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FAST MULTILEVEL AUGMENTATION METHODS WITH COMPRESSION TECHNIQUE FOR SOLVING ILL-POSED INTEGRAL EQUATIONS

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ABSTRACT. In this paper, multilevel augmentation methods with compression technique are developed for solving ill-posed integral equations. The methods are based on the combination of Lavrentiev regularization and multiscale Galerkin methods, and lead to fast solutions of the discrete equations. We provide *a priori* error analysis for the methods, propose an *a posteriori* regularization parameter choice strategy using compression technique, and establish optimal convergence rates for approximation solutions. Finally, numerical results are presented to illustrate the efficiency of the method.

1. Introduction. Many problems in science and engineering can be formulated as ill-posed Fredholm integral equations of the first kind. These equations are normally treated by regularization methods such as Tikhonov regularization and Lavrentiev regularization (see, for example, [**11, 12, 22**]). When solving the regularization equations by iteration methods or using discrepancy principles to determine the regularization parameters, we have to solve the regularization equations repeatedly, so developing an efficient fast solver for numerically solving such problems is an important and challenging task. This is what we try to do in this paper.

Multilevel methods are popular for the solution of well-posed problems such as Fredholm integral equation of the second kind (see, for example, [**1, 4, 5, 9, 21**] and the references cited therein). These methods have considerable advantages and have been becoming stan-

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dard approaches in applications. However, little is known about the behavior of multilevel methods when applied to the solution of ill-posed problems (cf. [**23**]). Some significant attempts have been made to develop multilevel methods for ill-posed problems. For examples, multilevel preconditioners for solving ill-posed equations were developed in [**13, 15, 16**]. Wavelet and multilevel algorithms for ill-posed problems were studied in [**10, 17, 18, 23, 24**]. The compression technique for self-regularization by projection methods was applied in [**14**]. In [**7, 8**] multilevel augmentation methods for solving ill-posed operator equations were developed by making use of the multiscale structure of the matrix representation of the operator. It is shown that multilevel methods lead to efficient, stable and accurate solvers for ill-posed problems, which are faster than standard methods and preserve the convergence rate. This paper continues the general theme of recent work in [**7**]. The multilevel augmentation method is developed by combining the compression technique. It will be shown by theoretical analysis and numerical experiment that the compression technique speeds up the multilevel augmentation method greatly, preserves the optimal convergence rate and does not ruin the *a posteriori* regularization parameter choice strategy presented in [**7**].

This paper is organized as follows. In Section 2, we describe the multiscale Galerkin method with compression technique for numerically solving Lavrentiev regularization equations of the first kind ill-posed integral equations and present convergence analysis for approximate solutions. We develop the multilevel augmentation method in Section 3 by using the multilevel decomposition of the truncated operator and provide *a priori* error estimates. In Section 4, we develop the *a posteriori* regularization parameter choice strategy presented in [**7**] by using the compression scheme described in Section 2. Optimal convergence rate is established for the multilevel augmentation solution obtained by using the *a posteriori* regularization parameter. Finally in Section 5, numerical results are presented to illustrate the efficiency of the method and confirm the theoretical results of this paper.

2. Fast multiscale Galerkin methods for Lavrentiev regularization. In this section, we describe the fast multiscale Galerkin method for solving ill-posed integral equations of the first kind via Lavrentiev regularization and present the convergence rate for the truncation scheme.

Suppose that $E \subset \mathbf{R}^d$ is a bounded domain and $\mathbf{X} := L^2(E)$ with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let $\mathcal{A} : \mathbf{X} \to \mathbf{X}$ be a linear and positive semi-definite compact operator, that is, $\langle \mathcal{A}x, x \rangle \geq 0$, for all $x \in \mathbf{X}$. We consider the operator equation of the first kind

$$\mathcal{A}u = f, \tag{2.1}$$

where $f \in \mathbf{X}$ is given, $u \in \mathbf{X}$ is the unknown to be determined, and the operator $\mathbf{A}$ is defined by

$$(\mathcal{A}u)(s) = \int_E K(s,t)u(t)\,dt, \quad s \in E,$$

with the continuous kernel $K(\cdot, \cdot) \in C(E \times E)$.

Since $\mathcal{A}$ is a compact operator defined on an infinity dimensional space, equation (2.1) is ill-posed. For $f \in R(\mathcal{A})$, we let $\widehat{u} \in \mathbf{X}$ denote the unique minimum norm solution of equation (2.1), that means

$$\mathcal{A}\widehat{u} = f \quad \text{and} \quad \|\widehat{u}\| = \inf\{\|v\| : \mathcal{A}v = f, v \in \mathbf{X}\}. \tag{2.2}$$

In fact the accurate data f may not be known and instead we have a noisy data $f^\delta \in \mathbf{X}$ satisfying

$$\|f^\delta - f\| \leq \delta, \tag{2.3}$$

where $\delta > 0$ is a given small number. In general, the solution of (2.1) does not continuously depend on the right-hand side, so we cannot expect that the solution of (2.1) with f replaced by f^δ is closed to $\widehat{u}$ even if δ is small enough. The Lavrentiev regularization method is one of the popular methods to obtain a stable approximation of equation (2.1). That is, for $\alpha > 0$, we solve the equation

$$(\alpha\mathcal{I} + \mathcal{A})u_\alpha^\delta = f^\delta. \tag{2.4}$$

Since $\mathcal{A}$ is positive semi-definite, we have

$$\|(\alpha\mathcal{I} + \mathcal{A})^{-1}\| \leq \alpha^{-1}. \tag{2.5}$$

Thus for any given $\alpha > 0$, equation (2.4) has a unique solution $u_\alpha^\delta \in \mathbf{X}$. It is well known that

$$\lim_{\substack{\alpha \to 0 \\ \delta\alpha^{-1} \to 0}} \|u_\alpha^\delta - \widehat{u}\| = 0.$$

Moreover, if $\widehat{u} \in R(\mathcal{A}^\nu)$ with $0 < \nu \leq 1$, then

$$\|u_\alpha - \hat{u}\| \leq c_\nu \|w\| \alpha^\nu, \quad \text{and} \quad \|u_\alpha^\delta - u_\alpha\| \leq \delta \alpha^{-1}. \tag{2.6}$$

where $\widehat{u} = \mathcal{A}^\nu w$ and c_ν is a constant depending only on ν (see, for example, [**12**]).

Now we describe the multiscale Galerkin method for solving the regularization equation (2.4). First we need a multiscale partition of the set E. Let $\mathbf{N} := \{1, 2, \dots\}$, $\mathbf{N}_0 := \{0, 1, 2, \dots\}$, $\mathbf{Z}_n := \{0, 1, \dots, n-1\}$, and let $\mu > 1$ be a positive integer. The multiscale partition of E consists of a family partitions $\{E_i : i \in \mathbf{N}_0\}$ having the properties that

$$E_i := \{E_{ij} : j \in \mathbf{Z}_{e(i)}\}, \quad i \in \mathbf{N}_0,$$
$$\bigcup_{j \in \mathbf{Z}_{e(i)}} E_{ij} = E, \qquad \text{meas}\,(E_{ij} \cap E_{ij'}) = 0, \quad j, j' \in \mathbf{Z}_{e(i)}, \quad j \neq j';$$

and

$$\text{meas}\,(E_{ij}) \sim d_i^d,\ e(i) \sim \mu^i,\ d_i \sim \mu^{-i/d}, \quad i \in \mathbf{N}_0,$$

where $e(i)$ denotes the cardinality of E_i, and $d_i := \max\{d(E_{ij}) : j \in \mathbf{Z}_{e(i)}\}$. Moreover, the sets E_{ij} are all star-shaped, i.e., E_{ij} contains a point such that the line segment connecting this point and any other point in it is contained in itself.

We next describe the multiscale space decomposition. For each $n \in \mathbf{N}_0$, let $\mathbf{X}_n$ be the piecewise polynomial space associated with the partition E_n with total degree less than k, $k \in \mathbf{N}$. Then we have $\overline{\bigcup_{n \in \mathbf{N}_0} \mathbf{X}_n} = \mathbf{X}$, $\mathbf{X}_n \subset \mathbf{X}_{n+1}$, $n \in \mathbf{N}_0$, and $s(n) := \dim \mathbf{X}_n \sim \mu^n$. For each $i \in \mathbf{N}$, let $\mathcal{W}_i$ be the orthogonal complement of $\mathbf{X}_{i-1}$ in $\mathbf{X}_i$, then $w(i) := \dim \mathcal{W}_i \sim \mu^i$. For a fixed $\ell \in \mathbf{N}$ and any $m \in \mathbf{N}_0$, we have the multiscale decomposition

$$\mathbf{X}_{\ell+m} = \mathbf{X}_\ell \oplus^\perp \mathcal{W}_{\ell+1} \oplus^\perp \cdots \oplus^\perp \mathcal{W}_{\ell+m}. \tag{2.7}$$

For each $n \in \mathbf{N}_0$, we let $\mathcal{P}_n : \mathbf{X} \to \mathbf{X}_n$ be the linear orthogonal projection, we have that there exists a positive constant c such that for any $u \in H^k(E)$,

$$\|u - \mathcal{P}_n u\| \leq c d_n^k \|u\|_k. \tag{2.8}$$

Let $\mathcal{A}_n := \mathcal{P}_n\mathcal{A}\mathcal{P}_n$, $f_n^\delta := \mathcal{P}_n f^\delta$, the Galerkin method for solving Lavrentiev regularization equation (2.4) is to find $u_{\alpha,n}^\delta \in \mathbf{X}_n$ such that

$$(\alpha\mathcal{I} + \mathcal{A}_n)u_{\alpha,n}^\delta = f_n^\delta. \tag{2.9}$$

Since $\mathcal{A}_n$ is also positive semi-definite, we have (cf. [**7**])

$$\|(\alpha\mathcal{I} + \mathbf{A}_n)^{-1}\| \le \alpha^{-1}, \tag{2.10}$$

and equation (2.9) has a unique solution.

To develop fast multiscale Galerkin algorithms for solving equation (2.9), we require the multiscale bases for $\mathbf{X}_n$, denoted by $\{w_{i,j} : (i,j) \in \mathcal{U}_n\}$ where $\mathcal{U}_n = \{(i,j) : j \in \mathbf{Z}_{w(i)}, i \in \mathbf{Z}_{n+1}\}$, have the following properties.

(i) There exist positive integers ρ and r such that for every $i > r$ and $j \in \mathbf{Z}_{w(i)}$ written in the form $j = \nu\rho + s$, where $s \in \mathbf{Z}_\rho$ and $\nu \in \mathbf{N}_0$, $w_{i,j}(x) = 0$, when $x \notin E_{i-r,\nu}$. This means that $\operatorname{supp} w_{i,j} \subset S_{i,j} := E_{i-r,\nu}$.

(ii) For any (i,j), $(i',j') \in \mathcal{U}_n$, $\langle w_{i,j}, w_{i',j'}\rangle = \delta_{(i,j),(i',j')}$. This leads to that for any polynomial p of total degree less than k, $\langle w_{i,j}, p\rangle = 0$.

(iii) There is a positive constant c such that for any $(i,j) \in \mathcal{U}_n$, $\|w_{i,j}\|_\infty \le c\mu^{i/2}$.

We remark that the construction of such bases can be seen in [**3, 5, 20**].

With the bases $\{w_{i,j} : (i,j) \in \mathcal{U}_n\}$ for space $\mathbf{X}_n$, the multiscale Galerkin scheme for solving equation (2.4) is to find $u_{\alpha,n}^\delta = \sum_{(i,j)\in\mathcal{U}_n} u_{i,j}^\delta(\alpha) w_{i,j} \in \mathbf{X}_n$ such that

$$\sum_{(i,j)\in\mathcal{U}_n} u_{i,j}^\delta(\alpha)\left(\alpha\langle w_{i,j}, w_{i',j'}\rangle + \langle \mathcal{A}w_{i,j}, w_{i',j'}\rangle\right) = \left\langle f_n^\delta, w_{i',j'}\right\rangle, \quad (i',j') \in \mathcal{U}_n. \tag{2.11}$$

Denote

$$\begin{aligned}
\mathbf{E}_n(\alpha) &:= \alpha\left[\langle w_{i,j}, w_{i',j'}\rangle\right]_{(i,j),(i',j')\in\mathcal{U}_n},\\
\mathbf{A}_n &:= \left[\langle \mathbf{A}w_{i,j}, w_{i',j'}\rangle\right]_{(i,j),(i',j')\in\mathcal{U}_n},\\
\mathcal{U}_n^\delta(\alpha) &:= \left[u_{ij}^\delta(\alpha)\right]_{(i,j)\in\mathcal{U}_n},\\
\mathbf{F}_n^\delta &:= \left[\left\langle f_n^\delta, w_{i',j'}\right\rangle\right]_{(i',j')\in\mathcal{U}_n},
\end{aligned}$$

then equation (2.11) can be written as

$$(\mathbf{E}_n(\alpha) + \mathbf{A}_n)\mathcal{U}_n^\delta(\alpha) = \mathbf{F}_n^\delta. \tag{2.12}$$

We now describe the matrix compression scheme for solving the equation (2.12). To do this we require the assumption that $K(\cdot,\cdot) \in C^k(E \times E)$, and there exists a positive constant c such that

$$\left| D_s^\alpha D_t^\beta K(s,t) \right| \le c, \quad s,t \in E,\ |\alpha| = |\beta| = k. \tag{2.13}$$

Some related lemmas are outlined in the following, which are similar to the case using Tikhonov regularization in [**2**]. We omit their proofs.

Lemma 2.1. *If condition* (2.13) *holds, then there exists a constant* c *independent of* n *such that*

$$|\langle \mathcal{A} w_{i,j}, w_{i',j'} \rangle| \le c\,(d_i d_{i'})^{k+(d/2)}.$$

This lemma shows that most entries of $\mathbf{A}_n$ are so small that they can be neglected without affecting the overall accuracy of the approximation. To compress the matrix, we partition $\mathbf{A}_n$ into a block matrix $\mathbf{A}_n = [\mathbf{A}_{i'i}]_{i',i \in Z_{n+1}}$, where $\mathbf{A}_{i'i} = [\langle \mathcal{A} w_{i,j}, w_{i',j'} \rangle]_{j' \in \mathbf{Z}_{w(i')}, j \in \mathbf{Z}_{w(i)}}$, and truncate each block by setting

$$\widetilde{\mathbf{A}}_{i'i} := \begin{cases} \mathbf{A}_{i'i} & i + i' \le n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.14}$$

This leads to a truncated matrix $\widetilde{\mathbf{A}}_n := \left[\widetilde{\mathbf{A}}_{i'i}\right]_{i',i \in \mathbf{Z}_{n+1}}$. The next lemma shows how sparse the matrix $\widetilde{\mathbf{A}}_n$ is.

Lemma 2.2. *If the matrix* $\widetilde{\mathbf{A}}_n$ *is obtained by truncation strategy* (2.14), *then*

$$\mathcal{N}(\widetilde{\mathbf{A}}_n) = \mathcal{O}((n+1)s(n)),$$

where $\mathcal{N}(\mathbf{A})$ *denotes the number of nonzero elements in the matrix* $\mathbf{A}$.

Adopting the truncation strategy (2.14), the linear system (2.12) can be written as

$$\left(\mathbf{E}_n(\alpha)+\widetilde{\mathbf{A}}_n\right)\widetilde{\mathcal{U}}_n^\delta(\alpha)=\mathbf{F}_n^\delta, \tag{2.15}$$

where $\widetilde{\mathcal{U}}_n^\delta(\alpha)=\left[\widetilde{u}_{ij}^\delta(\alpha):(i,j)\in\mathcal{U}_n\right]\in\mathbf{R}^{s(n)}$.

Let $\widetilde{\mathcal{A}}_n:\mathbf{X}\to\mathbf{X}_n$ be the truncated operator relative to the truncated matrix $\widetilde{\mathbf{A}}_n$ defined by

$$\widetilde{\mathcal{A}}_n x=\sum_{(i',j'),(i,j)\in\mathcal{U}_n}\widetilde{A}_{i'j',ij}\left\langle x,w_{i,j}\right\rangle w_{i',j'}, \tag{2.16}$$

where $\widetilde{A}_{i'j',ij}$ are the entries of $\widetilde{\mathbf{A}}_n$. The linear system (2.15) can be written as the operator form

$$\left(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_n\right)\widetilde{u}_{\alpha,n}^\delta=f_n^\delta, \tag{2.17}$$

where $\widetilde{u}_{\alpha,n}^\delta=\sum_{(i,j)\in\mathcal{U}_n}\widetilde{u}_{ij}^\delta(\alpha)w_{i,j}\in\mathbf{X}_n$.

In order to analyze the convergence of the truncated multiscale Galerkin scheme (2.17), we need the following estimates (cf. [**2, 7**]).

Lemma 2.3. *If condition* (2.13) *holds, then there is a positive constant* $\widetilde{c}$ *such that for any* $n\in\mathbf{N}$

$$\|\mathcal{A}_n-\widetilde{\mathcal{A}}_n\|\le\widetilde{c}(n+1)\mu^{-nk/d}, \tag{2.18}$$

and

$$\|\mathcal{A}-\widetilde{\mathcal{A}}_n\|\to 0 \quad \text{as } n\to\infty. \tag{2.19}$$

Lemma 2.4. *If condition* (2.13) *holds, then for* $0<c_0<1$, $\alpha>0$, *there exists a positive integer* N *such that for any* $n\in\mathbf{N}$, $n>N$,

$$\left\|\left(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_n\right)^{-1}\right\|\le\frac{1}{(1-c_0)\alpha}. \tag{2.20}$$

To analyze the convergence rate of the approximate solution $\widetilde{u}^{\delta}_{\alpha,n}$, we require the following hypotheses (cf. [**7**]).

(H-1) For some $\nu \in (0,1]$, $\widehat{u} \in R(\mathcal{A}^{\nu})$, i.e., there is a $w \in \mathbf{X}$ such that $\widehat{u} = \mathcal{A}^{\nu} w$.

(H-2) There exists a sequence $\{\theta_n : \ n \in \mathbf{N}_0\}$ satisfying

$$\sigma_0 \le \frac{\theta_{n+1}}{\theta_n} \le 1, \quad \text{and} \quad \lim_{n\to\infty} \theta_n = 0, \tag{2.21}$$

for some constant $\sigma_0 \in (0,1)$, such that when $n \ge N_0$,

$$\|(\mathcal{I} - \mathcal{P}_n)\,\mathcal{A}^{\nu}\| \le a_{\nu}\theta_n^{\nu}, \quad 0 < \nu \le 2, \tag{2.22}$$

and

$$\|\mathcal{A}\,(\mathcal{I} - \mathcal{P}_n)\| \le a_1 \theta_n, \tag{2.23}$$

where N_0 is a positive integer and a_{ν}, $0 < \nu \le 2$, are positive constants depending only on ν.

We now present the convergence rate for the truncation scheme of Lavrentiev regularization.

Theorem 2.5. *Assume that hypotheses* (H-1), (H-2) *and condition* (2.13) *hold. Let $\widehat{u}$ be the unique minimum norm solution of equation* (2.1), *and let $\widetilde{u}^{\delta}_{\alpha,n}$ be the solution of* (2.17). *Then for $c_0 \in (0,1)$ and $\alpha > 0$, there exists a positive integer N such that for $n \in \mathbf{N}$, $n > N$,*

$$\|\widehat{u} - \widetilde{u}^{\delta}_{\alpha,n}\| \le \frac{1}{1-c_0}\left(c_{\nu}\|w\|\alpha^{\nu} + \frac{\delta}{\alpha} + \widetilde{c}\|\widehat{u}\|\frac{(n+1)\mu^{-nk/d}}{\alpha} + a_1 a_{\nu}\|w\|\frac{\theta_n^{1+\nu}}{\alpha}\right), \tag{2.24}$$

where $\widetilde{c}$ is the constant appearing in Lemma 2.3.

Proof. Let u_{α} and $\widetilde{u}_{\alpha,n}$ be the solutions of equations

$$(\alpha\mathcal{I} + \mathcal{A})u_{\alpha} = f, \tag{2.25}$$

and

$$(\alpha\mathcal{I} + \widetilde{\mathcal{A}}_n)\widetilde{u}_{\alpha,n} = \mathcal{P}_n f, \tag{2.26}$$

respectively. It follows from (2.2), (2.25) and (2.26) that

$$(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_n)(u_\alpha-\widetilde{u}_{\alpha,n}) = (\widetilde{\mathcal{A}}_n-\mathcal{A})(u_\alpha-\widehat{u})+(\widetilde{\mathcal{A}}_n-\mathcal{A}_n)\widehat{u}+\mathcal{P}_n\mathcal{A}(\mathcal{P}_n-\mathcal{I})\widehat{u},$$

which with (H-1) yields that

$$\begin{aligned} u_\alpha - \widetilde{u}_{\alpha,n} = (\alpha\mathcal{I}+\widetilde{\mathcal{A}}_n)^{-1}\big[&(\widetilde{\mathcal{A}}_n-\mathcal{A})(u_\alpha-\widehat{u}) \\ &+ (\widetilde{\mathcal{A}}_n-\mathcal{A}_n)\widehat{u} - \mathcal{P}_n\mathcal{A}(\mathcal{I}-\mathcal{P}_n)\mathcal{A}^\nu w\big]. \end{aligned} \tag{2.27}$$

By Lemmas 2.3 and 2.4, for $c_0 \in (0,1)$, $\alpha > 0$, there exists a positive integer N such that when $n > N$,

$$\|\widetilde{\mathcal{A}}_n - \mathcal{A}\| \le c_0\alpha, \qquad \|(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_n)^{-1}\| \le \frac{1}{(1-c_0)\alpha}, \tag{2.28}$$

and there exists a constant $\widetilde{c} > 0$, such that

$$\|\widetilde{\mathcal{A}}_n - \mathcal{A}_n\| \le \widetilde{c}(n+1)\mu^{-nk/d}. \tag{2.29}$$

Using (2.6), (2.28), (2.29) and (H-2), we conclude from (2.27) that

$$\begin{aligned} \|u_\alpha - \widetilde{u}_{\alpha,n}\| \le\ & \|(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_n)^{-1}\|\big[\|\widetilde{\mathcal{A}}_n-\mathcal{A}\|\|u_\alpha-\widehat{u}\| \\ & + \|\widetilde{\mathcal{A}}_n-\mathcal{A}_n\|\|\widehat{u}\| \\ & + \|\mathcal{P}_n\|\|\mathcal{A}(\mathcal{I}-\mathcal{P}_n)\|\|(\mathcal{I}-\mathcal{P}_n)\mathcal{A}^\nu\|\|w\|\big] \\ \le\ & \frac{c_0}{1-c_0}c_\nu\|w\|\alpha^\nu + \frac{\widetilde{c}\|\widehat{u}\|}{1-c_0}\frac{(n+1)\mu^{-nk/d}}{\alpha} + \frac{a_1a_\nu\|w\|}{1-c_0}\frac{\theta_n^{1+\nu}}{\alpha}. \end{aligned} \tag{2.30}$$

On the other hand,

$$\begin{aligned} \|\widetilde{u}_{\alpha,n} - \widetilde{u}^\delta_{\alpha,n}\| &= \|(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_n)^{-1}\mathcal{P}_n(f-f^\delta)\| \\ &\le \|(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_n)^{-1}\|\|\mathcal{P}_n\|\|f-f^\delta\| \le \frac{1}{1-c_0}\frac{\delta}{\alpha}. \end{aligned} \tag{2.31}$$

Substituting estimates (2.6), (2.30) and (2.31) into the inequality

$$\|\widehat{u}-\widetilde{u}^\delta_{\alpha,n}\| \le \|\widehat{u}-u_\alpha\| + \|u_\alpha-\widetilde{u}_{\alpha,n}\| + \|\widetilde{u}_{\alpha,n}-\widetilde{u}^\delta_{\alpha,n}\|$$

completes the proof of this theorem. □

3. Multilevel augmentation algorithms using the truncated operator. In this section, we develop the multilevel augmentation method by using the truncated operator, and provide *a priori* error analysis.

Recalling that we have the space decomposition (2.7), we define $\mathcal{Q}_{n+1} := \mathcal{P}_{n+1} - \mathcal{P}_n$, $n \in \mathbf{N}_0$, and write the solution $\widetilde{u}^{\delta}_{\alpha,n} \in \mathbf{X}_n$ of equation (2.17) with $n = \ell + m$ as

$$\widetilde{u}^{\delta}_{\alpha,\ell+m} = \sum_{j=\mathbf{Z}_{m+1}} (\widetilde{u}^{\delta}_{\alpha,\ell+m})_j = [(\widetilde{u}^{\delta}_{\alpha,\ell+m})_0, (\widetilde{u}^{\delta}_{\alpha,\ell+m})_1, \ldots, (\widetilde{u}^{\delta}_{\alpha,\ell+m})_m]^T,$$

where $(\widetilde{u}^{\delta}_{\alpha,\ell+m})_0 \in \mathbf{X}_\ell$ and $(\widetilde{u}^{\delta}_{\alpha,\ell+m})_j \in \mathcal{W}_{\ell+j}$, for $j = 1, 2, \ldots, m$. We also identity the function $f^{\delta}_{\ell+m}$ with the vector form

$$f^{\delta}_{\ell+m} = [\mathcal{P}_\ell f^\delta, \mathcal{Q}_{\ell+1} f^\delta, \ldots, \mathcal{Q}_{\ell+m} f^\delta]^T,$$

and the operator $\widetilde{\mathcal{A}}_{\ell+m}$ with the matrix form

$$\widetilde{\mathcal{A}}_{\ell,m} := \begin{bmatrix} \mathcal{P}_\ell \widetilde{\mathcal{A}}_{\ell+m} \mathcal{P}_\ell & \mathcal{P}_\ell \widetilde{\mathcal{A}}_{\ell+m} \mathcal{Q}_{\ell+1} & \cdots & \mathcal{P}_\ell \widetilde{\mathcal{A}}_{\ell+m} \mathcal{Q}_{\ell+m} \\ \mathcal{Q}_{\ell+1} \widetilde{\mathcal{A}}_{\ell+m} \mathcal{P}_\ell & \mathcal{Q}_{\ell+1} \widetilde{\mathcal{A}}_{\ell+m} \mathcal{Q}_{\ell+1} & \cdots & \mathcal{Q}_{\ell+1} \widetilde{\mathcal{A}}_{\ell+m} \mathcal{Q}_{\ell+m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{Q}_{\ell+m} \widetilde{\mathcal{A}}_{\ell+m} \mathcal{P}_\ell & \mathcal{Q}_{\ell+m} \widetilde{\mathcal{A}}_{\ell+m} \mathcal{Q}_{\ell+1} & \cdots & \mathcal{Q}_{\ell+m} \widetilde{\mathcal{A}}_{\ell+m} \mathcal{Q}_{\ell+m} \end{bmatrix}.$$

As in [**6, 7**], we split the operator $\widetilde{\mathcal{A}}_{\ell,m}$ into the sum of two operators

$$\widetilde{\mathcal{A}}_{\ell,m} = \widetilde{\mathcal{A}}^L_{\ell,m} + \widetilde{\mathcal{A}}^H_{\ell,m},$$

where $\widetilde{\mathcal{A}}^L_{\ell,m} := \mathcal{P}_\ell \widetilde{\mathcal{A}}_{\ell+m} \mathcal{P}_{\ell+m}$ and $\widetilde{\mathcal{A}}^H_{\ell,m} := (\mathcal{P}_{\ell+m} - \mathcal{P}_\ell) \widetilde{\mathcal{A}}_{\ell+m} \mathcal{P}_{\ell+m}$, which have matrix forms

$$\widetilde{\mathcal{A}}^L_{\ell,m} := \begin{bmatrix} \mathcal{P}_\ell \widetilde{\mathcal{A}}_{\ell+m} \mathcal{P}_\ell & \mathcal{P}_\ell \widetilde{\mathcal{A}}_{\ell+m} \mathcal{Q}_{\ell+1} & \ldots & \mathcal{P}_\ell \widetilde{\mathcal{A}}_{\ell+m} \mathcal{Q}_{\ell+m} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix},$$

and

$$\widetilde{\mathcal{A}}^H_{\ell,m} := \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \mathcal{Q}_{\ell+1} \widetilde{\mathcal{A}}_{\ell+m} \mathcal{P}_\ell & \mathcal{Q}_{\ell+1} \widetilde{\mathcal{A}}_{\ell+m} \mathcal{Q}_{\ell+1} & \cdots & \mathcal{Q}_{\ell+1} \widetilde{\mathcal{A}}_{\ell+m} \mathcal{Q}_{\ell+m} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{Q}_{\ell+m} \widetilde{\mathcal{A}}_{\ell+m} \mathcal{P}_\ell & \mathcal{Q}_{\ell+m} \widetilde{\mathcal{A}}_{\ell+m} \mathcal{Q}_{\ell+1} & \ldots & \mathcal{Q}_{\ell+m} \widetilde{\mathcal{A}}_{\ell+m} \mathcal{Q}_{\ell+m} \end{bmatrix}.$$

For a given parameter $\alpha > 0$, we set

$$\widetilde{\mathcal{B}}_{\ell,m}(\alpha) := \mathcal{I} + \alpha^{-1}\widetilde{\mathcal{A}}^L_{\ell,m} \text{ and } \widetilde{\mathcal{C}}_{\ell,m}(\alpha) := \alpha^{-1}\widetilde{\mathcal{A}}^H_{\ell,m}. \tag{3.1}$$

Thus equation (2.17) with $n = \ell + m$ can be written as

$$[\widetilde{\mathcal{B}}_{\ell,m}(\alpha) + \widetilde{\mathcal{C}}_{\ell,m}(\alpha)]\widetilde{u}^\delta_{\alpha,\ell+m} = \alpha^{-1} f^\delta_{\ell+m}.$$

The multilevel augmentation scheme for solving (2.17) can be described as follows (cf. [**6, 7**]).

Algorithm 3.1. (Truncated multilevel augmentation algorithm).

Step 1. *For a fixed* $\ell > 0$, *solve* (2.17) *with* $n = \ell$ *exactly to obtain* $\widetilde{u}^\delta_{\alpha,\ell} \in \mathbf{X}_\ell$.

Step 2. *Set* $\widetilde{u}^\delta_{\alpha,\ell,0} := \widetilde{u}^\delta_{\alpha,\ell}$ *and compute* $\widetilde{\mathcal{B}}_{\ell,0}(\alpha)$ *and* $\widetilde{\mathcal{C}}_{\ell,0}(\alpha)$.

Step 3. *For* $m \in \mathbf{N}$, *suppose that* $\widetilde{u}^\delta_{\alpha,\ell,m-1} \in \mathbf{X}_{\ell+m-1}$ *has been obtained and do the following.*

- *Augment* $\widetilde{B}_{\ell,m-1}(\alpha)$ *and* $\widetilde{\mathcal{C}}_{\ell,m-1}(\alpha)$ *to form* $\widetilde{\mathcal{B}}_{\ell,m}(\alpha)$ *and* $\widetilde{\mathcal{C}}_{\ell,m}(\alpha)$, *respectively.*

- *Augment* $\widetilde{u}^\delta_{\alpha,\ell,m-1}$ *to form*

$$\overline{u}^\delta_{\alpha,\ell,m} := \begin{bmatrix} \widetilde{u}^\delta_{\alpha,\ell,m-1} \\ 0 \end{bmatrix} \in \mathbf{X}_{\ell+m}.$$

- *Solve* $\widetilde{u}^\delta_{\alpha,\ell,m} := [(\widetilde{u}^\delta_{\alpha,\ell,m})_0, (\widetilde{u}^\delta_{\alpha,\ell,m})_1, \dots, (\widetilde{u}^\delta_{\alpha,\ell,m})_m]^T$ *with* $(\widetilde{u}^\delta_{\alpha,\ell,m})_0 \in \mathbf{X}_\ell$ *and* $(\widetilde{u}^\delta_{\alpha,\ell,m})_j \in \mathcal{W}_{\ell+j}$, $j = 1, 2, \dots, m$ *from equation*

$$\widetilde{\mathcal{B}}_{\ell,m}(\alpha)\widetilde{u}^\delta_{\alpha,\ell,m} = \alpha^{-1} f^\delta_{\ell+m} - \widetilde{\mathcal{C}}_{\ell,m}(\alpha)\overline{u}^\delta_{\alpha,\ell,m}. \tag{3.2}$$

In the remainder of this section, we will give an *a priori* error analysis for Algorithm 3.1. We already have the estimate of $\|\widehat{u} - \widetilde{u}^\delta_{\alpha,\ell+m}\|$ in Theorem 2.5. In the following we estimate $\|\widetilde{u}^\delta_{\alpha,\ell+m} - \tilde{u}^\delta_{\alpha,\ell,m}\|$. To do this, we let

$$\widetilde{\gamma}^\delta_{\alpha,n} := \frac{2 - c_0}{1 - c_0}\frac{\delta}{\alpha} + \frac{(a_1 a_\nu + 2a_{1+\nu})\|w\|}{1 - c_0}\frac{\theta_n^{1+\nu}}{\alpha} + \frac{\widetilde{c}\|\widehat{u}\|}{1 - c_0}\frac{(n+1)\mu^{-nk/d}}{\alpha}, \tag{3.3}$$

where $c_0 \in (0,1)$ is the constant appearing in Lemma 2.4, and $\widetilde{c}$ is the constant appearing in Lemma 2.3. Since the sequence $\{\theta_n : n \in \mathbf{N}_0\}$ satisfies condition (2.21), we have

$$\frac{\widetilde{\gamma}^{\delta}_{\alpha,n}}{\widetilde{\gamma}^{\delta}_{\alpha,n+1}} \le \sigma := \max\left\{\frac{1}{\sigma_0^{1+\nu}}, \mu^{k/d}\right\}. \tag{3.4}$$

We will present the estimate of $\|\widetilde{u}^{\delta}_{\alpha,l+\ell} - \widetilde{u}^{\delta}_{\alpha,\ell,m}\|$ in Theorem 3.5. Before that, we give three lemmas, which are the preparation for the proof of the theorem.

Lemma 3.2. *If condition* (2.13) *holds, then for any* $\alpha > 0$, $\|\widetilde{\mathcal{C}}_{\ell,m}(\alpha)\| \to 0$ *as* $\ell \to \infty$ *uniformly for* $m \in \mathbf{N}_0$.

Proof. We use Lemma 2.3 to prove this lemma. Since

$$\widetilde{\mathcal{C}}_{\ell,m}(\alpha) = \frac{1}{\alpha}\left[(\mathcal{P}_{\ell+m} - \mathcal{P}_\ell)\mathcal{A}_{\ell+m} + (\mathcal{P}_{\ell+m} - \mathcal{P}_\ell)(\widetilde{\mathcal{A}}_{\ell+m} - \mathcal{A}_{\ell+m})\right],$$

we have

$$\|\widetilde{\mathcal{C}}_{\ell,m}(\alpha)\| \le \frac{1}{\alpha}\left[\|(\mathcal{P}_{\ell+m} - \mathcal{I})\mathcal{A}\| + \|(\mathcal{I} - \mathcal{P}_\ell)\mathcal{A}\| + 2\|\widetilde{\mathcal{A}}_{\ell+m} - \mathcal{A}_{\ell+m}\|\right]. \tag{3.5}$$

Because of the compactness of $\mathbf{A}$ and the pointwise convergence of $\mathcal{P}_n$ to $\mathcal{I}$,

$$\|(\mathcal{I} - \mathcal{P}_\ell)\mathcal{A}\| \to 0, \quad \text{as } \ell \to \infty,$$

which with Lemma 2.3 and equality (3.5) completes the proof. ◻

Lemma 3.3. *If condition* (2.13) *holds, then there exists a positive integer* N *such that for* $\ell > N$ *and* $m \in \mathbf{N}_0$, $\widetilde{\mathcal{B}}^{-1}_{\ell,m}(\alpha)$ *exists and*

$$\|\widetilde{\mathcal{B}}^{-1}_{\ell,m}(\alpha)\| \le \frac{1}{1 - c_0 - \|\widetilde{\mathcal{C}}_{\ell,m}(\alpha)\|}. \tag{3.6}$$

Proof. By Lemma 2.3, there exists a positive integer $N \in \mathbf{N}$ such that for all $\ell > N$, $m \in \mathbf{N}_0$,

$$\|\widetilde{\mathcal{A}}_{\ell+m} - \mathcal{A}\| \le c_0\alpha.$$

Thus,

$$\|\alpha\mathcal{I} + \widetilde{\mathcal{A}}_{\ell+m}\| \geq \|\alpha\mathcal{I} + \mathcal{A}\| - \|\mathcal{A} - \widetilde{\mathcal{A}}_{\ell+m}\| \geq (1 - c_0)\alpha.$$

Then it follows that

$$\|\widetilde{\mathcal{B}}_{\ell,m}(\alpha)\| = \left\| \frac{1}{\alpha}\left(\alpha\mathcal{I} + \widetilde{\mathcal{A}}_{\ell+m}\right) - \widetilde{\mathcal{C}}_{\ell,m}(\alpha) \right\| \geq 1 - c_0 - \|\widetilde{\mathcal{C}}_{\ell,m}(\alpha)\|, \tag{3.7}$$

which with Lemma 3.2 leads to the result of this lemma. □

Lemma 3.4. *If conditions* (H-1), (H-2) *and* (2.13) *hold, then there exists a positive constant N such that for $n > N$,*

$$\|\widetilde{u}^{\delta}_{\alpha,n} - u^{\delta}_{\alpha}\| \leq \widetilde{\gamma}^{\delta}_{\alpha,n}. \tag{3.8}$$

Proof. It follows from (2.4) and (2.17) that

(3.9)
$$\begin{aligned}\widetilde{u}^{\delta}_{\alpha,n} - u^{\delta}_{\alpha} &= (\alpha\mathcal{I} + \widetilde{\mathcal{A}}_n)^{-1}\mathcal{P}_n f^{\delta} - (\alpha\mathcal{I} + \mathcal{A})^{-1} f^{\delta} \\ &= (\alpha\mathcal{I} + \widetilde{\mathcal{A}}_n)^{-1}\mathcal{P}_n (f^{\delta} - f) - (\alpha\mathcal{I} + \mathcal{A})^{-1}(f^{\delta} - f) \\ &\quad + (\alpha\mathcal{I} + \widetilde{\mathcal{A}}_n)^{-1}(\mathcal{P}_n - \mathcal{I}) f \\ &\quad + \left[(\alpha\mathcal{I} + \widetilde{\mathcal{A}}_n)^{-1} - (\alpha\mathcal{I} + \mathcal{A})^{-1}\right] f.\end{aligned}$$

It can be seen from the above equality that we can obtain the estimate of $\|\widetilde{u}^{\delta}_{\alpha,n} - u^{\delta}_{\alpha}\|$ by estimating $\|(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_n)^{-1}\mathcal{P}_n(f^{\delta}-f)\|$, $\|(\alpha\mathcal{I}+\mathcal{A})^{-1}(f^{\delta}-f)\|$, $\|(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_n)^{-1}(\mathcal{P}_n-\mathcal{I})f\|$ and $\|[(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_n)^{-1} - (\alpha\mathcal{I}+\mathcal{A})^{-1}]f\|$. We will estimate them respectively in the following.

From Lemma 2.4, (2.3) and (2.5) we have

$$\|(\alpha\mathcal{I} + \widetilde{\mathcal{A}}_n)^{-1}\mathcal{P}_n(f^{\delta} - f)\| \leq \frac{\delta}{(1 - c_0)\alpha}, \tag{3.10}$$

and

$$\|(\alpha\mathcal{I} + \mathcal{A})^{-1}(f^{\delta} - f)\| \leq \frac{\delta}{\alpha}. \tag{3.11}$$

Using (H-1) and (H-2), we obtain

$$\|(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_n)^{-1}(\mathcal{P}_n-\mathcal{I})f\| \le \frac{a_{1+\nu}\|w\|}{1-c_0}\cdot\frac{\theta_n^{1+\nu}}{\alpha}. \tag{3.12}$$

Next we estimate $\|[(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_n)^{-1}-(\alpha\mathcal{I}+\mathcal{A})^{-1}]f\|$. Since

$$(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_n)^{-1}-(\alpha\mathcal{I}+\mathcal{A})^{-1}=(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_n)^{-1}(\mathcal{A}-\widetilde{\mathcal{A}}_n)(\alpha\mathcal{I}+\mathcal{A})^{-1},$$

we conclude that

$$\begin{aligned}\|[(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_n)^{-1}-(\alpha\mathcal{I}+\mathcal{A})^{-1}]f\| &\le \|(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_n)^{-1}(\mathcal{A}-\mathcal{A}_n)(\alpha\mathcal{I}+\mathcal{A})^{-1}\mathcal{A}^{1+\nu}w\| \\ &\quad+\|(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_n)^{-1}(\mathcal{A}_n-\widetilde{\mathcal{A}}_n)(\alpha\mathcal{I}+\mathcal{A})^{-1}\mathcal{A}\widehat{u}\|.\end{aligned} \tag{3.13}$$

By (H-2),

$$\|(\mathcal{A}\mathcal{A}_n)\mathcal{A}^{\nu}\| \le (a_1a_\nu+a_{1+\nu})\theta_n^{1+\nu}. \tag{3.14}$$

Noting that $(\alpha\mathcal{I}+\mathcal{A})^{-1}\mathcal{A}^{\nu}=\mathcal{A}^{\nu}(\alpha\mathcal{I}+\mathcal{A})^{-1}$, by Lemma 2.4 and equality (3.14), we have

$$\|(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_n)^{-1}(\mathcal{A}-\mathcal{A}_n)(\alpha\mathcal{I}+\mathcal{A})^{-1}\mathcal{A}^{1+\nu}w\| \le \frac{(a_1a_\nu+a_{1+\nu})\|w\|}{1-c_0}\cdot\frac{\theta_n^{1+\nu}}{\alpha}. \tag{3.15}$$

From Lemmas 2.3 and 2.4 we obtain

$$\|(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_n)^{-1}(\mathcal{A}_n-\widetilde{\mathcal{A}}_n)(\alpha\mathcal{I}+\mathcal{A})^{-1}\mathcal{A}\widehat{u}\| \le \frac{\widetilde{c}\|\widehat{u}\|}{1-c_0}\cdot\frac{(n+1)\mu^{-nk/d}}{\alpha}. \tag{3.16}$$

Combining (3.10)–(3.13), (3.15) and (3.16) yields the desired estimate of this lemma. □

The following theorem gives the estimate of $\|\widetilde{u}^{\delta}_{\alpha,\ell+m}-\widetilde{u}^{\delta}_{\alpha,\ell,m}\|$, which is similar to Proposition 3.3 in [**7**]. However, we don't need the condition on α as Proposition 3.3 in [**7**], and the proof is different.

Theorem 3.5. *If conditions* (H-1), (H-2) *and* (2.13) *hold, then there exists a positive integer* N *such that for* $\ell > N$, $m \in \mathbf{N}_0$,

$$\|\widetilde{u}^{\delta}_{\alpha,\ell+m} - \widetilde{u}^{\delta}_{\alpha,\ell,m}\| \le \widetilde{\gamma}^{\delta}_{\alpha,\ell+m}. \tag{3.17}$$

Proof. We conclude the estimate (3.17) similar to that in [**6**] by induction on m. When $m = 0$, $\widetilde{u}^{\delta}_{\alpha,\ell,0} = \widetilde{u}^{\delta}_{\alpha,\ell}$, thus the estimate holds. Suppose that (3.17) holds for $m = r - 1$; we come to prove that it holds for $m = r$.

It follows from (3.2) with $m = r$ and (2.17) with $n = \ell + r$ that

$$\widetilde{\mathcal{B}}_{\ell,r}(\alpha)(\widetilde{u}^{\delta}_{\alpha,\ell+r} - \widetilde{u}^{\delta}_{\alpha,\ell,r}) = \widetilde{\mathcal{C}}_{\ell,r}(\alpha)(\overline{u}^{\delta}_{\alpha,\ell,r} - \widetilde{u}^{\delta}_{\alpha,\ell+r}). \tag{3.18}$$

Using Lemma 3.3 and noting $\overline{u}^{\delta}_{\alpha,\ell,r} = \widetilde{u}^{\delta}_{\alpha,\ell,r-1}$, we conclude that there is an integer N_1 such that, when $\ell > N_1$,

$$\begin{aligned}\|\widetilde{u}^{\delta}_{\alpha,\ell+r} - \widetilde{u}^{\delta}_{\alpha,\ell,r}\| &= \|\widetilde{\mathcal{B}}^{-1}_{\ell,r}(\alpha)\widetilde{\mathcal{C}}_{\ell,r}(\alpha)(\widetilde{u}^{\delta}_{\alpha,\ell,r-1} - \widetilde{u}^{\delta}_{\alpha,\ell+r})\| \\ &\le \frac{\|\widetilde{\mathcal{C}}_{\ell,r}(\alpha)\|}{1 - c_0 - \|\widetilde{\mathcal{C}}_{\ell,r}(\alpha)\|}\big(\|\widetilde{u}^{\delta}_{\alpha,\ell,r-1} - \widetilde{u}^{\delta}_{\alpha,\ell+r-1}\| \\ &\qquad + \|\widetilde{u}^{\delta}_{\alpha,\ell+r-1} - u^{\delta}_{\alpha}\|r \\ &\qquad + \|u^{\delta}_{\alpha} - \widetilde{u}^{\delta}_{\alpha,\ell+r}\|\big).\end{aligned} \tag{3.19}$$

Noting that Lemma 3.4 leads to

$$\|\widetilde{u}^{\delta}_{\alpha,\ell+r-1} - u^{\delta}_{\alpha}\| \le \widetilde{\gamma}^{\delta}_{\alpha,\ell+r-1}, \quad \text{and} \quad \|u^{\delta}_{\alpha} - \widetilde{u}^{\delta}_{\alpha,\ell+r}\| \le \widetilde{\gamma}^{\delta}_{\alpha,\ell+r},$$

and (3.4) yields

$$\widetilde{\gamma}^{\delta}_{\alpha,\ell+r-1} \le \sigma\widetilde{\gamma}^{\delta}_{\alpha,\ell+r},$$

we obtain from (3.19) and the induction hypothesis that

$$\|\widetilde{u}^{\delta}_{\alpha,\ell+r} - \widetilde{u}^{\delta}_{\alpha,\ell,r}\| \le \frac{\|\widetilde{\mathcal{C}}_{\ell,r}(\alpha)\|}{1 - c_0 - \|\widetilde{\mathcal{C}}_{\ell,r}(\alpha)\|}(1 + 2\sigma)\widetilde{\gamma}^{\delta}_{\alpha,\ell+r}. \tag{3.20}$$

By Lemma 3.2, there exists a positive integer $N \ge N_1$ such that for $\ell > N$ and $r \in \mathbf{N}$,

$$\frac{\|\widetilde{\mathcal{C}}_{\ell,r}(\alpha)\|}{1 - c_0 - \|\widetilde{\mathcal{C}}_{\ell,r}(\alpha)\|} \le \frac{1}{1 + 2\sigma}. \tag{3.21}$$

Combining (3.20) and (3.21) we conclude that when $\ell > N$,

$$\|\widetilde{u}^{\delta}_{\alpha,\ell+r} - \widetilde{u}^{\delta}_{\alpha,\ell,r}\| \le \widetilde{\gamma}^{\delta}_{\alpha,\ell+r},$$

which completes the proof. □

According to Theorem 2.5 and Theorem 3.5, we get the following theorem.

Theorem 3.6. *If conditions* (H-1), (H-2) *and* (2.13) *hold, then for given* $c_0 \in (0,1)$ *and* $\alpha > 0$*, there exists a positive integer* N *such that for* $\ell > N$*,* $m \in \mathbf{N}_0$*,*

$$\begin{aligned}\|\widehat{u} - \widetilde{u}^{\delta}_{\alpha,\ell,m}\| \le \frac{1}{1-c_0}\bigg[& c_\nu \|w\|\alpha^\nu + (3-c_0)\frac{\delta}{\alpha} \\ & + 2(a_1 a_\nu + a_{1+\nu})\|w\|\frac{\theta^{1+\nu}_{\ell+m}}{\alpha} \\ & + 2\widetilde{c}\|\widehat{u}\|\frac{(\ell+m+1)\mu^{-(\ell+m)k/d}}{\alpha}\bigg].\end{aligned}$$

4. A posteriori parameter choice strategy. In this section, we develop the *a posteriori* regularization parameter choice strategy presented in [**7**] by using the truncated operator described in Section 2.

We first consider an auxiliary operator equation. For fixed ℓ, $m \in \mathbf{N}$, we consider the equation

$$(\alpha\mathcal{I} + \mathcal{A})\widetilde{\widetilde{u}}^{\delta}_{\alpha} = \widetilde{u}^{\delta}_{\alpha,\ell,m}, \tag{4.1}$$

where $\widetilde{\widetilde{u}}^{\delta}_{\alpha} \in \mathbf{X}$, and $\widetilde{u}^{\delta}_{\alpha,\ell,m}$ is the solution using the multilevel augmentation algorithm for equation (2.17). Obviously equation (4.1) has a unique solution which depends on ℓ, m. The truncated Galerkin method for equation (4.1) is to solve the equation

$$(\alpha\mathcal{I} + \widetilde{\mathcal{A}}_{\ell+i})\widetilde{\widetilde{u}}^{\delta}_{\alpha,\ell+i} = \mathcal{P}_{\ell+i}\widetilde{u}^{\delta}_{\alpha,\ell,m}, \quad i = 0, 1, \dots, m. \tag{4.2}$$

We use the notation $\widetilde{\widetilde{u}}^{\delta}_{\alpha,\ell,i}$ to denote the multilevel augmentation solution of the above equation.

Let

$$\widetilde{\widetilde{\gamma}}^{\delta}_{\alpha,n} := c_1\frac{\delta}{\alpha^2} + (c_2 a_1 a_\nu + c_3 a_{1+\nu})\|w\| \cdot \frac{\theta_n^{1+\nu}}{\alpha^2} + c_2\widetilde{c}\|\widehat{u}\| \cdot \frac{(n+1)\mu^{-nk/d}}{\alpha^2}, \tag{4.3}$$

where

$$c_1 := \frac{4-c_0-c_0^2}{(1-c_0)^2}, \quad c_2 := \frac{3-c_0}{(1-c_0)^2}, \quad c_3 := \frac{5-c_0}{(1-c_0)^2},$$

$\widetilde{c}$ and c_0 are constants appearing in Lemmas 2.3 and 2.4, respectively.

Proposition 4.1. *Let $\alpha > 0$ and $\delta > 0$. If conditions* (H-1), (H-2) *and* (2.13) *hold, then there exists a positive integer N such that for $\ell > N$, $m \in \mathbf{N}_0$,*

$$\|\widetilde{\widetilde{u}}^{\delta}_{\alpha,\ell+m} - \widetilde{\widetilde{u}}^{\delta}_{\alpha}\| \le \widetilde{\widetilde{\gamma}}^{\delta}_{\alpha,\ell+m}, \tag{4.4}$$

and for $i = 0, 1, \ldots, m$,

$$\|\widetilde{\widetilde{u}}^{\delta}_{\alpha,\ell,i} - \widetilde{\widetilde{u}}^{\delta}_{\alpha,\ell+i}\| \le \widetilde{\widetilde{\gamma}}^{\delta}_{\alpha,\ell+i}. \tag{4.5}$$

Proof. It follows from (4.1) and (4.2) that

$$\begin{aligned} \widetilde{\widetilde{u}}^{\delta}_{\alpha,\ell+m} - \widetilde{\widetilde{u}}^{\delta}_{\alpha} &= (\alpha\mathcal{I} + \widetilde{\mathcal{A}}_{\ell+m})^{-1}\widetilde{u}^{\delta}_{\alpha,\ell,m} - (\alpha\mathcal{I} + \mathcal{A})^{-1}\widetilde{u}^{\delta}_{\alpha,\ell,m} \\ &= (\alpha\mathcal{I} + \widetilde{\mathcal{A}}_{\ell+m})^{-1}(\widetilde{u}^{\delta}_{\alpha,\ell,m} - \widetilde{u}^{\delta}_{\alpha,\ell+m}) \\ &\quad - (\alpha\mathcal{I} + \mathcal{A})^{-1}(\widetilde{u}^{\delta}_{\alpha,\ell,m} - \widetilde{u}^{\delta}_{\alpha,\ell+m}) \\ &\quad + [(\alpha\mathcal{I} + \widetilde{\mathcal{A}}_{\ell+m})^{-1} - (\alpha\mathcal{I} + \mathcal{A})^{-1}]\widetilde{u}^{\delta}_{\alpha,\ell+m}. \end{aligned} \tag{4.6}$$

First we estimate $\|(\alpha\mathcal{I} + \widetilde{\mathcal{A}}_{\ell+m})^{-1}(\widetilde{u}^{\delta}_{\alpha,\ell,m} - \widetilde{u}^{\delta}_{\alpha,\ell+m})\|$. By Lemma 2.4 and Theorem 3.5, there exists an $N_1 \in \mathbf{N}$ such that for $\ell > N_1$, $m \in \mathbf{N}_0$,

$$\|(\alpha\mathcal{I} + \widetilde{\mathcal{A}}_{\ell+m})^{-1}\| \le \frac{1}{(1-c_0)\alpha}, \quad \|\widetilde{u}^{\delta}_{\alpha,\ell+m} - \widetilde{u}^{\delta}_{\alpha,\ell,m}\| \le \widetilde{\gamma}^{\delta}_{\alpha,\ell+m}. \tag{4.7}$$

Therefore, when $\ell > N_1$ and $m \in \mathbf{N}_0$,

$$\|(\alpha\mathcal{I} + \widetilde{\mathcal{A}}_{\ell+m})^{-1}(\widetilde{u}^{\delta}_{\alpha,\ell,m} - \widetilde{u}^{\delta}_{\alpha,\ell+m})\| \le \frac{1}{(1-c_0)\alpha} \cdot \widetilde{\gamma}^{\delta}_{\alpha,\ell+m}. \tag{4.8}$$

By (2.5) and (4.7), for $\ell > N_1$, $m \in \mathbf{N}_0$,

$$\|(\alpha\mathcal{I}+\mathcal{A})^{-1}(\widetilde{u}^\delta_{\alpha,\ell,m} - \widetilde{u}^\delta_{\alpha,\ell+m})\| \le \frac{1}{\alpha}\widetilde{\gamma}^\delta_{\alpha,\ell+m}. \tag{4.9}$$

On the other hand, a simple computation yields that

$$\begin{aligned}
&[(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_{\ell+m})^{-1} - (\alpha\mathcal{I}+\mathcal{A})^{-1}]\widetilde{u}^\delta_{\alpha,\ell+m} \\
&\quad = (\alpha\mathcal{I}+\widetilde{\mathcal{A}}_{\ell+m})^{-1}(\mathcal{A}-\widetilde{\mathcal{A}}_{\ell+m})(\alpha\mathcal{I}+\mathcal{A})^{-1}(\widetilde{u}^\delta_{\alpha,\ell+m} - u^\delta_\alpha) \\
&\qquad + (\alpha\mathcal{I}+\widetilde{\mathcal{A}}_{\ell+m})^{-1}(\mathcal{A}-\widetilde{\mathcal{A}}_{\ell+m})(\alpha\mathcal{I}+\mathcal{A})^{-1}(u^\delta_\alpha - u_\alpha) \\
&\qquad + (\alpha\mathcal{I}+\widetilde{\mathcal{A}}_{\ell+m})^{-1}(\mathcal{A}-\widetilde{\mathcal{A}}_{\ell+m})(\alpha\mathcal{I}+\mathcal{A})^{-2}f.
\end{aligned} \tag{4.10}$$

By Lemma 2.3, there exists an $N_2 \in \mathbf{N}$ such that for $\ell > N_2$, $m \in \mathbf{N}_0$,

$$\|\mathcal{A} - \widetilde{\mathcal{A}}_{\ell+m}\| \le c_0\alpha, \tag{4.11}$$

which with (4.7), (2.5), (3.8) and (2.6) yields that when N is sufficiently large, for $\ell > N$, $m \in \mathbf{N}_0$,

$$\begin{aligned}
&\|(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_{\ell+m})^{-1}(\mathcal{A}-\widetilde{\mathcal{A}}_{\ell+m})(\alpha\mathcal{I}+\mathcal{A})^{-1}(\widetilde{u}^\delta_{\alpha,\ell+m} - u^\delta_\alpha)\| \\
&\qquad \le \frac{c_0}{(1-c_0)\alpha}\widetilde{\gamma}^\delta_{\alpha,\ell+m},
\end{aligned} \tag{4.12}$$

$$\begin{aligned}
&\|(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_{\ell+m})^{-1}(\mathcal{A}-\widetilde{\mathcal{A}}_{\ell+m})(\alpha\mathcal{I}+\mathcal{A})^{-1}(u^\delta_\alpha - u_\alpha)\| \\
&\qquad \le \frac{c_0}{1-c_0}\cdot\frac{\delta}{\alpha^2}.
\end{aligned} \tag{4.13}$$

We now estimate the last term of the right hand side of (4.10). Noting that $f = \mathcal{A}\widehat{u} = \mathcal{A}^{1+\nu}w$, by (4.7), (3.14) and (2.18), we have

$$\begin{aligned}
&\|(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_{\ell+m})^{-1}(\mathcal{A}-\widetilde{\mathcal{A}}_{\ell+m})(\alpha\mathcal{I}+\mathcal{A})^{-2}f\| \\
&\quad \le \|(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_{\ell+m})^{-1}\|\cdot\|(\mathcal{A}-\mathcal{A}_{\ell+m})\mathcal{A}^\nu\| \\
&\qquad \cdot\|(\alpha\mathcal{I}+\mathcal{A})^{-1}\|\cdot\|(\alpha\mathcal{I}+\mathcal{A})^{-1}\mathcal{A}\|\cdot\|w\| \\
&\qquad + \|(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_{\ell+m})^{-1}\|\cdot\|\mathcal{A}_{\ell+m}-\widetilde{\mathcal{A}}_{\ell+m}\| \\
&\qquad \cdot\|(\alpha\mathcal{I}+\mathcal{A})^{-1}\|\cdot\|(\alpha\mathcal{I}+\mathcal{A})^{-1}\mathcal{A}\|\cdot\|\widehat{u}\| \\
&\quad \le \frac{(a_1a_\nu + a_{1+\nu})\|w\|}{1-c_0}\cdot\frac{\theta^{1+\nu}_{\ell+m}}{\alpha^2} \\
&\qquad + \frac{\widetilde{c}\|\widehat{u}\|}{1-c_0}\cdot\frac{(\ell+m+1)\mu^{-(\ell+m)k/d}}{\alpha^2}.
\end{aligned} \tag{4.14}$$

Substituting (4.12), (4.13) and (4.14) into (4.10) we see that

$$\begin{aligned}(4.15)\quad &\|[(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_{\ell+m})^{-1}-(\alpha\mathcal{I}+\mathcal{A})^{-1}]\widetilde{u}^{\delta}_{\alpha,\ell+m}\|\\ &\le \frac{c_0}{(1-c_0)\alpha}\widetilde{\gamma}^{\delta}_{\alpha,\ell+m}+\frac{c_0}{1-c_0}\\ &\cdot\frac{\delta}{\alpha^2}+\frac{(a_1a_\nu+a_{1+\nu})\|w\|}{1-c_0}\cdot\frac{\theta^{1+\nu}_{\ell+m}}{\alpha^2}\\ &+\frac{\widetilde{c}\|\widehat{u}\|}{1-c_0}\cdot\frac{(\ell+m+1)\mu^{-(\ell+m)k/d}}{\alpha^2}.\end{aligned}$$

Finally, combining (4.6), (4.8), (4.9), (4.15) and (3.3) yields (4.4).

Noting that

$$\frac{\widetilde{\widetilde{\gamma}}^{\delta}_{\alpha,n}}{\widetilde{\widetilde{\gamma}}^{\delta}_{\alpha,n+1}}\le\sigma\quad\text{and}\quad \widetilde{\widetilde{u}}^{\delta}_{\alpha,\ell,0}=\widetilde{\widetilde{u}}^{\delta}_{\alpha,\ell}$$

a similar argument to Theorem 3.5 leads to the estimate (4.5). This completes the proof. □

Let

$$\Delta_\alpha:=\alpha^2(\alpha\mathcal{I}+\mathcal{A})^{-2}f,\qquad \widetilde{\Delta}^{\delta}_{\alpha,\ell,m}:=\alpha^2\widetilde{\widetilde{u}}^{\delta}_{\alpha,\ell,m}$$

and

$$\begin{aligned}\widetilde{D}(\delta,\theta_{\ell+m}):=\;&c_4\delta+(c_5a_1a_\nu+c_6a_{1+\nu})\|w\|\theta^{1+\nu}_{\ell+m}\\ &+c_5\widetilde{c}\|\widehat{u}\|(\ell+m+1)\mu^{-(\ell+m)k/d},\end{aligned}$$

where

$$(4.16)\qquad c_4:=\frac{7-2c_0-c_0^2}{(1-c_0)^2},\quad c_5:=\frac{6-2c_0}{(1-c_0)^2},\quad c_6:=\frac{10-2c_0}{(1-c_0)^2}.$$

We then estimate the difference between Δ_α and $\widetilde{\Delta}^{\delta}_{\alpha,\ell,m}$.

Proposition 4.2. *If conditions* (H-1), (H-2) *and* (2.13) *hold, then there exists a positive integer N such that for $\ell>N$, $m\in\mathbf{N}_0$, there hold*

$$(4.17)\qquad \|\widetilde{\Delta}^{\delta}_{\alpha,\ell,m}-\Delta_\alpha\|\le\widetilde{D}(\delta,\theta_{\ell+m}),$$

and

$$\|\Delta_\alpha\| \le c_\nu^{(1-\nu)/\nu}\|w\|\alpha^{1+\nu}. \tag{4.18}$$

Moreover, if

$$\|f^\delta\| > \widetilde{D}(\delta, \theta_{\ell+m}) + (b+1)\delta \tag{4.19}$$

for some $b > 0$*, then*

$$\liminf_{\alpha\to+\infty} \|\widetilde{\Delta}^\delta_{\alpha,\ell,m}\| > b\delta. \tag{4.20}$$

Proof. From (4.2), we have

$$\begin{aligned}\widetilde{\Delta}^\delta_{\alpha,\ell,m} - \Delta_\alpha = {}& \alpha^2(\widetilde{\widetilde{u}}^\delta_{\alpha,\ell,m} - \widetilde{\widetilde{u}}^\delta_{\alpha,\ell+m}) \\ &+ \alpha^2(\alpha\mathcal{I} + \widetilde{\mathcal{A}}_{\ell+m})^{-1}(\widetilde{u}^\delta_{\alpha,\ell,m} - \widetilde{u}^\delta_{\alpha,\ell+m}) \\ &+ \alpha^2(\alpha\mathcal{I} + \widetilde{\mathcal{A}}_{\ell+m})^{-1}(\widetilde{u}^\delta_{\alpha,\ell+m} - u_\alpha) \\ &+ \alpha^2[(\alpha\mathcal{I} + \widetilde{\mathcal{A}}_{\ell+m})^{-1} - (\alpha\mathcal{I} + \mathcal{A})^{-1}](\alpha\mathcal{I} + \mathcal{A})^{-1}f.\end{aligned} \tag{4.21}$$

By Proposition 4.1, Lemma 2.4 and Theorem 3.5, there exists an $N \in \mathbf{N}$ such that for $\ell > N$, $m \in \mathbf{N}_0$,

$$\|\widetilde{\widetilde{u}}^\delta_{\alpha,\ell,m} - \widetilde{\widetilde{u}}^\delta_{\alpha,\ell+m}\| \le \widetilde{\widetilde{\gamma}}^\delta_{\alpha,\ell+m}, \tag{4.22}$$

$$\|(\alpha\mathcal{I} + \widetilde{\mathcal{A}}_{\ell+m})^{-1}(\widetilde{u}^\delta_{\alpha,\ell+m} - \widetilde{u}^\delta_{\alpha,\ell,m})\| \le \frac{1}{(1-c_0)\alpha}\widetilde{\gamma}^\delta_{\alpha,\ell+m}. \tag{4.23}$$

It follows from (2.17) and (2.25) that

$$\begin{aligned}\widetilde{u}^\delta_{\alpha,\ell+m} - u_\alpha = {}& (\alpha\mathcal{I} + \widetilde{\mathcal{A}}_{\ell+m})^{-1}\mathcal{P}_{\ell+m}(f^\delta - f) \\ &+ (\alpha\mathcal{I} + \widetilde{\mathcal{A}}_{\ell+m})^{-1}(\mathcal{P}_{\ell+m} - \mathcal{I})f \\ &+ [(\alpha\mathcal{I} + \widetilde{\mathcal{A}}_{\ell+m})^{-1} - (\alpha\mathcal{I} + \mathcal{A})^{-1}]f.\end{aligned} \tag{4.24}$$

By Lemma 2.4 and (2.3), (H-1) and (H-2), we have

$$\|(\alpha\mathcal{I} + \widetilde{\mathcal{A}}_{\ell+m})^{-1}\mathcal{P}_{\ell+m}(f^\delta - f)\| \le \frac{1}{1-c_0}\cdot\frac{\delta}{\alpha}, \tag{4.25}$$

$$\begin{aligned}(4.26)\quad \|(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_{\ell+m})^{-1}(\mathcal{P}_{\ell+m}-\mathcal{I})f\| &= \|(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_{\ell+m})^{-1}(\mathcal{P}_{\ell+m}-\mathcal{I})\mathcal{A}^{1+\nu}w\| \\ &\leq \frac{1}{1-c_0}\cdot\frac{a_{1+\nu}\|w\|\theta_{\ell+m}^{1+\nu}}{\alpha}.\end{aligned}$$

From (3.15) and (3.16) we conclude

$$\begin{aligned}(4.27)\quad & \|[(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_{\ell+m})^{-1}-(\alpha\mathcal{I}+\mathcal{A})^{-1}]f\| \\ &\leq \|(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_{\ell+m})^{-1}(\mathcal{A}-\mathcal{A}_{\ell+m})\mathcal{A}^{\nu}(\alpha\mathcal{I}+\mathcal{A})^{-1}\mathcal{A}w\| \\ &\quad + \|(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_{\ell+m})^{-1}(\mathcal{A}_{\ell+m}-\widetilde{\mathcal{A}}_{\ell+m})(\alpha\mathcal{I}+\mathcal{A})^{-1}\mathcal{A}\widehat{u}\| \\ &\leq \frac{1}{(1-c_0)\alpha}\big[(a_1a_\nu+a_{1+\nu})\|w\|\theta_{\ell+m}^{1+\nu} \\ &\qquad + \widetilde{c}\|\widehat{u}\|(\ell+m+1)\mu^{-(\ell+m)k/d}\big].\end{aligned}$$

Similarly, there holds

$$\begin{aligned}(4.28)\quad & \|[(\alpha\mathcal{I}+\widetilde{\mathcal{A}}_{\ell+m})^{-1}-(\alpha\mathcal{I}+\mathcal{A})^{-1}](\alpha\mathcal{I}+\mathcal{A})^{-1}f\| \\ &\leq \frac{1}{(1-c_0)\alpha^2}\big[(a_1a_\nu+a_{1+\nu})\|w\|\theta_{\ell+m}^{1+\nu} \\ &\qquad + \widetilde{c}\|\hat{u}\|(\ell+m+1)\mu^{-(\ell+m)k/d}\big].\end{aligned}$$

Combining (4.24)–(4.27) yields that

$$\begin{aligned}(4.29)\quad \|\widetilde{u}_{\alpha,\ell+m}^{\delta}-u_\alpha\| \leq \frac{1}{1-c_0}\bigg[&\frac{\delta}{\alpha}+(a_1a_v+2a_{1+\nu})\|w\|\frac{\theta_{\ell+m}^{1+\nu}}{\alpha} \\ &+\widetilde{c}\|\widehat{u}\|\frac{(\ell+m+1)\mu^{-(\ell+m)k/d}}{\alpha}\bigg].\end{aligned}$$

The estimate (4.17) follows from (4.21)–(4.23), (4.28) and (4.29).

The proof of estimates (4.18) and (4.20) is similar to that of Proposition 4.2 in [**7**], which is omitted. □

We remark that if $\|f^\delta\| \geq c\delta$ with $c > c_4+1$, then condition (4.19) holds when ℓ is large enough, since

$$\begin{aligned}\|f^\delta\| - \widetilde{D}(\delta,\theta_{\ell+m}) - \delta &\geq (c-c_4-1)\delta - (c_5a_1a_\nu + c_6a_{1+\nu})\|w\|\theta_{\ell+m}^{1+\nu} \\ &\quad - c_5\widetilde{c}\|\widehat{u}\|(\ell+m+1)\mu^{-(\ell+m)k/d}.\end{aligned}$$

We now consider parameter choice strategy. As in [**7**], we let $\ell \in \mathbf{N}$, $\tau > 0$ and $d > 1$ be fixed, choose a positive number α_0 satisfying

$$\tau\theta_\ell \leq \alpha_0 \leq d\tau\theta_\ell, \tag{4.30}$$

and define a sequence α_n by the formula

$$\alpha_n = d\alpha_{n-1}, \quad n = 1, 2, \dots .$$

We replace $\Delta^\delta_{\alpha,\ell,m}$ in [**7**] by $\widetilde{\Delta}^\delta_{\alpha,\ell,m}$ defined above and get the following lemma (cf. [**7**]).

Lemma 4.3. *If conditions* (4.19) *hold for some constant $b > 0$, then there exists an $n_0 \in \mathbf{N}_0$ such that*

$$\|\widetilde{\Delta}^\delta_{\alpha_{n_0-1},\ell,m}\| \leq b\delta \leq \|\widetilde{\Delta}^\delta_{\alpha_{n_0},\ell,m}\|, \tag{4.31}$$

where $\|\widetilde{\Delta}^\delta_{\alpha_{-1},\ell,m}\| := 0$.

Now we present the algorithm for choosing an *a posteriori* regularization parameter.

Algorithm 4.4 (*A posteriori* regularization parameter choice). *Let $d > 1$, $b > c_4$ be fixed. Input positive integers* Startlevel *and* Endlevel.

Step 1. *For a given $\delta > 0$, choose a positive integer $\ell \in \mathbf{N}$, $\ell \geq$* Startlevel *such that $\theta_\ell \leq \widehat{c}\delta$ for some $\widehat{c} > 0$ and choose a constant α_0 such that* (4.30) *holds.*

Step 2. *Let $m =$* Endlevel *$- \ell$. When α_n has been defined, use Algorithm* 3.1 *to compute $\widetilde{u}^\delta_{\alpha_n,\ell,m}$ and $\widetilde{\Delta}^\delta_{\alpha_n,\ell,m}$. If $\|\widetilde{\Delta}^\delta_{\alpha_n,\ell,m}\| < b\delta$, we set $\alpha_{n+1} := d\alpha_n$, $n := n + 1$, and repeat this step. Otherwise, go to Step* 3.

Step 3. *Set $\widehat{\alpha} := \alpha_{n-1}$ and stop.*

According to Algorithm 4.4, the output $\widehat{\alpha}$ depends on ℓ, m and δ, and it satisfies

$$\tau\theta_\ell \leq \widehat{\alpha} \leq d\tau\theta_\ell \quad \text{and} \quad b\delta \leq \|\widetilde{\Delta}^\delta_{\widehat{\alpha},\ell,m}\|, \tag{4.32}$$

or

$$\widehat{\alpha} \geq \tau\theta_\ell \quad \text{and} \quad \|\widetilde{\Delta}^\delta_{\widehat{\alpha},\ell,m}\| \leq b\delta \leq \|\widetilde{\Delta}^\delta_{d\hat{\alpha},\ell,m}\|. \tag{4.33}$$

Note that $(\ell+m+1)\mu^{-(\ell+m)k/d} \to 0$ as $\ell \to +\infty$ uniformly for $m \in \mathbf{N}_0$, we have the following proposition which is similar to Proposition 4.5 in [**7**].

Lemma 4.5. *Suppose that hypotheses* (H-1), (H-2) *and* (2.13) *hold. Let* $\widehat{\alpha} := \widehat{\alpha}(\ell, m, \delta)$ *be chosen according to Algorithm 4.4. Then*

$$\lim_{\delta\to 0, \ell\to+\infty} \widehat{\alpha}(\ell, m, \delta) = 0, \tag{4.34}$$

uniformly for $m \in \mathbf{N}_0$.

We also quote the following technical result from [**7**].

Lemma 4.6. *Suppose that* $\alpha \geq \alpha' > 0$*. Let* u_α *denote the solution of* $(\alpha\mathcal{I} + \mathcal{A})u_\alpha = f$ *and* $\widehat{u}$ *denote the minimum norm solution of* $\mathcal{A}u = f$*. Then,*

$$\|u_\alpha - \widehat{u}\| \leq \|u_{\alpha'} - \widehat{u}\| + \frac{\Delta_\alpha}{\alpha'}. \tag{4.35}$$

We next present the optimal convergence rate for the multilevel augmentation solution obtained by using the truncated operator and the *a posteriori* regularization parameter choice strategy. The proof is similar to the proof of Theorem 4.7 in [**7**], and we present it briefly. The readers can refer to [**7**] for the detailed proof.

Theorem 4.7. *Suppose that hypotheses* (H-1), (H-2) *and* (2.13) *hold. Let* $\widehat{u}$ *be the minimum norm solution of equation* (2.1), *and let* $\widetilde{u}^\delta_{\widehat{\alpha},\ell,m}$ *be the approximate solution obtained by Algorithm* 3.1 *with* $\widehat{\alpha}$ *chosen according to Algorithm* 4.4. *If* $\|f^\delta\| > c\delta$ *with* $c > \overline{c} := (8 - 4c_0)/(1 - c_0)^2$*, then there exists a positive integer* N *and positive constants* c *independent of* δ *such that for* $\ell > N$ *and* $m \in \mathbf{N}_0$,

$$\|\widetilde{u}^\delta_{\widehat{\alpha},\ell,m} - \widehat{u}\| \leq c\delta^{\nu/(1+\nu)}.$$

Proof. Since $\|f^\delta\| > c\delta$, $c > \overline{c} = c_4 + 1$, according to the remark after Proposition 4.2 that there exists an $N_1 \in \mathbf{N}$ such that for $\ell > N_1$, $m \in \mathbf{N}_0$, condition (4.19) holds. So $\widehat{\alpha}$ can be chosen according to Algorithm 4.4.

By Theorem 3.6, there exists an $N \in \mathbf{N}$ with $N \geq N_1$ such that for $\ell > N$, $m \in \mathbf{N}_0$,

$$\begin{aligned} \|\widehat{u} - \widetilde{u}^\delta_{\widehat{\alpha},\ell,m}\| \leq \frac{1}{1-c_0}\bigg[& c_\nu \|w\| \widehat{\alpha}^\nu + (3-c_0)\frac{\delta}{\widehat{\alpha}} \\ & + 2(a_1 a_\nu + a_{1+\nu})\|w\| \frac{\theta^{1+\nu}_{\ell+m}}{\widehat{\alpha}} \\ & + \widetilde{c}\|\widehat{u}\| \frac{(\ell+m+1)\mu^{-(\ell+m)k/d}}{\widehat{\alpha}} \bigg]. \end{aligned} \tag{4.36}$$

In the case that (4.32) holds, by Algorithm 4.4 we have

$$\widehat{\alpha}^\nu \leq (d\tau\theta_\ell)^\nu \leq (\widehat{c}\, d\tau\delta)^\nu = (\widehat{c}\, d\tau)^\nu \delta^\nu. \tag{4.37}$$

It follows from (4.17) and (4.18) that, similar to the proof in [**7**], we obtain that

$$b\delta \leq \|\widetilde{\Delta}^\delta_{\widehat{\alpha},\ell,m}\| \leq \|\Delta_{\widehat{\alpha}}\| + \|\widetilde{\Delta}^\delta_{\widehat{\alpha},\ell,m} - \Delta_{\widehat{\alpha}}\| \leq c_\nu^{(1-\nu)/\nu}\|w\|\widehat{\alpha}^{1+\nu} + \widetilde{D}(\delta, \theta_{\ell+m}). \tag{4.38}$$

We choose the integer N large enough such that for $\ell > N$, there holds

$$\ell > \frac{\mu^{k/d}}{\mu^{k/d} - 1} \quad \text{and} \quad (\ell+1)\mu^{-\ell k/d} \leq c_1'\delta, \tag{4.39}$$

where c_1' satisfies $c_5\widetilde{c}\|\widehat{u}\|c_1' + c_4 < b$. Thus for $\ell > N$, $(\ell + m + 1)\mu^{-(\ell+m)k/d}$ decreases as ℓ increases. Combining (4.38) and (4.39) we get

$$b\delta \leq c_\nu^{(1-\nu)/\nu}\|w\|\widehat{\alpha}^{1+\nu} + c_4\delta + (c_5 a_1 a_\nu + c_6 a_{1+\nu})\|w\|\theta^{1+\nu}_{\ell+m} + c_5\widetilde{c}\|\widehat{u}\|c_1'\delta. \tag{4.40}$$

Let $c_2' := c_4 + c_1' c_5 \widetilde{c}\|\widehat{u}\|$. Then $c_2' < b$. It follows from equation (4.40) that

$$\frac{(b-c_2')\delta}{\widehat{\alpha}} \leq c_\nu^{(1-\nu)/\nu}\|w\|\widehat{\alpha}^\nu + (c_5 a_1 a_\nu + c_6 a_{1+\nu})\|w\|\frac{\theta^{1+\nu}_{\ell+m}}{\widehat{\alpha}}. \tag{4.41}$$

From (4.32), (4.37) and (4.41) we conclude that

$$\frac{\delta}{\widehat{\alpha}} \leq \frac{c_\nu^{(1-\nu)/\nu}}{b-c_2'}\|w\|(\widehat{c}d\tau)^\nu\delta^\nu + \frac{(c_5a_1a_\nu + c_6a_{1+\nu})\|w\|}{(b-c_2')\tau}\theta_{\ell+m}^\nu. \tag{4.42}$$

It follows from equation (H-2) and (4.32) that

$$\frac{\theta_{\ell+m}^{1+\nu}}{\widehat{\alpha}} = \frac{\theta_{\ell+m}^\nu}{\widehat{\alpha}} \cdot \theta_{\ell+m}^\nu \leq \frac{\theta_\ell}{\widehat{\alpha}} \cdot \theta_{\ell+m}^\nu \leq \frac{\theta_{\ell+m}^\nu}{\tau}. \tag{4.43}$$

It follows from equations (4.39) and (4.42) that when ℓ is large enough we have

$$\begin{aligned}\frac{(\ell+m+1)\mu^{-(\ell+m)k/d}}{\widehat{\alpha}} &\leq c_1' \cdot \frac{\delta}{\widehat{\alpha}} \\ &\leq c_1'\Bigg[\frac{c_\nu^{(1-\nu)/\nu}}{b-c_2'}\|w\|(\widehat{c}d\tau)^\nu\delta^\nu \\ &\qquad + \frac{(c_5a_1a_\nu + c_6a_{1+\nu})\|w\|}{(b-c_2')\tau}\theta_{\ell+m}^\nu\Bigg].\end{aligned} \tag{4.44}$$

Substituting (4.37), (4.42), (4.43) and (4.44) into (4.36), we obtain

$$\|\widehat{u} - \widetilde{u}_{\widehat{\alpha},\ell,m}^\delta\| \leq d_1\delta^\nu + d_2\theta_{\ell+m}^\nu, \tag{4.45}$$

where

$$d_1 := \frac{\|w\|}{1-c_0}\left[c_\nu(\widehat{c}d\tau)^\nu + \frac{3-c_0+\widetilde{c}c_1'\|\widehat{u}\|}{b-c_2'} \cdot c_\nu^{(1-\nu)/\nu}(\widehat{c}d\tau)^\nu\right],$$
$$d_2 := \frac{\|w\|}{1-c_0}\left[\frac{2(a_1a_\nu + a_{1+\nu})}{\tau} + \frac{(3-c_0+\widetilde{c}c_1'\|\widehat{u}\|)(c_5a_1a_\nu + c_6a_{1+\nu})}{(b-c_2')\tau}\right].$$

Without lost of generality we assume that $\delta \leq 1$. This leads to $\delta^\nu \leq \delta^{\nu/(1+\nu)}$.

We next consider the case that (4.33) holds. Let $\alpha' := \delta^{1/(1+\nu)} + \theta_{\ell+m}$. Then the equation

$$(\alpha')^\nu \leq 2^\nu(\delta^{\nu/(1+\nu)} + \theta_{\ell+m}^\nu) \tag{4.46}$$

holds.

If $\widehat{\alpha} \geq \alpha'$, it follows from (4.39) that

$$\frac{(\ell+m+1)\mu^{-(\ell+m)k/d}}{\widehat{\alpha}} \leq \frac{c_1'\delta}{\widehat{\alpha}} \leq c_1'\delta^{\nu/(1+\nu)}. \tag{4.47}$$

By Proposition 4.2, Lemma 4.6, Theorem 3.5 and equations (2.6), (4.29), (4.33), (4.39), (4.46), we can conclude (cf. [**7**]) that there exists an N large enough such that for $\ell > N$, $m \in \mathbf{N}_0$,

$$\begin{aligned}\|\widehat{u} - \widetilde{u}^{\delta}_{\widehat{\alpha},\ell,m}\| &\leq \|\widehat{u} - u_{\widehat{\alpha}}\| + \|u_{\widehat{\alpha}} - \widetilde{u}^{\delta}_{\widehat{\alpha},\ell+m}\| + \|\widetilde{u}^{\delta}_{\widehat{\alpha},\ell+m} - \widetilde{u}^{\delta}_{\widehat{\alpha},\ell,m}\| \\ &\leq d_1\delta^{\nu/(1+\nu)} + d_2\theta^{\nu}_{\ell+m},\end{aligned}$$

where

$$\begin{aligned} d_1 &:= 2^{\nu}c_{\nu}\|w\| + b + c_4 + c_1'c_5\widetilde{c}\|\widehat{u}\| + \frac{3-c_0}{1-c_0} + \frac{2c_1'\widetilde{c}\|\widehat{u}\|}{1-c_0}, \\ d_2 &:= \left[2^{\nu}c_{\nu} + c_5a_1a_{\nu} + c_6a_{1+\nu} + \frac{2(a_1a_{\nu}+2a_{1+\nu})}{1-c_0}\right]\|w\|. \end{aligned}$$

If $\widehat{\alpha} \leq \alpha'$, then by (4.46) the following equation holds:

$$\widehat{\alpha}^{\nu} \leq (\alpha')^{\nu} \leq 2^{\nu}(\delta^{\nu/(1+\nu)} + \theta^{\nu}_{\ell+m}). \tag{4.48}$$

It follows from equations (4.33), (4.39) and Proposition 4.2 that (cf. [**7**])

$$\frac{\delta}{\widehat{\alpha}} \leq c_3'\widehat{\alpha}^{\nu} + c_4'\theta^{\nu}_{\ell+m}, \tag{4.49}$$

where

$$c_3' := \frac{c_{\nu}^{(1-\nu)/\nu}\|w\|d^{1+\nu}}{b-c_2'}, \quad \text{and} \quad c_4' := \frac{(c_5a_1a_{\nu} + c_6a_{1+\nu})\|w\|}{(b-c_2')\tau}.$$

Combining (4.39), (4.48) and (4.36), we obtain

$$\|\widehat{u} - \widetilde{u}^{\delta}_{\widehat{\alpha},\ell,m}\| \leq d_1\delta^{\nu/(1+\nu)} + d_2\theta^{\nu}_{\ell+m},$$

where

$$d_1 := \frac{2^\nu}{1-c_0}[c_\nu \|w\| + (3-c_0)c_3' + c_1' c_3' \widetilde{c} \|\widehat{u}\|],$$

$$d_2 := \frac{1}{1-c_0}\big[c_\nu 2^\nu \|w\| + 2(a_1 a_\nu + a_{1+\nu})\|w\|/\tau + (3-c_0)(c_3' 2^\nu + c_4') + c_1' \widetilde{c} \|\widehat{u}\|(c_3' 2^\nu + c_4')\big].$$

Since $\theta_{\ell+m} \to 0$ uniformly as $\ell \to \infty$, we can choose N large enough such that for $\ell > N$, $m \in \mathbf{N}_0$,

$$\theta_{\ell+m} \le \delta^{1/(1+\nu)},$$

which with the above estimates of $\|\widehat{u} - \widetilde{u}^\delta_{\widehat{\alpha},\ell,m}\|$ leads to the conclusion of this theorem. □

5. Numerical results. In this section, we present numerical results to illustrate the efficiency of the algorithms and confirm the theoretical results described in previous sections.

We consider the problem of solving the integral equation

$$(\mathcal{K}u)(s) = f(s), \quad s \in E := [0,1], \tag{5.1}$$

where $\mathcal{K} : L^2(E) \to L^2(E)$ is a linear compact operator defined by

$$(\mathcal{K}u)(s) := \int_E K(s,t)u(t)\,dt, \quad s \in E, \tag{5.2}$$

with the kernel

$$K(s,t) := \frac{1}{2}\cos(-s+t) + \frac{1}{4}\sin(s+t) - \frac{1}{4}\sin(s+t+2),$$

and the right-hand side

$$f(s) := \frac{1}{8}\sin\left(\frac{2s-3}{2}\right) + \frac{1}{2}\cos\left(\frac{2s-1}{2}\right) - \frac{11}{32}\sin\left(\frac{2s+5}{2}\right) + \frac{1}{16}\cos\left(\frac{2s-9}{2}\right) + \frac{13}{32}\sin\left(\frac{2s+1}{2}\right) - \frac{3}{16}\cos\left(\frac{2s-5}{2}\right) + \frac{1}{16}\cos\left(\frac{2s+7}{2}\right) - \frac{1}{32}\sin\left(\frac{2s+13}{2}\right) - \frac{3}{16}\cos\left(\frac{2s+3}{2}\right) + \frac{3}{32}\sin\left(\frac{2s+9}{2}\right).$$

The unique solution of this problem is $\widehat{u}(t) = \cos((2t-1)/2) - (1/2)\sin((2t+5)/2) + (1/2)\sin((2t+1)/2)$. Since $\widehat{u} = \mathcal{K}w$ with $w = 1$, we have that $\widehat{u} \in R(\mathcal{K})$, which means $\nu = 1$. In this case, the optimal convergence rate should be $\delta^{1/2}$.

Let $\mathbf{X}_n$ be the space of piecewise linear polynomials on E with knots at $j/2^n$, $j = 1, 2, \dots, 2^n - 1$. As in [**6**], we decompose $\mathbf{X}_n$ into the form of the orthogonal direct sum of subspaces

$$\mathbf{X}_n = \mathbf{X}_0 \oplus^{\perp} \mathbf{W}_1 \oplus^{\perp} \cdots \oplus^{\perp} \mathbf{W}_n,$$

where $\mathbf{X}_0 = \mathbf{W}_0$ is the linear polynomial space on E, and for $i \in \mathbf{N}$, $\mathbf{W}_i$ is the orthogonal complement of $\mathbf{X}_{i-1}$ in $\mathbf{X}_i$. The basis for $\mathbf{W}_i$, $i = 2, 3, \dots$, can be constructed recursively once the basis for $\mathbf{W}_1$ is given (cf. [**3, 20**]).

We choose a basis for $\mathbf{X}_0$

$$w_{0,0}(t) := 1, \qquad w_{0,1}(t) := \sqrt{3}(2t-1), \quad t \in [0,1],$$

and a basis for $\mathbf{W}_1$

$$w_{1,0}(t) := \begin{cases} -6t+1 & t \in [0,1/2], \\ -6t+5 & t \in (1/2,1], \end{cases}$$
$$w_{1,1}(t) := \begin{cases} -4\sqrt{3}t + \sqrt{3} & t \in [0,1/2], \\ 4\sqrt{3}t - 3\sqrt{3} & t \in (1/2,1]. \end{cases}$$

The bases for subspaces $\mathbf{W}_i = \operatorname{span}\{w_{i,j} : j = 0, 1, \dots, 2^i - 1\}$ are recursively generated by

$$w_{i,j}(t) = \begin{cases} \sqrt{2}w_{i-1,j}(2t) & t \in [0,1/2], \\ 0 & t \in (1/2,1], \end{cases}$$
$$w_{i,2^{i-1}+j}(t) = \begin{cases} 0 & t \in [0,1/2], \\ \sqrt{2}w_{i-1,j}(2t-1) & t \in (1/2,1], \end{cases}$$

where $j \in \mathbf{Z}_{2^{i-1}}$.

We complete the computation on a PC with Intel(R) Celeron(R) 2.40 GHz CPU and 512 MB memory. We report in Table 1 the compression rate (Comp. Rate) and computational times measured in seconds for generating matrix $\mathbf{A}_n$ and matrix $\widetilde{\mathbf{A}}_n$ respectively, where compression rate is defined as the ratio of the number of the nonzero entries in $\widetilde{\mathbf{A}}_n$ to that of the full matrix $\mathbf{A}_n$, i.e., $\mathcal{N}(\widetilde{\mathbf{A}}_n)/s(n)^2$. It shows that when n gets larger the time saved by using our truncation strategy gets more.

TABLE 1. The time comparison between the generation of $\mathbf{A}_n$ and $\widetilde{\mathbf{A}}_n$.

n	Comp. Rate	Time for generating $\mathbf{A}_n$ (sec.)	Time for generating $\widetilde{\mathbf{A}}_n$ (sec.)
6	$6.250e-2$	0.1250	0.07800
7	$3.516e-2$	0.7500	0.3440
8	$1.953e-2$	5.641	1.563
9	$1.074e-2$	38.33	6.672
10	$5.859e-3$	287.8	26.30

TABLE 2. Numerical results for *a priori* parameter choice ($\alpha = \delta^{1/(1+\nu)} = \delta^{1/2},\ \ell = 5,\ m = 5$).

δ	$\delta^{1/2}$	$\|\widehat{u} - u^{\delta}_{\alpha,\ell+m}\|_{L^2}$	$\|\widehat{u} - \widetilde{u}^{\delta}_{\alpha,\ell,m}\|_{L^2}$	T_1 (sec.)	T_2 (sec.)
$1.000e-2$	$1.000e-1$	$2.038e-1$	$2.014e-1$	296.8	5.031
$5.000e-3$	$7.071e-2$	$1.487e-1$	$1.475e-1$	287.2	5.031
$1.250e-3$	$3.536e-2$	$7.739e-2$	$7.712e-1$	290.3	5.047
$5.000e-4$	$2.236e-2$	$4.975e-2$	$4.961e-2$	287.2	5.031
$3.125e-4$	$1.768e-2$	$3.956e-2$	$3.950e-2$	287.4	5.094
$7.813e-5$	$8.839e-3$	$2.002e-2$	$2.002e-2$	288.1	5.062
$1.953e-5$	$4.419e-3$	$1.009e-2$	$1.009e-2$	287.6	5.047
$1.000e-6$	$1.000e-3$	$2.303e-3$	$2.303e-3$	287.2	5.031

TABLE 3. Numerical results for *a posteriori* parameter choice ($d = 3.5$, $b = 8$, $\widehat{c} = 6$, $\tau = 1$, Startlevel$= 5$, Endlevel$= 10$).

δ	α_{post}	α_{prio}	$\Vert\widehat{u} - \widetilde{u}^{\delta}_{\alpha_{\text{post}},\ell,m}\Vert_{L^2}$	$\Vert\widehat{u} - u^{\delta}_{\alpha_{\text{prio}},\ell+m}\Vert_{L^2}$
$1.000e-2$	$2.538e-1$	$1.000e-1$	$3.748e-1$	$2.040e-1$
$5.000e-3$	$7.252e-2$	$7.071e-2$	$1.494e-1$	$1.488e-1$
$1.250e-3$	$7.252e-2$	$3.536e-2$	$1.349e-1$	$7.742e-2$
$5.000e-4$	$2.072e-2$	$2.236e-2$	$4.767e-2$	$4.979e-2$
$3.125e-4$	$2.072e-2$	$1.768e-2$	$4.391e-2$	$3.954e-2$
$7.813e-5$	$5.180e-3$	$8.839e-3$	$1.839e-2$	$2.003e-2$
$1.953e-5$	$4.533e-3$	$4.419e-3$	$1.025e-2$	$1.009e-2$
$1.000e-6$	$9.915e-4$	$1.000e-3$	$2.291e-3$	$2.304e-3$

In our numerical implementation, we choose a perturbed right-hand side f^{δ} with $\Vert f^{\delta} - f\Vert_{L^2} = \delta$, and an *a priori* parameter $\alpha = \delta^{1/1+\nu} = \delta^{1/2}$. The numerical results are presented in Table 2, where T_1 denotes the time for solving the linear system (2.12) by directly using LU factorization method and T_2 denotes the time for solving the linear system (2.15) by multilevel augmentation algorithm. The results shows that optimal convergence rate can be obtained by the *a priori* parameter choice using fast multilevel augmentation algorithm. Moreover, the multilevel augmentation method with truncation strategy is much more efficient than the direct method without truncation. We can also see from Table 1 and Table!2 that the fast multilevel augmentation algorithm is especially efficient for large scale computation.

Table 3 gives numerical results for the *a posteriori* parameter choice, where α_{post} stands for the *a posteriori* parameter and α_{prio} stands for the *a priori* parameter. The results shows optimal convergence rate can be obtained by the *a posteriori* parameter choice suggested in Section 4.

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