

**THE GENERALIZED EULER-MACLAURIN FORMULA  
FOR THE NUMERICAL SOLUTION OF  
ABEL-TYPE INTEGRAL EQUATIONS**

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**ABSTRACT.** An extension of the Euler-Maclaurin formula to singular integrals was introduced by Navot [12]. In this paper this result is applied to derive a quadrature rule for integral equations of the Abel type. We present a stability and convergence analysis and numerical results that are in good agreement with the theory. The method is particularly useful when combined with fast methods for evaluating integral operators.

**1. Introduction.** The generalized Abel equation is a weakly singular Volterra integral equation of the first kind which usually appears in the form

$$(1) \quad \int_0^t (t - \tau)^{-\alpha} k(t, \tau) g(\tau) d\tau = f(t).$$

Here  $0 \leq t \leq T$ ,  $0 < \alpha < 1$ , and the kernel  $k(\cdot, \cdot)$  is smooth and satisfies  $k(t, t) = 1$ . The following solvability result was given by Atkinson [1].

**Theorem 1.1.** *If  $f(t) = t^{1-\alpha-\beta} \tilde{f}(t)$  with  $\tilde{f} \in C^\infty[0, T]$  and  $\beta < 1$ , then (1) has a unique solution  $g$  which is of the form  $g(t) = t^{-\beta} \tilde{g}(t)$  and  $\tilde{g} \in C^\infty[0, T]$ .*

Numerical methods for (1) are usually based on collocation or product integration methods. This subject has been studied extensively and is reviewed in the monographs by Linz [6] and Brunner [2]. A more recent approach to discretizing (1) is the use of the convolution quadrature rules [9].

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The aforementioned discretization methods lead to a recurrence formula for the numerical approximations  $g_n$  of  $g(t_n)$  which are of the form

$$(2) \quad h^{1-\alpha} \sum_{j=1}^n w_{n,j} k(t_n, t_j) g_j = f(t_n), \quad n = 0, 1, 2, \dots,$$

where  $h$  is the stepsize,  $t_n = nh$  and  $w_{n,j}$  are quadrature weights.

The present paper examines another discretization method of (1) which is based upon the generalization of the Euler-Maclaurin expansion to integrals with algebraic singularities. The formula was first discovered by Navot [12] and further generalized in a series of papers, see e.g., [10, 11, 15]. We will use this result to obtain a discretization of (1) in the form

$$(3) \quad h \sum_{j=0}^{n-1} (t_n - t_j)^{-\alpha} k(t_n, t_j) g_j + h^{1-\alpha} \sum_{j=[n-p]_+}^n r_{n-j}^{(p)} k(t_n, t_j) g_j = f(t_n),$$

$n = 0, 1, 2, \dots$ . The first sum, or history part, is the usual composite trapezoidal rule. The second sum contains only a very small number of terms and can be considered as a correction to account for the singularity of the integrand at  $t = \tau$ . The parameter  $p$  controls the order of the method; we will show that for  $p = 0$  the order is  $O(h^{2-\alpha})$  and the scheme is stable for  $\alpha \in (0, 1)$ . When  $p = 1$  and  $p = 2$  the order is  $O(h^{3-\alpha})$  and  $O(h^{4-\alpha})$ , respectively, but the interval of  $\alpha$ 's for which the method is stable is reduced. For larger values of  $p$  the important case  $\alpha = 1/2$  is no longer stable, and therefore the resulting schemes are probably only of limited interest for solving first-kind equations.

A complication we encounter is that the remainder in the generalized Euler-Maclaurin expansion depends in a subtle way upon the interval length. To ensure that we obtain uniform bounds in  $t$  we must initially restrict the discussion to the case that the solution has sufficiently many vanishing derivatives at  $t = 0$ . To handle the general case, including singularities, we will discuss a correction of the righthand side to maintain the convergence rates. This procedure is similar to Lubich's approach to overcoming a similar issue for convolutional quadrature schemes [9]. We note however, that other techniques have been discussed to handle singularities, see e.g., [5].

The difference between the two recurrence formulas is that (2) contains quadrature weights that depend on  $n$  and  $j$ , whereas the history part in (3) does not. This simplifies matters significantly if fast summation methods are used to compute the history part in the recurrence formula efficiently. Fast methods are based on separating the  $t_n$  and  $t_j$  variables. In (3) this can be accomplished fairly easily, for instance, by Taylor expanding the kernel in (1). On the other hand, recurrence (2) involves in addition a separation of the  $n$ - and  $j$ -index of the weights, which does not seem to be practical. It is not our goal to discuss fast methods here. Instead, we refer to paper [14] where the  $p = 0$  method has been used as a time discretization for integral formulation of the heat equation.

It should be noted that the idea to use the generalized Euler-Maclaurin expansion to obtain discretizations of singular Volterra integral equations is not new, although it has received only very limited attention in the past. It appears that the first paper to use this result for Volterra equations of the second kind is Tao and Yong [13]. Later, the equation of the first kind was treated by transformation to an equation of the second kind, see [7, 8].

**2. The generalized Euler-Maclaurin expansion.** The result by Navot is a generalization of the Euler-Maclaurin formula for integrals with an algebraic singularity at one endpoint of the interval of integration [12]. Later, the formula was generalized and its derivation was simplified in various ways. Probably the most efficient tool is the Mellin transform, which was initiated by Verlinden and Haegemans [15]. The methodology in their paper applies for integrals over a semi-infinite interval, but can be easily extended to integrals over  $[0, 1]$  using neutralizer functions [11]. We will briefly recall this technique and then discuss arbitrary length intervals.

**Basic expansion for the semi-infinite interval.** The Mellin transform of a function  $\varphi$  is given by

$$\Phi(z) = \int_0^{\infty} x^{z-1} \varphi(x) dx$$

and the inverse transform is given by

$$\varphi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} x^{-z} \Phi(z) dz,$$

where  $c \in \mathbf{R}$  is chosen such that  $\Phi$  is analytic for  $\operatorname{Re}(z) \geq c$ .

Suppose  $\varphi(x) = x^{-\alpha} \tilde{\varphi}(x)$  where  $\tilde{\varphi} \in C^\infty[0, \infty)$  decays superalgebraically, that is,

$$\lim_{x \rightarrow \infty} x^k \tilde{\varphi}(x) = 0, \quad \text{for every } k \geq 0.$$

From the inverse transform it follows that

$$\begin{aligned} (4) \quad h \sum_{j=1}^{\infty} \varphi(jh) &= \frac{h}{2\pi i} \int_{c-i\infty}^{c+i\infty} \sum_{j=1}^{\infty} (jh)^{-z} \Phi(z) dz \\ &= \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \zeta(z) \Phi(z) h^{1-z} dz, \end{aligned}$$

where  $h > 0$  is the stepsize,  $c > 1$  and  $\zeta(\cdot)$  is the Riemann zeta function. Integration by parts leads to

$$(5) \quad \Phi(z) = \omega_s(z - \alpha) \int_0^\infty x^{z-\alpha+s} \tilde{\varphi}^{(s+1)}(x) dx, \quad s = 0, 1, \dots$$

where  $\omega_s(\cdot)$  is defined by

$$(6) \quad \omega_s(z) = \frac{(-1)^{s+1}}{z \cdot \dots \cdot (z+s)}.$$

The integral in (5) is an analytic function for  $\operatorname{Re}(z) > -(s+1-\alpha)$ . Furthermore,  $\operatorname{Res}(\zeta(z), z=1) = 1$ , so the integrand in (4) has residuals

$$\operatorname{Res}(\zeta(z)\Phi(z)h^{1-z}, z=1) = \Phi(1) = \int_0^\infty \varphi(x) dx$$

and

$$\operatorname{Res}(\zeta(z)\Phi(z)h^{1-z}, z=\alpha-s) = \frac{\tilde{\varphi}^{(s)}(0)}{s!} \zeta(\alpha-s) h^{s+1-\alpha}$$

for  $s = 0, 1, \dots$ . Thus we may move the integration contour in (4) into the left plane and obtain

$$(7) \quad h \sum_{j=1}^{\infty} \varphi(jh) = \int_0^{\infty} \varphi(x) dx + \sum_{s=0}^q \frac{\tilde{\varphi}^{(s)}(0)}{s!} \zeta(\alpha - s) h^{s+1-\alpha} + R^{(q)}(h)$$

for some integer  $q$ . This is the generalized Euler-Maclaurin expansion. The remainder is given by

$$(8) \quad R^{(q)}(h) = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \zeta(z) \Phi(z) h^{-z+1} dz,$$

where  $\alpha - q - 1 < c' < \alpha - q$ .

The remainder  $R^{(q)}$  can be estimated by

$$\left| \int_{c'-i\infty}^{c'+i\infty} \zeta(z) \Phi(z) h^{-z+1} dz \right| \leq \frac{h^{1-c'}}{2\pi} \int_{-\infty}^{\infty} |\zeta(c' + iy)| |\Phi(c' + iy)| dy.$$

From (5) it follows that  $|\Phi(c' + iy)|$  decays superalgebraically as  $y \rightarrow \pm\infty$ . On the other hand, it is known that for a fixed real part, the zeta function increases at most like a polynomial, see [15]. Therefore the integral on the righthand side converges and the remainder is  $O(h^{1-c'})$ .

**Expansion for the interval  $[0, t]$ .** We now derive a similar expansion for integrals of the type that appear in (1). Of course, the form of the generalized Euler-Maclaurin expansion for a finite interval is well known. However, in Volterra integral equations, the length is time dependent and enters the constants in the estimates. We will discuss this dependence in the following.

It turns out that we must assume that the solution is of the form  $g(\tau) = \tau^\gamma \tilde{g}(\tau)$  where  $\tilde{g}$  is smooth in  $[0, T]$  and  $\gamma$  is sufficiently large to ensure that the estimates are independent of time. Later, it will become clear that  $\gamma = p + 2 - \alpha$ , where  $p$  is the order of the method, defined in (3). By Theorem 1.1 this is equivalent to limiting the discussion to righthand sides of the form  $f(t) = t^{1-\alpha+\gamma} \tilde{f}(t)$ . Section 5 will describe a modified method that maintains its convergence properties for general, even singular solutions.

To simplify notations, we will write

$$\varphi(t, \tau) = k(t, \tau)g(\tau) \text{ and } \varphi(t, \tau) = \tau^\gamma \tilde{\varphi}(t, \tau)$$

where  $\tilde{\varphi} \in C^\infty[0, T]$  and  $\gamma > 0$  is a not yet specified parameter. Furthermore, we introduce a neutralizer function  $\chi(x)$  with the properties

$$(9) \quad \chi \in C^\infty(\mathbf{R}), \quad \chi(x) = 1, \quad x \leq \frac{1}{3}, \quad \chi(x) = 0, \quad x \geq \frac{2}{3}$$

and, in addition,

$$(10) \quad \chi(1-x) = 1 - \chi(x).$$

An example of such a function is

$$\chi(x) = \frac{1}{2} \left\{ 1 + \tanh \left( \frac{1}{x - (1/3)} - \frac{1}{(2/3) - x} \right) \right\}, \quad \frac{1}{3} < x < \frac{2}{3}.$$

Using these definitions the integrand in (1) can be written as the sum of two functions,

$$(t - \tau)^{-\alpha} \varphi(t, \tau) = \tau^\gamma \tilde{\phi}_0(t, \tau) + (t - \tau)^{-\alpha} \tilde{\phi}_1(t, t - \tau)$$

where

$$\tilde{\phi}_0(t, x) = \chi\left(\frac{x}{t}\right) (t - x)^{-\alpha} \tilde{\varphi}(t, x),$$

$$\tilde{\phi}_1(t, x) = \chi\left(\frac{x}{t}\right) (t - x)^\gamma \tilde{\varphi}(t, t - x).$$

Furthermore, we set  $\varphi_0(t, x) = x^\gamma \tilde{\phi}_0(t, x)$  and  $\varphi_1(t, x) = x^{-\alpha} \tilde{\phi}_1(t, x)$ . All these functions can be trivially extended to the semi-infinite interval  $x \geq 0$ . With these definitions it follows that

$$\int_0^t (t - \tau)^{-\alpha} \varphi(t, \tau) d\tau = \int_0^\infty \varphi_0(t, x) dx + \int_0^\infty \varphi_1(t, x) dx$$

and

$$\sum_{j=1}^{n-1} (t_n - t_j)^{-\alpha} \varphi(t_n, t_j) = \sum_{j=1}^{\infty} \varphi_0(t_n, t_j) + \sum_{j=1}^{\infty} \varphi_1(t_n, t_j).$$

We can apply the generalized Euler-Maclaurin formula to both sums and combine the expansions. The definitions of functions  $\tilde{\varphi}_1$  and  $\varphi$  imply that

$$\frac{\partial^s \tilde{\varphi}_1}{\partial x^s}(t, 0) = \frac{\partial^s \varphi}{\partial \tau^s}(t, t)(-1)^s.$$

If  $\gamma > q$  the derivatives up to order  $q$  of  $\phi_0(t, x)$  vanish for  $x = 0$ . Thus, the function will not contribute to the residuals and will only appear in the remainder.

(11)

$$\begin{aligned} h \sum_{j=1}^{n-1} (t_n - t_j)^{-\alpha} \varphi(t_n, t_j) &= \int_0^{t_n} (t_n - \tau)^{-\alpha} \varphi(t_n, \tau) d\tau \\ &+ \sum_{s=0}^q \frac{(-1)^s}{s!} \frac{\partial^s \varphi}{\partial \tau^s}(t_n, t_n) \zeta(\alpha - s) h^{s+1-\alpha} \\ &+ R_E^{(q)}(t_n, h), \end{aligned}$$

where the remainder is given by

$$(12) \quad R_E^{(q)}(t, h) = \frac{1}{2\pi i} \int_{c'-i\infty}^{c'+i\infty} \zeta(z) \left( \Phi_0(t, z) + \Phi_1(t, z) \right) h^{-z+1} dz$$

and  $\Phi_0(t, z)$  and  $\Phi_1(t, z)$  are the Mellin transforms of  $\phi_0(t, x)$  and  $\phi_1(t, x)$  with respect to the  $x$ -variable.

**Lemma 2.1.** *If  $g(t) = t^\gamma \tilde{g}(t)$  with  $\tilde{g} \in C^\infty[0, T]$ , then there is a smooth function  $I_E(t, h)$  such that the remainder is given by*

$$R_E^{(q)}(t, h) = t^{\gamma+c'-\alpha} h^{1-c'} I_E(t, h), \quad -q - 1 + \alpha < c' < -q + \alpha.$$

*Proof.* As in (5) we obtain from integration by parts

$$\Phi_1(t, z) = \omega_s(z - \alpha) \int_0^\infty x^{z-\alpha+s} \left( \frac{\partial}{\partial x} \right)^{s+1} \tilde{\varphi}_1(t, x) dx.$$

Changing variables  $x \rightarrow x/t$  and sorting out the powers of  $t$  show that

$$\Phi_1(t, z) = t^{z-\alpha+\gamma} \omega_s(z - \alpha) I_1^{(s)}(t, z),$$

where

$$I_1^{(s)}(t, z) = \int_0^{2/3} x^{z-\alpha+s} \left( \frac{\partial}{\partial x} \right)^{s+1} \chi(x)(1-x)^\gamma \tilde{\varphi}(t, t(1-x)) dx.$$

Likewise, we find for  $\Phi_0$  that

$$\Phi_0(t, z) = t^{z-\alpha+\gamma} \omega_s(z+\gamma) I_0^{(s)}(t, z),$$

where

$$I_0^{(s)}(t, z) = \int_0^{2/3} x^{z+\gamma+s} \left( \frac{\partial}{\partial x} \right)^{s+1} \chi(x)(1-x)^{-\alpha} \tilde{\varphi}(t, tx) dx.$$

By the properties of parameter-dependent integrals the functions  $I_0^{(s)}(t, z)$  and  $I_1^{(s)}(t, z)$  are smooth in  $t$  and in  $z$ . Furthermore, they are bounded in  $t \in [0, T]$  and  $\text{Im}(z)$ , when  $\text{Re}(z)$  is fixed. Hence, it follows from (12) that

$$\begin{aligned} R_E^{(q)}(t, h) &= \frac{1}{2\pi i} t^{\gamma+c'-\alpha} h^{1-c'} \int_{c'-i\infty}^{c'+i\infty} \zeta(z) \\ &\quad \times \left( \omega_s(z+\gamma) I_0^{(s)}(t, z) + \omega_s(z-\alpha) I_1^{(s)}(t, z) \right) (th)^{\text{Im}(z)} dz. \end{aligned}$$

The zeta function increases like a polynomial on the integration contour, whereas  $\omega_s$  decreases like  $|z|^{-s-1}$ . Thus we can choose  $s$  large enough such that the integral converges absolutely with bounds independent of  $t$  and  $h$ . Then the integral is a smooth function of all parameters.  $\square$

**Quadrature rule.** To derive a quadrature rule we retain the first  $p$  terms in expansion (11) and replace derivatives by finite difference approximations. The weights  $r_j^{(p)}$  are determined such that

$$\begin{aligned} \sum_{s=0}^p \zeta(\alpha-s) \frac{(-1)^s}{s!} \frac{\partial^s \varphi}{\partial \tau^s}(t, t) h^{1-\alpha+s} \\ = -h^{1-\alpha} \sum_{j=0}^p \varphi(t, t-hj) r_j^{(p)} + R_D^{(p)}(t, h). \end{aligned}$$



Here we set  $\varphi = 0$  if the second argument is negative. If  $\gamma > p + 1$ , then this extension is  $p + 1$ -times continuously differentiable. Thus we can apply the Taylor expansion to obtain a linear system for the  $r_j^{(p)}$

$$(13) \quad \sum_{j=0}^p j^s r_j^{(p)} = -\zeta(\alpha - s), \quad s = 0, \dots, p.$$

Note that the weights are independent of  $t$ . We list the weights for the important cases  $p = 0, 1, 2$ ,

$$\begin{aligned} r_0^{(0)} &= -\zeta(\alpha), \\ \begin{bmatrix} r_0^{(1)} \\ r_1^{(1)} \end{bmatrix} &= - \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \zeta(\alpha) \\ \zeta(\alpha - 1) \end{bmatrix}, \\ \begin{bmatrix} r_0^{(2)} \\ r_1^{(2)} \\ r_2^{(2)} \end{bmatrix} &= - \begin{bmatrix} 1 & -3/2 & 1/2 \\ 0 & 2 & -1 \\ 0 & -1/2 & 1/2 \end{bmatrix} \begin{bmatrix} \zeta(\alpha) \\ \zeta(\alpha - 1) \\ \zeta(\alpha - 2) \end{bmatrix}. \end{aligned}$$

In view of Lemma 3.5 below, it is necessary to express the remainder as the product of certain powers of  $h$  and  $t$  and a function that is sufficiently regular on the triangle

$$(14) \quad \Delta_T := \{(t, h) : 0 \leq h \leq t \leq T\}.$$

**Lemma 2.2.** *If  $g(t) = t^{p+2-\alpha}\tilde{g}(t)$  with  $\tilde{g} \in C^\infty[0, T]$ , the remainder  $R_D^{(p)}(t, h)$  is of the form*

$$(15) \quad R_D^{(p)}(t, h) = h^{p+2-\alpha} t^{1-\alpha} I_D^{(p)}(t, h),$$

where  $I_D^{(p)} \in C^\infty(\Delta_T)$ .

*Proof.* Using the integral form of the remainder in the Taylor expansion, we get

$$R_D^{(p)}(t, h) = h^{1-\alpha} \frac{(-1)^p}{p!} \sum_{j=0}^p r_j^{(p)} \int_0^{jh} \tau^p \frac{\partial \varphi^{p+1}}{\partial \tau^{p+1}}(t, t - h + \tau) d\tau.$$

Since  $\varphi(t, \tau) = \tau^{p+2-\alpha} \tilde{\varphi}(t, \tau)$  there is a smooth  $\hat{\varphi}$  such that  $\partial_\tau^{p+1} \varphi(t, \tau) = \tau_+^{1-\alpha} \hat{\varphi}(t, \tau)$ , where  $\tau_+$  is the positive part of  $\tau$ . Hence

$$R_D^{(p)}(t, h) = h^{1-\alpha} \frac{(-1)^p}{p!} \sum_{j=0}^p r_j^{(p)} \int_0^{jh} (t - jh + \tau)_+^{1-\alpha} \tau^p \hat{\varphi}(t, \tau) d\tau$$

Changing variables  $\tau \rightarrow h\tau$  leads to

$$R_D^{(p)}(t, h) = h^{p+2-\alpha} t^{1-\alpha} \frac{(-1)^p}{p!} \sum_{j=0}^p r_j^{(p)} I_j(t, h)$$

where

$$I_j(t, h) = \int_0^j \left(1 - \frac{h}{t}(j - \tau)\right)_+^{1-\alpha} \tau^p \hat{\varphi}(t, h\tau) d\tau,$$

which is a smooth and bounded function when  $t \geq h$ .  $\square$

**3. Stability analysis.** We first discuss the stability of discretizations of the  $p = 0$  rule for the constant kernel  $k(t, \tau) = 1$ . In this case the discretization rule leads to the semi-circulant matrix

$$(16) \quad -\zeta(\alpha)h^{1-\alpha}A = -\zeta(\alpha)h^{1-\alpha} \begin{bmatrix} a_0 & & & \\ a_1 & a_0 & & \\ a_2 & a_1 & a_0 & \\ & \ddots & \ddots & \ddots \end{bmatrix}$$

whose coefficients are

$$(17) \quad a_n = \begin{cases} 1 & n = 0, \\ -(n^{-\alpha})/(\zeta(\alpha)) & n \geq 1. \end{cases}$$

The characteristic function associated with a semi-circulant matrix with entries  $a_n$  is given by

$$a(z) = \sum_{n=0}^{\infty} a_n z^n.$$

The stability of recurrence (3) hinges on the properties of the inverse matrix which is another semi-circulant matrix whose coefficients are the expansion coefficients of  $a^{-1}(z)$ ,

$$(18) \quad a^{-1}(z) = \sum_{n=0}^{\infty} A_n z^n.$$

Here and in the following we will use the convention that capitalized coefficients denote the expansion coefficients of the reciprocal function. A frequently used tool is the following result which goes back to Hardy [4].

**Lemma 3.1.** *If  $c_0 = 1$ ,  $c_n > 0$ ,  $n > 0$ ,  $\sum_n c_n = \infty$  and*

$$(19) \quad \frac{c_{n+1}}{c_n} \geq \frac{c_n}{c_{n-1}}, \quad n > 0,$$

then

$$(20) \quad C_n \leq 0, \quad n > 0 \text{ and } \sum_{n=1}^{\infty} C_n = -1.$$

From the behavior of the zeta function in the interval  $(0, 1)$ , shown in Figure 1, it follows that the sequence  $a_n$  in (17) is indeed positive. Furthermore, (19) is fulfilled when  $-\zeta(\alpha) \geq 2^\alpha$ , which implies that  $\alpha$  must be in the interval  $(\bar{\alpha}, 1)$  with  $\bar{\alpha} = 0.4836\dots$ . In this case the inverse is a bounded operator and we have

$$(21) \quad A_n < 0, \quad n > 0, \text{ and } \|A\|_\infty = 2, \text{ when } \alpha \in (\bar{\alpha}, 1).$$

Unfortunately, there is no information for  $\alpha \in (0, \bar{\alpha})$ . Furthermore, in the subsequent analysis it is necessary to have asymptotic estimates of the  $A_n$ 's. More information on the  $A_n$ 's for all  $\alpha$  can be obtained from Eggermont's analysis of the product integration method for the Abel equation. We summarize the development in the appendix of the paper [3].

**Lemma 3.2.** *Let  $c_n$  be a sequence with  $c_0 = 1$ , and set*

$$(22) \quad g_n = \begin{cases} 1 - (1/2)c_1 & n = 1, \\ (1/2)(c_{n-1} - c_n) & n > 1. \end{cases}$$

If the  $g_n$  satisfy

$$(23) \quad g_n > 0, \quad \frac{g_n}{g_{n-1}} < \frac{g_{n+1}}{g_n} < 1, \quad n > 1,$$

and, in addition, if there is an  $\bar{s} \in \mathbf{R}$  and  $\beta \in [0, 1)$  such that

$$(24) \quad \sum_{n=1}^y c_n \sim \bar{s} y^\beta, \quad y \rightarrow \infty,$$

then

$$(25) \quad C_n \sim -\frac{\sin(\beta\pi)}{\bar{s}\pi} \frac{1}{n^{\beta+1}}.$$

Our main stability result depends upon Lemma 3.2. Note, however, no information on the sign of the  $C_n$ 's is given; thus, the conclusions from Lemma 3.1 will be still important later on.

**Theorem 3.3.** *The coefficients  $A_n$  in (18) satisfy*

$$(26) \quad A_n \sim \frac{1}{\pi} \sin((1-\alpha)\pi) (1-\alpha) \zeta(\alpha) \frac{1}{n^{2-\alpha}},$$

and  $\sum_{n=0}^{\infty} A_n = 0.$

*Proof.* We first establish that the coefficients  $a_n$  of (17) satisfy the conditions of Lemma 3.2. For this sequence, the  $g_n$ 's are given by

$$(27) \quad g_n = \begin{cases} 1 + (1/2)\zeta(\alpha) & n = 1, \\ -(1/2)\zeta(\alpha)((n-1)^{-\alpha} - n^{-\alpha}) & n \geq 1, \end{cases}$$

The positiveness of the  $g_n$ 's follows easily from the fact that  $-\zeta(\alpha)$  increases from  $1/2$  to  $\infty$  in the interval  $(0, 1)$ , see Figure 1. To verify the second hypothesis of (23), consider the case  $n = 2$  first, which is equivalent to

$$(28) \quad \frac{1 - 2^{-\alpha}}{-2\zeta(\alpha) - 1} < \frac{2^{-\alpha} - 3^{-\alpha}}{1 - 2^{-\alpha}}.$$

This must be verified by computing function values. The validity of this inequality is illustrated in Figure 1. When  $n > 2$  the validity of (23) for the  $g_n$ 's defined in (27) follows from elementary calculus.

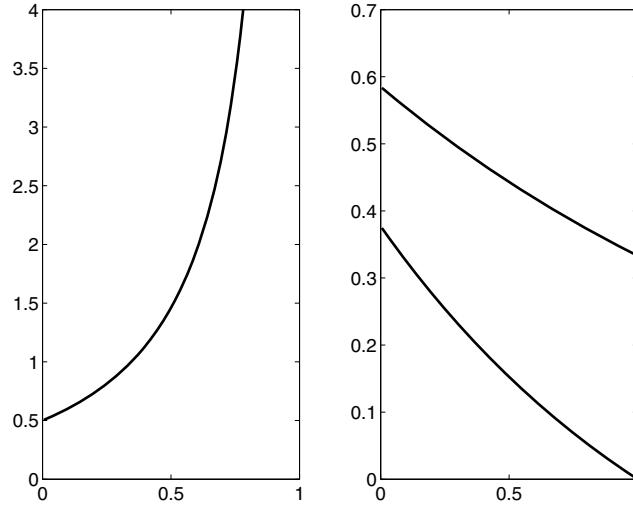


FIGURE 1. Graph of the function  $-\zeta(\cdot)$  (left) and the ratios in (23) when  $n = 2$  (right, bottom) and  $n = 3$  (right, top).

From (17) it also follows that

$$\sum_{n=1}^y a_n \sim \frac{-1}{\zeta(\alpha)(1-\alpha)} y^{1-\alpha}$$

hence the exponent in (24) is  $\beta = 1 - \alpha$  which implies the asymptotic estimate for the  $A_n$ 's.

Since  $\sum_n |A_n|$  converges, it follows that  $a^{-1}(z)$  is continuous on the closed unit disk, and by the Abel limit theorem it follows that

$$\sum_{n=0}^{\infty} A_n = \lim_{x \rightarrow 1^-} a^{-1}(x) = 0. \quad \square$$

The following two lemmas will be necessary for the stability analysis.

**Lemma 3.4.** *Let  $\Phi \in C^1[0, T]$ ,  $t_j = hj$ . Then*

$$(29) \quad \sum_{j=0}^n A_{n-j} a_j \Phi(t_j) = \delta_{n,0} \Phi(t_n) + O(h).$$

*Proof.* Write

$$\sum_{j=0}^n A_{n-j} a_j \Phi(t_j) = \sum_{j=0}^n A_{n-j} a_j \Phi(t_n) - \sum_{j=0}^{n-1} A_{n-j} a_j (\Phi(t_n) - \Phi(t_j)).$$

The first term is the convolution product of  $\{a_j\}_j$  and its inverse and accounts for the first term in the righthand side of (29). The first term in the second sum is  $O(h)$ . To estimate the remainder of the second term, note that it follows from the generalized Euler-Maclaurin formula that

$$h \sum_{j=1}^{n-1} (t_n - t_j)^{-1+\alpha} t_j^{-\alpha} = O\left(\int_0^{t_n} (t_n - \tau)^{-1+\alpha} \tau^{-\alpha} d\tau\right) = O(t_n).$$

Hence, using  $|\Phi(t_n) - \Phi(t_j)| \leq C(t_n - t_j)$  and Theorem 3.3,

$$\left| \sum_{j=1}^{n-1} A_{n-j} (\Phi(t_n) - \Phi(t_j)) a_j \right| \leq Ch^2 \sum_{j=1}^{n-1} (t_n - t_j)^{-1+\alpha} t_j^{-\alpha} = O(h).$$

This proves the assertion.  $\square$

**Lemma 3.5.** *Let  $\Phi \in C^1(\Delta_T)$ . Then for any  $\lambda > 0$*

$$(30) \quad \sum_{j=1}^n A_{n-j} t_j^\lambda \Phi(t_j, h) = O(t_n^{\lambda-1+\alpha} h^{1-\alpha}).$$

*Proof.* For simplicity of notations, we omit the second argument from  $\Phi$ . Setting  $\Psi(t) = t^\lambda \Phi(t)$  we have

$$\begin{aligned} |t_n^\lambda \Phi(t_n) - t_j^\lambda \Phi(t_j)| &= |\Psi(t_n) - \Psi(t_j)| \\ &= \left| \int_{t_j}^{t_n} \Psi'(\tau) d\tau \right| \leq (t_n - t_j) \max_{t_j \leq \tau \leq t_n} |\Psi'(\tau)|. \end{aligned}$$

It follows from the chain rule that, if  $j \leq n$ ,

$$|t_n^\lambda \Phi(t_n) - t_j^\lambda \Phi(t_j)| \leq \begin{cases} C(t_n - t_j)t_n^{\lambda-1} & \lambda \geq 1, \\ C(t_n - t_j)t_j^{\lambda-1} & \lambda < 1. \end{cases}$$

Because of  $\sum_j A_j = 0$ ,

$$\begin{aligned} (31) \quad \sum_{j=1}^n A_{n-j} t_j^\lambda \Phi(t_j) &= \sum_{j=1}^n A_{n-j} t_n^\lambda \Phi(t_n) - \sum_{j=1}^{n-1} A_{n-j} (t_n^\lambda \Phi(t_n) - t_j^\lambda \Phi(t_j)) \\ &= \sum_{j=n-1}^{\infty} A_j t_n^\lambda \Phi(t_n) - \sum_{j=1}^{n-1} A_{n-j} (t_n^\lambda \Phi(t_n) - t_j^\lambda \Phi(t_j)). \end{aligned}$$

For  $\lambda \geq 1$  we estimate using Theorem 3.3

$$\begin{aligned} \left| \sum_{j=1}^n A_{n-j} t_j^\lambda \Phi(t_j) \right| &\leq C_1 t_n^\lambda \sum_{j=n}^{\infty} j^{-2+\alpha} + C_2 t_n^{\lambda-1} h \sum_{j=1}^{n-1} (n-j)^{-1+\alpha} \\ &\leq C (t_n^\lambda n^{-1+\alpha} + t_n^{\lambda-1} h n^\alpha) \leq C t_n^{\lambda-1+\alpha} h^{1-\alpha}. \end{aligned}$$

For the case  $0 < \lambda < 1$  note that

$$h \sum_{j=1}^{n-1} (t_n - t_j)^{-1+\alpha} t_j^{\lambda-1} = O\left(\int_0^{t_n} (t_n - \tau)^{-1+\alpha} \tau^{\lambda-1} d\tau\right) = O(t_n^{\lambda-1+\alpha}).$$

Hence, beginning with (31)

$$\begin{aligned} \left| \sum_{j=1}^n A_{n-j} t_j^\lambda \Phi(t_j) \right| &\leq C_1 t_n^\lambda \sum_{j=n+1}^{\infty} j^{-2+\alpha} + C_2 h^{2-\alpha} \sum_{j=1}^n (t_n - t_j)^{-1+\alpha} t_j^{\lambda-1} \\ &\leq C t_n^{\lambda-1+\alpha} h^{1-\alpha}. \end{aligned}$$

Thus the assertion has been shown in both cases.  $\square$

**Higher order methods.** In the case  $p > 1$  and a constant kernel  $k(t, \tau) = 1$  discretization (3) leads to the system

$$Ag + \widehat{R}^{(p)} g = -\frac{h^{-1+\alpha}}{\zeta(\alpha)} f$$

where  $A$  is defined in (16) and  $\widehat{R}^{(p)}$  is a semi-circulant matrix with coefficients

$$(32) \quad \widehat{r}_j^{(p)} = \begin{cases} -(r_0^{(p)}/\zeta(\alpha)) - 1 & j = 0, \\ -(r_j^{(p)}/\zeta(\alpha)) & 1 \leq j \leq p, \\ 0 & j > p. \end{cases}$$

To determine stability we factor

$$A + \widehat{R}^{(p)} = A(I + A^{-1}\widehat{R}^{(p)})$$

and investigate whether the second factor has a continuous inverse. To that end, set  $S^{(p)} = A^{-1}\widehat{R}^{(p)}$  and find the values of  $\alpha$  and  $p$  for which the coefficients of  $S^{(p)}$  satisfy

$$(33) \quad \|S^{(p)}\|_\infty = \sum_{n=0}^{\infty} |s_n^{(p)}| < 1.$$

If (33) is satisfied, then  $I + A^{-1}\widehat{R}^{(p)}$  has a continuous inverse in  $l_\infty$ . The following lemma gives a necessary condition for stability.

**Lemma 3.6.** *For every  $p \geq 0$  there is an  $\alpha_p \in (0, 1)$  such that condition (33) is satisfied for  $\alpha \in (\alpha_p, 1)$ .*

*Proof.* If  $p = 0$ , then  $S^{(0)} = 0$  and there is nothing to show, and hence we may set  $\alpha_0 = 0$ . If  $p > 0$  and  $\alpha > \bar{\alpha}$  we have from (21) that  $\|A^{-1}\|_\infty = 2$ . Hence  $\|S^{(p)}\|_\infty = \|A^{-1}\widehat{R}^{(p)}\|_\infty \leq 2\|\widehat{R}^{(p)}\|_\infty$ . Furthermore, it follows from (13) and (32) that

$$(34) \quad \widehat{r}_j^{(p)} = -\frac{1}{\zeta(\alpha)} \sum_{s=1}^p m_s^{(p)} (-1)^s \zeta(\alpha - s)$$

where  $m_s^{(p)}$  are the columns of  $[M^{(p)}]^{-1}$ , and  $M_{s,j}^{(p)} = j^s$ . Since the zeta function is only singular when  $\alpha = 1$ , it follows that for any  $p$  fixed  $\|r^{(p)}\|_1 \rightarrow 0$  as  $\alpha \rightarrow 1$ . Hence there is an  $\alpha_p < 1$  for which  $\|S^{(p)}\|_\infty < 1$ .  $\square$

Using the technique of the proof we compute values of  $\alpha_1$  and  $\alpha_2$  that guarantee stability. When  $p = 1$ , we have

$$\begin{bmatrix} \widehat{r}_0^{(1)} \\ \widehat{r}_1^{(1)} \end{bmatrix} = -\frac{1}{\zeta(\alpha)} \begin{bmatrix} \zeta(\alpha - 1) \\ -\zeta(\alpha - 1) \end{bmatrix}.$$



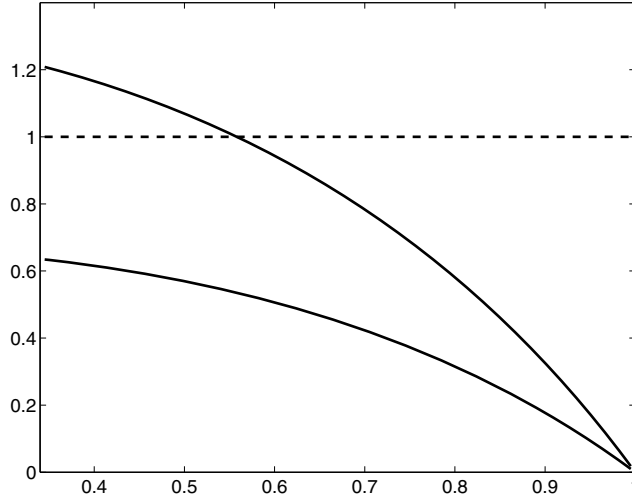


FIGURE 2. Graph of the functions  $s_1$  (bottom) and  $s_2$  (top).

When  $\alpha > \bar{\alpha}$  we can conclude that

$$(35) \quad \|S^{(1)}\|_\infty \leq \|A^{-1}\|_\infty \|\widehat{R}^{(1)}\|_\infty = 4 \frac{\zeta(\alpha - 1)}{\zeta(\alpha)} =: s_1(\alpha).$$

As shown in Figure 2,  $s_1(\alpha)$  is always less than unity, but (35) is valid only if  $\alpha \in (\bar{\alpha}, 1)$ ; hence, we have  $\alpha_1 = \bar{\alpha} = 0.4836\dots$

When  $p = 2$ , we have

$$\begin{bmatrix} \widehat{r}_0^{(2)} \\ \widehat{r}_1^{(2)} \\ \widehat{r}_2^{(2)} \end{bmatrix} = -\frac{1}{\zeta(\alpha)} \begin{bmatrix} 3/2 & 1/2 \\ -2 & -1 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} \zeta(\alpha - 1) \\ -\zeta(\alpha - 2) \end{bmatrix}.$$

Figure 2 shows the graph of  $s_2(\alpha) := -2/(\zeta(\alpha))\|\widehat{r}^{(2)}\|_1$ . From the graph it is evident that  $\alpha \in (\alpha_2, 1)$  with  $\alpha_2 = 0.558\dots$  is necessary for stability.

**4. Discretization error.** The quadrature error consists of two parts, the error introduced by truncating the Euler-Maclaurin expansion and the error introduced by replacing the derivatives in the leading terms of the expansion by finite differences. Setting  $q = p + 1$  in (11), the  $m$ th time step, the quadrature error is

$$(36) \quad d_m := \frac{(-1)^{p+1}}{(p+1)!} \frac{\partial^{p+1} \varphi}{\partial \tau^{p+1}}(t_m, t_m) \zeta(\alpha - p - 1) h^{p+2-\alpha} \\ + R_E^{(p+1)}(t_m, h) + R_D^{(p)}(t_m, h).$$

Subtracting equations (1) and (3) shows that the discretization errors  $e_j = g(t_j) - g_j$  and the quadrature error  $d_m$  are related by

$$(37) \quad h \sum_{j=0}^{m-1} (t_m - t_j)^{-\alpha} k(t_m, t_j) e_j + h^{1-\alpha} \sum_{j=0}^p r_j^{(p)} k(t_m, t_{m-j}) e_{m-j} = d_m.$$

Equation (37) is equivalent to

$$(38) \quad \sum_{j=0}^m a_{m-j} k(t_m, t_j) e_j + \sum_{j=m-p}^m \hat{r}_{m-j}^{(p)} k(t_m, t_j) e_j = -\frac{h^{\alpha-1}}{\zeta(\alpha)} d_m$$

where the coefficients  $\hat{r}_j^{(p)}$  are defined in (32).

To estimate the error we convolve both sides of the equation with the sequence  $\{A_n\}_n$ . For the first term in the lefthand side, we calculate

$$\begin{aligned} \phi_n^{(1)} &= \sum_{m=0}^n A_{n-m} \sum_{j=0}^m a_{m-j} k(t_m, t_j) e_j \\ &= \sum_{j=0}^n \sum_{m=j}^n A_{n-m} a_{m-j} k(t_m, t_j) e_j \\ &= \sum_{j=0}^n \sum_{m=0}^{n-j} A_{n-j-m} a_m k(t_{m+j}, t_j) e_j. \end{aligned}$$

Lemma 3.4 applied to the inner sum with  $\Phi(\tau) = k(\tau + t_j, t_j)$  shows that there are coefficients  $\kappa_{n,j}^{(1)} = O(1)$  such that

$$(39) \quad \phi_n^{(1)} = e_n + h \sum_{j=0}^n \kappa_{n,j}^{(1)} e_j.$$

Proceeding in a similar manner shows that the second term satisfies

$$\phi_n^{(2)} = \sum_{j=0}^n \sum_{m=0}^p A_{n-j-m} \widehat{r}_m^{(p)} k(t_{m+j}, t_j) e_j.$$

In the above sum one has  $k(t_{m+j}, t_j) = k(t_j, t_j) + O(h) = 1 + O(h)$ . Thus there are coefficients  $\kappa_{n,j}^{(2)} = O(1)$  such that

$$(40) \quad \phi_n^{(2)} = \sum_{j=0}^n s_{n-j}^{(p)} e_j + h \sum_{j=0}^n \kappa_{n,j}^{(2)} e_j,$$

where  $s_j^{(p)}$  are the coefficients of  $S^{(p)} = A^{-1} \widehat{R}^{(p)}$ . In matrix notation, (39) and (40) can be combined as

$$(41) \quad (I + S^{(p)}) e + h K e = h^{\alpha-1} A^{-1} d,$$

where  $K$  is a lower triangular matrix with entries

$$K_{n,j} = \kappa_{n,j}^{(1)} + \kappa_{n,j}^{(2)}$$

which are bounded independently of  $h$ . We continue with an estimate for the righthand side.

**Lemma 4.1.** *If  $\gamma = 2 + p - \alpha$  the coefficients of  $\tilde{e} = h^{\alpha-1} A^{-1} d$  are bounded by*

$$\tilde{e}_n = O(h^{p+2-\alpha}).$$

*Proof.* The coefficients of  $\tilde{e} = h^{\alpha-1} A^{-1} d$  are given by

$$\tilde{e}_n = h^{\alpha-1} \sum_{j=1}^n A_{n-j} d_j.$$

In view of (36) we have to analyze the effect of the convolution on three parts, all of which are in a form that fit the hypotheses of Lemma 3.5.

For the first part note that the  $p + 1$ st derivative of  $\varphi$  is of the form  $t^{1-\alpha}I(t)$ , where  $I(\cdot)$  is smooth; hence,

$$h^{\alpha-1} \sum_{j=1}^n A_{n-j} t_j^{1-\alpha} I(t_j) h^{p+2-\alpha} = O(h^{p+2-\alpha} t^0) = O(h^{p+2-\alpha}).$$

The estimate of the second part follows from Lemma 2.1. In that lemma, we may set, for instance,  $c' = -p - 1$ , then

$$\begin{aligned} h^{\alpha-1} \sum_{j=1}^n A_{n-j} R_E^{(p+1)}(t_j, h) &= O\left(h^{1-c'} t_n^{\gamma+c'-1}\right) = O\left(h^{p+2} t_n^{\gamma-p-2}\right) \\ &= O\left(h^{p+2-\alpha} t_n^{\gamma-p-2+\alpha}\right) \left(\frac{h}{t_n}\right)^\alpha = O\left(h^{p+2-\alpha}\right) \frac{1}{n^\alpha}. \end{aligned}$$

Hence this part is  $O(h^{p+2-\alpha})$  multiplied by a decaying term. For the third part of (36) use Lemma 2.2

$$h^{\alpha-1} \sum_{j=1}^n A_{n-j} R_D^{(p)}(t_j, h) = h^{p+1} \sum_{j=1}^n A_{n-j} t_j^{1-\alpha} I_D^{(p)}(t_j, h) = O(h^{p+2-\alpha}).$$

Adding the three parts together completes the proof.  $\square$

If  $\alpha$  and  $p$  are such that  $\|I + S^{(p)}\|_\infty$  is bounded independently of  $n$ , then (41) is equivalent to

$$(42) \quad e + h \left(I + S^{(p)}\right)^{-1} K e = \left(I + S^{(p)}\right)^{-1} \tilde{e}.$$

Here the matrix has entries bounded independently of  $h$  and the righthand side has the same asymptotic estimate as  $\tilde{e}$ . Our final result follows directly from the discrete Gronwall lemma.

**Theorem 4.2.** *If  $\alpha_p < \alpha < 1$  and  $\gamma = p + 2 - \alpha$ , then the discretization error satisfies the asymptotic estimate  $e_n = O(h^{p+2-\alpha})$ .*

**5. The general case.** In Section 2 we have introduced a quadrature rule which has the desired order only if the integrand is  $O(\tau^{p+2-\alpha})$  as  $\tau \rightarrow 0$ . We will write this rule in the form

$$\int_0^{t_n} (t_n - \tau)^{-\alpha} \phi(t_n, \tau) d\tau = \sum_{j=1}^n \phi(t_n, t_j) w_{n-j} + O(h^{p+2-\alpha}),$$

where the weights are

$$w_j = h^{1-\alpha} \begin{cases} r_0^{(p)} & j = 0, \\ j^{-\alpha} + r_{p-j}^{(p)} & 1 \leq j \leq p, \\ j^{-\alpha} & j > p. \end{cases}$$

We will show how to modify this rule if  $\phi$  is replaced by  $\tau^{-\beta} \phi(t, \tau)$ , where  $\beta < 1$  and  $\phi$  is a smooth function in both variables that has no conditions on the behavior near  $\tau = 0$ . The modification depends on the Taylor expansion of  $\phi$  at  $\tau = 0$

$$(43) \quad \phi_q(t, \tau) = \sum_{s=0}^q \frac{1}{s!} \frac{\partial^s \phi}{\partial \tau^s}(t, 0) \tau^s,$$

which will be subtracted from  $\phi$  to obtain a function of order  $\tau^{q+1-\beta}$ . The quadrature rule is applied to the difference and the rest is integrated analytically, as shown in the following calculation

$$\begin{aligned} & \int_0^{t_n} (t_n - \tau)^{-\alpha} \tau^{-\beta} \phi(t_n, \tau) d\tau \\ &= \int_0^{t_n} (t_n - \tau)^{-\alpha} \tau^{-\beta} \left( \phi(t_n, \tau) - \phi_q(t_n, \tau) \right) d\tau \\ &+ \int_0^{t_n} (t_n - \tau)^{-\alpha} \tau^{-\beta} \phi_q(t_n, \tau) d\tau \\ &\approx \sum_{j=1}^n t_j^{-\beta} \left( \phi(t_n, t_j) - \phi_q(t_n, t_j) \right) w_{n-j} \\ &+ \int_0^{t_n} (t_n - \tau)^{-\alpha} \tau^{-\beta} \phi_q(t_n, \tau) d\tau \\ &= \sum_{j=1}^n t_j^{-\beta} \phi(t_n, t_j) w_{n-j} + c_n. \end{aligned}$$

This is the original quadrature rule with a correction term. Since only polynomials appear in this term, the integral can be evaluated in closed form

$$\begin{aligned} c_n &= \int_0^{t_n} (t_n - \tau)^{-\alpha} \tau^{-\beta} \phi_p(t_n, \tau) d\tau - \sum_{j=1}^n t_j^{-\beta} \phi_p(t_n, t_j) w_{n-j} \\ &= \sum_{s=0}^q \frac{1}{s!} \frac{\partial^s \phi}{\partial \tau^s}(t_n, 0) \left\{ \int_0^{t_n} (t_n - \tau)^{-\alpha} \tau^{s-\beta} d\tau - \sum_{j=1}^n t_j^{s-\beta} w_{n-j} \right\} \\ &= \sum_{s=0}^q \frac{1}{s!} \frac{\partial^s \phi}{\partial \tau^s}(t_n, 0) \left\{ E_s^{\alpha, \beta} t_n^{1+s-\alpha-\beta} - \sum_{j=1}^n t_j^{s-\beta} w_{n-j} \right\}. \end{aligned}$$

Here,  $E_s^{\alpha, \beta}$  denotes the Euler integral

$$E_s^{\alpha, \beta} = \int_0^1 (1-x)^{-\alpha} x^{s-\beta} dx = \frac{\Gamma(1-\alpha)\Gamma(1+s-\beta)}{\Gamma(2+s-\alpha-\beta)}.$$

We now give a recipe to solve the Abel integral equation with the general hand side as it appears in Theorem 1.1.

1. Compute the coefficients  $g_i$ ,  $i = 0, \dots, q$ , in the expansion of the solution  $\tilde{g}(\tau) = g_0 + g_1\tau + g_2\tau^2 + \dots$ . These follow directly from the expansion coefficients of the kernel and the righthand side. Set  $\tilde{g}_q(t) = (g_0 + g_1\tau + \dots + g_q\tau^q)$  and  $\hat{g}_q(t) = g(t) - \tau^{-\beta}\tilde{g}_q(t)$ .

2. Subtract  $\tau^{-\beta}\tilde{g}_q(t)$  in (1) and solve the equivalent Abel equation

$$\int_0^t (t-\tau)^{-\alpha} k(t, \tau) \hat{g}(\tau) d\tau = f(t) - \int_0^t (t-\tau)^{-\alpha} k(t, \tau) \tau^{-\beta} \tilde{g}_q(\tau) d\tau,$$

with  $\hat{g}(\tau) = O(\tau^{1+q-\beta})$  as the new unknown. To apply rule (3),  $q$  must satisfy  $1+q-\beta \geq p+2-\alpha$ ; hence, the order of the Taylor expansion in (43) is the smallest integer  $q$  with

$$q \geq p+1+\beta-\alpha.$$

3. In every time step, compute the correction factors  $c_n$  and evaluate the integral on the righthand side with the corrected quadrature rule.

The additional error introduced in this procedure comes from replacing the integral on the righthand side with the corrected quadrature rule. The analysis of this error parallels the discussion of Sections 2 and 4 and is therefore omitted.

**6. Numerical example.** Our test problem is the integral equation

$$\int_0^t (t - \tau)^{-\alpha} \exp(t - \tau) g(\tau) d\tau = \exp(t) t^{1-\alpha-\beta} \sum_{k=0}^5 E_k^{\alpha,\beta} t^k$$

with known solution  $g(t) = \exp(t) \sum_{k=0}^5 t^{k-\beta}$ . We solve this equation in the interval  $[0, 1]$  with stepsize  $h = 1/N$  and  $\beta = 0.5$ .

TABLE 1. Maximum errors and orders of convergence when  $p = 0$ .

N	$\alpha = 0.15$		$\alpha = 0.5$		$\alpha = 0.85$	
	error	order	error	order	error	order
10	0.10025		0.188351		0.162072	
20	0.027132	1.8856	0.067565	1.4791	0.074338	1.1245
40	0.007534	1.8483	0.024040	1.4908	0.033800	1.1371
80	0.0020919	1.8487	0.008525	1.4958	0.015302	1.1433
160	0.0005805	1.8494	0.003018	1.4979	0.006913	1.1464
320	0.0001611	1.8497	0.001068	1.4989	0.003119	1.1480

TABLE 2. Maximum errors and orders of convergence when  $p = 1$ .

N	$\alpha = 0.15$		$\alpha = 0.5$		$\alpha = 0.85$	
	error	order	error	order	error	order
10	0.05261		0.01707		0.01496	
20	0.20980	-1.996	0.003117	2.4534	0.003496	2.0974
40	8.17868	-5.285	0.0005597	2.4776	0.0008020	2.1239
80	37781	-12.17	9.972e-05	2.4889	0.0001823	2.1370
160	2.384e+12	-25.91	1.770e-05	2.4944	4.127e-05	2.1435
320	2.749e+28	-53.36	3.134e-06	2.4972	9.320e-06	2.1467

TABLE 3. Maximum errors and orders of convergence when  $p = 2$ .

N	$\alpha = 0.15$		$\alpha = 0.5$		$\alpha = 0.85$	
	error	order	error	order	error	order
10	2.1507		0.004426		0.001596	
20	170.16	-6.305	0.0003550	3.6401	0.0001871	3.0927
40	6.022e+06	-15.11	6.273e-06	5.8223	2.148e-05	3.1225
80	4.267e+16	-32.72	1.013e-06	2.6301	2.442e-06	3.1369
160	1.212e+37	-67.94	1.044e-07	3.2788	2.763e-07	3.1439
320	5.534e+78	-138.3	9.241e-09	3.4981	3.118e-08	3.1473

Tables 1–3 display the convergence of the error for different values of  $\alpha$  and  $p$ , all of which are in good agreement with the theoretical analysis. In particular, the method is unstable for small values of  $\alpha$  when  $p \geq 1$ , but otherwise the order- $h^{p+2-\alpha}$  convergence is clearly visible. When  $p = 2$  and  $\alpha = 0.5$  the method still converges at the expected rate even though this parameter combination is not in the range with guaranteed stability that is implied by Figure 2. In Figure 3 we further illustrate the stability of  $\alpha = 0.5$  for different  $p$  by displaying the coefficients of the semi-circulant matrices  $(A + \widehat{R}^{(p)})^{-1}$ . It appears that after some transient effects the entries have the same asymptotic behavior when  $p = 0$ ,  $p = 1$  and  $p = 2$ . For larger values of  $p$  the coefficients blow up and the method is unstable.

Figure 4 shows the errors for  $\alpha = 0.5$  and  $p = 2$  as a function of time. Even though the analytical solution is singular at  $t = 0$ , the convergence happens at the expected rate and the error is small near the singularity.

To put the results presented into the right perspective, it should be noted that it is possible to derive stable rules of even higher order based on convolutional quadrature. In fact, Lubich presents results for a fourth-order scheme [9]. However, the point here was to introduce quadrature rules of the simpler form (3) which is advantageous, for instance, for the fast evaluation of time-dependent boundary integral operators.



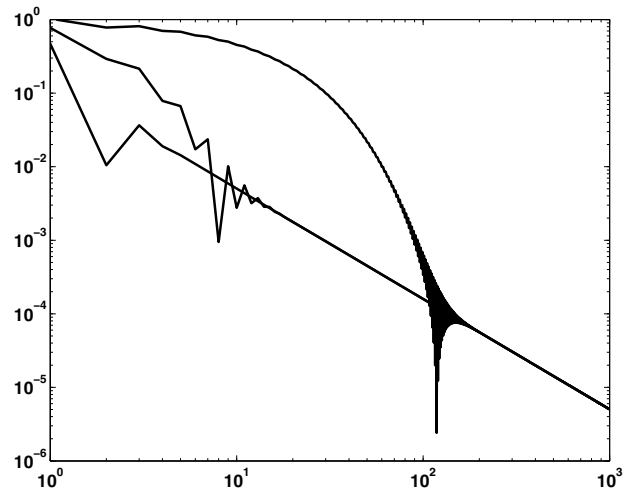


FIGURE 3. Magnitude of the coefficients of  $(A + \widehat{R}^{(p)})^{-1}$  for  $\alpha = 0.5$  and  $p = 0$ ,  $p = 1$  and  $p = 2$ .

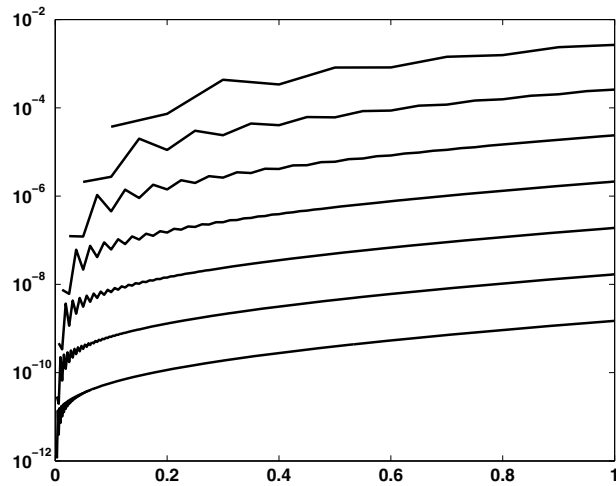


FIGURE 4. Errors for  $\alpha = 0.5$ ,  $\beta = 0.5$ ,  $p = 2$ ,  $N = 10, 20, \dots, 640$  as a function of  $t$ .

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