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MODIFIED TIKHONOV REGULARIZATION FOR NONLINEAR ILL-POSED PROBLEMS IN BANACH SPACES

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ABSTRACT. We present a variant of Tikhonov regularization for nonlinear ill-posed problems in Banach spaces, where the convergence rate $O(\delta)$ for the Bregman distance is obtained under the same conditions as this rate is achieved for standard Tikhonov regularization. However, in this variant the regularization parameter can be chosen a-priori and independently from the condition on the exact solution.

1. Introduction. We consider nonlinear ill-posed problems

$$F(x) = y, \tag{1.1}$$

where $F : \mathcal{D}F \subset X \to Y$ is a nonlinear bounded operator between Banach spaces. In practice only noisy data y^δ are available, where δ denotes the noise level. Throughout this paper we will assume that $\|y - y^\delta\| \le \delta$.

Due to the ill-posedness, one has to use regularization methods to obtain stable approximations for an exact solution $x^\dagger$ of problem (1.1). A widely used method is Tikhonov regularization, where the regularized solution, x_α^δ, is a minimizer of the functional

$$\frac{1}{p}\|F(x) - y^\delta\|^p + \alpha R(x), \quad \alpha > 0, \tag{1.2}$$

where $R(x)$ is a penalty term.

This method is well understood if F is an operator between Hilbert spaces, $p = 2$ and $R(x) = \|x - x_*\|^2$ (see, e.g., [**3**]). It turns out that in several situations Tikhonov regularization in Hilbert spaces does not yield good results, since it has the tendency to smooth the

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solutions. This is of course not appropriate if one knows that the exact solution has jumps or a sparse structure. Better results are then achieved when the Tikhonov functional is considered in Banach spaces. Usually the penalty term is then either chosen as $R(x) = \|x\|^q$ or $R(x) = \sum_{i=1}^{\infty} w_i |\langle x, \phi_i \rangle|^q$, where X is a Hilbert space and $\{\phi_i\}$ is a frame. The latter is the usual choice for problems with sparsity constraints. In some papers, where sparsity is treated, q is allowed to be between 0 and 1. Since we need convexity of our functional R (see (A2) below), we have to restrict q such that $q \geq 1$. Sparsity then means q close to 1.

For stability and convergence questions we refer the reader to [**14**] and the many references therein. Usually, convergence is considered in the Bregman distance. Convergence results with respect to the norm in X are only available if the Bregman distance may be estimated from below by some power of the norm. Methods for how to actually minimize the functional in (1.2) have been considered, e.g., in [**1, 2, 7**].

As for Tikhonov regularization in Hilbert spaces also in Banach spaces the convergence of the regularized solutions towards the exact solution as the noise level δ goes to 0 can be arbitrarily slow. Convergence rates can be obtained only if $x^\dagger$ satisfies some source conditions and if the regularization parameter is chosen appropriately.

For instance, if $x^\dagger$ satisfies (A10), then it was shown (see, e.g., [**14**]) that the rate $O(\delta)$ is obtained for the Bregman distance $D_\xi(x_\alpha^\delta, x^\dagger)$ provided that $\alpha \sim \delta^{p-1}$.

Improved rates were derived under additional assumptions on the solution $x^\dagger$ and/or the spaces X and Y in, e.g., [**4, 6, 11, 12, 13**]. For rate results in Hilbert spaces see [**3**].

All results have one thing in common: the regularization parameter α has to be chosen in a specific way to obtain the rate. Unfortunately, this specific choice depends on the source conditions satisfied by the exact solution $x^\dagger$. But usually, it is not known what source conditions are satisfied.

Therefore, for Tikhonov regularization in Hilbert spaces a lot of interest was put in developing so called a-posteriori parameter rules that always yield optimal convergence rates without having to know any information about the exact solution (see [**3**]).

A-posteriori parameter rules have in common that a nonlinear problem has to be solved to find the correct regularization parameter. For linear equations (1.1) in Hilbert spaces this might be okay since the minimizer of (1.2) for a fixed value $\alpha > 0$ can be found by solving just one linear equation. However, for nonlinear equations (1.1) this is quite some work since the calculation of a minimizer for a fixed α is already quite involved.

Therefore, in [**9, 10**] a modification of the Tikhonov functional was presented that allows to find regularized solutions converging with the desired rates by only solving one nonlinear problem. In the next section, we extend this method to regularization in Banach spaces. Finally, we apply the results to nonlinear integral equations.

2. Modified Tikhonov regularization. We approximate exact solutions of problem (1.1) by a solution $x_\beta^{\delta,\eta}$ ($\beta > 0$, $\delta, \eta \geq 0$) of the problem

$$\begin{aligned} f_\beta^\delta(x_\beta^{\delta,\eta}) &\leq f_\beta^\delta(x) + \eta, \quad \text{for all } x \in \Delta, \\ f_\beta^\delta(x) &:= \Big| \|F(x) - y^\delta\| - \delta \Big|^p + \beta R(x). \end{aligned} \tag{2.1}$$

The parameter η reflects the fact that, in general, a minimizer of the (modified) Tikhonov functional f_β^δ cannot be calculated exactly.

The question is what conditions have to be satisfied by the operator F and the functional R so that one can guarantee existence of such solutions, stability and convergence results. A rather general answer to this question has been given in the book [**14**] for the minimizers x_α^δ of (1.2). We ask for somewhat stronger conditions that guarantee norm convergence and not only weak convergence. These conditions are:

(A1) X and Y are reflexive Banach spaces with norms $\|\cdot\|_X$ and $\|\cdot\|_Y$, respectively. We will omit space indices whenever it is clear from the context what norm is meant. X^* and Y^* denote the dual spaces of X and Y with dual forms $\langle\cdot,\cdot\rangle_{Y^*,Y}$ and $\langle\cdot,\cdot\rangle_{X^*,X}$, respectively. Again we omit space indices.

(A2) The functional $R : X \to [0, \infty]$ is convex and weakly sequentially lower semi-continuous.

(A3) The operator $F : \mathcal{D}(F) \subset X \to Y$ is weakly sequentially closed and continuous.

(A4) $\mathcal{D} := \mathcal{D}(F) \cap \mathcal{D}(R)$ is nonempty and convex, where $\mathcal{D}(R) := \{x \in X : R(x) \neq \infty\}$.

(A5) Equation (1.1) has a solution in $\mathcal{D}$. Note that then it also has an R-minimizing solution $x^\dagger$ (see [**14**, Theorem 3.25]), i.e., $R(x^\dagger) = \min\{R(x) : F(x) = y\}$.

(A6) Every sequence $\{x_k\}$ in $\mathcal{D}$ satisfying that $R(x_k)$ and $\|F(x_k)\|$ are bounded has a weakly convergent subsequence.

(A7) There is an element $x_* \in \mathcal{D}$ such that $R(x_*) < R(x)$ for all $x \in \mathcal{D}$ with $F(x) = y$.

(A8) The exponent p in (1.2) satisfies $1 \leq p < \infty$.

(A9) For every R-minimizing solution $x^\dagger$

$$x_k \rightharpoonup x^\dagger \ \wedge \ R(x_k) \to R(x^\dagger) \Rightarrow x_k \to x^\dagger.$$

(A10) (see [**14**, Assumption 3.34] There is an R-minimizing solution $x^\dagger$, an element $\xi \in \partial R(x^\dagger)$, and constants $c_1 \in [0,1)$, $c_2 \geq 0$, $\rho > 0$ such that

$$\langle \xi, x^\dagger - x \rangle \leq c_1 D_\xi(x, x^\dagger) + c_2 \|F(x) - F(x^\dagger)\|$$

for all $x \in \mathcal{D}$ with $R(x) \leq R(x^\dagger) + \rho$ and $\|F(x) - F(x^\dagger)\| \leq \rho$. Here,

$$D_\xi(x, x^\dagger) := R(x) - R(x^\dagger) - \langle \xi, x - x^\dagger \rangle$$

denotes the Bregman distance.

Remark 2.1. Note that if X is a Hilbert space and if $R(x) = \|x - x_*\|^2$, then (A7) is equivalent to $F(x_*) \neq y$. Moreover, $\xi = 2(x^\dagger - x_*)$ and $D_\xi(x, x^\dagger) = \|x - x^\dagger\|^2$.

Assumption (A9) is satisfied if R is totally convex and if every R-minimizing solution $x^\dagger$ has a nonempty subdifferential $\partial R(x^\dagger)$ (compare the proof of [**14**, Proposition 3.32]). If $R(x) = \|x\|^q$, $q \geq 1$, assumption (A9) is equivalent to the so-called *Radon-Riesz property* stating that $x_k \rightharpoonup x^\dagger$ and $\|x_k\| \to \|x^\dagger\|$ imply that $x_k \to x^\dagger$. This property typically holds in every L^r-space with $1 < r < \infty$ (cf. [**8**]).

According to [**14**, Proposition 3.35] Assumption (A10) is satisfied if F is Gateaux-differentiable in $x^\dagger$ and if there are constants $c_F \geq 0$ and $\rho > 0$ such that

$$\xi = F'(x^\dagger)^{\#} w \in \partial R(x^\dagger), \quad w \in Y^*, \qquad c_F \|w\| < 1,$$

and

$$\|F(x) - F(x^\dagger) - F'(x^\dagger)(x - x^\dagger)\| \le c_F D_\xi(x, x^\dagger)$$

for all $x \in \mathcal{D}$ with $R(x) \le R(x^\dagger) + \rho$ and $\|F(x) - F(x^\dagger)\| \le \rho$. Here $F'(x^\dagger)^\# : Y^* \to X^*$ denotes the Banach space adjoint of $F'(x^\dagger)$.

Let us turn to the question of existence of solutions $x_\beta^{\delta,\eta}$. Obviously, such solutions exist for $\eta > 0$. We will show that under conditions (A1)–(A7) the functional f_β^δ has a minimizer $x_\beta^\delta := x_\beta^{\delta,0}$ if $\delta \ge 0$ is sufficiently small.

For the proof of this result and the convergence proofs we need the following lemma.

Lemma 2.2. *Let assumptions* (A1)–(A7) *hold. Then there exists an element $x^\delta \in \mathcal{D}$ minimizing the functional $R(x)$ on the set $M^\delta := \{x \in \mathcal{D} : \|F(x) - y^\delta\| \le \delta\}$. Moreover, if $x_* \notin M^\delta$ with x_* as in* (A7) *which is the case for $\delta > 0$ sufficiently small, then*

$$\|F(x^\delta) - y^\delta\| = \delta.$$

Proof. Note that $M^\delta \neq \varnothing$, since $x^\dagger \in M^\delta$. The existence of the minimizing element x^δ immediately follows from conditions (A1)–(A6) and the weak lower semi-continuity of the norm.

Due to (A7), $F(x_*) \neq y$. Hence, $x_* \notin M^\delta$ for $\delta > 0$ sufficiently small. Let us now assume that $x_* \notin M^\delta$ and that $\|F(x^\delta) - y^\delta\| < \delta$. Then, due to the continuity of F and the convexity of $\mathcal{D}$, there is a $t \in [0, 1)$ such that $\|F(x_t) - y^\delta\| \le \delta$, where $x_t := tx^\delta + (1-t)x_*$. Thus, $x_t \in M^\delta$ and $R(x_t) \le tR(x^\delta) + (1-t)R(x_*) < R(x^\delta)$ contradicting the definition of x^δ. Therefore, the assumption $\|F(x^\delta) - y^\delta\| < \delta$ was wrong. This shows that $\|F(x^\delta) - y^\delta\| = \delta$ whenever $x_* \notin M^\delta$. □

Theorem 2.3. *Let assumptions* (A1)–(A8) *hold. Then the functional f_β^δ defined by* (2.1) *has a minimizer $x_\beta^\delta \in \mathcal{D}$ if $x_* \notin M^\delta$. Moreover, $\|F(x_\beta^\delta) - y^\delta \ge \delta$.*

Proof. Since $x_* \notin M^\delta$, it follows by Lemma 2.2 and (A8) that

$$f_\beta^\delta(x^\delta) = \beta R(x^\delta) \le f_\beta^\delta(x), \quad \text{for all } x \in M^\delta.$$

Let us assume that x^δ is not a minimizer of f_β^δ, otherwise we are done. Then obviously $\rho := \inf\{f_\beta^\delta(x) : x \in \mathcal{D}\} < f_\beta^\delta(x^\delta)$ and there is a sequence $\{x_k\}$ in $\mathcal{D}$ such that $f_\beta^\delta(x_k) \to \rho$ as $k \to \infty$ and $R(x_k) < R(x^\delta) - \varepsilon$ for some $\varepsilon > 0$ and $x_k \notin M^\delta$, i.e., $\|F(x_k) - y^\delta\| > \delta$.

Now assumptions (A1)–(A4) and (A6) imply that there is a subsequence (again denoted by $\{x_k\}$) and an element $\overline{x} \in \mathcal{D}$ such that $x_k \rightharpoonup \overline{x}$, $F(x_k) \to F(\overline{x})$, and $R(\overline{x}) \le R(x^\delta) - \varepsilon$. Hence, $\overline{x} \notin M^\delta$.

Since $\|F(x_k) - y^\delta\| > \delta$ and $\|F(\overline{x}) - y^\delta\| > \delta$, by the weak lower semicontinuity of the norm and (A2) we obtain that $f_\beta^\delta(\overline{x}) \le \lim_{k\to\infty} f_\beta^\delta(x_k)$ $= \rho \le f_\beta^\delta(\overline{x})$. Thus, $\overline{x}$ is a minimizer. □

Weak stability of the minimizers x_β^δ can be shown as in [**14**, Theorem 3.23]. Strong stability holds if assumption (A9) not only holds for R-minimizing solutions but for all $x \in \mathcal{D}$.

In the next theorem we answer the question concerning convergence of the regularized solutions $x_\beta^{\delta,\eta}$ towards an R-minimizing solution.

Theorem 2.4. *Let assumptions* (A1)–(A9) *hold and assume that* $\delta \to 0$*,* $\beta \to 0$ *and* $\eta = o(\beta)$*. Then every sequence* $x_{\beta_k}^{\delta_k,\eta_k}$ *has a convergent subsequence. The limit of every convergent subsequence is an* R*-minimizing solution. If, in addition, the* R*-minimizing solution* $x^\dagger$ *is unique, then*

$$x_\beta^{\delta,\eta} \to x^\dagger \quad \textit{as} \quad \delta \to 0 \quad \textit{and} \quad \beta \to 0.$$

Proof. Let $\delta > 0$ be so small that $x_* \notin M^\delta$. Then Lemma 2.2 implies that

$$f_\beta^\delta(x_\delta^\beta) \le f_\beta^\delta(x^\delta) + \eta = \beta R(x^\delta) + \eta \le \beta R(x^\dagger) + \eta.$$

The rest of the proof follows the lines of [**14**, Theorem 3.26] noting that (A9) yields strong convergence. □

Note that in the theorem above β only has to go to 0 to get convergence of the regularized solutions $x_\beta^{\delta,\eta}$. No restriction with respect to δ is needed. In comparison, to get convergence of the regularized solutions x_α^δ, the minimizers of (1.2), one needs that $\delta^p/\alpha \to$

0, i.e., α is not allowed to go to 0 too fast. Moreover, it immediately follows from the proof that $\|F(x_\beta^{\delta,\eta}) - y\| = O(\delta)$ if $\beta = O(\delta^p)$.

We now turn to convergence rates. It was mentioned in the introduction that under condition (A10) it was shown that $D_\xi(x_\alpha^\delta, x^\dagger) = O(\delta)$ if $\alpha \sim \delta^{p-1}$. We will show in the next theorem that this is also true for the regularized solutions $x_\beta^{\delta,\eta}$.

Theorem 2.5. *Let assumptions* (A1)–(A10) *hold and assume that* $\beta = O(\delta^p)$ *and* $\eta = O(\beta\delta)$. *Then*

$$D_\xi(x_\beta^{\delta,\eta}, x^\delta) = O(\delta) \quad \textit{and} \quad \|F(x_\beta^{\delta,\eta}) - y\| = O(\delta).$$

Proof. Let $\delta > 0$ be so small that $x_* \notin M^\delta$. Then (2.1) and Lemma 2.2 imply that

$$\Big|\|F(x_\beta^{\delta,\eta}) - y^\delta\| - \delta\Big|^p + \beta R(x_\beta^{\delta,\eta}) \le \beta R(x^\delta) + \eta \le \beta R(x^\dagger) + \eta.$$

Hence, $\|F(x_\beta^{\delta,\eta}) - y = O(\delta)$ and $R(x_\beta^{\delta,\eta}) \le R(x^\dagger) + O(\delta)$. Together with (A10) we now obtain that

$$\begin{aligned} D_\xi(x_\beta^{\delta,\eta}, x^\dagger) &= R(x_\beta^{\delta,\eta}) - R(x^\dagger) - \langle \xi, x_\beta^{\delta,\eta} - x^\dagger \\ &\le O(\delta) - \langle \xi, x_\beta^{\delta,\eta} - x^\dagger \rangle \\ &\le O(\delta) + c_1 D_\xi(x_\beta^{\delta,\eta}, x^\dagger) + c_2 \|F(x_\beta^{\delta,\eta}) - y\|. \end{aligned}$$

Since $c_1 < 1$, this yields that $D_\xi(x_\beta^{\delta,\eta}, x^\dagger) = O(\delta)$. □

Once more we want to emphasize that as for regularization in Hilbert spaces the choice of β does not depend on the source conditions as it is the case for the a-priori choices of α. For standard Tikhonov regularization the power of how α has to depend on δ is strongly linked to the source condition satisfied by the exact solution. In this variant β only has to go to 0 fast enough.

Since for linear problems in a Hilbert space setting the modified method (2.1) is equivalent to standard Tikhonov regularization where the regularization parameter is chosen according to a modified discrepancy principle (cf. [**10**, Remark 2.7]) and since the discrepancy principle does not yield better rates than $D_\xi(x_\alpha^\delta, x^\dagger) = \|x_\alpha^\delta - x^\dagger\|^2 = O(\delta)$, even if the exact solution is smooth enough, also in Banach spaces this approach will most probably not yield the enhanced rates in [**6, 11**] under stronger source conditions.

Once more, we want to emphasize that rates can only be obtained under condition (A10). This condition is usually only satisfied if the exact solution is smooth enough (see next section) and, therefore, will usually never hold for solutions with jumps, unless the smoothing property of the operator F is very weak and hence the problem is only very mildly ill-posed.

3. Application to nonlinear integral equations. In this section we apply the results above to nonlinear ill-posed integral operators. For the theory of Tikhonov regularization applied to linear Fredholm integral equations of the first kind in Hilbert spaces see, e.g., [**5**].

As Banach spaces we choose $X = W_0^{1,q}[0,1]$ and $Y = L^r[0,1]$, $1 < q \leq 2$, $1 < r < \infty$, with norms

$$\|x\|_X := \left(\int_0^1 |x'(s)|^q \, ds \right)^{1/q}$$

and

$$\|y\|_Y := \left(\int_0^1 |y(t)|^r \, dt \right)^{1/r}.$$

We consider nonlinear operators $F : X \to Y$ defined by

$$F(x)(t) := \int_0^1 k(s,t,x(s)) \, ds.$$

The kernel k is assumed to be differentiable with respect to the third variable (this differentiation is denoted by the operator D_3) satisfying

$$|D_3 k(s,t,u_1) - D_3 k(s,t,u_2)| \leq g_1(s,t) g_2(c) |u_1 - u_2| \qquad \text{for all } |u_1|, |u_2| \leq c,$$

where $g_1 \in L^r([0,1]^2)$ and $g_2 : \mathbf{R}_0^+ \to \mathbf{R}_0^+$ is monotonically increasing.

In the Tikhonov functional (2.1) we choose $p \geq 1$ and $R(x) := (1/q)\|x - x_*\|_X^q$.

To be able to apply Theorem 2.5 we have to check if conditions (A1)–(A10) are satisfied. It is easy to show that conditions (A1)–(A4), (A6), (A8), and (A9) hold. F even satisfies a stronger condition than (A3), it maps weakly into strongly convergent sequences. If we assume that $F(x) = y$ has a solution and that $F(x_*) \neq y$, then also (A5) and (A7) hold.

We turn to condition (A10): first of all note that F is Frechét-differentiable with derivative

$$(F'(x)h)(t) = \int_0^1 D_3 k(s, t, x(s))h(s)\, ds.$$

Moreover,

$$\|F(z) - F(x) - F'(x)(z - x) \leq \frac{1}{2}\|g_1\|_{L^r}\, g_2(c)\, \|z - x\|_{L^\infty}^2 \qquad \text{for all } \|x\|_{L^\infty}, \|z\|_{L^\infty} \leq c.$$

For all elements $x \in X$ the subdifferential $\partial R(x)$ consists of one single element and it holds that (see [**14**, Example 10.27])

$$\partial R(x) = -(|x' - x'_*|^{q-1}\operatorname{sgn}(x' - x'_*))' \in (W_0^{1,q})^*.$$

Noting that for $q \leq 2$ (see, e.g., [**11**])

$$\begin{aligned} D_\xi(z, x) &= \frac{1}{q}\|z'\|_{L^q}^q - \frac{1}{2}\|x'\|_{L^q}^q - \langle |x'|^{q-1}\operatorname{sgn}(x'), z' - x' \\ &\geq c\|z' - x'^2_{L^q} = c\|z - x\|^2 \end{aligned}$$

for some $c \geq 0$ and that $\|z - x\| \geq \|z - x\|_{L^\infty}$, we may conclude from Remark 2.1 that condition (A10) is satisfied if

$$\begin{gathered} \xi = \partial R(x^\dagger) = F'(x^\dagger)^\# w, \\ w \in Y^* = L^{r_*}[0, 1] \quad \frac{1}{r} + \frac{1}{r_*} = 1, \end{gathered}$$

with $\|w\|$ sufficiently small or equivalently that

$$\text{(3.1)} \quad -(|(x^\dagger)'(s) - x'_*(s)|^{q-1}\operatorname{sgn}((x^\dagger)'(s) - x'_*(s)))' = \int_0^1 D_3 k(s, t, x^\dagger(s)))w(t)\, dt.$$

We give an interpretation of this condition for a specific choice of k, namely for

$$k(s,t,x) := \overline{k}(s,t)\,x^3 \quad \text{with} \quad \overline{k}(s,t) := \begin{cases} s(1-t) & s \le t, \\ t(1-s) & s \ge t. \end{cases}$$

The operator F with this kernel is a so-called Hammerstein integral operator. Noting that $Kw \in W_0^{1,r_*} \cap W^{2,r_*}$ and $(Kw)'' = -w$, where

$$Kw(s) = \int_0^1 \overline{k}(s,t)w(t)\,dt,$$

condition (3.1) is then equivalent to

$$\begin{aligned} a \in W^{2,q}, &\qquad a'(0) = 0 = a'(1), \\ \frac{a'}{(x^\dagger)^2} \in W^{2,r_*}, &\qquad \frac{a'(0)}{(x^\dagger(0))^2} = 0 = \frac{a'(1)}{(x^\dagger(1))^2}, \\ w = \left(\frac{a'}{3(x^\dagger)^2}\right)'', &\qquad a = |(x^\dagger)' - x_*'|^{q-1}\mathrm{sgn}\,((x^\dagger)' - x_*'). \end{aligned}$$

This means that condition (3.1) can only be satisfied if $x^\dagger$ and x_* are sufficiently smooth and if x_* approximates $x^\dagger$ asymptotically well in points where $x^\dagger$ is 0.

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