 d\mu_{rj}(\theta)\int_\theta^{t\wedge\tau_k+\theta}|x(s)|^{\alpha_j}\,ds - \int_0^{t\wedge\tau_k}|x(s)|^{\alpha_j}\,ds \\ &\le \int_{-\tau}^0 d\mu_{rj}(\theta)\int_{-\tau}^{t\wedge\tau_k}|x(s)|^{\alpha_j}\,ds - \int_0^{t\wedge\tau_k}|x(s)|^{\alpha_j}\,ds \\ &\le \int_{-\tau}^0 |\xi(\theta)|^{\alpha_j}\,d\theta.\end{aligned}$$

Hence,

$$k\mathbf{P}(\sigma_k \le t) \le \mathbf{E}V(\xi(0), r(0)) + \check{K}_0 t + \sum_{j=1}^m \check{K}_j \int_{-\tau}^0 |\xi(\theta)|^{\alpha_j}\,d\theta =: K_t,$$

where K_t is independent of k. For any $t > 0$, letting $k \to \infty$,

$$\lim_{k\to\infty}\mathbf{P}(\sigma_k \le t) \le \lim_{k\to\infty}\frac{K_t}{k} = 0,$$

which shows that $x(t)$ is the global solution of equation (1.1).

Step 2. *Proof of* (3.3). Inequality (3.5) together with the generalized Itô formula applied $e^{\varepsilon t}V(x(t), r(t))$ gives

$$\begin{aligned}&e^{\varepsilon t}\mathbf{E}V(x(t), r(t)) \\ &= V(\xi(0), r(0)) + \mathbf{E}\int_0^t e^{\varepsilon s}[\mathcal{L}V(x_s, r(s)) + \varepsilon V(x(s), r(s))]\,ds \\ &\le V(\xi(0), r(0)) \\ &\quad + \mathbf{E}\int_0^t e^{\varepsilon s}\Big[K_{r0} + \sum_{j=1}^m K_{rj}\Big(\int_{-\tau}^0 |x(s+\theta)|^{\alpha_j}\,d\mu_{rj}(\theta) - e^{\varepsilon\tau}|x(s)|^{\alpha_j}\Big)\Big]\,ds.\end{aligned}$$

By the Fubini theorem, we have the following estimate

$$\begin{aligned}
&\int_0^t e^{\varepsilon s}\int_{-\tau}^0 |x(s+\theta)|^{\alpha_j}\,d\mu_{rj}(\theta)\,ds-\int_0^t e^{\varepsilon(s+\tau)}|x(s)|^{\alpha_j}\,ds\\
&\quad=\int_{-\tau}^0 d\mu_{rj}(\theta)\int_\theta^{t+\theta} e^{\varepsilon(s-\theta)}|x(s)|^{\alpha_j}\,ds-\int_0^t e^{\varepsilon(s+\tau)}|x(s)|^{\alpha_j}\,ds\\
&\quad\le\int_{-\tau}^0 d\mu_{rj}(\theta)\int_{-\tau}^t e^{\varepsilon(s+\tau)}|x(s)|^{\alpha_j}\,ds-\int_0^t e^{\varepsilon(s+\tau)}|x(s)|^{\alpha_j}\,ds\\
&\quad=\int_{-\tau}^0 e^{\varepsilon(\theta+\tau)}|\xi(\theta)|^{\alpha_j}\,d\theta.
\end{aligned}$$

It therefore follows that

$$\begin{aligned}
e^{\varepsilon t}\mathbf{E}V(x(t),r(t))&\le V(\xi(0),r(0))+\varepsilon^{-1}\check{K}_0(e^{\varepsilon t}-1)\\
&\qquad+\sum_{j=1}^m \check{K}_j\eta^{-1}\int_{-\tau}^0 e^{\varepsilon(\theta+\tau)}|\xi(\theta)|^{\alpha_j}\,d\theta,
\end{aligned}$$

which implies that

$$\limsup_{t\to\infty}\mathbf{E}V(x(t))\le\varepsilon^{-1}\check{K}_0.$$

The desired assertion (3.3) follows by setting $\overline{K}=\varepsilon^{-1}\check{K}_0$.

Step 3. *Boundedness of the moment average in time.* Inequality (3.6) together with the generalized Itô formula applied $V(x(t),r(t))$ yields

$$\begin{aligned}
\widehat{b}\int_0^t \mathbf{E}|x(s)|^{\alpha+p}ds&\le \mathbf{E}V(x(t),r(t))+\hat{b}\int_0^t\mathbf{E}|x(s)|^{\alpha+p}ds\\
&\le V(\xi(0),r(0))\\
&\quad+\mathbf{E}\int_0^t\mathcal{L}V(x_s,r(s))ds+b_r\int_0^t\mathbf{E})|x(s)|^{\alpha+p}ds\\
&\le V(\xi(0),r(0))\\
&\quad+\mathbf{E}\int_0^t\Big[K_{r0}+\sum_{j=1}^m K_{rj}\Big(\int_{-\tau}^0|x(s+\theta)|^{\alpha_j}d\mu_{rj}(\theta)\\
&\qquad\qquad-|x(s)|^{\alpha_j}\Big)\Big]ds\\
&\le V(\xi(0),r(0))+\check{K}_0t+\sum_{j=1}^m\check{K}_j\int_{-\tau}^0|\xi(\theta)|^{\alpha_j}d\theta,
\end{aligned}$$

which implies that

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{E}|x(s)|^{\alpha+p}ds \le \widehat{b}^{-1}\check{K}_0.$$

The desired assertion (1.6) follows by setting $K_p^* = \widehat{b}^{-1}\check{K}_0$. □

4. Main results. To continue our discussion, we place additional assumptions on f and g. For any $\varphi \in C([-\tau,0];\mathbf{R}^n)$ satisfying $\varphi(0)\neq 0$ and $i\in\mathbf{S}$, we impose the following polynomial conditions on f and g.

(H1) There exist positive definite matrix $Q_i \in \mathbf{R}^{n\times n}$, constants α, κ_i, $\overline{\kappa}_i \ge 0$ and a probability measure μ_i on $[-\tau,0]$ such that

$$|\varphi(0)|^{-2}\varphi^T(0)Q_i f(\varphi,i) \le \kappa_i|\varphi(0)|^{\alpha} + \overline{\kappa}_i\int_{-\tau}^0 |\varphi(\theta)|^{\alpha}d\mu_i(\theta) + o(|\varphi(0)|^{\alpha}).$$

(H2) There exist $\beta>0$, λ_i, $\overline{\lambda}_i \ge 0$ and a probability measure ν_i on $[-\tau,0]$ such that

$$|\varphi(0)|^{-1}|g(\varphi,i)| \le \lambda_i|\varphi(0)|^{\beta} + \overline{\lambda}_i\int_{-\tau}^0 |\varphi(\theta)|^{\beta}d\nu_i(\theta) + o(|\varphi(0)|^{\beta}).$$

(H3) For positive definite matrix $Q_i \in \mathbf{R}^{n\times n}$, there exist constants b_i, $\sigma_i \ge 0$ and a probability measure $\overline{\nu}_i$ on $[-\tau,0]$ such that

$$\begin{aligned}|\varphi(0)|^{-4}&[\varphi^T(0)Q_i g(\varphi,i)]^2\\ &\ge b_i|\varphi(0)|^{2\beta} - \sigma_i\int_{-\tau}^0 |\varphi(\theta)|^{2\beta}d\overline{\nu}_i(\theta) + o(|\varphi(0)|^{2\beta}).\end{aligned}$$

(F1) There exist positive definite matrix $Q_i \in \mathbf{R}^{n\times n}$, constants $b_i>0$, $\alpha,\sigma_i \ge 0$ and a probability measure μ_i on $[-\tau,0]$ such that

$$\varphi^T(0)Q_i f(\varphi,i) \le -b_i|\varphi(0)|^{\alpha+2} + \sigma_i\int_{-\tau}^0 |\varphi(\theta)|^{\alpha+2}d\mu_i(\theta) + o(|\varphi(0)|^{\alpha+2}).$$

(F2) There exist constants $\beta>0$, $\lambda_i,\overline{\lambda}_i \ge 0$ and a probability measure ν_i on $[-\tau,0]$ such that

$$|g(\varphi,i)| \le \lambda_i|\varphi(0)|^{\beta} + \overline{\lambda}_i\int_{-\tau}^0 |\varphi(\theta)|^{\beta}d\nu_i(\theta) + o(|\varphi(0)|^{\beta}).$$

If $f(\varphi,i)$ and $g(\varphi,i)$ are replaced by $f(x,i)$ and $g(x,i)$, the above conditions may be rewritten as

(H1′) There exist a positive definite matrix $Q_i \in \mathbf{R}^{n\times n}$ and constants α, $\kappa_i \geq 0$ such that

$$x^T Q_i f(x,i) \leq \kappa_i |x|^{\alpha+2} + o(|x|^{\alpha+2}).$$

(H2′) There exist $\beta > 0$, $\lambda_i \geq 0$ such that

$$|g(x,i)| \leq \lambda_i |x|^{\beta+1} + o(|x|^{\beta+1}).$$

(H3′) For the positive definite matrix $Q_i \in \mathbf{R}^{n\times n}$, there exist constants b_i, $\beta > 0$ such that

$$[x^T Q_i g(x,i)]^2 \geq b_i |x|^{2\beta+4} + o(|x|^{2\beta+4}).$$

(F1′) There exist a positive definite matrix $Q_i \in \mathbf{R}^{n\times n}$ and constants $b_i > 0$, $\alpha \geq 0$ such that

$$x^T Q_i f(x,i) \leq -b_i |x|^{\alpha+2} + o(|x|^{\alpha+2}).$$

(F2′) There exist constants $\beta > 0$, $\lambda_i \geq 0$ such that

$$|g(x,i)| \leq \lambda_i |x|^{\beta} + o(|x|^{\beta}).$$

For any $x \in \mathbf{R}^n$ and $i \in \mathbf{S}$, if

$$V(x,i) = (x^T Q_i x)^{p/2}, \tag{4.1}$$

where $p > 0$ and $Q_i \in \mathbf{R}^{n\times n}$ is a positive definite matrix, we have

$$\widehat{q}|x|^p \leq V(x,i) \leq \|\check{Q}\||x|^p, \tag{4.2}$$

where $\|\check{Q}\| = \max_{i\in\mathbf{S}} \|Q_i\|$. By (2.5),

$$\begin{aligned} \mathcal{L}V(\varphi,i) + \varepsilon V(\varphi(0),i) &= p(\varphi^T(0)Q_i\varphi(0))^{p/2-1}\varphi^T(0)Q_i f(\varphi,i) \\ &\quad + \frac{p}{2}(\varphi^T(0)Q_i\varphi(0))^{p/2-1} g^T(\varphi,i)Q_i g(\varphi,i) \\ &\quad + \frac{p(p-2)}{2}(\varphi^T(0)Q_i\varphi(0))^{p/2-2}[\varphi^T(0)Q_i g(\varphi,i)]^2 \\ &\quad + \sum_{j=0}^{N}\gamma_{ij}(\varphi^T(0)Q_j\varphi(0))^{p/2} \\ &\quad + \varepsilon(\varphi^T(0)Q_i\varphi(0))^{q/2} \\ &=: I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{4.3}$$

Lemma 4.1. *Let* $I = \sum_{j=0}^{N} \gamma_{ij}(x^T Q_j x)^{p/2}$. *Then*

$$I \le M_p |x|^p, \tag{4.4}$$

where

$$M_p := \max_i \Big(\gamma_{ii} q_i + \sum_{j \ne i} \gamma_{ij} \|Q_j\| \Big) \ge 0. \tag{4.5}$$

Proof. Inequality (4.4) is obtained from (2.1) directly. We only need to prove $M_p \ge 0$. Without loss of generality, assume $\|Q_1\| \le \|Q_2\| \le \cdots \le \|Q_N\|$. Thus,

$$\begin{aligned} M_p &\ge \gamma_{11} q_1 + \sum_{j>1} \gamma_{1j} \|Q_j\| \\ &\ge \gamma_{11} q_1 + \sum_{j>1} \gamma_{1j} \|Q_1\| \\ &\ge q_1 \sum_j \gamma_{1j} = 0, \end{aligned} \tag{4.6}$$

as desired. □

We can now state one of our main results in this paper.

Theorem 4.2. *Let Assumption* (3.1) *hold. Under conditions* (H1)–(H3), *if* $\alpha < 2\beta$ *and*

$$2 b_i R_i^{-2} > 2\sigma_i + q_i \|Q_i\| (\lambda_i + \bar{\lambda}_i)^2, \tag{4.7}$$

where $R_i = \|Q_i\|/q_i$, *then for any initial data* $\xi \in C([-\tau, 0]; \mathbf{R}^n)$, *equation* (1.1) *almost surely admits a unique global solution and this solution is ultimately bounded. For any* $p \in (0, 2)$ *satisfying*

$$(2-p)(b_i R_i^{p/2-2} - \sigma_i) > q_i \|Q_i\| (\lambda_i + \overline{\lambda}_i)^2, \tag{4.8}$$

this global solution has property (1.4) *and there exists a constant* $K^*_{2\beta+p}$ *such that*

$$\limsup_{t \to \infty} \frac{1}{t} \int_0^t \mathbf{E}|x(s)|^{2\beta+p} ds \le K^*_{2\beta+p}. \tag{4.9}$$

Proof. Choosing $V(x,i) = (x^T Q_i x)^{p/2}$, to apply Lemma 3.2, we need to estimate I_1–I_5 appearing in (4.3). For $p \in (0,2)$, by condition (H1) and the Young inequality, we have

$$\begin{aligned}
I_1 &\le p(\varphi^T(0)Q_i\varphi(0))^{(p/2)-1}|\varphi(0)|^2 \\
&\quad \times \left[\kappa_i|\varphi(0)|^\alpha + \overline{\kappa}_i \int_{-\tau}^0 |\varphi(\theta)|^\alpha d\mu_i(\theta) + o(|\varphi(0)|^\alpha)\right] \\
&\le p q_i^{(p/2)-1}|\varphi(0)|^p \Big(\kappa_i|\varphi(0)|^\alpha + \overline{\kappa}_i \int_{-\tau}^0 |\varphi(\theta)|^\alpha d\mu_i(\theta)\Big) + o(|\varphi(0)|^{\alpha+p}) \\
&= p q_i^{(p/2)-1}\Big(\kappa_i|\varphi(0)|^{\alpha+p} + \overline{\kappa}_i \int_{-\tau}^0 |\varphi(0)|^p|\varphi(\theta)|^\alpha d\mu_i(\theta)\Big) \\
&\qquad + o(|\varphi(0)|^{\alpha+p}) \\
&\le p q_i^{(p/2)-1}\Big(\kappa_i|\varphi(0)|^{\alpha+p} + \frac{\overline{\kappa}_i p}{\alpha+p}|\varphi(0)|^{\alpha+p} \\
&\qquad\qquad + \frac{\alpha\overline{\kappa}_i}{\alpha+p}\int_{-\tau}^0 |\varphi(\theta)|^{\alpha+p} d\mu_i(\theta)\Big) + o(|\varphi(0)|^{\alpha+p}).
\end{aligned}$$

Recall the elementary inequality: for any $x, y \ge 0$ and $u \in (0,1)$,

$$(x+y)^2 \le \frac{x^2}{u} + \frac{y^2}{1-u}. \tag{4.10}$$

For any $u, \delta_i \in (0,1)$, by condition (H2), the Hölder inequality and the Young inequality, it follows that

$$\begin{aligned}
I_2 &\le \frac{p}{2}\|Q_i\| q_i^{(p/2)-1}|\varphi(0)|^{p-2}|g(\varphi,i)|^2 \\
&\le \frac{p}{2}\|Q_i\| {q_i}^{(p/2)-1}|\varphi(0)|^p \Big[\lambda_i|\varphi(0)|^\beta \\
&\qquad\qquad + \overline{\lambda}_i \int_{-\tau}^0 |\varphi(\theta)|^\beta d\nu_i(\theta) + o(|\varphi(0)|^\beta)\Big]^2 \\
&\le \frac{p}{2u}\|Q_i\| {q_i}^{(p/2)-1}|\varphi(0)|^p \Big[\lambda_i|\varphi(0)|^\beta + \overline{\lambda}_i \int_{-\tau}^0 |\varphi(\theta)|^\beta d\nu_i(\theta)\Big]^2 \\
&\quad + o(|\varphi(0)|^{2\beta+p}) \\
&\le \frac{p}{2u}\|Q_i\| q_i^{(p/2)-1}\left[\frac{\lambda_i^2|\varphi(0)|^{2\beta+p}}{\delta_i} + \frac{\overline{\lambda}_i^2|\varphi(0)|^p}{1-\delta_i}\left(\int_{-\tau}^0 |\varphi(\theta)|^\beta d\nu_i(\theta)\right)^2\right]
\end{aligned}$$

$$
\begin{aligned}
&+ o(|\varphi(0)|^{2\beta+p}) \\
&\le \frac{p}{2u}\|Q_i\| {q_i}^{\frac{p}{2}-1}\left[\frac{\lambda_i^2|\varphi(0)|^{2\beta+p}}{\delta_i} + \frac{\overline{\lambda}_i^2|\varphi(0)|^p}{1-\delta_i}\int_{-\tau}^0 |\varphi(\theta)|^{2\beta} d\nu_i(\theta)\right] \\
&+ o(|\varphi(0)|^{2\beta+p}) \\
&\le \frac{p}{2u}\|Q_i\| {q_i}^{(p/2)-1}\left[\frac{\lambda_i^2|\varphi(0)|^{2\beta+p}}{\delta_i} + \frac{\overline{\lambda}_i^2}{(1-\delta_i)(2\beta+p)}\right. \\
&\qquad\qquad \left.\times \Big(p|\varphi(0)|^{2\beta+p} + 2\beta\int_{-\tau}^0 |\varphi(\theta)|^{2\beta+p} d\nu_i(\theta)\Big)\right] \\
&+ o(|\varphi(0)|^{2\beta+p}).
\end{aligned}
$$

By condition (H3), the Hölder inequality and the Young inequality,

$$
\begin{aligned}
I_3 &\le \frac{p(p-2)}{2}(\varphi^T(0)Q_i\varphi(0))^{(p/2)-2}|\varphi(0)|^4 \\
&\quad \times \Big[b_i|\varphi(0)|^{2\beta} - \sigma_i\int_{-\tau}^0 |\varphi(\theta)|^{2\beta} d\overline{\nu}_i(\theta) + o(|\varphi(0)|^{2\beta})\Big] \\
&\le \frac{p(p-2)}{2}\Big(b_i\|Q_i\|^{(p/2)-2}|\varphi(0)|^{2\beta+p} \\
&\quad - \sigma_i q_i^{(p/2)-2}\int_{-\tau}^0 |\varphi(0)|^p|\varphi(\theta)|^{2\beta} d\overline{\nu}_i(\theta)\Big) + o(|\varphi(0)|^{2\beta+p}) \\
&\le \frac{p(p-2)}{2}\Big[b_i\|Q_i\|^{(p/2)-2}|\varphi(0)|^{2\beta+p} \\
&\quad - \frac{\sigma {q_i}^{(p/2)-2}}{2\beta+p}\Big(p|\varphi(0)|^{2\beta+p} + 2\beta\int_{-\tau}^0 |\varphi(\theta)|^{2\beta+p} d\overline{\nu}_i(\theta)\Big)\Big] \\
&\qquad + o(|\varphi(0)|^{2\beta+p}).
\end{aligned}
$$

Noting that $I_4 + I_5 = o(|\varphi(0)|^{2\beta+p})$ by Lemma 4.1, substituting these estimates into (4.3) yields

$$
\begin{aligned}
&(4.11)\quad \mathcal{L}V(\varphi,i) + \varepsilon V(\varphi(0),i) \\
&\le H_i(\varphi(0)) + \frac{\alpha p\overline{\kappa}_i}{\alpha+p} q_i^{(p/2)-1}\Big(\int_{-\tau}^0 |\varphi(\theta)|^{\alpha+p} d\mu_i(\theta) - e^{\varepsilon\tau}|\varphi(0)|^{\alpha+p}\Big) \\
&\quad + \frac{\beta\overline{\lambda}_i^2 p {q_i}^{(p/2)-1}\|Q_i\|}{u(1-\delta)(2\beta+p)}\Big(\int_{-\tau}^0 |\varphi(\theta)|^{2\beta+p} d\nu_i(\theta) - e^{\varepsilon\tau}|\varphi(0)|^{2\beta+p}\Big) \\
&\quad + \frac{\sigma_i\beta p(2-p){q_i}^{(p/2)-2}}{2\beta+p}\Big(\int_{-\tau}^0 |\varphi(\theta)|^{2\beta+p} d\overline{\nu}_i(\theta) - e^{\varepsilon\tau}|\varphi(0)|^{2\beta+p}\Big),
\end{aligned}
$$

where

$$\begin{aligned}H_i(x) &= pq_i{}^{(p/2)-1}\left(\kappa_i + \frac{\overline{\kappa}_i(\alpha e^{\varepsilon\tau}+p)}{\alpha+p}\right)|x|^{\alpha+p} \\ &\quad + \frac{p}{2u}\|Q_i\|q_i{}^{(p/2)-1}\left(\frac{\lambda^2}{\delta_i} + \frac{\overline{\lambda}_i^2(2\beta e^{\varepsilon\tau}+p)}{(1-\delta_i)(2\beta+p)}\right)|x|^{2\beta+p} \\ &\quad + \frac{p(p-2)}{2}\|Q_i\|^{(p/2)-2}\left(b_i - \sigma_i R_i^{2-(p/2)}\frac{2\beta e^{\varepsilon\tau}+p}{2\beta+p}\right)|x|^{2\beta+p} \\ &\quad + o(|x|^{2\beta+p}) + o(|x|^{\alpha+p}).\end{aligned}$$

Noting that $\alpha < 2\beta$, $H_i(x)$ may be rewritten as

$$\begin{aligned}H_i(x) &= -\frac{p(2-p)}{2}\|Q_i\|^{(p/2)-2}\Bigg[b_i - \sigma_i R_i^{2-\frac{p}{2}}\frac{2\beta e^{\varepsilon\tau}+p}{2\beta+p} \\ &\qquad - \frac{q_i\|Q_i\|R_i^{2-\frac{p}{2}}}{u(2-p)}\left(\frac{\lambda_i^2}{\delta_i} + \frac{\overline{\lambda}_i^2(2\beta e^{\varepsilon\tau}+p)}{(1-\delta_i)(2\beta+p)}\right)\Bigg]|x|^{2\beta+p} + o(|x|^{2\beta+p}) \\ &=: -\frac{p(p-2)}{2}\|Q_i\|^{(p/2)-2}h_i(\varepsilon,u,\delta_i)|x|^{2\beta+p} + o(|x|^{2\beta+p}),\end{aligned}$$

where

$$\begin{aligned}h_i(\varepsilon,u,\delta_i) &= b_i - \sigma_i R_i^{2-(p/2)}\frac{2\beta e^{\varepsilon\tau}+p}{2\beta+p} \\ &\qquad - \frac{q_i\|Q_i\|R_i^{2-(p/2)}}{u(2-p)}\left(\frac{\lambda_i^2}{\delta_i} + \frac{\overline{\lambda}_i^2(2\beta e^{\varepsilon\tau}+p)}{(1-\delta_i)(2\beta+p)}\right).\end{aligned}$$

Clearly,

$$h_i\left(0,1,\frac{\lambda_i}{\lambda_i+\overline{\lambda}_i}\right) = b_i - \sigma_i R_i^{2-(p/2)} - \frac{q_i\|Q_i\|R_i^{2-(p/2)}}{2-p}(\lambda_i+\overline{\lambda}_i)^2. \tag{4.12}$$

By condition (4.7),

$$\lim_{p\to 0} h_i\left(0,1,\frac{\lambda}{\lambda_i+\overline{\lambda}_i}\right) = b_i - \sigma_i R_i^2 - \frac{q_i\|Q_i\|R_i^2}{2}(\lambda_i+\overline{\lambda}_i)^2 > 0.$$

Choosing sufficiently small p and ε and letting $u \to 1$ and $\delta_i = \lambda_i/(\lambda_i+\overline{\lambda}_i)$ such that $h_i(\varepsilon,u,\delta_i) > 0$ (we assume $\lambda_i, \overline{\lambda}_i > 0$). When

λ_i or $\overline{\lambda}_i = 0$, the computation is direct. By Lemma 2.1, there exists a constant $\overline{H}_i$ such that $H_i(x) \le \overline{H}_i$, which implies that (4.3) satisfies the condition (3.2). Lemma 3.2 shows that for any initial data $\xi \in C([-\tau, 0]; \mathbf{R}^n)$, equation (1.1) almost surely admits a unique global solution and this solution has property (1.5).

For the given $p \in (0, 2)$ satisfying the condition (4.8) and $h_i(\varepsilon, u, \delta_i) > 0$, Lemma 3.2 shows that for any initial data $\xi \in C([-\tau, 0]; \mathbf{R}^n)$, the solution has properties (3.3) and (4.9). Noting that $\widehat{q}|x|^p \le V(x(t), r(t))$, inequality (3.3) implies inequality (1.4). By Lemma 2.2, (1.4) gives stochastically ultimate boundedness, as desired. □

Applying Theorem 4.2 to equation (1.2), where conditions (H2) and (H3) are replaced by (H2′) and (H3′), we give the following result.

Corollary 4.3. *Let Assumption* 3.1 *hold. Under conditions* (H1), (H2′) *and* (H3′), *if* $\alpha < 2\beta$ *and*

$$2b_i R_i^{-2} > q_i \|Q_i\| \lambda_i^2, \tag{4.13}$$

then for any initial data $\xi \in C([-\tau, 0]; \mathbf{R}^n)$, *equation* (1.2) *almost surely admits a unique global solution, and this solution is ultimately bounded. For all* $p \in (0,\ 2)$ *satisfying*

$$(2 - p) b_i R_i^{(p/2)-2} > q_i \|Q_i\| \lambda_i^2,$$

this global solution has properties (1.4) *and* (4.9).

By Theorem 4.2, we can also obtain a similar result for equation (1.3). We omit it since it has the same expression as Theorem 4.2.

Observing the proof of Theorem 4.2, we find that it is key for boundedness of $H_i(x)$, which depends on the term

$$-\frac{b_i p(2 - p)}{2} \|Q_i\|^{(p/2)-2} h_i(\varepsilon, u, \delta_i) |x|^{2\beta+p} \tag{4.14}$$

for $p \in (0, 2)$. This term is from condition (H3), which shows that stochastic perturbation intensity g plays an important role in guaranteeing existence of the global solution when this perturbation strongly depends on the solution $x(t)$ in the sense $\alpha < 2\beta$. If we impose conditions (F1) and (F2) on f, then f can also play a similar role. This idea may be described as the following theorem.

Theorem 4.4. *Let Assumption* 3.1 *hold. Under conditions* (F1) *and* (F2), *if* $\alpha > 2\beta - 2 > 0$ *and*

$$b_i > \sigma_i, \tag{4.15}$$

then for any initial data $\xi \in C([-\tau, 0]; \mathbf{R}^n)$, *equation* (1.1) *almost surely admits a global solution* $x(t)$, *and this solution is ultimately bounded and there exists a constant* $\overline{p} > 2$ *such that for any* $p \in (2, \overline{p})$, *this solution has properties* (1.4) *and* (1.6).

Proof. We estimate I_1–I_5 appeared in (4.3) using conditions (F1) and (F2). For $p > 2$, condition (F1) and the Young inequality yield

$$\begin{aligned} I_1 &\le p(\varphi^T(0)Q_i\varphi(0))^{(p/2)-1}\Big[-b_i|\varphi(0)|^{\alpha+2} \\ &\qquad\qquad + \sigma_i \int_{-\tau}^0 |\varphi(\theta)|^{\alpha+2} d\mu_i(\theta) + o(|\varphi|^{\alpha+2})\Big] \\ &\le -b_i p q_i^{(p/2-)1}|\varphi(0)|^{\alpha+p} + p\sigma_i\|Q_i\|^{(p/2)-1} \\ &\qquad \times \int_{-\tau}^0 |\varphi(0)|^{p-2}|\varphi(\theta)|^{\alpha+2} d\mu_i(\theta) + o(|\varphi(\theta)|^{\alpha+p}) \\ &\le -b_i p q_i^{(p/2)-1}|\varphi(0)|^{\alpha+p} + \frac{p(p-2)\sigma_i}{\alpha+p}\|Q_i\|^{(p/2)-1}|\varphi(0)|^{\alpha+p} \\ &\quad + \frac{p(\alpha+2)\sigma_i}{\alpha+p}\|Q_i\|^{(p/2)-1}\int_{-\tau}^0 |\varphi(\theta)|^{\alpha+p} d\mu_i(\theta) + o(|\varphi(0)|^{\alpha+p}). \end{aligned}$$

Condition (F2), the Hölder inequality and the Young inequality, for any $u, \delta_i \in (0,1)$, plus inequality (4.10) yield

$$\begin{aligned} I_2 + I_3 &\le \frac{p}{2}\|Q_i\|^{(p/2)}|\varphi(0)|^{p-2}|g(\varphi,i)|^2 \\ &\quad + \frac{p(p-2)}{2} q_i{}^{-1}\|Q_i\|^{(p/2)+1}|\varphi(0)|^{p-2}|g(\varphi,i)|^2 \\ &= \frac{pm_i}{2}\|Q_i\|^{(p/2)-1}|\varphi(0)|^{p-2}|g(\varphi,i)|^2 \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{pm_i}{2}\|Q_i\|^{(p/2)-1}|\varphi(0)|^{p-2} \\
&\quad \times \left[\lambda_i|\varphi(0)|^{\beta} + \bar{\lambda}_i\int_{-\tau}^{0}|\varphi(\theta)|^{\beta}d\nu_i(\theta) + o(|\varphi(0)|^{\beta})\right]^2 \\
&\leq \frac{pm_i}{2u}\|Q_i\|^{(p/2)-1}\left(\frac{\lambda_i^2|\varphi(0)|^{2\beta+p-2}}{\delta_i}\right. \\
&\qquad\qquad \left. + \frac{\overline{\lambda}_i^2}{1-\delta_i}\int_{-\tau}^{0}|\varphi(0)|^{p-2}|\varphi(\theta)|^{2\beta}d\nu_i(\theta)\right) \\
&\quad + o(|\varphi(0)|^{2\beta+p-2}) \\
&\leq \frac{pm_i}{2u}\|Q_i\|^{(p/2)-1}\left[\frac{\lambda_i^2|\varphi(0)|^{2\beta+p-2}}{\delta_i} + \overline{\lambda}_i^2\frac{(p-2)|\varphi(0)|^{2\beta+p-2}}{(1-\delta_i)(2\beta+p-2)}\right. \\
&\qquad \left. + \frac{2\beta\overline{\lambda}_i^2}{(1-\delta_i)(2\beta+p-2)}\int_{-\tau}^{0}|\varphi(\theta)|^{2\beta+p-2}d\nu_i(\theta)\right] \\
&\quad + o(|x|^{2\beta+p-2}),
\end{aligned}
$$

where $m_i = \|Q_i\|[1 + R_i(p-2)]$. Noting that $I_4 + I_5 = o(|\varphi(0)|^{\alpha+p})$, substituting these estimates into (4.3) gives

$$
\begin{aligned}
&\mathcal{L}V(\varphi, i) + \varepsilon V(\varphi(0), i) \\
&\qquad \leq H_i^*(\varphi(0)) + \frac{p(\alpha+2)\sigma_i}{\alpha+p}\|Q_i\|^{(p/2)-1} \\
&\qquad\quad \times \left(\int_{-\tau}^{0}|\varphi(\theta)|^{\alpha+p}d\mu_i(\theta) - e^{\varepsilon\tau}|\varphi(0)|^{\alpha+p}\right) \\
&\qquad\quad + \frac{\beta p\overline{\lambda}^2 m_i\|Q_i\|^{(p/2)-1}}{u(1-\delta_i)(2\beta+p-2)} \\
&\qquad\quad \times \left(\int_{-\tau}^{0}|\varphi(\theta)|^{2\beta+p-2}d\nu_i(\theta) - e^{\varepsilon\tau}|\varphi(0)|^{2\beta+p-2}\right),
\end{aligned}
$$

where

$$
\begin{aligned}
H_i^*(x) &= -pq_i{}^{(p/2)-1}\left[b_i - \sigma_i R_i^{(p/2)-1}\frac{(\alpha+2)e^{\varepsilon\tau}+p-2}{\alpha+p}\right]|x|^{\alpha+p} \\
&\quad + \frac{pm_i\|Q_i\|^{(p/2)-1}}{2u}\left(\frac{\lambda_i^2}{\delta_i} + \frac{\overline{\lambda}_i^2(2\beta e^{\varepsilon\tau}+p-2)}{(1-\delta_i)(2\beta+p-2)}\right)|x|^{2\beta+p-2} \\
&\quad + o(|x|^{2\beta-2+p}) + o(|x|^{\alpha+p}).
\end{aligned}
$$

Noting that $\alpha > 2\beta - 2 > 0$, $H_i^*(x)$ may be rewritten as

$$\begin{aligned} H_i^*(x) &= -pq_i{}^{(p/2)-1}\left[b_i - \sigma_i R_i^{(p/2)-1}\frac{(\alpha+2)e^{\varepsilon\tau}+p-2}{\alpha+p}\right]|x|^{\alpha+p} \\ &\quad + o(|x|^{\alpha+p}) \\ &=: -ph_i^*(\varepsilon)|x|^{\alpha+p} + o(|x|^{\alpha+p}). \end{aligned}$$

Letting $p \to 2^+$, by condition (4.15), $h_i^*(0) = b_i - \sigma_i > 0$. For sufficiently small $\varepsilon > 0$, there exists a constant $\overline{p} > 2$ such that for any $p \in (2, \overline{p})$, $h_i^*(\varepsilon) > 0$, which implies that there exists a constant $\overline{H}_i^*$ such that $H_i^*(x) \le \overline{H}_i^*$ by Lemma 2.1. This shows that (4.3) satisfies condition (3.2). Lemma 3.2 gives the desired result by the same process as the proof of Theorem 4.2. □

When $\sigma_i = 0$, condition (F1) may be rewritten as condition (F1′). Applying Theorem 4.4 to equation (1.3), in which condition (F1) is replaced by condition (F1′), yields the following result.

Corollary 4.5. *Let Assumption* 3.1 *hold. Under conditions* (F1′) *and* (F2)*, if* $\alpha > 2\beta - 2 > 0$*, then for any initial data* $\xi \in C([-\tau, 0]; \mathbf{R}^n)$*, equation* (1.3) *almost surely admits a global solution* $x(t)$ *and this solution is ultimately bounded. Moreover, for any* $p > 2$*, this global solution has properties* (1.4) *and* (1.6)*.*

Proof. In condition (F1′), $b_i > 0$ implies condition (4.15) since $\sigma_i = 0$, then Theorem 4.4 gives that there exists a unique global solution to equation (1.3) and this solution is ultimately bounded. We employ the proof of Theorem 4.4, for any $p > 2$, to conclude that this global solution has properties (1.4) and (1.6). □

Comparing Theorem 4.2 with Theorem 4.4, we may find the following interesting phenomena. In Theorem 4.2, stochastic perturbation intensity g plays an important role in guaranteeing existence of the global solution, which shows that the stochastic noise may suppress growth of the solution when the intensity of this noise is strongly dependent on the solution in the sense $\alpha < 2\beta$. The idea of suppression of the stochastic noise recently attracts the increasing attention in population systems, which shows that the stochastic perturbation may has positive effect on the population dynamics (cf. [**3, 17**]). If the intensity of this noise is weakly dependent on the solution in the sense $\alpha > 2\beta - 2 > 0$ (note that notations of α, β in the conditions (H) and (F) may be different, but $\alpha < 2\beta$ and $\alpha > 2\beta - 2 > 0$ can still measure the intensity of environmental noise), the deterministic coefficient f plays a crucial role to determine existence of the global solution and its asymptotic properties, which shows robustness of equation (1.1) for stochastic perturbation. This idea also appears in some stochastic population dynamic models (for example, [**4, 20**]).

5. One-dimensional case. In this section, we consider the following scalar polynomial stochastic functional differential equation with Markovian switching

$$dx(t) = f(x_t, r(t))\,dt + g(x_t, r(t))\,dw(t), \tag{5.1}$$

where

$$\begin{aligned} f(\varphi, i) &= \sum_{\varpi + j \le m} a^i_{\varpi j} \int_{-\tau}^0 \varphi^{\varpi}(0)\varphi^j(\theta) d\mu^i_{\varpi j}(\theta), \\ g(\varphi, i) &= \sum_{k+l \le n} b^i_{kl} \int_{-\tau}^0 \varphi^k(0)\varphi^l(\theta)\, d\nu^i_{kl}(\theta), \end{aligned} \tag{5.2}$$

and ϖ, j, k, l are nonnegative integers, $a^i_{\varpi j}$ and b^i_{kl} are constants, $\mu^i_{\varpi j}$ and ν^i_{kl} are probability measures on $[-\tau, 0]$. We establish the following two results for functionals f and g.

Lemma 5.1. *For functions f and g in* (5.2), *there exist probability measures μ_i and ν_i on $[-\tau, 0]$ such that*

$$|f(\varphi, i)| \le const\Big(1 + |\varphi(0)|^m + \int_{-\tau}^0 |\varphi(\theta)|^m d\mu_i(\theta)\Big), \tag{5.3}$$

$$|g(\varphi, i)| \le const\Big(1 + |\varphi(0)|^n + \int_{-\tau}^0 |\varphi(\theta)|^n d\nu_i(\theta)\Big). \tag{5.4}$$

Proof. Applying Lemma 2.1 and the Young inequality yields

$$\begin{aligned}|f(\varphi,i)| &\le |a^i_{00}| + \sum_{0<j+\varpi\le m} |a^i_{j\pi}| \int_{-\tau}^0 |\varphi(0)|^j |\varphi(\theta)|^{\varpi} d\mu^i_{j\varpi}(\theta)\\ &\le |a^i_{00}|\\ &\quad + \sum_{0<j+\varpi\le m} |a^i_{j\varpi}| \int_{-\tau}^0 \frac{j|\varphi(0)|^{j+\varpi} + \varpi|\varphi(\theta)|^{j+\varpi}}{j+\varpi}\, d\mu^i_{j\varpi}(\theta)\\ &\le const \sum_{k=0}^m \Big[|\varphi(0)|^k + \int_{-\tau}^0 |\varphi(\theta)|^k d\Big(\sum_{0<j+\varpi\le m} \mu^i_{j\varpi}\Big)\Big]\\ &\le const\Big(1 + |\varphi(0)|^m + \int_{-\tau}^0 |\varphi(\theta)|^m d\mu_i(\theta)\Big),\end{aligned}$$

where $\mu_i = c\sum_{0<j+\varpi<m} \mu^i_{j\varpi}$ and c is a given positive constant to guarantee that μ_i is a probability measure. By the same process, (5.4) may also be obtained, as desired. □

By the same technique, we can also obtain the following result.

Lemma 5.2. *For functional f in (5.2), if $m > 0$ and $a^i_{(j-1)1} = a^i_{(j-2)2} = \cdots = a^i_{0m} = 0$, for any given $\varepsilon > 0$,*

$$f(\varphi,i) \le a^i_{m0}|\varphi(0)|^m + \varepsilon\Big(|\varphi(0)|^m + \int_{-\tau}^0 |\varphi(\theta)|^m d\mu_i(\theta)\Big) + const,$$

and

$$|f(\varphi,i)| \le (|a^i_{m0}| + \varepsilon)|\varphi(0)|^m + \varepsilon\int_{-\tau}^0 |\varphi(\theta)|^m d\mu_i(\theta) + const,$$

where μ_i is a probability measure on $[-\tau, 0]$.

To apply Theorems 4.2 and 4.4 to equation (5.1), we need to verify conditions (H1)–(H3) and (F1) and (F2). Since equation (5.1) is a scalar equation, we choose $Q_i = 1$, which implies that $q_i = R_i = 1$, $M_p = 0$.

Let $m \geq 1$ and $a^i_{0\varpi} = 0$ $(0 \leq \varpi \leq m)$. By Lemma 5.1, we have

$$\varphi^{-1}(0)|f(\varphi, i)| \leq const\Big(1 + |\varphi(0)|^{m-1} + \int_{-\tau}^{0} |\varphi(\theta)|^{m-1} d\mu_i(\theta)\Big), \tag{5.5}$$

which implies that f satisfies condition (H1) with $\alpha = m - 1 \geq 0$.

Let $n \geq 1$, $b^i_{0l} = 0$ for $(1 \leq l \leq n)$. Equation (5.4) in Lemma 5.1 gives that g satisfies condition (H2). Here $\beta = n - 1$ when $n > 1$ and $\beta \in (0, 1)$ is arbitrary when $n = 1$.

Let $n > 1$, $b^i_{n0} \neq 0$, $b^i_{(n-1)1} = b^i_{(n-2)2} = \cdots = b^i_{1(n-1)} = b^i_{0n} = 0$ and $b^i_{0l} = 0$ $(0 \leq l \leq n)$. It follows that

$$\begin{aligned}
&\varphi^{-4}(0)[\varphi(0)g(\varphi, i)]^2 \\
&\quad = \Big(b^i_{n0}\varphi^{n-1}(0) + \sum_{k+l \leq n-1, k \geq 1, l \geq 0} b^i_{kl}\varphi^k(0) \int_{-\tau}^{0} \varphi^l(\theta)\, d\nu^i_{kl}(\theta)\Big)^2 \\
&\quad = [b^i_{n0}\varphi^{n-1}(0)]^2 - h(\varphi, i) \\
&\quad \geq (b^i_{n0})^2\varphi^{2n-2}(0) - \varepsilon\varphi^{2n-2}(0) \\
&\qquad - \varepsilon \int_{-\tau}^{0} \varphi^{2n-2}(\theta)\, d\nu_i(\theta) - const,
\end{aligned}$$

where $h(\varphi, i)$ is a polynomial function with order $2n - 3$ and ν_i is a probability measure on $[-\tau, 0]$, which implies that condition (H3) is satisfied with $b = (b^i_{n0})^2 - \varepsilon$, $\sigma_i = \varepsilon$ and $\beta = n - 1 > 0$.

Let m be an odd integer. For any $\varepsilon > 0$, by the Young inequality and Lemma 2.1, we have the following estimate

$$\begin{aligned}
\varphi(0)f(\varphi, i) &= a^i_{m0}\varphi^{m+1}(0) \\
&\quad + \sum_{j=1}^{m} a^i_{(m-j)j} \int_{-\tau}^{0} \varphi^{m-j+1}(0)\varphi^j(\theta)\, d\mu^i_{(m-j)j}(\theta) \\
&\quad + \sum_{\varpi+j<m} a^i_{j\varpi} \int_{-\tau}^{0} \varphi^{j+1}(0)\varphi^{\varpi}(\theta)\, d\mu^i_{\varpi j}(\theta) \\
&\leq a^i_{m0}\varphi^{m+1}(0) \\
&\quad + \frac{1}{m+1} \sum_{j=1}^{m} |a^i_{(m-j)j}| \Big[(m - j + 1)\varphi^{m+1}(0)
\end{aligned} \tag{5.6}$$

$$
\begin{aligned}
&\quad + j \int_{-\tau}^{0} \varphi^{m+1}(\theta)\, d\mu^i_{(m-j)j}(\theta)\Big] \\
&+ \varepsilon\Big(\varphi^{m+1}(0) + \int_{-\tau}^{0} \varphi^{m+1}(\theta)\, d\nu_i(\theta)\Big) + const \\
&=: -b_i\varphi^{m+1}(0) + \sigma_i \int_{-\tau}^{0} \varphi^{m+1}(\theta)\, d\bar{\mu}_i(\theta) + const,
\end{aligned}
$$

where $\overline{\mu}_i$ is a probability measure on $[-\tau, 0]$ and

$$
\begin{aligned}
-b_i &= a^i_{m0} + \varepsilon + \frac{1}{m+1}\sum_{j=1}^{m}(m-j+1)|a^i_{(m-j)j}|, \\
\sigma &= \varepsilon + \frac{1}{m+1}\sum_{j=1}^{m} j|a^i_{(m-j)j}|.
\end{aligned}
\tag{5.7}
$$

Now condition (4.15) may be rewritten as

$$
-a^i_{m0} > \sum_{j=1}^{m} |a^i_{(m-j)j}| + 2\varepsilon.
$$

By the arbitrary property of ε, we have

$$
-a^i_{m0} > \sum_{j=1}^{m} |a^i_{(m-j)j}|. \tag{5.8}
$$

Under this condition, $b_i > 0$, which implies that condition (F1) is satisfied with $\alpha = m - 1 \geq 0$. Note that it is necessary to require m to be an odd number since we can not guarantee that the solution is positive.

It is obvious that g satisfies condition (F2) with $\beta = n$ when $n \geq 1$.

We employ Theorem 4.2 and Theorem 4.4 to establish the following result.

Theorem 5.3. (i) *If* $1 \leq m < 2n - 1$, $b^i_{(n-1)1} = b^i_{(n-2)2} = \cdots = b^i_{0n} = a^i_{0\pi} = b^i_{0l} = 0$ $(0 \leq \pi \leq m, 0 \leq l \leq n)$, $b^i_{n0} \neq 0$ *and* $|b^i_{kl}|(i \in \mathbf{S}, k + l \leq n)$ *is sufficiently small, then for any initial* *data $\xi \in C([-\tau, 0]; \mathbf{R}^n)$, there exists a unique global solution $x(t, \xi)$ to equation (5.1) and this solution is ultimately bounded. Moreover, for sufficiently small $p > 0$, this solution $x(t, \xi)$ satisfies (1.4) and*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t E|x(t,\xi)|^{2n+p-2} ds \le K,$$

where K is a positive constant independent of ξ.

(ii) *If $m > 2n-1$ is an odd number, $p \ge 2$ and condition (5.8) holds, then for any initial data $\xi \in C([-\tau, 0]; \mathbf{R}^n)$, there exists a unique global solution $x(t, \xi)$ to equation (5.1) and this solution is ultimately bounded. Moreover, there exists a $\bar{p} > 2$ such that for any $p \in (2, \overline{p})$, $x(t, \xi)$ has properties (1.4) and*

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t E|x(t,\xi)|^{m+p-1} ds \le K,$$

where K is a positive constant independent of ξ.

Let us now discuss an example to show our result.

Example 5.1. Consider the scalar stochastic functional differential equation

$$dx(t) = \left[-a_r x^3(t) + b_r x^2(t)\int_{-1}^0 x(t+\theta)d\theta + c_r\right] dt + \left[k_r x(t) + l_r \int_{-1}^0 x(t+\theta)d\theta + m_r\right] dw(t),$$

where $i \in \mathbf{S}$ and a_i, b_i, c_i, k_i, l_i, m_i are constants.

By case (ii) of Theorem 5.3, if $a_i > |b_i|$ ($i \in \mathbf{S}$), which implies condition (5.8) holds, there exists a unique global solution to equation (5.9) and this solution is ultimately bounded. Moreover, for all $p > 2$, this solution has property (1.4) and there exists a constant K independent of the initial data such that

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \mathbf{E}|x(s)|^{2+p} ds \le K.$$

6. A special stochastic functional differential equation. Let us consider the n-dimensional stochastic functional equation

$$dx(t) = |x(t)|^\gamma \bigg[\bigg(a_r + A_r x(t) + \int_{-\tau}^{0} B_r(\theta) x(t+\theta)\, d\mu_r(\theta) \bigg) dt + h(x(t), r(t))\, dw(t) \bigg],$$

which is a special case of equation (1.2) and where $a_i \in \mathbf{R}^n$, $A_i \in \mathbf{R}^{n\times n}$, $B_i(\theta) \in C([-\tau,0];\mathbf{R}^{n\times n})$, μ_i is a probability measure on $[-\tau,0]$ and $h(x,i) : \mathbf{R}^n \times \mathbf{S} \to \mathbf{R}^n$ satisfies the local Lipschitz condition. Define

$$\begin{aligned} f(\varphi,i) &= |\varphi(0)|^\gamma \bigg(a_i + A_i\varphi(0) + \int_{-\tau}^{0} B_i(\theta)\varphi(\theta)\, d\mu_i(\theta) \bigg), \\ g(x,i) &= |x|^\gamma h(x,i). \end{aligned} \tag{6.1}$$

Letting $m_i = \sup_{\theta\in[-\tau,0]} \|B_i(\theta)\|$ and applying the Young inequality give

$$\begin{aligned} &|\varphi(0)|^{-2}\varphi^T(0) f(\varphi,i) \\ &\quad\le |\varphi(0)|^{\gamma-1} \bigg| a_i + A_i\varphi(0) + \int_{-\tau}^{0} B_i(\theta)\varphi(\theta)\, d\mu_i(\theta) \bigg| \\ &\quad\le \|A_i\| |\varphi(0)|^\gamma + m_i \int_{-\tau}^{0} |\varphi(0)|^{\gamma-1} |\varphi(\theta)|\, d\mu_i(\theta) + o(|\varphi(0)|^\gamma) \\ &\quad\le \bigg(\|A_i\| + \frac{\gamma-1}{\gamma} m_i \bigg) |\varphi(0)|^\gamma + \frac{m_i}{\gamma} \int_{-\tau}^{0} |\varphi(\theta)|^\gamma d\mu_i(\theta) \\ &\qquad + o(|\varphi(0)|^\gamma), \end{aligned} \tag{6.2}$$

which shows that f satisfies condition (H1) with $Q_i = I, \alpha = \gamma$ when $\gamma \ge 1$, where I represents the identity matrix. Applying Corollary 4.3 to (6.1) gives

Theorem 6.1. *Let $\gamma \ge 1$. If there exist constants $\beta > (\gamma-1)\vee(\gamma/2)$, $b_i > 0$ and $\lambda_i \in [0, \sqrt{2b_i})$ such that*

$$|h(x,i)| \le \lambda_i |x|^{\beta-\gamma+1} + o(|x|^{\beta-\gamma+1}), \tag{6.3}$$

$$[x^T h(x,i)]^2 \ge b_i |x|^{2(\beta-\gamma+2)} + o(|x|^{2(\beta-\gamma+2)}), \tag{6.4}$$

then for any initial data $\xi \in C([-\tau, 0]; \mathbf{R}^n)$*, equation* (6.1) *almost surely admits a unique global solution and this solution is ultimately bounded. For any* $p \in (0, \min_{i\in\mathbf{S}}\{2 - \lambda_i^2/b\})$*, this solution further has properties* (1.4) *and* (4.9).

Proof. Inequality (6.2) shows that f satisfies condition (H1). We then test conditions (H2′) and (H3′). By conditions (6.3) and (6.4), we have

$$\begin{aligned} |x|^{-1}|g(x,i)| &\le |x|^{\gamma-1}[\lambda_i|x|^{\beta-\gamma+1} + o(|x|^{\beta-\gamma+1})] \\ &= \lambda_i|x|^{\beta} + o(|x|^{\beta}) \\ |x|^{-4}[x^T g(x,i)]^2 &\ge |x|^{2\gamma-4}[b_i|x|^{2\beta-2\gamma+4} + o(|x|^{2(\beta-\gamma+2)})] \\ &= b_i|x|^{2\beta} + o(|x|^{2\beta}). \end{aligned}$$

Note that condition $\beta > (\gamma - 1) \vee (\gamma/2)$ implies condition $0 < \alpha < 2\beta$ and $\lambda_i \in [0, \sqrt{2b_i})$ implies condition (4.13). By Corollary 4.3, equation (6.1) almost surely admits a unique global solution and this solution is ultimately bounded. Noting that $\overline{\lambda}_i = 0$, for any $p \in (0, \min_{i\in\mathbf{S}}\{2 - \lambda_i^2/b\})$, the solution of equation (6.1) has properties (1.4) and (4.9), as desired. □

Let $Q_i \in \mathbf{R}^{n\times n}$ be a positive definite matrix. Let

$$s_i = \lambda_{\max}(Q_iA_i + A_i^TQ_i) \tag{6.5}$$

denote the biggest eigenvalues of the symmetric matrix $Q_iA_i + A_i^TQ_i$. For any $\varepsilon > 0$, applying the Young inequality yields

$$\begin{aligned} &\varphi^T(0)Q_if(\varphi,i) \\ &\quad = |\varphi(0)|^{\gamma}\varphi^T(0)Q_i\Big(A_i\varphi(0) + \int_{-\tau}^0 B_i(\theta)\varphi(\theta)\,d\mu_i(\theta)\Big) \\ &\qquad + o(|\varphi(0)|^{\gamma+2}) \\ &\quad \le |\varphi(0)|^{\gamma}\Big(\varphi^T(0)Q_iA_i\varphi(0) \\ &\qquad\quad + m_i\|Q_i\|\int_{-\tau}^0 |\varphi(0)||\varphi(\theta)|\,d\mu_i(\theta)\Big) + o(|\varphi(0)|^{\gamma+2}) \\ &\quad \le \frac{s_i}{2}|\varphi(0)|^{\gamma+2} + \frac{m_i\|Q_i\|}{r+2}\Big[(r+1)|\varepsilon_i\varphi(0)|^{\gamma+2} \end{aligned}$$

$$+\varepsilon_i^{-(\alpha+1)(\alpha+2)}\int_{-\tau}^{0}|\varphi(\theta)|^{\alpha+2}d\mu_i(\theta)\Big]+o(|\varphi(0)|^{\alpha+2})$$
$$=: -b_i|\varphi(0)|^{\gamma+2}+\sigma_i\int_{-\tau}^{0}|\varphi(\theta)|^{\alpha+2}d\mu_i(\theta)+o(|\varphi(0)|^{\gamma+2}),$$

where

$$-b_i=\frac{\varepsilon_i}{2}+\frac{m_i\|Q_i\|(\gamma+1)}{\gamma+2}\varepsilon_i^{\alpha+2},\quad \sigma_i=\frac{m_i\|Q_i\|}{r+2}\varepsilon_i^{-(\alpha+1)(\alpha+2)}. \tag{6.6}$$

Choosing $\varepsilon_i=m_i^{1/(\alpha+2)^2}$, condition $b_iR_i^{1-p/2}>\sigma_i$ may be rewritten as

$$-s_i>2m_i\|Q_i\|R_i^{p-2/2(\gamma+2)},$$

which implies that

$$-\lambda_{\max}(Q_iA_i+A_i^TQ_i)>2m_i\|Q_i\|\left(\frac{\|Q_i\|}{q_i}\right)^{(p-2)/2(\gamma+2)}. \tag{6.7}$$

This condition also implies $b_i>0$, which shows that f satisfies condition (F1) with $\alpha=\gamma$. For any $\beta\in(0,1+\gamma/2)$, which implies that $\gamma>2\beta-2$, by Theorem 4.4, it follows that

Theorem 6.2. *Let $\gamma\ge 0$. If there exist $\beta\in(0,1+\gamma/2)$ and $\lambda_i\ge 0$ such that*

$$|h(x,i)|\le\lambda_i|x|^{\beta}+o(|x|^{\beta}), \tag{6.8}$$

and a positive definite matrix Q_i such that condition (6.7) holds, then for any initial data ξ, equation (6.1) almost surely admits a unique global solution and this solution is ultimately bounded. Moreover, there exists a constant $\overline{p}>2$ such that for any $p\in(2,\overline{p})$, this solution has property (1.4) and there exists a constant $K^*_{\gamma+p}$ such that

$$\limsup_{t\to\infty}\frac{1}{t}\int_0^t\mathbf{E}|x(s)|^{\gamma+p}ds\le K^*_{\gamma+p}.$$

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