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A CLASS OF INTEGRAL EQUATIONS AND INDEX TRANSFORMATIONS RELATED TO THE MODIFIED AND INCOMPLETE BESSEL FUNCTIONS

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ABSTRACT. We consider a parametric family of integral equations of the first kind, which can be treated as index transformations and generalize classical Kontorovich-Lebedev transformation and related operators. The kernel of these equations is associated with the modified and incomplete Bessel functions and their derivatives with respect to an index. For certain kernels general solutions are found by using Sneddon's operational proof of the inversion formula for the Kontorovich-Lebedev transformation.

1. Introduction. Let f, g be complex-valued measurable functions defined on $\mathbf{R}_+$ and $\mathbf{R}$, respectively. We will deal here with the following integral equation of the first kind

$$\int_0^\infty S_n(x,y) f(x) \frac{dx}{x} = g(y), \quad y \in \mathbf{R}, \tag{1.1}$$

and the kernel $S_n(x,y)$ is defined by the formula

$$S_n(x,y) = \frac{1}{2} \int_{-\infty}^\infty e^{-x \cosh u} P_n(u) e^{iuy}\, du, \quad n \in \mathbf{N}_0. \tag{1.2}$$

Here $P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ is a polynomial of degree $n \in \mathbf{N}_0$ with real-valued coefficients. A function $g(y)$ in (1.1) is given

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and $f(x)$ is to be determined. When $n = 0$, $a_0 = 1$ we invoke [**5**, Vol. I, relation (2.4.18.4)] or [**6**, (6-1-2)] and we come immediately to the value $S_0(x,y) = K_{iy}(x)$, where $K_{iy}(x)$ is the modified Bessel function of the second kind [**1**] with respect to the pure imaginary index iy given by the integral

$$K_{iy}(x) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x\cosh u + iyu}\, du. \tag{1.3}$$

Hence equation (1.1) becomes

$$\int_0^\infty K_{iy}(x) f(x) \frac{dx}{x} = g(y), \tag{1.4}$$

and it is called the Kontorovich-Lebedev integral equation or transformation (cf. [**6, 7, 8, 10**]). As is known, the modified Bessel function $K_\nu(z)$ satisfies the differential equation

$$z^2 \frac{d^2u}{dz^2} + z\frac{du}{dz} - (z^2 + \nu^2)u = 0, \tag{1.5}$$

for which it is the solution that remains bounded as z tends to infinity on the real line. It has the asymptotic behavior (see [**1**, relations (9.6.8), (9.6.9), (9.7.2)])

$$K_\nu(z) = \left(\frac{\pi}{2z}\right)^{1/2} e^{-z}[1 + O(1/z)], \quad z \to \infty, \tag{1.6}$$

and near the origin

$$K_\nu(z) = O\left(z^{-|\mathrm{Re}\nu|}\right), \quad z \to 0, \tag{1.7}$$

$$K_0(z) = -\log z + O(1), \quad z \to 0. \tag{1.8}$$

Kernel (1.2) generates two more kernels, which will be studied below, namely its real and imaginary parts written as

$$\mathrm{Re}\, S_n(x,y) = \int_0^\infty e^{-x\cosh u} P_{n_e}(u) \cos uy\, du, \tag{1.9}$$

$$\mathrm{Im}\, S_n(x,y) = \int_0^\infty e^{-x\cosh u} P_{n_o}(u) \sin uy\, du, \tag{1.10}$$

where $P_{n_e}(u)$, $P_{n_o}(u)$ are even and odd parts of the polynomial $P_n(u) = P_{n_e}(u) + P_{n_o}(u)$.

We will extend and motivate Sneddon's formal operational proof [**6**, Ch. 6] of the inversion for the Kontorovich-Lebedev transformation (1.4) on integral equations of type (1.1) with certain kernels (1.2). For instance, we will use it to find a solution of the following integral equation

$$\int_0^\infty M_{iy}(x) f(x) \frac{dx}{x} = g(y), \quad y > 0, \tag{1.11}$$

where $M_{iy}(x)$ is a special function related to Bessel's functions and represented by the integral [**4, 9**]

$$M_{iy}(x) = \int_0^\infty e^{-x\cosh u} \sin yu \, du, \quad x > 0, \tag{1.12}$$

which occurs in calculation of impedance. We will also show below that general kernels (1.9), (1.10) are connected in some sense with the incomplete Bessel functions (see, for instance, in [**3**]). In particular, based on the modified Bessel function $K_\nu(z)$ according to [3] it has

$$K_\nu(z, \omega) = K_\nu(z) - J(z, \nu, \omega), \tag{1.13}$$

where

$$J(z, \nu, \omega) = \int_0^\omega e^{-z\cosh u} \cosh \nu u du, \quad \mathrm{Re} z > 0. \tag{1.14}$$

Our main goal is to find sufficient solvability conditions for the integral equation (1.1) and describe its solutions applying the Sneddon operational method. It is based in turn, on the use of the operational properties for the Fourier and sine Fourier transforms

$$(Ff)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^\infty f(t) e^{itx} \, dt, \tag{1.15}$$

$$(F_s f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(t) \sin tx \, dt, \tag{1.16}$$

and the Laplace transform

$$(Lf)(x) = \int_0^\infty e^{-xt} f(t)\, dt. \tag{1.17}$$

2. Properties of the kernel $S_n(x,y)$. Let us consider some elementary properties of the kernel (1.2), which will be employed in the sequel. First we observe that integral (1.2) and its derivatives of any order with respect to y are convergent absolutely and uniformly by $y \in \mathbf{R}$. Thus $S_n(x,y)$ is an infinitely differentiable function with respect to y and we easily have

(2.1)

$$\frac{\partial^k}{\partial y^k} S_n(x,y) = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x\cosh u}(iu)^k P_n(u) e^{iuy}\, du, \quad k \in \mathbf{N}_0, \quad x > 0.$$

Meanwhile, taking (1.3) we differentiate it similarly with respect to y to get for each $m \in \mathbf{N}_0$ and $x > 0$

$$\frac{\partial^m}{\partial y^m} K_{iy}(x) = \frac{1}{2}\int_{-\infty}^{\infty} (iu)^m e^{-x\cosh u + iyu}\, du. \tag{2.2}$$

Therefore kernels $S_n(x,y)$, $K_{iy}(x)$ belong to the space $C^m(\mathbf{R})$ of functions, which are m times continuously differentiable by y for any $m \in \mathbf{N}_0$. Moreover, it is not difficult to obtain the following representations of the kernel (1.2) and its derivatives

$$S_n(x,y) = \sum_{m=0}^{n} (-i)^m a_m \frac{\partial^m}{\partial y^m} K_{iy}(x), \tag{2.3}$$

$$\frac{\partial^k}{\partial y^k} S_n(x,y) = \sum_{m=0}^{n} (-i)^m a_m \frac{\partial^{m+k}}{\partial y^{m+k}} K_{iy}(x). \tag{2.4}$$

Furthermore, for all $k \in \mathbf{N}_0$ we deduce from (2.1) the uniform estimate with respect to y

(2.5)
$$\begin{aligned}\left|\frac{\partial^k}{\partial y^k} S_n(x,y)\right| &\le \frac{1}{2}\int_{-\infty}^{\infty} e^{-x\cosh u}|u|^k |P_n(u)|\,du \\ &\le e^{-x}\int_0^{\infty} e^{-2x\sinh^2(u/2)} u^k(|P_{n_e}(u)| + |P_{n_o}(u)|)\,du \\ &\le e^{-x}\int_0^{\infty} e^{-(xu^2/2)} u^k(|P_{n_e}(u)| + |P_{n_o}(u)|)\,du \\ &\le \frac{e^{-x}}{2}\sum_{m=0}^{n} |a_m| \left(\frac{x}{2}\right)^{-(m+k+1)/2} \Gamma\left(\frac{m+k+1}{2}\right), \\ &\qquad x > 0,\end{aligned}$$

where $\Gamma(z)$ is the Euler gamma-function [**1**]. On the other hand, we observe that, for each $x > 0$, the integrand in (2.2) is an entire function. Therefore, in the closed strip $\{z \in \mathbf{C},\ |\operatorname{Im} z| \le \delta\}$ where $\delta \in (0, (\pi/2))$, taking into account that

$$\lim_{|\operatorname{Re} z|\to\infty} \int_0^{\pm\delta} (i(\operatorname{Re} z + i\beta))^m e^{-x\cosh(\operatorname{Re} z + i\beta) + iy(\operatorname{Re} z + i\beta)}\,d\beta = 0,$$

we can shift the contour of integration in (2.2) along the upper (lower) half-strip when $y > 0$ ($y < 0$). Consequently we obtain,

$$\frac{\partial^m}{\partial y^m} K_{iy}(x) = \frac{1}{2}\int_{\pm i\delta-\infty}^{\pm i\delta+\infty} (iz)^m e^{-x\cosh z + iyz}\,dz \tag{2.6}$$

and

(2.7)
$$\begin{aligned}\left|\frac{\partial^m}{\partial y^m} K_{iy}(x)\right| &\le \frac{e^{-\delta|y|}}{2}\int_{-\infty}^{\infty} (|u| + |\delta|)^m e^{-x\cos\delta\cosh u}\,du \\ &< e^{-\delta|y| - x\cos\delta}\int_0^{\infty}\left(u + \frac{\pi}{2}\right)^m e^{-(xu^2\cos\delta)/2}\,du \\ &= e^{-\delta|y| - x\cos\delta} \\ &\times \left(\int_0^{\pi/2} + \int_{\pi/2}^{\infty}\right)\left(u + \frac{\pi}{2}\right)^m e^{-(xu^2\cos\delta)/2}\,du\end{aligned}$$

$$\leq \frac{1}{2} e^{-\delta|y|-x\cos\delta} \Big[\pi^{m+1} + 2^{(3m+1)/2} \times (x\cos\delta)^{-(m+1)/2} \Gamma\Big(\frac{m+1}{2}\Big)\Big],$$

for any $x > 0$, $y \in \mathbf{R}$, $0 \leq \delta < (\pi/2)$. Thus, returning to (2.4), we have accordingly the estimate

(2.8)
$$\left|\frac{\partial^k}{\partial y^k} S_n(x,y)\right| \leq \sum_{m=0}^{n} \left| a_m \frac{\partial^{m+k}}{\partial y^{m+k}} K_{iy}(x)\right| \leq \frac{1}{2} e^{-\delta|y|-x\cos\delta} \sum_{m=0}^{n} |a_m| \Big[\pi^{m+k+1} + 2^{(3(m+k)+1)/2} \times (x\cos\delta)^{-(m+k+1)/2} \Gamma\Big(\frac{m+k+1}{2}\Big)\Big],$$

where $x > 0$, $y \in \mathbf{R}$, $0 \leq \delta < (\pi/2)$, $k \in \mathbf{N}_0$. Meanwhile, the kernel (1.2) can be treated as the Fourier transform of the absolutely integrable function $e^{-x\cosh u} P_n(u) \in L_1(\mathbf{R})$ for each $x > 0$. Moreover, fixing δ from the interval $(0, \pi/2)$ we see that $S_n(x,y) \in L_1(\mathbf{R})$ by y via (2.8). Thus inverting the Fourier transform we obtain the equality

$$e^{-x\cosh u} P_n(u) = \frac{1}{\pi} \int_{-\infty}^{\infty} S_n(x,y) e^{-iuy}\, dy. \tag{2.9}$$

For kernels (1.9), (1.10) we have similarly their cosine and sine Fourier transformations, respectively,

(2.10)
$$\sqrt{\frac{2}{\pi}} \int_0^\infty \operatorname{Re} S_n(x,y) \cos uy\, dy = \sqrt{\frac{\pi}{2}} e^{-x\cosh u} P_{n_e}(u),$$

(2.11)
$$\sqrt{\frac{2}{\pi}} \int_0^\infty \operatorname{Im} S_n(x,y) \sin uy\, dy = \sqrt{\frac{\pi}{2}} e^{-x\cosh u} P_{n_0}(u).$$

Finally in this section we represent functions (1.9), (1.10) in terms of the incomplete Bessel functions. Precisely, with the interchange of the order of integration via Fubini's theorem we get plainly the representations

(2.12)
$$\operatorname{Re} S_n(x,y) = \int_0^\infty \frac{dP_{n_e}(\omega)}{d\omega} K_{iy}(x,\omega)\, d\omega + P_{n_e}(0) K_{iy}(x),$$

(2.13)
$$\operatorname{Im} S_n(x,y) = \int_0^\infty \frac{dP_{n_0}(\omega)}{d\omega} M_{iy}(x,\omega)\, d\omega,$$

where $K_{iy}(x,\omega)$ is defined by the integral (see (1.13), (1.14))

$$K_{iy}(x,\omega) = \int_\omega^\infty e^{-x\cosh u} \cos uy\, du,$$

and $M_{iy}(x,\omega)$ is an incomplete M-function (1.12)

$$M_{iy}(x,\omega) = \int_\omega^\infty e^{-x\cosh u} \sin uy\, du.$$

3. Sneddon's operational solutions of the equations (1.4), (1.11). In this section we will motivate the Sneddon operational method [**6**, Ch. 6] to find a solution of the Kontorovich-Lebedev integral equation (1.4) and will solve in a similar manner equation (1.11). To do this we will formulate sufficient conditions on the right-hand sides of equations (1.4), (1.11) in order to seek solutions in a certain class and guarantee their existence and uniqueness.

By $L_1(\Omega;\rho(x)\,dx)$ we denote, as usual, the Banach space of summable functions with respect to the Lebesgue measure $\rho(x)\,dx$ and the norm

$$||f|| = \int_\Omega |f(x)|\rho(x)\, dx < \infty.$$

We will say that $f(x)$ belongs to the class $\mathcal{L}_m(\mathbf{R}_+) \subset L_1(\mathbf{R}_+)$ if f satisfies the condition

$$f \in L_1\left((0,1); x^{-(m+3)/2}\, dx\right) \cap L_1\left((1,\infty); dx\right), \quad m \in \mathbf{N}_0.$$

We will prove the following

Theorem 1. *Let* $f \in \mathcal{L}_m(\mathbf{R}_+)$. *Then the right-hand side of the Kontorovich-Lebedev integral equation* (1.4) $g(y)$ *is necessarily even and belongs to* $C^m(\mathbf{R})$. *Moreover, if* $g(y) \in L_1(\mathbf{R}; |y|e^{(\pi-\delta)|y|}\,dy)$, $\delta \in (0, \pi/2)$ *and satisfies the condition*

(3.1)
$$\max_{k=0,1,\dots,[(m+1)/4]} \int_0^\infty e^{[(m+1)/2]x} \left| \int_{-\infty}^\infty \frac{ye^{iy(x+\log 2)}}{\Gamma(k-iy+1)} g(y)\,dy \right| dx < \infty,$$

then there exists a unique solution of equation (1.4) *in the class* $\mathcal{L}_m(\mathbf{R}_+)$ *given by the formula*

$$f(x) = \frac{1}{\pi i} \int_{-\infty}^\infty y I_{-iy}(x) g(y)\,dy. \tag{3.2}$$

Proof. Indeed, the right-hand side $g(y)$ of equation (1.4) is even (see (1.3)) and belongs to $C^m(\mathbf{R})$, $m \in \mathbf{N}_0$. The latter fact can be shown by using (2.7) and the possibility to differentiate under the integral sign in (1.4). Precisely, this is because of the estimates

$$\begin{aligned}
|g^{(m)}(y)| &\le \int_0^\infty \left| \frac{\partial^m K_{iy}(x)}{\partial y^m} f(x) \right| \frac{dx}{x} \le \frac{1}{2} e^{-\delta|y|} \\
&\quad \times \left(\int_0^1 + \int_1^\infty \right) \Bigg[\pi^{m+1} + 2^{(3m+1)/2} (x\cos\delta)^{-(m+1)/2} \\
&\qquad\qquad \times \Gamma\left(\frac{m+1}{2}\right) \Bigg] e^{-x\cos\delta} |f(x)| \frac{dx}{x} \\
&\le C_{m,\delta} e^{-\delta|y|} \left(\int_0^1 x^{-(m+3)/2} |f(x)|\,dx + \int_1^\infty |f(x)|\,dx \right) < \infty,
\end{aligned}$$

where $C_{m,\delta} > 0$ is a constant. Furthermore, we get that $g \in L_1(\mathbf{R})$. Therefore, applying the Fourier transform (1.15) to both sides of (1.4) we invert the order of integration via the estimate above. The inner integral is calculated as the result of the inverse Fourier transform, which is taken in (1.3). Thus we arrive at the equality

$$\int_0^\infty e^{-x\cosh u} f(x) \frac{dx}{x} = \frac{1}{\pi} \int_{-\infty}^\infty g(y) e^{-iyu}\,dy. \tag{3.3}$$

Making an elementary substitution $\cosh u = p$ in (3.3) it becomes

$$\int_0^\infty e^{-px} f(x) \frac{dx}{x} = \frac{1}{\pi} \int_{-\infty}^\infty g(y) e^{-iy \log\left(p+\sqrt{p^2-1}\right)}\, dy, \quad p > 1. \tag{3.4}$$

But $g \in L_1(\mathbf{R}; y\, dy)$ too. This circumstance and the condition $f \in \mathcal{L}_m(\mathbf{R}_+)$ give the possibility to differentiate through in (3.4) with respect to p. Hence we obtain

$$\int_0^\infty e^{-px} f(x)\, dx = \frac{i}{\pi} \int_{-\infty}^\infty y g(y) \frac{e^{-iy \log(p+\sqrt{p^2-1})}}{\sqrt{p^2-1}}\, dy. \tag{3.5}$$

Now we appeal to the value of the integral (see [**5**, Vol. II, relation (2.15.3.1)])

$$\frac{e^{-iy \log(p+\sqrt{p^2-1})}}{\sqrt{p^2-1}} = \int_0^\infty e^{-px} I_{iy}(x)\, dx, \quad p > 1, \tag{3.6}$$

where $I_{iy}(t)$ is the modified Bessel function of the first kind [**1**], which is represented by the series

$$I_{iy}(x) = \sum_{n=0}^\infty \frac{(x/2)^{2n+iy}}{n!\Gamma(n+iy+1)}. \tag{3.7}$$

Substituting (3.6) into (3.5), we change the order of integration in the right-hand side of (3.5) to obtain

$$\int_0^\infty e^{-px} f(x)\, dx = \int_0^\infty e^{-px} \frac{1}{\pi i} \int_{-\infty}^\infty y I_{-iy}(x) g(y)\, dy\, dx. \tag{3.8}$$

Canceling the Laplace transformation (1.17) in both sides of (3.8) by virtue of the uniqueness theorem for the Laplace transform of summable functions [**2**] we arrive at the unique solution (3.2) of the Kontorovich-Lebedev integral equation (1.4) in the L_1-sense, i.e., for almost all $x > 0$. Our goal now is to motivate the change of the integration order in the right-hand side of (3.8) and to justify the fact that our solution is indeed from the class $\mathcal{L}_m(\mathbf{R}_+)$. To do this we begin to estimate the modified Bessel function (3.7). We deduce

$$
\begin{aligned}
|I_{iy}(x)| &\le \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{n!|\Gamma(n+iy+1)|} \\
&= \frac{1}{|\Gamma(1/2+iy)|} \sum_{n=0}^{\infty} \frac{(x/2)^{2n}|B(iy+1/2,n+1/2)|}{n!\Gamma(n+1/2)} \\
&\le \frac{\sqrt{\pi}}{|\Gamma(1/2+iy)|} + \sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{(n!)^2} \\
&\le \frac{\sqrt{\pi}}{|\Gamma(1/2+iy)|} \left(\sum_{n=0}^{\infty} \frac{(x/2)^{n}}{n!} \right)^2 \\
&= e^{x}\sqrt{\cosh \pi y} \le e^{x+\pi|y|/2}.
\end{aligned}
$$

We note here that $B(a,b)$ is the Euler beta-function [**1**]. Consequently, since $g \in L_1(\mathbf{R}; |y|e^{(\pi-\delta)|y|}\,dy)$, $\delta \in (0,\pi/2)$ and the following iterated integral is absolutely convergent

$$
\begin{aligned}
&\int_0^\infty e^{-px} \int_{-\infty}^{\infty} |yg(y)I_{iy}(x)|\,dy\,dx \\
&\qquad \le \int_0^\infty e^{-(p-1)x}\,dx \int_{-\infty}^{\infty} |y|e^{\pi|y|/2}|g(y)|\,dy \\
&\qquad < \int_0^\infty e^{-(p-1)x}\,dx \int_{-\infty}^{\infty} |y|e^{(\pi-\delta)|y|}|g(y)|\,dy < \infty,\ p>1,
\end{aligned}
$$

we justify immediately the interchange of the order of integration in the right-hand side of (3.8) by the Fubini theorem.

In order to complete the proof of the theorem we show finally that solution (3.2) belongs to the class $\mathcal{L}_m(\mathbf{R}_+)$. But first we write it in a different form. Making use of the formula [**1**], [**5**, Vol. 2]

$$
K_{iy}(x) = \frac{\pi}{2i\sinh \pi y}\left[I_{-iy}(x) - I_{iy}(x)\right],
$$

and the evenness of g we find the classical form of the solution (3.2) for the Kontorovich-Lebedev integral equation (1.4) (cf. [**6, 7, 10**])

$$
f(x) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} y \sinh \pi y K_{iy}(x) g(y)\,dy. \tag{3.9}
$$

Hence via (2.7) it follows

$$|f(x)| < C_{m,\delta}\ e^{-x\cos\delta} \int_{-\infty}^{\infty} |y| e^{(\pi-\delta)|y|} |g(y)|\, dy < \infty, \quad x \geq 1,$$

where $C_{m,\delta} > 0$ is a constant and $\delta \in (0, \pi/2)$. Therefore $f(x) \in L_1((1,\infty); dx)$. On the other hand, returning to the form (3.2) for the solution we substitute (3.7) into the integral and split it in two sums. Hence with elementary substitutions we deduce

$$\begin{aligned} (3.10)\quad & \int_0^1 x^{-(m+3)/2} |f(x)|\, dx \\ & \leq \frac{1}{\pi} \int_0^1 x^{-(m+3)/2} \left| \sum_{n=0}^{[(m+1)/4]} \frac{(x/2)^{2n}}{n!} \int_{-\infty}^{\infty} \frac{y e^{iy\log(2/x)}}{\Gamma(n-iy+1)} g(y)\, dy \right| dx \\ & \quad + \frac{1}{\pi} \int_0^1 x^{-(m+3)/2} \left| \int_{-\infty}^{\infty} \sum_{n=[(m+1)/4]+1}^{\infty} \frac{(x/2)^{2n} y e^{iy\log(2/x)}}{n!\Gamma(n-iy+1)} g(y)\, dy \right| dx \\ & = I_1 + I_2. \end{aligned}$$

However (see (3.1)),

$$\begin{aligned} I_1 & \leq \frac{1}{\pi} \sum_{n=0}^{[(m+1)/4]} \frac{1}{2^{2n} n!} \\ & \quad \times \int_0^{\infty} e^{([(m+1)/2]-2n)x} \left| \int_{-\infty}^{\infty} \frac{y e^{iy(x+\log 2)}}{\Gamma(n-iy+1)} g(y)\, dy \right| dx \\ & \leq A_m \max_{n=0,1,\dots,[(m+1)/4]} \int_0^{\infty} e^{[(m+1)/2]x} \\ & \quad \times \left| \int_{-\infty}^{\infty} \frac{y e^{iy(x+\log 2)}}{\Gamma(n-iy+1)} g(y)\, dy \right| dx < \infty, \end{aligned}$$

where $A_m > 0$ is a constant. Finally we estimate I_2. We have

$$I_2 \le \frac{1}{\sqrt{\pi}} \sum_{n=[(m+1)/4]+1}^{\infty} \frac{1}{2^{2n}(n!)^2} \int_0^{\infty} e^{((m+1/2)-2n)x}\, dx \times \int_{-\infty}^{\infty} |yg(y)| e^{|y|\pi/2}\, dy$$
$$= \frac{1}{\sqrt{\pi}} \sum_{n=[(m+1)/4]+1}^{\infty} \frac{1}{2^{2n}(n!)^2(2n-(m+1/2))} \int_{-\infty}^{\infty} |yg(y)| e^{|y|\pi/2}\, dy$$
$$< B_m \int_{-\infty}^{\infty} |yg(y)| e^{(\pi-\delta)|y|}\, dy, \quad \delta \in \left(0, \frac{\pi}{2}\right),$$

where $B_m > 0$ is a constant. Thus via (3.10) we confirm that solution (3.2) belongs to $L_1((0,1); x^{-(m+3/2)}\, dx)$ as well and we complete the proof of Theorem 1.

An operational solution of the equation (1.11) will be treated in the class $\mathcal{M}_m(\mathbf{R}_+)$, i.e., f satisfies the condition

$$f \in L_1\left((0,1); x^{-(m+3)/2}\, dx\right) \cap L_1\left((1,\infty); e^{-x} x^{-3/2}\, dx\right), \quad m \in \mathbf{N}_0.$$

Theorem 2. *Let $f \in \mathcal{M}_m(\mathbf{R}_+)$. Then the right-hand side of integral equation (1.11) $g(y)$ necessarily belongs to $C^m(\mathbf{R}_+)$. Moreover, if $g \in L_1(\mathbf{R}_+; ye^{\pi y/2}\, dy)$ and satisfies the condition*

$$\max_{n=0,1,\dots,[(m+1)/4]} \int_0^{\infty} e^{[(m+1)/2]x} \times \left| \int_0^{\infty} \mathrm{Re}\left[\frac{e^{iy(x+\log 2)}}{\Gamma(n-iy+1)}\right] yg(y)\, dy \right| dx < \infty, \tag{3.11}$$

then there exists a unique solution of equation (1.11) in the class $\mathcal{M}_m(\mathbf{R}_+)$ given by the formula

$$f(x) = -\frac{2}{\pi} \int_0^{\infty} y \mathrm{Re}\,[I_{iy}(x)] g(y)\, dy, \quad x > 0. \tag{3.12}$$

Proof. Differentiating in (1.12) with respect to y we easily get the estimate similar to (2.5)

(3.13)
$$\begin{aligned}\left|\frac{\partial^m}{\partial y^m} M_{iy}(x)\right| &= \left|\int_0^\infty e^{-x\cosh u} u^m \sin\left(yu + \frac{\pi}{2}m\right) du\right| \\ &\le \int_0^\infty e^{-x\cosh u} u^m \, du \\ &= e^{-x} \int_0^\infty e^{-2x\sinh^2(u/2)} u^m \, du \\ &\le e^{-x} \int_0^\infty e^{-(xu^2)/2} u^m \, du \\ &= \frac{e^{-x}}{2} \left(\frac{x}{2}\right)^{-(m+1)/2} \Gamma\left(\frac{m+1}{2}\right), \quad x > 0.\end{aligned}$$

Therefore if $f \in \mathcal{M}_m(\mathbf{R}_+)$, then we find from (1.11) and (3.13)

$$\begin{aligned}|g^{(m)}(y)| &= \left|\int_0^\infty \frac{\partial^m}{\partial y^m} M_{iy}(x) f(x) \frac{dx}{x}\right| \\ &\le \int_0^\infty \left|\frac{\partial^m}{\partial y^m} M_{iy}(x) f(x)\right| \frac{dx}{x} \\ &\le 2^{(m-1)/2} \Gamma\left(\frac{m+1}{2}\right) \int_0^\infty e^{-x} |f(x)| x^{-(m+3)/2} \, dx \\ &\le 2^{(m-1)/2} \Gamma\left(\frac{m+1}{2}\right) \left(\int_0^1 |f(x)| x^{-(m+3)/2} \, dx \right. \\ &\qquad \left. + \int_1^\infty |f(x)| e^{-x} x^{-3/2} \, dx\right) < \infty,\end{aligned}$$

which implies $g \in C^m(\mathbf{R}_+)$. But $g \in L_1(\mathbf{R}_+; ye^{\pi y/2}\, dy)$. Therefore it belongs to $L_1(\mathbf{R}_+)$. Applying the sine Fourier transform (1.16) to both sides of (1.11) we have

(3.14)
$$\begin{aligned}(F_s g)(u) &= \sqrt{\frac{2}{\pi}} \int_0^\infty g(y) \sin yu \, dy \\ &= \sqrt{\frac{2}{\pi}} \lim_{\lambda\to\infty} \int_0^\lambda \sin yu \int_0^\infty M_{iy}(x) f(x) \frac{dx}{x} \, dy.\end{aligned}$$

The change of the order of integration in the right-hand side of (3.14) is possible via the uniform convergence with respect to $y \in [0, \lambda]$ of the inner integral (see the estimates above). Hence we obtain

$$(3.15) \qquad (F_s g)(u) = \sqrt{\frac{2}{\pi}} \lim_{\lambda \to \infty} \int_0^\infty \left(\int_0^\lambda M_{iy}(x) \sin yu \, dy \right) f(x) \frac{dx}{x}.$$

But integral (1.12) converges absolutely and uniformly with respect to $y \in \mathbf{R}_+$. Therefore,

$$\int_0^\lambda M_{iy}(x) \sin yu \, dy = \frac{1}{2} \int_0^\infty e^{-x \cosh t} \left(\frac{\sin \lambda(u-t)}{u-t} - \frac{\sin \lambda(u+t)}{u+t} \right) dt.$$

Substituting the right-hand side of the latter equality into (3.15) we change the order of integration for each λ, u by Fubini's theorem. This fact can be motivated by the condition $f \in \mathcal{M}_m(\mathbf{R}_+)$, the estimate

$$\int_0^\infty e^{-x \cosh t} \left| \frac{\sin \lambda(u-t)}{u-t} - \frac{\sin \lambda(u+t)}{u+t} \right| dt \leq 2\lambda \int_0^\infty e^{-x \cosh t} \, dt = 2\lambda K_0(x),$$

where $K_0(x)$ is the modified Bessel function of the index zero, and asymptotic formulas (1.6), (1.8). Denoting by

$$(3.16) \qquad h(t) = \int_0^\infty e^{-x \cosh t} f(x) \frac{dx}{x},$$

we come out with the equality (see (3.15))

$$(3.17) \qquad \int_0^\infty g(y) \sin yu \, dy = \lim_{\lambda \to \infty} \frac{1}{2} \left[\int_0^\infty h(t) \frac{\sin \lambda(u-t)}{u-t} \, dt - \int_0^\infty h(t) \frac{\sin \lambda(u+t)}{u+t} \, dt \right].$$

However, since $f \in \mathcal{M}_m(\mathbf{R}_+)$ it follows (see (1.6), (1.8)) that $f \in L_1(\mathbf{R}_+; K_0(x)x^{-1}dx)$. Moreover,

$$|h(t)| \leq \int_0^\infty e^{-x \cosh t_0} |f(x)| \frac{dx}{x} < \infty, \; t \geq t_0 > 0,$$

which means that $h(t)$ is continuous for $t > 0$. Further, (see (1.3))

$$\int_0^\infty |h(t)|\,dt \le \int_0^\infty dt \int_0^\infty e^{-x\cosh t}|f(x)|\frac{dx}{x} = \int_0^\infty K_0(x)|f(x)|\frac{dx}{x} < \infty,$$

and this yields that $h \in L_1(\mathbf{R}_+)$. Hence

$$\lim_{\lambda\to\infty} \int_0^\infty h(t)\frac{\sin\lambda(u+t)}{u+t}\,dt = \lim_{\lambda\to\infty} \int_u^\infty h(t-u)\frac{\sin\lambda t}{t}\,dt = 0, \quad u > 0$$

by virtue of the Riemann-Lebesgue lemma. Now fixing small $\delta > 0$ and splitting the first integral, the equality (3.17) becomes

(3.18)
$$\int_0^\infty g(y)\sin yu\,dy = \lim_{\lambda\to\infty} \frac{1}{2}\left[\int_{|t-u|\le\delta} + \int_{|t-u|>\delta}\right] h(t)\frac{\sin\lambda(u-t)}{u-t}\,dt,$$

and in the same manner we obtain

$$\lim_{\lambda\to\infty} \int_{|t-u|>\delta} h(t)\frac{\sin\lambda(u-t)}{u-t}\,dt = 0, \quad u > 0.$$

Further,

$$\int_{|t-u|\le\delta} h(t)\frac{\sin\lambda(u-t)}{u-t}\,dt = \int_{-\delta}^{\delta} h(u-t)\frac{\sin\lambda t}{t}\,dt = \int_{-\delta}^{\delta} \frac{h(u-t)-h(u)}{t}\sin\lambda t\,dt + h(u)\int_{-\delta}^{\delta}\frac{\sin\lambda t}{t}\,dt.$$

But, in the meantime for each $u > 0$

$$h(u)\int_{-\delta}^{\delta}\frac{\sin\lambda t}{t}\,dt = h(u)\int_{-\delta\lambda}^{\delta\lambda}\frac{\sin t}{t}\,dt \to \pi h(u), \quad \lambda\to\infty.$$

Finally from (3.16) and the above estimates we find that $h(t) \in C^1(\mathbf{R}_+)$ and its derivative is bounded on any compact set of $\mathbf{R}_+$. Consequently, the Dini condition from the theory of Fourier integrals in L_1 is satisfied and

$$\lim_{\lambda\to\infty}\int_{-\delta}^{\delta}\frac{h(u-t)-h(u)}{t}\sin\lambda t\,dt=0.$$

Hence, passing to the limit in the right-hand side of (3.18) it takes the form

$$\frac{2}{\pi}\int_0^\infty g(y)\sin yu\,dy=\int_0^\infty e^{-x\cosh u}f(x)\frac{dx}{x}. \tag{3.19}$$

As in the proof of the previous theorem we make the substitution $\cosh u = p$ in (3.19) to write

$$\int_0^\infty e^{-px}f(x)\frac{dx}{x}=\frac{2}{\pi}\int_0^\infty g(y)\sin\left(y\log\left(p+\sqrt{p^2-1}\right)\right)\,dy,\quad p>1. \tag{3.20}$$

Since $g\in L_1(\mathbf{R};ydy)$ and $f\in\mathcal{M}_m(\mathbf{R}_+)$ we differentiate through in (3.20) with respect to p to deduce

$$\int_0^\infty e^{-px}f(x)dx=-\frac{2}{\pi}\int_0^\infty yg(y)\frac{\cos\left(y\log\left(p+\sqrt{p^2-1}\right)\right)}{\sqrt{p^2-1}}\,dy. \tag{3.21}$$

The value of integral (3.6) yields the equality

$$\frac{\cos\left(y\log\left(p+\sqrt{p^2-1}\right)\right)}{\sqrt{p^2-1}}=\int_0^\infty e^{-px}\mathrm{Re}\,[I_{iy}(x)]\,dx.$$

Substituting this into (3.21) we change the order of integration in the right-hand side of (3.21) by Fubini's theorem, motivating this as in the proof of Theorem 1. Thus we obtain

$$\int_0^\infty e^{-px}f(x)\,dx=-\int_0^\infty e^{-px}\frac{2}{\pi}\int_0^\infty yg(y)\mathrm{Re}\,[I_{iy}(x)]\,dy\,dx. \tag{3.22}$$

One can cancel the Laplace transformation (1.17) in both sides of (3.22) by virtue of the uniqueness theorem for the Laplace transform of summable functions [**2**], which is zero at least at the countable set of points $p_l=p_0+lq_0$, $p_0>1$, $q_0>0$, $l=0,1,2,\ldots$. Consequently, we arrive at the unique solution (3.12) of the integral equation (1.11) almost for all $x > 0$. In order to show that our solution is indeed from the class $\mathcal{M}_m(\mathbf{R}_+)$ we have as in the proof of Theorem 1

$$\int_1^\infty e^{-x}x^{-3/2}\int_0^\infty y\,|\mathrm{Re}[I_{iy}(x)]g(y)|\,dy\,dx \leq \int_1^\infty x^{-3/2}\,dx\int_0^\infty y|g(y)|e^{\pi y/2}\,dy < \infty.$$

Therefore $f(x) \in L_1((1,\infty); e^{-x}x^{-3/2}\,dx)$. Meanwhile, by using condition (3.11) we derive

$$\begin{aligned}
&\frac{2}{\pi}\int_0^1 x^{-(m+3)/2}\left|\int_0^\infty y\mathrm{Re}\,[I_{iy}(x)]g(y)\,dy\right|dx \\
&\quad\leq \frac{2}{\pi}\int_0^1 x^{-(m+3)/2} \\
&\qquad\times\left|\sum_{n=0}^{[(m+1)/4]}\frac{x^{2n}}{n!}\int_0^\infty y\mathrm{Re}\left[\frac{e^{iy\log(2/x)}}{\Gamma(n-iy+1)}\right]g(y)\,dy\right|dx \\
&\qquad+\frac{2}{\pi}\int_0^1 x^{-(m+3)/2}\left|\int_0^\infty\sum_{n=[(m+1)/4]+1}^\infty\frac{x^{2n}}{n!}\right. \\
&\qquad\qquad\left.\times\,\mathrm{Re}\left[\frac{e^{iy\log(2/x)}}{\Gamma(n-iy+1)}\right]g(y)y\,dy\right|dx \\
&\quad\leq C_m\max_{n=0,1,\dots,[(m+1)/4]}\int_0^\infty e^{[(m+1)/2]x} \\
&\qquad\qquad\times\left|\int_0^\infty\mathrm{Re}\left[\frac{e^{iy(x+\log 2)}}{\Gamma(n-iy+1)}\right]yg(y)\,dy\right|dx \\
&\qquad+D_m\int_0^\infty y|g(y)|e^{y\pi/2}\,dy<\infty,
\end{aligned}$$

where C_m, $D_m > 0$ are constants. Therefore solution (3.12) belongs to $L_1((0,1); x^{-(m+3)/2}\,dx)$. Theorem 2 is proved.

4. A solution of the general equation (1.1). An example. In this section we will find solutions of the equation (1.1) and its particular case in the closed form. We will seek them in the class $\mathcal{L}_{m+n}(\mathbf{R}_+)$ (see

Theorem 1), where $m \in \mathbf{N}_0$ is related to the order of the derivative of a function g and $n \in \mathbf{N}$ is defined by the kernel (1.2). We have

Theorem 3. *Let $f \in \mathcal{L}_{m+n}(\mathbf{R}_+)$ and let a polynomial $P_n(z)$ in (1.2) have no zeros in the closed horizontal strip $\mathcal{D} = \{z \in \mathbf{C} : |\mathrm{Im}\, z| \le \pi\}$. Then the right-hand side $g(y)$ of the general equation (1.1) is necessarily from $C^m(\mathbf{R})$. Moreover, if $g \in L_1(\mathbf{R}; |y|e^{(\pi-\delta)|y|}\, dy)$, $\delta \in (0, (\pi/2))$ and the Fourier convolution*

$$(g * k_n)(\tau) \equiv s(\tau) = \int_{-\infty}^{\infty} g(y) k_n(\tau - y)\, dy, \tag{4.1}$$

with

$$k_n(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{ixu}}{P_n(u)}\, du, \tag{4.2}$$

satisfies condition (3.1), where m is substituted by $m + n$, there exists a unique solution of equation (1.1) in the class $\mathcal{L}_{m+n}(\mathbf{R}_+)$ given by the formula

$$f(x) = \int_{-\infty}^{\infty} \widehat{S}_n(x, y) g(y)\, dy, \tag{4.3}$$

where the kernel $\widehat{S}_n(x, y)$ is defined as

$$\widehat{S}_n(x, y) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \tau I_{-i\tau}(x) k_n(\tau - y)\, d\tau. \tag{4.4}$$

Proof. The right-hand side $g(y)$ of equation (1.1) belongs to $C^m(\mathbf{R})$. Indeed, we use (2.8) and the condition $f \in \mathcal{L}_{m+n}(\mathbf{R}_+)$ to obtain

$$\begin{aligned} |g^{(m)}(y)| &\le \int_0^{\infty} \left| \frac{\partial^m S_n(x, y)}{\partial y^m} f(x) \right| \frac{dx}{x} \\ &\le B_{m,\delta} e^{-\delta|y|} \bigg(\int_0^1 x^{-(m+n+3)/2} |f(x)|\, dx \\ &\qquad\qquad + \int_1^{\infty} |f(x)|\, dx \bigg) < \infty, \end{aligned}$$

where $B_{m,\delta} > 0$ is a constant. Since via the latter estimate $g \in L_1(\mathbf{R})$, we apply the Fourier transform (1.15) to both sides of (1.1), and we invert the order of integration by Fubini's theorem. Hence, appealing to (2.9), and taking into account that $P_n(u)$ in (1.2) has no real zeros we come out with the equality

$$\int_0^\infty e^{-x\cosh u} f(x)\frac{dx}{x} = \frac{1}{\pi}\int_{-\infty}^\infty g(y)\frac{e^{-iyu}}{P_n(u)}\,dy. \tag{4.5}$$

Applying again the Fourier transform to both sides of (4.5), we use representation (1.3) to arrive at the Kontorovich-Lebedev integral equation like (1.4). Namely, we deduce

$$\int_0^\infty K_{i\tau}(x)f(x)\frac{dx}{x} = \frac{1}{2\pi}\int_{-\infty}^\infty du\int_{-\infty}^\infty g(y)\frac{e^{i(\tau-y)u}}{P_n(u)}\,dy. \tag{4.6}$$

But conditions of the theorem imply that n has to be even. Furthermore, the estimate

$$\int_{-\infty}^\infty du\int_{-\infty}^\infty \left|g(y)\frac{e^{i(\tau-y)u}}{P_n(u)}\right| dy \le \int_{-\infty}^\infty \frac{du}{|P_n(u)|}\int_{-\infty}^\infty |g(y)|\,dy < \infty$$

allows us to invert the order of integration in the right-hand side of the equation (4.6) due to Fubini's theorem. Consequently, it becomes

$$\int_0^\infty K_{i\tau}(x)f(x)\frac{dx}{x} = (g * k_n)(\tau), \tag{4.7}$$

where the right-hand side of (4.7) is defined by (4.1), (4.2). In the meantime, since $P_n(z)$ has no zeros in the strip $\mathcal{D} = \{z \in \mathbf{C} : |\operatorname{Im} z| \le \pi\}$, we use the Cauchy theorem to shift the contour of integration into the upper (when $x > 0$) or lower (when $x < 0$) half-strip and to establish the following estimate of kernel (4.2)

$$|k_n(x)| = \frac{1}{2\pi}\left|\int_{-\infty\pm i\pi}^{\infty\pm i\pi}\frac{e^{ixz}}{P_n(z)}\,dz\right| \le \frac{e^{-\pi|x|}}{2\pi}\int_{-\infty}^\infty \frac{1}{|P_n(u\pm i\pi)|}\,du$$
$$\le \text{const.}\, e^{-\pi|x|}.$$

Hence, as we observe from the corresponding condition (3.1) and the estimate

$$
\begin{aligned}
(4.8)\qquad \int_{-\infty}^{\infty} |y|e^{(\pi-\delta)|y|} \int_{-\infty}^{\infty} |g(t)k_n(y-t)|\,dt\,dy& \\
= \int_{-\infty}^{\infty} |g(t)| \int_{-\infty}^{\infty} |u+t|e^{(\pi-\delta)|u+t|}|k_n(u)|\,du\,dt& \\
\le \int_{-\infty}^{\infty} e^{(\pi-\delta)|t|}|g(t)| \int_{-\infty}^{\infty} (|u|+|t|)e^{(\pi-\delta)|u|}|k_n(u)|\,du\,dt& \\
\le \text{const.} \int_{-\infty}^{\infty} e^{(\pi-\delta)|t|}|g(t)|\,dt \int_{-\infty}^{\infty} |u|e^{-\delta|u|}\,du& \\
+ \text{const.} \int_{-\infty}^{\infty} |t|e^{(\pi-\delta)|t|}|g(t)|\,dt \int_{-\infty}^{\infty} e^{-\delta|u|}\,du < \infty&
\end{aligned}
$$

the right-hand side of (4.7) satisfies conditions of Theorem 1. Therefore, we get the following solution

$$
f(x) = \frac{1}{\pi i} \int_{-\infty}^{\infty} \tau I_{-i\tau}(x)(g * k_n)(\tau)\,d\tau, \tag{4.9}
$$

which belongs to the class $\mathcal{L}_{m+n}(\mathbf{R}_+)$. Substituting the convolution expression (4.1) into (4.9) and changing the order of integration, which is motivated by estimate (4.8) and Fubini's theorem, we find that solution (4.9) can be written by formula (4.3). Theorem 3 is proved. □

Remark 1. We could add in Theorem 3 the case $n = 0$, extending the definition of convolution (4.1) by considering not only regular distributions, but also convolutions with $k_0(x)$, which, in turn, represents a delta-function. Then (4.5) leads immediately to the solution of the Kontorovich-Lebedev equation (1.4).

Let us demonstrate that solution (4.3) can be represented in another form. Indeed, it is not difficult to see from (4.7) that convolution (4.1) is even with respect to τ. Hence via (3.9) we obtain from (4.7)

$$
f(x) = \frac{1}{\pi^2} \int_{-\infty}^{\infty} \tau e^{\pi\tau} K_{i\tau}(x) \int_{-\infty}^{\infty} g(y)k_n(\tau - y)\,dy\,d\tau. \tag{4.10}
$$

Changing the order of integration as above we write solution (4.10) as

$$
f(x) = \int_{-\infty}^{\infty} \widehat{\mathcal{S}}_n(x,y)g(y)\,dy, \tag{4.11}
$$

where

$$\widehat{\mathcal{S}}_n(x,y) = \frac{1}{\pi^2}\int_{-\infty}^{\infty} \tau e^{\pi\tau} K_{i\tau}(x) k_n(\tau - y)\, d\tau. \tag{4.12}$$

Meanwhile integral (4.2) can be calculated with the use of the residue theorem. Precisely, due to conditions of Theorem 3, zeros of $P_n(z)$ are situated symmetrically about the real line and out of the strip $\mathcal{D}$. Therefore $n = 2l$, $l \in \mathbf{N}$, and we have l different complex zeros $z_1, z_2, \ldots, z_l$ and their corresponding conjugates with the multiplicities $m_1, m_2, \ldots, m_l$ such that $m_1 + m_2 + \cdots + m_l = l$. Consequently, by virtue of the residue theorem

$$k_n(x) = i\sum_{p=1}^{l} \mathrm{Res}_{z=z_p,\ \mathrm{Im}\, z_p > \pi}\left[\frac{e^{ixz}}{P_n(z)}\right], \quad x \geq 0, \tag{4.13}$$

and

$$k_n(x) = i\sum_{p=1}^{l} \mathrm{Res}_{z=\overline{z_p}}\left[\frac{e^{ixz}}{P_n(z)}\right], \quad x < 0. \tag{4.14}$$

But

$$\begin{aligned}
\mathrm{Res}_{z=z_p}\left[\frac{e^{ixz}}{P_n(z)}\right] &= \frac{1}{(m_p-1)!}\lim_{z\to z_p}\frac{d^{m_p-1}}{dz^{m_p-1}}\left[\frac{(z-z_p)^{m_p}}{P_n(z)}e^{ixz}\right] \\
&= \frac{1}{(m_p-1)!}\lim_{z\to z_p} e^{ixz}\sum_{r=0}^{m_p-1}\binom{m_p-1}{r} \\
&\qquad \times \frac{d^{m_p-r-1}}{dz^{m_p-r-1}}\left[\frac{(z-z_p)^{m_p}}{P_n(z)}\right](ix)^r \\
&= e^{ixz_p}\mathcal{P}_{m_p-1}(x), \quad x \geq 0,
\end{aligned}$$

where $\mathcal{P}_{m_p-1}(x)$ is a polynomial of degree $m_p - 1$. Combining with (4.13) we get the expression for $k_n(x)$

$$k_n(x) = i\sum_{p=1}^{n/2} e^{ixz_p}\mathcal{P}_{m_p-1}(x), \quad x \geq 0, \quad \mathrm{Im}\, z_p > \pi.$$

Analogously, for case (4.14) we derive

$$k_n(x) = i\sum_{p=1}^{n/2} e^{ix\overline{z_p}}\widehat{\mathcal{P}}_{m_p-1}(x), \quad x < 0.$$

Substituting the values $k_n(x)$ into (4.12) we find the corresponding formula for the kernel of the solution (4.11).

Finally let us consider an example of equation (1.1) and its solution. Let $n = 2$ and $P_2(u) = u^2 + 4\pi^2$. Then, from (4.2), we immediately obtain $k_2(x) = (1/4\pi)e^{-2\pi|x|}$. Thus the kernel (4.13) can be written by the integral

$$\widehat{\mathcal{S}}_2(x, y) = \frac{1}{4\pi^3}\int_{-\infty}^{\infty} \tau e^{\pi(\tau-2|\tau-y|)} K_{i\tau}(x)\, d\tau. \tag{4.15}$$

In the meantime, with the integration by parts in (1.3) it is not difficult to obtain that

$$yK_{iy}(x) = x\mathrm{Im}\left[\int_0^{\infty} e^{-x\cosh u+iyu}\sinh u\, du\right].$$

Substituting the latter integral into (4.16) and changing the order of integration, we calculate an elementary inner integral and we come out with the following value of the kernel (4.15) for the corresponding solution (4.11)

$$\widehat{\mathcal{S}}_2(x, y) = \frac{xe^{\pi y}}{\pi^2}\int_0^{\infty} e^{-x\cosh u}\frac{(3\pi^2 + u^2)\sin uy + 2\pi u\cos uy}{(\pi^2 + u^2)(9\pi^2 + u^2)}\sinh u\, du.$$

On the other hand, due to (2.3), the kernel (1.2) for this case is equal to

$$S_2(x, y) = 4\pi^2 K_{iy}(x) - \frac{\partial^2}{\partial y^2}K_{iy}(x)$$

and equation (1.1) can be transformed to the ordinary differential equation in terms of the Kontorovich-Lebedev operator (1.4) denoted by $K_{iy}[f]$

$$4\pi^2 K_{iy}[f] - \frac{\partial^2}{\partial y^2}K_{iy}[f] = g(y). \tag{4.16}$$

It is easily seen that the respective homogeneous equation has nontrivial solutions. Taking into account the evenness of the transformation $K_{iy}[f]$ with respect to y we arrive at the following solutions

$$K_{iy}[f] = \lambda \cosh 2\pi y, \tag{4.17}$$

where λ is a constant. Therefore a general solution of equation (4.17) is not unique and does not belong to the class $\mathcal{L}_{m+2}(\mathbf{R}_+)$. The Kontorovich-Lebedev equation (4.17), in turn, can be solved in spaces of distributions, and its solution is given accordingly by

$$f(x) = \lim_{T\to\infty} \frac{\lambda}{2\pi^2} \int_0^T y \frac{\sinh 4\pi y}{\cosh \pi y} K_{iy}(x)\, dy,$$

where the latter limit is taken in $\mathcal{D}'(\mathbf{R}_+)$ (see [**11**]).

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