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AN INTEGRO-DIFFERENTIAL EQUATION MODEL FOR THE SPREAD OF ALCOHOL ABUSE

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ABSTRACT. The alcohol abuse level of individuals in a population as it depends on resilience and peer influence is considered in this paper. Several simple models are studied as well as an integro-differential equation model which is derived using coarse graining from a pre-existing discrete network system. The connection structure of the discrete system tends to be richer than that of the integro-differential equation model; however, the continuum problem can be studied analytically using traveling wave, perturbation and phase plane techniques.

The analysis presented in this paper suggests that, in both the discrete network and integro-differential models, nearly alcoholic or highly sober individuals are relatively unaffected by peer pressure, and this aspect of the models leads to an inertia in the spread of alcohol abuse or sobriety depending on the connectivity, initial conditions and resilience of the population. A related but different model is introduced that avoids this inertia.

A treatment scheme had also been developed for the discrete network system. A continuum version for the integro-differential model is provided here.

1. Introduction. Alcohol abuse is an important social problem in the world today. The papers [**2, 17**] introduce a discrete network model with a bistable-type rate function for the spread and treatment of this condition along with computational studies. Here, in order to examine the bistable mechanism analytically, we introduce several simple new models as well as a continuum model adaptation involving an integro-differential equation (IDE). We study the IDE using perturbation and traveling wave techniques as well as scientific computing.

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There are many situations where peer pressure has an influence. Connections between criminals increases ability through mentoring (see also [**6**]). Obesity studies reveal that peer influence is significant in its spread [**5**]. We direct the reader to [**2, 17**] for a full discussion of the ideas behind the models we study as well as references to the Sociology literature. Essentially, we follow the evolution over time of the alcohol abuse level of individuals or averages of individuals. The dynamics depend on the resilience or susceptibility of the population members and their social networks. Quoting from [**2**], the mechanism in these models "has the capacity to simulate the effect of moving alcohol abusers into non-abusers."

Basic differential equation model. The following ordinary differential equation,

$$\frac{dv_i}{dt} = v_i(1 - v_i)(n_i - r_i), \tag{1}$$

has the bistable rate function used in the discrete network system and forms a starting point for our work in this paper (It was the also the starting point for the work in [**2, 17**]). Equation (1) models the dynamics of the probability function $v_i = v_i(t)$ of an individual i at time t having alcohol-related problems (e.g., hospital admission, arrest, auto accident or complaint with authorities). We are assuming the level of a person's alcohol related problems can be quantified. We will use the shorthand terms alcoholic health, sobriety, or alcoholic (for the v_i). Individuals are sober if $v_i \cong 0$ or alcoholic if $v_i \cong 1$. The function $n_i = n_i(t)$ measures the peer pressure influence by taking a weighted average health,

$$n_i(t) = \sum_j w_{ij} v_j(t),$$

where the $w_{ij} > 0$ measure the strength of the connection between individuals i and j. We assume the w_{ij} are normalized with $\sum_j w_{ij} = 1$ for all i. Except for the two patch models described in Section 2, we will assume symmetry, that $w_{ij} = w_{ji}$ for all i and j. Each r_i is the resilience of the individual i with $0 \leq r_i \leq 1$. The closer r_i is to 1 the less susceptible the individual is to becoming an alcoholic. We generally assume that resilience is a constant r in this paper. The initial condition

$$v_i(0) = g_i \text{ for all } i \tag{2}$$

is needed to fully specify the problem.

Discrete network model. In [**2, 17**] a fully discrete network model with synchronous updating is considered where the change in the alcoholic health of the ith individual from the mth to the $(m+1)$st time step in a large network is modeled by

$$v_i^{m+1} - v_i^m = \lambda v_i^m (1 - v_i^m)(n_i^m - r_i) \quad \text{where} \quad n_i^m = \sum_j w_{ij} v_j^m. \tag{3}$$

Here, v_ℓ^m is the alcoholic health of individual ℓ at time t_m and $\lambda > 0$ is a rate constant. If we assume the length of the steps is small and re-scale time, the differential equation (1) emerges as an approximation from the fully discrete equation (3).

The spread of disease on networks is a topic of current research (see, for instance, [**9, 13, 16**]). In [**2, 17**], one focus is on a comparison of network dynamics when the wiring or connections are either defined as small world or random. The research in these papers suggests the following relationship:

$$\lim_{m\to\infty} v_i^m = \begin{cases} 1 & \text{if } \overline{v}_0 > \overline{r} \\ 0 & \text{if } \overline{v}_0 < \overline{r} \end{cases} \tag{4}$$

where $\overline{v}_0$ is the average initial alcohol index and average resilience is $\overline{r}$. When $\overline{v}_0 > \overline{r}$ the population evolves to a state consisting primarily of alcohol abusers, while, if the reverse is true, $\overline{v}_0 < \overline{r}$, then the population becomes predominantly sober.

A treatment scheme is also considered in [**2**]; they suppose that a small percent of the population with high alcohol indices can be treated. These unhealthy individuals are removed for a specified period of time and then returned to the general population with their index set to half of their resilience level. In one computation, it is found that if, roughly, 7% of the population is treated on a regular basis, then the entire population eventually becomes sober.

Outline. In Section 2 we consider a set of simple problems motivated by the discrete network model. This set includes a single variable problem, a two population differential equation system and an integro-differential equation model where peer pressure is computed by a simple average. We also introduce a new switch model that has similar dynamics to the model in [**2**] but uses different mathematical mechanisms.

In Section 3, by rewriting the n-term as a convolution of the w and v functions, we form an integro-differential equation model on the real line. The goal of this work is to place the discrete models in a continuum framework and examine, via traveling waves, how the spread of abuse relates to the other parameters and, in particular, r. The new switch model is also examined in this way and has robust traveling waves.

Finally, in Section 5, a treatment regime that mimics the one in [**2**] is introduced and studied in a simplified situation.

Throughout, we assume that there exist unique solutions to our various differential and integral equation problems. We have not proved this fully but note that a proof of existence of short time solutions is provided in [**15**].

2. Models inspired by the discrete network system.

In this section we look at three simple models where the long time behavior in (4) can be shown directly. We also introduce a new model using a switch with a Heaviside function that has similar dynamics to those of the bistable model.

Single variable bistable model: The simple model we discuss in this section was the motivation for the discrete system in [**2, 17**]. If all individuals were the same (homogeneous) and were coupled in an all-to-all manner with equal connection strength; then, (1) would simplify to the initial value problem

$$v' = v(1-v)(v-r) \quad \text{with} \quad v(0) = v_0.$$

Noting that 0, 1 and r are the fixed points, the rate function $v(1-v)(v-r)$ indicates that the solution v will either tend to 0 (sober state) if $v_0 < r$ or 1 (alcoholic state) if $v_0 > r$. This result is consistent with (4).

Two patch model: Another very simple model would have two patches or populations. As is done in the computations in [**2**], we consider having a large population of moderate drinkers and a small group of individuals with a strong tendency toward alcohol abuse. To accomplish this, we examine the initial value problem consisting of a system of two ordinary differential equations

$$v_1' = v_1(1-v_1)(kv_1+(1-k)v_2-r_1), \quad 0 \le k \le 1, \tag{5}$$

$$v_2' = v_2(1-v_2)(kv_1+(1-k)v_2-r_2) \tag{6}$$

where

$$v_1(0) = v_1^0 \quad \text{and} \quad v_2(0) = v_2^0. \tag{7}$$

In this case we are considering two homogeneous populations. A wide range of dynamics are possible in this model. To narrow the focus, we consider the case where v_2 represents the index of a small set of individuals who have a strong tendency toward alcohol abuse while v_1 represents the rest of the population. We further assume that r_1 and r_2 are both $O(1)$ but $r_2 \ll r_1$. To represent the small influence of the alcohol abuser's population, $k \cong 1$. Examining the phase plane in this special case (see Figure 1) there are fixed points at $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$, $(r_1/k, 0)$ and $((r_1-(1-k))/k, 1)$. When we restrict to trajectories within the box $(0,1)\times(0,1)$ we find that only $(0,0)$ and $(1,1)$ are attractive.

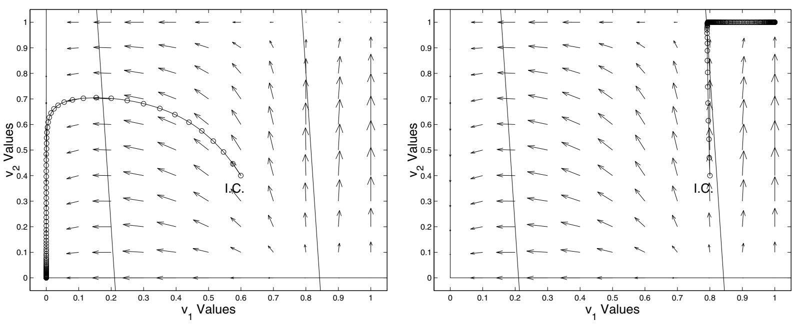
Figure 1 illustrates the two typical outcomes that depend on the initial condition (7). Only if the majority of the population is initially abusive (v_1^0 close to 1) will the entire population tend to become alcoholic (Figure 1B). Otherwise the entire population will become sober (tend to $(0,0)$, Figure 1A). Once again, we find a result which is consistent with (4).

Average peer influence model: Here we suppose there are J individuals which are all connected with weight $w_i = 1/J$ and have the same resilience r. This model involves the following differential equation

$$\frac{dv_i}{dt} = v_i(1-v_i)\,(n_i - r)\,, \qquad n_i = \frac{1}{J}\sum_{j=1}^{J} v_j, \quad i = 1,\dots,J. \tag{8}$$

Thinking of $v_i = v(x_i)$ where $x_i = i/J$, we consider the following integro-differential equation on an interval $[0,1]$,

$$\frac{\partial v}{\partial t} = v(1-v)(n-r) \quad \text{with} \quad n(t) = \int_0^1 v(x,t)\,dx$$



A. **B.**

FIGURE 1. Phase plane for the system (5)–(7) with $k_1 = 0.95$, $k_2 = 0.05$, $r_1 = 0.8$, and $r_2 = 0.2$. Nullclines and direction arrows are displayed as well as a solution trajectory. **A.** Initial condition $v_1^0 = 0.6$ and $v_2^0 = 0.4$. **B.** Initial condition $v_1^0 = 0.8$ and $v_2^0 = 0.4$.

as an approximation of (8). The advantage of this simple model is that we can now show

$$\lim_{t\to\infty} v(x,t) = \begin{cases} 1 & \text{if } \int_0^1 v_0\,dx > r, \\ 0 & \text{if } \int_0^1 v_0\,dx < r, \end{cases} \tag{9}$$

which is similar to (4).

We proceed by viewing $n(t)$ as a known function and solve for v at a specific point $\bar{x}$. This approach allows us to see the integro-differential equation above as separable and we obtain,

$$\int_{v_0(\overline{x})}^{v(\overline{x},t)} \frac{du}{u(1-u)} = N(t),$$

where $N(t) = \int_0^t (n(s) - r)\,ds$. We find that

$$v(\overline{x},t) = \left(1 + \frac{1 - v_0(\overline{x})}{v_0(\overline{x})} \exp(-N(t))\right)^{-1}, \tag{10}$$

which shows $0 \le v \le 1$ if we assume that $0 < v_0(\overline{x}) < 1$ for all $\overline{x} \in [0,1]$. We note that we can create an ordinary differential equation for the function n;

$$\frac{d}{dt}\left[\int_0^1 v\,dx\right] = \left[\int_0^1 v(1-v)\,dx\right](n - r)$$

or

$$\frac{d}{dt}(n-r) = \beta(n-r) \quad \text{with} \quad \beta(t) = \int_0^1 v(1-v)\,dx.$$

Note that $\beta(t) \geq 0$. If we think of β as a nonnegative but given function we can solve the linear differential equation for $n-r$ (in terms of β) and obtain

$$n(t) - r = (n(0) - r)\exp\left(\int_0^t \beta(s)\,ds\right). \tag{11}$$

There are now two conclusions. The first comes when we assume $n(0) > r$ so that $n(t) - r \geq n(0) - r$ and then

$$\begin{aligned}\exp(-N(t)) &= \exp\left(-\int_0^t (n(s) - r)\,ds\right)\\ &\leq \exp\left(-(n(0)-r)t\right) \longrightarrow 0 \text{ as } t \to +\infty\end{aligned}$$

which then allows us to conclude from (10) that $v(\overline{x}, t) \to 1$ as $t \to +\infty$. One can use the same reasoning to argue that, if $n(0) < r$, then $v(\overline{x}, t) \to 0$ as $t \to +\infty$. This now proves (9).

New switch model: In this section we offer a model that has a *switch* or Heaviside function depending on the sign of the $n_i - r_i$ function. This model has the same long time behavior and dynamics as the bistable system (3). We will see in the next section that it has a more robust movement of abuse/sobriety.

We choose

$$\frac{dv_i}{dt} = -H(r_i - n_i)v_i - H(n_i - r_i)(v_i - 1) \quad \text{with} \quad n_i(t) = \sum_j w_{ij}v_j(t).$$

Here, $H = H(s)$ is the Heaviside function; $H(s) = 1$ if $s > 0$ and $H(s) = 0$ if $s \leq 0$. Note that if $n_i < r_i$ the equation becomes $dv_i/dt = -v_i$ so $v_i \to 0$ (sobriety). If $n_i > r_i$, then $dv_i/dt = -(v_i - 1)$ and $v_i \to 1$ (alcohol abuse). In either case v_i will not be greater than 1 or less than 0 which is the same as the bistable model.

Since $H(r_i - n_i) + H(n_i - r_i) = 1$ we find that

$$\frac{dv_i}{dt} = -v_i + H(n_i - r_i).$$

To compare to the bistable model (3), we consider the following fully discrete system:

$$(12)\quad v_i^{m+1} = v_i^m + \Delta t \left[-v_i^m + H(n_i^m - r_i)\right] \quad \text{where} \quad n_i^m = \sum_\ell w_{i\ell} v_\ell^m.$$

We compared (3) and (12) by creating a network of 180 individuals consisting of 30 "caves"–sets of 6 individuals initially connected in an all-to-all manner–and, then, reconnected globally with probability $\rho = 0.3$ to form a small world network (this procedure to generate small world networks is explained in [**13**]). For each system, (3) and (12), 150 steps were taken with $\Delta t = \lambda = T/150$ where $T = 15$. The r_j were chosen randomly with mean 0.5 and standard deviation 1/6 (We also truncated these randomly chosen r_j's so that $0 \leq r_j \leq 1$.). The initial conditions were also chosen randomly with means v_{Mean} and standard deviation 1/6 (and truncated so $0 \leq v_i^0 \leq 1$.). Figure 2 has the plot of the final means of the individuals. These results suggest that, at least in the long time calculations, the switch model (12) leads to outcomes that are similar to those found in the bistable model from [**2**].

3. Integro-differential equation model.

In this section we develop a continuum model from the network model (see [**11, 14**] for other examples of the coarse-graining we use in this section). We are interested in traveling wave solutions (TWS) and their wave speeds as well as steady state solutions. Motivated by the discrete network model we propose the following integro-differential equation model:

$$(13)\qquad \frac{\partial v}{\partial t} = v(1-v)(n-r).$$

Here $v = v(x,t)$ measures an averaged health of individuals at a spatial location x and time t. Individuals are healthy if $v \cong 0$ and highly alcohol dependent if $v \cong 1$. The function $n = n(x,t)$ measures the influence of neighboring individuals and is a convolution of the connection function w with v;

$$n(x,t) = (w * v)\,(x,t) = \int_{-\infty}^{\infty} w(x-y) v(y,t)\, dy.$$

Here, $w > 0$ is a weight or footprint function with $\int_R w\, ds = 1$. Assuming $w = w(s)$, of course, imposes a translation invariance on

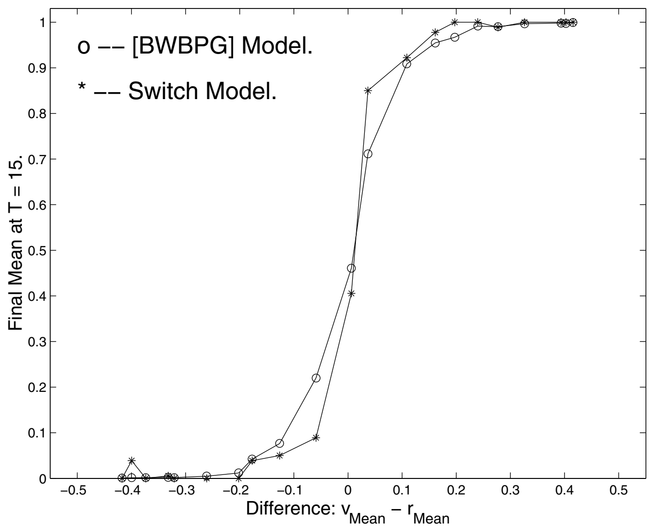


FIGURE 2. Long time evolution of fully discrete models (3) and (12) with $\Delta t = \lambda = T/15$, respectively. The final mean of the alcohol index for each population at $T = 15$ time units is plotted versus the difference between the mean of the initial population alcohol index v_{Mean} and the mean population resilience r_{Mean}.

the connection scheme that does not exist in the discrete models. We assume $0 \leq r \leq 1$ is constant. The initial condition

$$v(\cdot, 0) = v_0. \tag{14}$$

is needed to fully specify the problems.

The transition from the network model (2) to the IDE problem (13)–(14) follows by a standard coarse graining argument (see, for example, [**14**]). We suppose the individuals are arranged on the real axis where individual i is positioned at $x_i = i\Delta x$, $0 < \Delta x \ll 1$, and we assume the connections are modeled by the "footprint" function w. If we set $w_{ij} = \Delta x\, w(x_i - x_j)$ then we can approximate the $n_i(t)$ in (1) by an integral;

$$\begin{aligned} n_i &= \sum_j w_{ij} v_j = \sum_j \Delta x\, w(x_i - x_j) v(x_j, \cdot) \\ &\cong \int w(x_i - y) v(y, \cdot)\, dy = n(x_i, \cdot). \end{aligned}$$

We note that this model (as well as (1) or (3)) has the feature that as individuals become more extreme in their ways (either sobriety or alcohol abuse), they are less susceptible to peer influence due to the $v(1-v)$ factor.

Note that any function consisting of segments or points with values of 0 or 1 will form a steady state solution or standing wave for the IDE (13). This could be considered a degeneracy and motivates adding the regularization term Dv_{xx} where $D > 0$. This term can be thought of as providing communication between neighboring nodes. So, we consider

$$\frac{\partial v}{\partial t} = -v(v-1)(n-r) + D\frac{\partial^2 v}{\partial x^2}, \tag{15}$$

and the initial condition (14).

Chen [**3**] has proved the existence and exponential stability of traveling waves for the regularized problem (14)–(15). We now seek to estimate the speed of these waves as they depend on the regularization parameter D.

Perturbation estimate of wave speed. Here, we take advantage of the fact that

$$U_0(z) = \frac{1}{2}\left[1 + \tanh\left(\frac{z}{2\sqrt{2D}}\right)\right] \text{ satisfies } -DU_0'' = -\phi_{1/2}(U_0),$$

where

$$\phi_{1/2}(U_0) = U_0(U_0 - 1)(U_0 - 1/2)$$

(see [**10**], for instance). We note also that

$$U_0(z) = \frac{1}{1 + \exp(-z/\sqrt{2D})}.$$

So, we look for a TWS of (15) with $v(x,t) = U(x-ct)$ and $z = x - ct$;

$$-cU' - DU'' = -U(U-1)(n(U) - r)$$

where we further specify

$$r = \frac{1}{2} + \varepsilon \quad \text{and} \quad n(U) = U + \varepsilon q(U).$$

We are assuming that our convolution term $n = n(U)$ is well approximated by the function U with an error term $\varepsilon q(U)$. We also assume that ε is positive but small while D is moderately sized and independent of ε. We make the perturbation assumption that $c = 0 + c_1\varepsilon$ and then have

$$-DU'' + \phi_{1/2}(U) = \varepsilon\left[c_1U' - U(U-1)(q(U)-1)\right].$$

We let $U = U_0 + \varepsilon U_1 + \cdots$ and find that

$$\begin{aligned} -DU_0'' + \phi_{1/2}(U_0) + \varepsilon(-DU_1'' + \phi_{1/2}'(U_0)U_1) \\ = \varepsilon\left[c_1U_0' - U_0(U_0-1)(q(U_0)-1)\right] + O(\varepsilon^2). \end{aligned}$$

We assume that $U_1(z) \to 0$ as $z \to \pm\infty$. At $O(1)$ the equation is satisfied by U_0 while at $O(\varepsilon)$ we have

$$-DU_1'' + \phi_{1/2}'(U_0)U_1 = c_1U_0' - U_0(U_0-1)(q(U_0)-1).$$

Multiplying the equation by U_0' and integrating-by-parts twice over $\mathbf{R}$ we find, using again that $-DU_0'' + \phi_{1/2}(U_0) = 0$,

$$\begin{aligned} 0 &= \int_{\mathbf{R}} U_1\left(-DU_0'' + \phi_{1/2}(U_0)\right)' \, dz \\ &= \int_{\mathbf{R}} U_0'\left[c_1U_0' - U_0(U_0-1)(q(U_0)-1)\right] \, dz \end{aligned}$$

from which we can obtain a formula for the wave speed coefficient

$$c_1 = \frac{\int_{\mathbf{R}} U_0(U_0-1)(q(U_0)-1)U_0' \, dz}{\int_{\mathbf{R}} (U_0')^2 \, dz}.$$

A key observation is that $q(U_0)$ is an odd function since U_0 is odd. Thus, since U_0' and $U_0(1-U_0)$ are even, we must have

$$\int_{\mathbf{R}} U_0(1-U_0)q(U_0)U_0'\,dz = 0.$$

So,

$$c_1 = \frac{\int_{\mathbf{R}} U_0(1-U_0)U_0'\,dz}{\int_{\mathbf{R}}(U_0')^2\,dz}.$$

If we also change variables to $z = \sqrt{2D}y$ letting $\widetilde{U}_0(y) = U_0(\sqrt{2D}y)$, then

$$c_1 = \sqrt{2D}\frac{\int_{\mathbf{R}} \widetilde{U}_0(1-\widetilde{U}_0)\widetilde{U}_0'\,dy}{\int_{\mathbf{R}}(\widetilde{U}_0')^2\,dy}.$$

But, $\widetilde{U}_0(y) = (1+\exp(-y))^{-1}$ which helps us determine that

$$\int_{\mathbf{R}} \widetilde{U}_0(1-\widetilde{U}_0)\widetilde{U}_0'\,dy = 1/6$$

and

$$\int_{\mathbf{R}} (\widetilde{U}_0')^2\,dy = 1/6.$$

Combining these evaluated integrals with our expansion for c_1 gives

$$c_{Pert} = \varepsilon c_1 = \sqrt{2D}\varepsilon = \sqrt{2D}(r-1/2). \tag{16}$$

This formula suggests that waves will move from left-to-right if $r > 1/2$ and right-to-left if $r < 1/2$. It also predicts that wave speeds tend to 0 as $D \to 0$. Figure 3C has a plot of this relation.

In the next section we will use scientific computation to examine these waves.

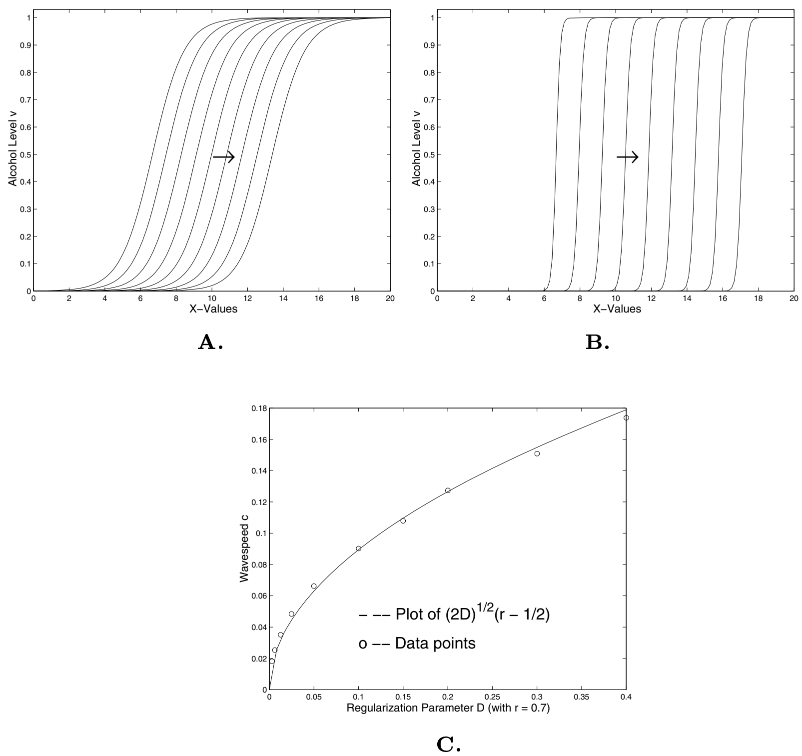


FIGURE 3. Numerical approximation of wave shapes. The waves pictured above will move in the opposite direction if $r < 1/2$. The wave speeds were computed by comparing the x-locations and times of the points where the 9 waves pictured took on the value $1/2$. Averaging the eight speeds provided the estimate of the computed velocity. **A.** IDE problem (15), (17) and (14) with parameters $r = 0.7$, $D = 0.4$ and $\alpha = 4.0$. Resulting wave speed was $c = 0.1738$. **B.** IDE problem (15), (17) and (14) with parameters $r = 0.7$, $D = 0.00625$ and $\alpha = 4.0$. Resulting wave speed was $c = 0.0261$. **C.** Wave speeds from perturbation (16) and computations (like those in (A) and (B)) as they depend on the regularization parameter D.

Computations on the IDE Model: We use finite difference computations to study the full regularized IDE problem (15) and (17). To simplify the IDE computation we take $w(s) = \alpha e^{-\alpha|s|}/2$ where $\alpha > 0$. Note that $\int_{\mathbf{R}} w\, ds = 1$ and observe that this allows us to interpret n as a Green's function:

$$n'' - \alpha^2 n = -\alpha^2 v. \tag{17}$$

This approach offers the opportunity to study a system of partial differential equations (15) and (17) instead of an IDE (see, for instance, [**1**] for the inspiration for this idea. Of course, for general convolution integrals, FFT's would be used to speed the computations). So, the problem (15), (14) and (17) was solved by a finite difference scheme with the implicit Euler method for time stepping. Figures 3A and 3B reveal typical traveling waves that were found in the computations. Waves like the ones pictured (with $v \to 0$ at $-\infty$ and $v \to 1$ at $+\infty$) move in the opposite direction (right-to-left) if $r < 1/2$.

Comparison of Figures 3A and 3B reveals that as D gets smaller the wave speeds decrease and the wave shapes become squarer and closer to the degenerate $0/1$ standing states.

In Figure 3C we have plotted computed wave speeds and the estimated speed function c_{Pert}.

Standing waves in diffusion approximation: It is often useful to consider a localized footprint and approximate the IDE by a single partial differential equation (PDE). If, as above, we take $w(s) = \alpha e^{-\alpha|s|}/2$, but, here, assume $\alpha \gg 1$ then, using Taylor's theorem

$$n(x,t) = \frac{\alpha}{2}\int_{\mathbf{R}} v(x+z,t)e^{-\alpha|s|}\,ds \cong v(x,t) + \alpha^{-2}v_{xx}(x,t).$$

Here we expanded v to its second derivative term around x using Taylor's theorem and dropped the $O(\alpha^{-4})$ term. So, we consider the following PDE approximation of the regularized IDE, (15) and (14),

$$\begin{aligned} v_t - [D + \sigma b(v)]\, v_{xx} &= -\phi(v), \quad \sigma = \alpha^{-2}, \\ b(v) &= v(1-v), \quad \text{and} \\ \phi(v) &= v(v-1)(v-r). \end{aligned} \tag{18}$$

The advantage of this simplification is that we can use phase plane analysis to study the traveling wave shapes and speeds. We also can derive formulas for the standing wave solutions (SWS) even in cases where $r \neq 1/2$.

Phase plane analysis: A system of two differential equations can be developed to find TWS of (18) and (14). Here we set $v(x,t) = U(x-ct)$, label $z = x - ct$, and obtain, for (18),

$$-cU' - \sigma b(U)U'' = -\phi(U) + DU''. \tag{19}$$

We then define a two-variable system of ordinary differential equations:

$$U' = V \quad \text{and} \quad V' = \frac{-cV + \phi(U)}{D + \sigma b(U)}, \tag{20}$$

which has fixed points $(0,0)$, $(r,0)$ and $(1,0)$. Letting

$$\vec{F}(U,V) = \begin{pmatrix} V \\ (-cV + \phi(U))/(D + \sigma b(U)) \end{pmatrix};$$

then

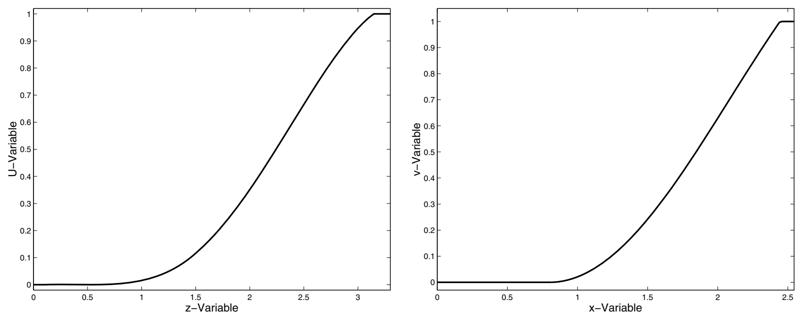
$$\vec{F}'(U_{FP}, V_{FP}) = \begin{bmatrix} 0 & 1 \\ \phi'(U_{FP})/(D + \sigma b(U_{FP})) & -c/(D + \sigma b(U_{FP})) \end{bmatrix}$$

since $-cV + \phi(U) = 0$ at the fixed points which we label (U_{FP}, V_{FP}). We can now determine the type, stability, and local eigenvectors near each fixed point. For the fixed point $(0,0)$ we found that it is a saddle point with eigenvectors

$$\vec{\xi}_{\pm} = \begin{pmatrix} 1 \\ \mu_{\pm} \end{pmatrix} \quad \text{and} \quad \mu_{\pm}(0,0) = \frac{-c \pm \sqrt{c^2 + 4rD}}{2D}.$$

We also found that $(1,0)$ was a saddle point and $(r,0)$ was an unstable spiral if c was sufficiently small.

We search for a TWS where $c > 0$ and $r > 1/2$. We expect $U \to 0$ as $z \to -\infty$ and $U \to 1$ as $z \to +\infty$. To determine this, we examined the UV phase plane and used a shooting method on the system (20). We guessed c and used an ODE solver to compute a numerical approximation to (20) starting near the fixed point $(0,0)$ along an unstable eigenvector. We iterated using the bisection method until a value of c was found where the corresponding trajectory converged to the final fixed point along a stable eigenvector direction near the fixed point $(1,0)$. We found a trajectory connecting $(0,0)$ to $(1,0)$ which led to the TWS displayed in Figure 4A.



A. **B.**

FIGURE 4. Wave shapes from approximations of the solutions to the PDE (18). **A.** Plot of the traveling wave shape for the case when $r = 0.7$, $D = .002$ and $\sigma = 2/3$ obtained by the phase plane analysis. Resulting speed was $c = 0.05372$. **B.** Plot of standing wave obtained for $r = 0.7$ and $\sigma = 2/3$ using the formula (22).

Standing waves: We start with the steady state equation, noting that $\phi(v) = -b(v)(v - r)$, we have

$$-\sigma b(v)v'' = b(v)(v - r).$$

There are two cases; for $r > 1/2$ we assume $v \equiv 0$ for $x < 0$ and $v \equiv 1$ for all $x > \overline{x}$ where $\overline{x} > 0$. Then on $[0, \overline{x}]$ we consider

$$\sigma v'' + v = r \tag{21}$$

and, from viewing the computed TWS in Figure 4A, we require $v(0) = v'(0) = 0$ and $v(\overline{x}) = 1$. The solution of (21) with these conditions is

$$v(x) = r\left[1 - \cos\left(\frac{x}{\sqrt{\sigma}}\right)\right] \quad \text{with} \quad \overline{x} = \sqrt{\sigma}\arccos\left(1 - \frac{1}{r}\right). \tag{22}$$

In the case when $r < 1/2$ we look for a solution with $v \equiv 0$ for $x < \overline{x}$ and $v \equiv 1$ for all $x > 0$ where $\overline{x} > 0$. Here, we require v to satisfy (21), $v(\overline{x}) = 0$, $v'(0) = 0$ and $v(0) = 1$. In this case we find on $[\overline{x}, 0]$ that

$$v(x) = (1 - r)\cos\left(\frac{x}{\sqrt{\sigma}}\right) + r \quad \text{with } \overline{x} = \sqrt{\sigma}\arccos\left(\frac{r}{r - 1}\right). \tag{23}$$

A plot of the standing wave for $r > 1/2$, (22), is provided in Figure 4B. Note that the shape of the slow moving wave in figure 4A is similar. Computations with the full time-dependent model (18) using our implicit Euler method and either of the SWS, (22) or (23), supported the notion that these formulas are steady states.

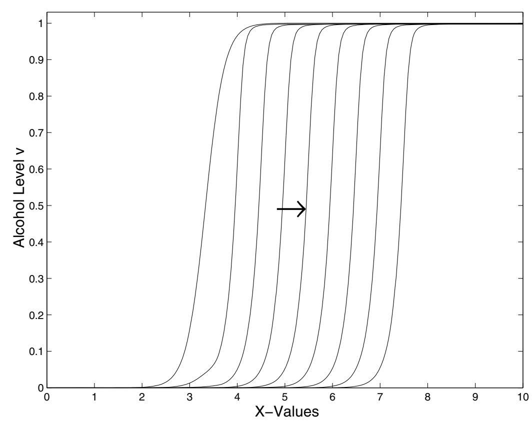


FIGURE 5. Dynamics of the switch IDE model (24). Computed traveling waves are displayed with $r = 0.7$ and $\alpha = 4$. The approximate wave speed in this computation was 0.2053.

Traveling waves in the switch model: As we have seen, the current IDE model and its PDE approximation have standing waves in situations ($r \neq 1/2$) when one might expect traveling waves. They also exhibit decreasing wave speeds as the regularization parameter $D \to 0$. The v and $v - 1$ factors in the bistable model (1)–(2), that we have studied thus far, lead to greatly diminished peer influence for individuals that are severe alcoholics or nearly sober. In this section we look for traveling waves in the integro-differential equations for the *switch* model (12). Here, we consider

$$v_t = -v + H(n - r) \quad \text{where } n = w * v. \tag{24}$$

If we then take the convolution with footprint function w we have

$$n_t = -n + w * H(n - r).$$

From [**11**] we know that there exists a traveling wave for n. We can then recover a traveling wave for v by solving the equation $n = w * v$.

We developed a finite difference implicit Euler approximation to solutions of (24) similar to those used in Section 3 (choosing n as a solution of (17), etc.). Again, we used the specific weight function $w(s)$ and the associated differential equation (17). Figure 5 displays a sample computation that features a TWS.

4. IDE treatment. We also consider a treatment regime for (13)–(14) that is available to the worst alcohol abusers ($v > 1 - p$ where $0 < p << 1$). One way to incorporate this involves adding the following term to the right side of (13):

$$T(x,t) = -\frac{1}{\tau_S} m(x,t)\left(v(x,t) - \frac{r(x)}{2}\right), \quad \tau_S > 0, \tag{25}$$

where

$$\frac{\partial m}{\partial t} = \frac{1}{\tau_T} H(v - (1-p))\,(1-m) - \frac{m}{\tau_R}, \quad \tau_R > 0 \text{ and } \tau_T > 0. \tag{26}$$

Here, H is the Heaviside function as described in the previous section. Parameter τ_T measures the time to impose a treatment regimen, τ_R measures the rehabilitation time and $1/\tau_S$ measures the strength of the treatment. If the "treatment" is imposed (case where $v > 1 - p$), the worst alcohol abusers will have $m \to 1$ in $O(\tau_T)$ time which will invoke the treatment $T(x,t)$, a force driving $v \to r/2$ if τ_S is sufficiently small. Once the individuals being treated are healthier, $v < 1 - p$, the treatment will be removed in $O(\tau_R)$ since $m \to 0$.

Below we examine a special case.

Homogeneous treatment: In this section we assume all individuals in (13) start at the same abuse level; that is v_0 is constant in (14). If we also assume r is constant, then the IDE model with treatment (25)–(26) reduces to a differential equation system where the solutions $v = v(t)$ and $w = w(t)$ satisfy

$$\begin{aligned} v' &= \left[-\phi(v) - \frac{1}{\tau_S} m\left(v - \frac{r}{2}\right)\right] \\ m' &= \frac{1}{\tau_T} H(v-(1-p))(1-m) - \frac{1}{\tau_R} m. \end{aligned}$$

with $v(0) = v_0$ and $m(0) = m_0$. A major advantage of this formulation is the opportunity to use phase planes.

We discuss the results displayed in Figures 6 and 7. In the $r = 0.2$ and $p = 0.2$ case (Figure 6), the population with low resilience is in

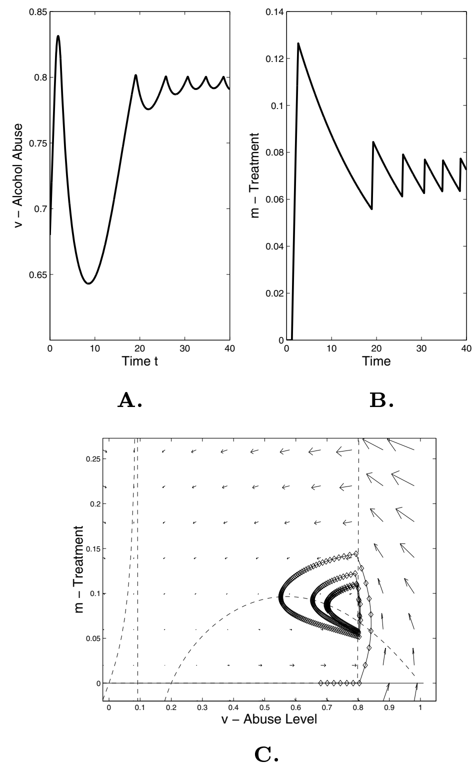


FIGURE 6. Results for homogeneous treatment model when $r = 0.2$, $p = 0.2$, $\tau_T = 10$, $\tau_S = 0.5$ and $\tau_R = 20$. Here we took $(v_0, m_0) = (0.68, 0)$. **A:** Dynamics of the population index $v = v(t)$. **B:** Dynamics of the treatment variable $m = m(t)$. **C:** Phase plane showing the nullclines, direction arrows and solution trajectory. Here the center fixed point is a stable spiral.

and out of treatment and remains at a high consumption level ($v \cong 0.8$). The trajectory tends to the stable spiral fixed point. This situation resembles the behaviors of addictive personalities, and it is like the celebrity behavior that frequently appears in the tabloids.

With r still set at 0.2 but p increased to 0.4 (Figure 7), we see that the population becomes sober. In this case the treatment is invoked for a larger class of individuals (those with $v > 0.6$). Here, the fixed point is an unstable spiral and the trajectory tends to the fixed point at $(0, 0)$. Note that this behavior matches the trends shown in Braun et al. [**2**, Figure 6, page 8].

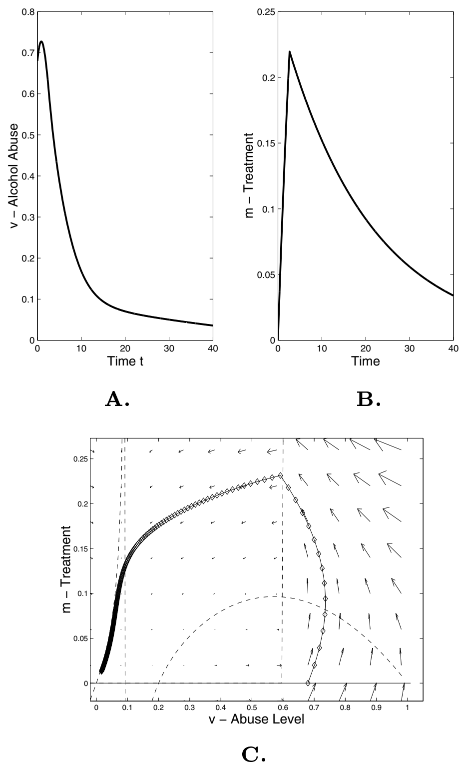


FIGURE 7. Results for homogeneous treatment model when $r = 0.2$, $p = 0.4$, $\tau_T = 10$, $\tau_S = 0.5$ and $\tau_R = 20$. Here we took $(v_0, m_0) = (0.78, 0)$. **A:** Dynamics of the population index $v = v(t)$. **B:** Dynamics of the treatment variable $w = w(t)$. **C:** Phase plane showing the nullclines, direction arrows and solution trajectory. Here the center fixed point is an unstable spiral type.

5. Discussion. In this paper we have investigated a wide range of idealized models for the spread of alcohol abuse which are motivated directly from the discrete network model with bistable rate function introduced in [**2**].

Computations in [**2**] show that the populations tend to become abusive ($v \cong 1$) if, initially, the average abuse level $\overline{v}_0$ is greater than their average resilience $\overline{r}$. If the opposite is true, initially, the population tends toward sobriety (This is the same as (4)). In Section 2, we introduced three simple but related models where this behavior can be shown directly. In Section 3, an IDE problem for the bistable model was introduced. Computations, a perturbation analysis and phase plane analysis on a regularized version of the new equation model as well as the existence of standing waves in the non-regularized model suggest that there is an inertia in the bistable model and it may not have traveling wave solutions.

We also introduced a switch model that has many of the same features as the bistable model. This new switch model, however, has robust traveling waves as can be shown computationally and via known results from the literature.

The results in Sections 2 and 3 for the bistable models, at first glance, appear to be contradictory. However, we expect that the presence of noise and randomness in the discrete network system would tend to break the inertia caused by the v and $(1-v)$ functions.

In Section 5 we developed a continuum treatment strategy and displayed its efficacy in a simple homogenous case.

There are many future directions. We note that the degeneracy caused by the v and $1-v$ functions in (13) have some resemblance to those studied in [**12**]. The approach in [**12**] could clarify the dynamics. We are interested in extending the model (13)–(14) to more realistic situations with data in the style of the work in [**8**].

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