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A NOTE ON THE POLYNOMIAL APPROXIMATION OF VERTEX SINGULARITIES IN THE BOUNDARY ELEMENT METHOD IN THREE DIMENSIONS

ALEXEI BESPALOV

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ABSTRACT. We study polynomial approximations of vertex singularities of the type $r^\lambda |\log r|^\beta$ on three-dimensional surfaces. The analysis focuses on the case when $\lambda > -\frac{1}{2}$. This assumption is a minimum requirement to guarantee that the above singular function is in the energy space for boundary integral equations with hypersingular operators. Furthermore, such strong vertex singularities may appear in solutions to boundary integral formulations of time-harmonic problems of electromagnetism in Lipschitz domains. Thus, the approximation results for such singularities are needed for the error analysis of boundary element methods in three dimensions. Moreover, to our knowledge, the approximation of strong singularities ($-\frac{1}{2} < \lambda \le 0$) by high-order polynomials is missing in the existing literature. In this note we prove an estimate for the error of polynomial approximation of the above vertex singularities on quasi-uniform meshes discretising a polyhedral surface. The estimate gives an upper bound for the error in terms of the mesh size h and the polynomial degree p.

1. Introduction. In this note we analyse polynomial approximations of vertex singularities inherent to solutions of boundary integral equations (BIE) on a Lipschitz polyhedral surface Γ. In particular, denoting by r the distance to a vertex of Γ, we study approximations

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of singularities of the type $r^\lambda|\log r|^\beta$ under a minimum assumption on λ ensuring that this singular function is in the space $H^{1/2}(\Gamma)$ (the energy space for the BIE with hypersingular operator on Γ; see §2 for definitions of the Sobolev spaces on Γ).

It is well known that solutions to BIE on piecewise smooth surfaces exhibit a singular behaviour in neighbourhoods of edges and vertices of the surface. In [**17, 18**] explicit formulas are given to specify this behaviour for polyhedral and piecewise plane open surfaces. In particular, it has been shown that solutions of BIE can be decomposed into a number of singular functions and a smooth remainder. Moreover, taking enough singularity terms in this decomposition, one can obtain the smooth remainder as regular as needed. Let r be the distance to a vertex v of Γ and let ρ be the distance to one of the edges $e \subset \partial\Gamma$ such that $\bar{e} \ni v$. Then typical singularities are:

(i) vertex singularities of the type $r^\lambda|\log r|^{\beta_1}$;

(ii) edge singularities of the type $\rho^\gamma|\log\rho|^{\beta_2}$;

(iii) combined edge-vertex singularities of the type $r^{\lambda-\gamma}\rho^\gamma|\log r|^{\beta_3}$;

here, λ and γ are real parameters to be specified below and β_i ($i = 1,\dots,3$) are non-negative integers.

The admissible values of λ and γ depend on the problem under consideration. Let us consider the following model problem: *Find* $u \in H^{1/2}(\Gamma)$ *such that*

$$\langle Wu, v\rangle = \langle f, v\rangle \quad \forall\ v \in H^{1/2}(\Gamma). \tag{1.1}$$

Here, $f \in H^{-1/2}(\Gamma)$ is a given functional, W is the hypersingular operator

$$Wu(x) := -\frac{1}{4\pi}\frac{\partial}{\partial n_x}\int_\Gamma u(y)\frac{\partial}{\partial n_y}\frac{1}{|x-y|}\,dS_y, \quad W\colon H^{1/2}(\Gamma) \to H^{-1/2}(\Gamma);$$

$\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_{L^2(\Gamma)}$ denotes the extension of the $L^2(\Gamma)$-inner product by duality, and $H^{-1/2}(\Gamma)$ is the dual space of $H^{1/2}(\Gamma)$.

As it follows from [**18**], for *sufficiently smooth* given f, the singularity exponents λ and γ satisfy

$$\lambda \ge \lambda_1 > 0 \quad \text{and} \quad \gamma \ge \gamma_1 > 1/2. \tag{1.2}$$

We note that in the case of an open surface Γ, the energy space for problem (1.1) is $\tilde{H}^{1/2}(\Gamma)$ and for sufficiently smooth given f there hold

$$\lambda \geq \lambda_1 > 0 \quad \text{and} \quad \gamma \geq \gamma_1 \geq 1/2. \tag{1.3}$$

Thus, conditions (1.2) (or (1.3)) appear, if singularities in the solution to problem (1.1) are caused solely by the geometry of the surface. However, for singular right-hand sides f in (1.1), it may also occur that

$$\lambda \geq \lambda_1 > -1/2 \quad \text{and} \quad \gamma \geq \gamma_1 > 0, \tag{1.4}$$

which are the minimum requirements to ensure $u \in H^{1/2}(\Gamma)$.

As an example, let us consider the model integral equation $Wu = f$ on the plane open surface Γ with smooth boundary $\partial\Gamma$. Assume additionally that $\Gamma \subset Ox_1x_2$, $O \in \partial\Gamma$, and $f = r^{\lambda-1}$, where $r = (x_1^2 + x_2^2)^{1/2}$ and $\lambda \in (-\frac{1}{2}, 0)$. To recover the behaviour of the solution u, we apply the Fourier transform $\mathcal{F}$ to both sides of the equation and recall that W is a pseudodifferential operator of order $+1$ with the principal symbol $|\xi| = (\xi_1^2 + \xi_2^2)^{1/2}$, $\xi = (\xi_1, \xi_2) \in \mathbf{R}^2 \backslash \{O\}$ (see [**11, 16**]). Then using a formula for the Fourier transform of power functions in r (see [**12**, p. 363]) and omitting lower order terms in $|\xi|$, we find

$$\mathcal{F}(u) = \frac{1}{|\xi|}\,\mathcal{F}(r^{\lambda-1}) = C(\lambda)\,|\xi|^{-\lambda-2}.$$

Applying now the inverse Fourier transform it is easy to see that the leading singularity in the solution u is due to the singular right-hand side function and this singularity is of order r^λ with $\lambda \in (-\frac{1}{2}, 0)$.

However, the main motivation to consider strong vertex singularities is the fact that such singularities naturally appear in solutions to BIE stemming from time-harmonic problems of electromagnetism in domains with piecewise smooth boundaries. It is known that solutions to the latter problems (e.g., boundary value problems for time-harmonic Maxwell equations in domains with edges and corners) are vector fields whose components exhibit singularities analogous to those in (i)–(iii) (see [**10**]). Using these results and a trace argument, we studied the behaviour of the solution to the electric field integral equation (EFIE) at edges and corners of piecewise smooth (open or closed)

Lipschitz surfaces and derived explicit formulas for edge, vertex, and combined edge-vertex singularities (see [**8**]). In particular, the leading vertex singularities are of the type

$$\mathbf{curl}\Big(r^{\lambda_1}\,|\log r|^{\beta_1}\Big) + \Big(r^{\lambda_2}\,|\log r|^{\bar\beta_1},\, r^{\lambda_2}\,|\log r|^{\bar\beta_1}\Big), \quad \lambda_1, \lambda_2 > -1/2,$$

and the leading edge singularities are of the type

$$\mathbf{curl}\Big(\rho^{\gamma_1}\,|\log \rho|^{\beta_2}\Big) + \Big(\rho^{\gamma_2}\,|\log \rho|^{\bar\beta_2},\, \rho^{\gamma_2}\,|\log \rho|^{\bar\beta_2}\Big), \quad \gamma_1, \gamma_2 \geq 1/2$$

(here, r, ρ are the same as in (i)–(iii) above and β_i, $\bar\beta_i$ $(i = 1, 2)$ are integers). Moreover, it has been shown in [**8**] that the analysis of polynomial approximations of such singular vector fields in the corresponding energy space (which is either $\mathbf{H}^{-1/2}(\mathrm{div}, \Gamma)$ or $\tilde{\mathbf{H}}^{-1/2}(\mathrm{div}, \Gamma)$) can be reduced to the analysis of scalar singularities (i)–(iii) in Sobolev spaces $H^{1/2}(\Gamma)$ or $\tilde{H}^{1/2}(\Gamma)$. Thus, approximation results for scalar singularities (i)–(iii) in these Sobolev spaces (i.e., in the same framework as for the BIE with hypersingular operator) are critical for the error analysis of the boundary element method (BEM) for the EFIE.

In the framework of the p-version of the BEM, approximations of singularities (i)–(iii) were first analysed in [**15**] under assumptions (1.2) on λ, γ. These assumptions guarantee that all singular functions (i)–(iii) are $H^1(\Gamma)$-regular. Due to this fact, a rigorous analysis of polynomial approximations of these singularities in $L^2(\Gamma)$ and in $H^1(\Gamma)$ was performed in [**15**]. Then, using interpolation between these spaces, the paper culminated in the optimal a priori error estimate for the p-version of the BEM with hypersingular operator on a (closed) polyhedral surface Γ (for smooth right-hand side f).

Later, in [**4**], we extended the results of [**15**] to the case of open surfaces, where singularity exponents λ, γ satisfy (1.3). Moreover, our analysis in [**4**] for the p-BEM and then in [**6**] for the hp-BEM with quasi-uniform meshes, covers the least regular cases (λ, γ satisfying (1.4)), but only for edge and vertex-edge singularities (on both open and closed piecewise plane surfaces). In these cases the corresponding singularities are not in $H^1(\Gamma)$ and one cannot apply the results of [**15**]. To the author's knowledge, the analysis of the high-order polynomial approximations of the least regular vertex singularities is missing in the existing literature. With this note we aim to fill this gap. As in [**4, 6**], we perform the error analysis on a scale of fractional order Sobolev spaces. However, in contrast to [**4**], the analysis of p-approximations in this note relies on explicit definitions of corresponding norms by the K-method of interpolation.

We also note that for BIE with weakly singular operators, where the energy space is $H^{-1/2}(\Gamma)$ (or $\tilde{H}^{-1/2}(\Gamma)$, if Γ is an open surface), the minimum assumptions for singularity parameters are

$$\lambda \geq \lambda_1 > -3/2 \quad \text{and} \quad \gamma \geq \gamma_1 > -1.$$

Polynomial approximations of singularities (i)–(iii) under these minimum assumptions were studied in [**5, 7**] in the context of the p-BEM and the hp-BEM with quasi-uniform meshes.

The rest of the paper is organised as follows. In the next section we introduce a quasi-uniform mesh discretising a Lipschitz polyhedral surface, define corresponding sets of piecewise polynomials, recall definitions of Sobolev spaces and norms, and collect several auxiliary results. Section 3 is focused on p-approximations of vertex singularities on a separate element of the fixed size. Then in §4 we prove the main result (Theorem 4.1), which states an error estimate (in terms of the mesh parameter h and polynomial degree p) for the approximation of vertex singularities by piecewise polynomials on quasi-uniform meshes.

2. Preliminaries. Throughout the paper, Γ denotes a Lipschitz polyhedral surface with plane faces $\Gamma^{(i)}$ and straight edges. In what follows, $h > 0$ and $p \geq 1$ will always specify the mesh parameter and a polynomial degree, respectively. We will denote by C a generic positive constant which does not depend on h and p.

For any bounded domain $\Omega \subset \mathbf{R}^n$ we will denote $\rho_\Omega = \sup\{\mathrm{diam}(B);$ B is a ball in $\Omega\}$. By $A \simeq B$ we mean that A is equivalent to B, i.e., there exists a constant $C > 0$ such that $C\,B \leq A \leq C^{-1}B$ where B and A may depend on a parameter (usually h or p) but C does not.

Let $\mathcal{M} = \{\Delta_h\}$ be a family of meshes $\Delta_h = \{\Gamma_j;\ j = 1, \dots, J\}$ on Γ, where Γ_j are open triangles or parallelograms such that $\bar{\Gamma} = \cup_{j=1}^{J} \bar{\Gamma}_j$. For any $\Gamma_j \in \Delta_h$ we will denote $h_j = \mathrm{diam}(\Gamma_j)$ and $\rho_j = \rho_{\Gamma_j}$. Let $h = \max_j h_j$. In this paper we will consider a family $\mathcal{M}$ of quasi-uniform meshes Δ_h on Γ. This implies the existence of positive constants $\sigma_1,\ \sigma_2$

independent of h such that for any $\Gamma_j \in \Delta_h$ and arbitrary $\Delta_h \in \mathcal{M}$

$$h \le \sigma_1 h_j, \qquad h_j \le \sigma_2 \rho_j. \tag{2.1}$$

Let $Q = (0,1)^2$ and $T = \{(x_1, x_2);\ 0 < x_1 < 1, 0 < x_2 < x_1\}$ be the reference square and triangle, respectively. Then for any $\Gamma_j \in \Delta_h$ one has $\Gamma_j = M_j(K)$, where M_j is an affine mapping with Jacobian $|J_j| \simeq h_j^2$ and $K = Q$ or T as appropriate.

Further, $\mathcal{P}_p(I)$ denotes the set of polynomials of degree $\le p$ on an interval $I \subset \mathbf{R}$. Moreover, $\mathcal{P}_p^1(T)$ is the set of polynomials on T of total degree $\le p$, and $\mathcal{P}_p^2(Q)$ is the set of polynomials on Q of degree $\le p$ in each variable. Let $K \subset \mathbf{R}^2$ be an arbitrary triangle or parallelogram, and let $K = M(T)$ or $K = M(Q)$ with an invertible affine mapping M. Then by $\mathcal{P}_p(K)$ we will denote the set of polynomials v on K such that $v \circ M \in \mathcal{P}_p^1(T)$ if K is a triangle and $v \circ M \in \mathcal{P}_p^2(Q)$ if K is a parallelogram (in particular, we will use this notation for $K = Q$ and $K = T$). For given p, we then consider the space of continuous, piecewise polynomials on the mesh $\Delta_h \in \mathcal{M}$,

$$S^{hp}(\Gamma) := \{v \in C^0(\Gamma);\ v|_{\Gamma_j} \in \mathcal{P}_p(\Gamma_j),\ j = 1, \dots, J\}.$$

Let us recall definitions of the Sobolev spaces and norms. First, for $t \in \mathbf{R}$ we define the Sobolev space $H^t(\mathbf{R}^n)$ ($n \ge 1$) in the usual way, via Fourier transform (see, e.g., [**14**]). This space is equipped with the norm

$$\|u\|_{H^t(\mathbf{R}^n)} = \left\|(1 + |\xi|^2)^{t/2}\, \hat{u}\right\|_{L_2(\mathbf{R}^n)}.$$

Here $\hat{u}(\xi) = (2\pi)^{-n/2} \int\limits_{\mathbf{R}^n} u(x) e^{-i\, x\cdot\xi} dx$ denotes the Fourier transform of the function u, $x = (x_1, \dots, x_n)$, $\xi = (\xi_1, \dots, \xi_n)$, $x \cdot \xi = x_1\xi_1 + \dots + x_n\xi_n$, and $L_2(\mathbf{R}^n)$ is the usual Lebesgue space of square integrable functions on $\mathbf{R}^n$ with the standard norm $\|\cdot\|_{L_2(\mathbf{R}^n)}$.

Then for a Lipschitz domain $\Omega \subseteq \mathbf{R}^n$ we set

$$H^t(\Omega) = \{u = \varphi|_\Omega;\ \varphi \in H^t(\mathbf{R}^n)\} \ \text{ with norm}$$
$$\|u\|_{H^t(\Omega)} = \inf_{\substack{\varphi \in H^t(\mathbf{R}^n) \\ u = \varphi|_\Omega}} \|\varphi\|_{H^t(\mathbf{R}^n)}$$

and

$$\tilde{H}^t(\Omega) = \{u \in H^t(\mathbf{R}^n);\ \operatorname{supp} u \subseteq \bar{\Omega}\} \text{ with norm}$$
$$\|u\|_{\tilde{H}^t(\Omega)} = \|u\|_{H^t(\mathbf{R}^n)}.$$

For any $t \in \mathbf{R}$ the space $H^t(\Omega)$ is the dual space of $\tilde{H}^{-t}(\Omega)$ with $L_2(\Omega) = H^0(\Omega) = \tilde{H}^0(\Omega)$ as pivot space. When Ω is bounded and $t > 0$ we will also use the space $H^t_0(\Omega)$ being the closure of $C_0^\infty(\Omega)$ with respect to the norm in $H^t(\Omega)$.

Note that $H^t(\Omega) = \tilde{H}^t(\Omega) = H^t_0(\Omega)$ if $0 \le t < \frac{1}{2}$, and $\tilde{H}^t(\Omega) = H^t_0(\Omega)$ if $t - \frac{1}{2}$ is not an integer (see [**13**]). Moreover, in the latter case, the norms $\|\cdot\|_{\tilde{H}^t(\Omega)}$ and $\|\cdot\|_{H^t(\Omega)}$ are equivalent.

The Sobolev spaces satisfy the interpolation property (see [**3**]): let $t_1, t_2 \in \mathbf{R}$, $t_1 < t_2$, and $t = (1-\theta)t_1 + \theta t_2$ for $0 < \theta < 1$, then

$$H^t(\Omega) = \big(H^{t_1}(\Omega), H^{t_2}(\Omega)\big)_\theta \qquad \text{and} \qquad \tilde{H}^t(\Omega) = \Big(\tilde{H}^{t_1}(\Omega), \tilde{H}^{t_2}(\Omega)\Big)_\theta.$$

Here we use the real K-method of interpolation where, for two normed spaces A_0 and A_1, the interpolation space $(A_0, A_1)_\theta$ $(0 < \theta < 1)$ is equipped with the norm

$$\|a\|_{(A_0,A_1)_{\theta,2}} := \left(\int_0^\infty t^{-2\theta} \inf_{a=a_0+a_1} (\|a_0\|^2_{A_0} + t^2\,\|a_1\|^2_{A_1})\,\frac{dt}{t}\right)^{1/2}.$$

If Ω is a bounded Lipschitz domain, then the following equivalent norms can be defined for spaces $H^t(\Omega)$, $t \ge 0$ (we will use the same notation $\|\cdot\|_{H^t(\Omega)}$ for these norms). First, we set

$$\|u\|_{H^0(\Omega)} = \|u\|_{L_2(\Omega)}.$$

If $t \ge 1$ is an integer, then

$$\|u\|^2_{H^t(\Omega)} = \|u\|^2_{H^{t-1}(\Omega)} + |u|^2_{H^t(\Omega)}$$

where

$$|u|^2_{H^t(\Omega)} = \int_\Omega |D^t u(x)|^2\,dx.$$

Here, $|D^t u(x)|^2 = \sum_{|\alpha|=t} |D^\alpha u(x)|^2$ in the usual notation with multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ and with respect to Cartesian coordinates $x = (x_1, \ldots, x_n)$. For a positive non-integer $t = m + \sigma$ with integer $m \geq 0$ and $0 < \sigma < 1$, the norm in $H^t(\Omega)$ is defined as

$$\|u\|^2_{H^t(\Omega)} = \|u\|^2_{H^m(\Omega)} + |u|^2_{H^t(\Omega)}$$

with semi-norm

$$|u|^2_{H^t(\Omega)} = \sum_{|\alpha|=m} \int_\Omega \int_\Omega \frac{|D^\alpha u(x) - D^\alpha u(y)|^2}{|x-y|^{n+2\sigma}}\, dx\, dy.$$

We also note that if $\Omega \subset \mathbf{R}^n$ is a bounded Lipschitz domain, then its boundary $\partial\Omega$ is locally the graph of a Lipschitz function. Since the Sobolev spaces H^t for $|t| \leq 1$ are invariant under Lipschitz (i.e., $C^{0,1}$) coordinate transformations, the spaces $H^t(\partial\Omega)$ with $|t| \leq 1$ can be defined in the usual way via a partition of unity subordinate to a finite family of local coordinate patches (see [**1, 14**]). Due to such a definition, the properties of Sobolev spaces on Lipschitz domains in $\mathbf{R}^n$ carry over to Sobolev spaces on Lipschitz surfaces. If $\tilde{\Gamma}$ is an open surface in $\mathbf{R}^n$, then the Sobolev spaces $H^t(\tilde{\Gamma})$, $\tilde{H}^t(\tilde{\Gamma})$ for $|t| \leq 1$ and $H^t_0(\tilde{\Gamma})$ for $0 < t \leq 1$ are constructed in terms of the above Sobolev spaces $H^t(\partial\Omega)$, where $\partial\Omega$ in this case is a closed Lipschitz surface which contains $\tilde{\Gamma}$ (see [**14**]).

Now let us collect several technical lemmas. We will need the following scaling result.

Lemma 2.1. *Let K^h and K be two open subsets of $\mathbf{R}^n$ such that $K^h = M(K)$ under an invertible affine mapping M. Let* $\operatorname{diam} K^h \simeq \rho_{K^h} \simeq h$ *and* $\operatorname{diam} K \simeq \rho_K \simeq 1$. *If $u \in H^m(K^h)$ with integer $m \geq 0$, then $\hat{u} = u \circ M \in H^m(K)$ and there exists a positive constant C depending on m but not on h or u such that*

$$|\hat{u}|_{H^m(K)} \leq C h^{m-\frac{n}{2}} |u|_{H^m(K^h)}. \tag{2.2}$$

Analogously for any $\hat{u} \in H^m(K)$ there holds

$$|u|_{H^m(K^h)} \leq C h^{\frac{n}{2}-m} |\hat{u}|_{H^m(K)}. \tag{2.3}$$

Moreover, if $\hat{u} \in H^s(K)$ *with real* $s \in [0, m]$, *then*

$$C_1 h^{\frac{n}{2}} \|\hat{u}\|_{H^s(K)} \le \|u\|_{H^s(K^h)} \le C_2 h^{\frac{n}{2}-s} \|\hat{u}\|_{H^s(K)}. \tag{2.4}$$

For the proof of (2.2), (2.3) see [**9**, Theorem 3.1.2]. Inequalities (2.4) then follow by interpolation (see [**1**, Lemma 4.3]).

The next theorem states the *hp*-approximation result for piecewise smooth functions.

Theorem 2.1. *Let* $m > 1$. *Assume that* $u \in H^1(\Gamma)$ *and* $u \in H^m(\Gamma^{(i)})$ *for any face* $\Gamma^{(i)}$ *of* Γ. *Then there exists* $u_{hp} \in S^{hp}(\Gamma)$ *such that for* $s \in [0, 1]$

$$\|u - u_{hp}\|_{H^s(\Gamma)} \le C h^{\mu - s} p^{-(m-\tilde{s})} \left(\sum_i \|u\|^2_{H^m(\Gamma^{(i)})} \right)^{1/2},$$

where $\mu = \min\{m, p+1\}$ *and*

$$\tilde{s} = \begin{cases} 1/2 & \text{if } \ s \in [0, 1/2), \\ 1/2 + \varepsilon, \ \varepsilon > 0 & \text{if } \ s = 1/2, \\ s & \text{if } s \in (1/2, 1]. \end{cases} \tag{2.5}$$

This result has been proved in [**6**, Proposition 4.1] for a plane open surface Γ. The proof of Theorem 2.1 repeats the arguments from [**6**] and is skipped.

The following two lemmas have been also proved in [**6**], cf. Lemma 3.4 and Lemma 3.5 therein.

Lemma 2.2. *Let* K^h *be a triangle (respectively, a parallelogram) satisfying the assumptions of Lemma* 2.1, *and let* l^h *be a side of* K^h *with vertices* v_1, v_2. *Let* $w_{hp} \in \mathcal{P}_p(l^h)$ *be such that* $w_{hp}(v_1) = w_{hp}(v_2) = 0$, *and* $\|w_{hp}\|_{L_2(l^h)} \le f(h, p)$. *Then there exists* $u_{hp} \in \mathcal{P}_{2p+1}(K^h)$ *(respectively,* $u_{hp} \in \mathcal{P}_p(K^h)$*) such that* $u_{hp} = w_{hp}$ *on* l^h, $u_{hp} = 0$ *on* $\partial K^h \backslash l^h$, *and for* $0 \le s \le 1$

$$\|u_{hp}\|_{H^s(K^h)} \le C\, h^{1/2-s}\, p^{-1+2s}\, f(h, p).$$

Lemma 2.3. *Let $\Delta_h = \{\Gamma_j\}$ be a quasi-uniform mesh on Γ. Then for $0 < s < 1$*

$$\|u\|^2_{H^s(\Gamma)} \ge \sum_j \|u\|^2_{H^s(\Gamma_j)} \quad \forall\ u \in H^s(\Gamma),$$

and for $1/2 < s < 1$ there holds

$$\|u\|^2_{H^s(\Gamma)} \le C \sum_j \left(h_j^{-2s} \|u\|^2_{L_2(\Gamma_j)} + |u|^2_{H^s(\Gamma_j)} \right) \quad \forall\ u \in H^s(\Gamma). \tag{2.6}$$

The positive constant C in (2.6) *is independent of u and the mesh Δ.*

3. p-approximation on a separate element of the fixed size. We start with a model situation on the reference square $Q = (0,1)^2$. This will lead us to the p-approximation result on a separate element (either a triangle or a parallelogram) of the fixed size.

For the model situation, let $\kappa > 1$ and denote $S_\kappa = \{x \in Q;\ \kappa^{-1}x_1 < x_2 < \kappa x_1\}$. We consider the following singular function over the square Q:

$$u(r,\theta) = r^\lambda |\log r|^\beta \chi(r) w(\theta), \tag{3.1}$$

where (r,θ) denote local polar coordinates with origin at $(0,0)$, $\lambda > -1/2$, $\beta \ge 0$ is an integer, $w(\theta)$ is sufficiently smooth, and χ is a C^∞ cut-off function satisfying

$$\chi(r) = 1 \ \text{ for } \ 0 \le r \le \delta/2 \ \text{ and } \ \chi(r) = 0 \ \text{ for } \ r \ge \delta. \tag{3.2}$$

Here, $\delta \in (0,1)$ is small enough. If $\lambda = 0$, we will assume that β is a positive integer, so that the function u has only a logarithmic singularity in this case. Observing that $u \in H^s(S_{\kappa_0})$ for $\kappa_0 > 1$ and for any $s \in [0, \lambda+1)$, we study polynomial approximations of u. We emphasize again that for $\lambda \in (-1/2, 0]$ the function u is not $H^1(\Gamma)$-regular, and one cannot apply the results of [**2, 4, 15**], were λ was assumed to be positive.

Theorem 3.1. *Let u be given by* (3.1). *Then there exists a sequence $u_p \in \mathcal{P}^2_{p+2}(Q)$, $p = 1, 2, \ldots$, such that $u_p = 0$ at the origin* $(0,0)$,

$$\|u - u_p\|_{H^s(S_{\kappa_0})} \le C\, p^{-2(\lambda+1-s)} (1 + \log p)^\beta, \qquad 0 \le s < \min\{1, \lambda+1\}, \tag{3.3}$$

and for any straight line $\ell \ni (0,0)$ *there holds*

$$\|u-u_p\|_{L_2(\ell\cap\bar S_{\kappa_0})} \le C\,p^{-2(\lambda+1/2)}\,(1+\log p)^{\beta}. \tag{3.4}$$

Although the results of [**2**] and [**15**] cannot be applied directly, we will use the approach developed in these papers (see, in particular, Theorem 5.1 in [**2**] and Theorem 8.1 in [**15**]). First, we extend u smoothly from S_{κ_0} to $S_\kappa \supset S_{\kappa_0}$. This can be done by multiplying (3.1) by a C^∞ cut-off function $\tilde\chi(\theta)$ such that for $\kappa > \kappa_0$

$$\begin{aligned}&\tilde\chi(\theta)=1 \ \text{ for } \ \arctan\kappa_0^{-1}\le\theta\le\arctan\kappa_0,\\ &\tilde\chi(\theta)=0 \ \text{ for } \ \theta\le\arctan\kappa^{-1} \ \text{ and } \ \theta\ge\arctan\kappa.\end{aligned}$$

We will retain the notation u for the extended function. Let

$$\xi(x_1,x_2)=(x_1-\kappa x_2)(\kappa x_1-x_2)=r^2\Phi_1(\theta)$$

and

$$u_0(x_1,x_2)=\frac{u(x_1,x_2)}{\xi(x_1,x_2)}=r^{\lambda-2}|\log r|^{\beta}\chi(r)\Phi_2(\theta),$$

where $\Phi_2(\theta)$ is smooth. Introducing a cut-off function ω such that

$$\omega\in C^\infty(\mathbf{R}),\quad \omega(z)=0 \ \text{ for } \ z\le 1,\quad \omega(z)=1 \ \text{ for } \ z\ge 2, \tag{3.5}$$

we define for a small $\Delta\in(0,1)$

$$\omega^\Delta(r)=\omega\left(\tfrac{r}{\Delta}\right),\quad \tilde\omega^\Delta(r)=1-\omega^\Delta(r),\quad r\ge 0.$$

Then we decompose u_0 as

$$u_0(x)=\frac{u(x)}{\xi(x)}=u_0(x)\omega^\Delta(r)+u_0(x)\tilde\omega^\Delta(r)=:v_0(x)+w_0(x). \tag{3.6}$$

The function v_0 in (3.6) is smooth and vanishes for $0\le r\le\Delta$. Moreover, for any non-negative integers k and l there exists a positive constant $C(k+l)$ independent of Δ such that for $(x_1,x_2)\in Q$ and for $i=1,2$

$$\left|\frac{\partial^{k+l}v_0}{\partial x_1^k\partial x_2^l}\right|\le C(k+l)\begin{cases}0 & \text{for } \ 0<r<\Delta,\\ x_i^{\lambda-2-k-l}|\log\Delta|^{\beta} & \text{otherwise.}\end{cases} \tag{3.7}$$

Polynomial approximations of functions satisfying (3.7) (and not necessarily having the explicit form given above) were investigated in [**2**] when proving Theorem 5.1 therein, and were also studied in [**15**, Theorem 8.1]. The estimate for the approximation error in the $H^1(S_\kappa)$-norm immediately follows from [**2**], while [**15**] gives also the estimate in the $L_2(S_\kappa)$-norm and then, by interpolation, in the norm of $H^s(S_\kappa)$ with $0 \le s \le 1$. Moreover, [**15**, Lemma 8.2] estimates the approximation error in the $L_2(\ell \cap S_\kappa)$-norm, where ℓ is the line $x_1 = \tilde{\kappa} x_2$ ($\kappa_0^{-1} \le \tilde{\kappa} \le \kappa_0$). We summarise the mentioned results in the following lemma.

Lemma 3.1. *Let* $\Delta = p^{-2}$. *If* v_0 *satisfies* (3.7), *then there exists a sequence* $v_p \in \mathcal{P}^2_{p+2}(Q)$, $p = 1, 2, \ldots,$ *such that* $v_p(0,0) = 0$ *and for any* $0 \le s \le 1$

$$\|\xi v_0 - v_p\|_{H^s(S_{\kappa_0})} \le C\, p^{-2(\lambda+1-s)}\, (1 + \log p)^\beta.$$

Moreover,

$$\|\xi v_0 - v_p\|_{L_2(\ell \cap \bar{S}_{\kappa_0})} \le C\, p^{-2(\lambda+1/2)}\, (1 + \log p)^\beta,$$

where ℓ *denotes the line* $x_1 = \tilde{\kappa} x_2$ ($\kappa_0^{-1} \le \tilde{\kappa} \le \kappa_0$).

The function w_0 in (3.6) has a small support, $\operatorname{supp} w_0 \subset \bar{K}_\Delta = \{x \in \bar{S}_\kappa;\ 0 \le r \le 2\Delta\}$. In the next lemma we show that the function ξw_0 being approximated by zero leads to the same estimates as in (3.3, 3.4).

Lemma 3.2. *Let* $\Delta = p^{-2}$. *Then for* $0 \le s < \min\{1, \lambda + 1\}$

$$\|\xi w_0\|_{H^s(S_{\kappa_0})} \le C\, p^{-2(\lambda+1-s)}\, (1 + \log p)^\beta. \tag{3.8}$$

$$\|\xi w_0\|_{L_2(\ell \cap \bar{S}_{\kappa_0})} \le C\, p^{-2(\lambda+1/2)}\, (1 + \log p)^\beta, \tag{3.9}$$

where ℓ *is the same as in Lemma* 3.1.

Proof. First, we prove (3.8) for $s = 0$. For sufficiently small $\Delta > 0$ one has (hereafter, $\theta_1 = \arctan \kappa^{-1},\ \theta_2 = \arctan \kappa$)

$$\begin{aligned} \|\xi w_0\|^2_{L_2(S_{\kappa_0})} &\le \|\xi w_0\|^2_{L_2(S_\kappa)} = \|\xi w_0\|^2_{L_2(K_\Delta)} \\ &\le C \int_0^{2\Delta} \int_{\theta_1}^{\theta_2} r^{2\lambda} |\log r|^{2\beta}\, r\, d\theta dr \le C \Delta^{2\lambda+2} |\log \Delta|^{2\beta}, \\ &\qquad \lambda > -1, \end{aligned} \tag{3.10}$$

where $C > 0$ is independent of Δ. Let $0 < s < \min\{1, \lambda + 1\}$. Then

$$\begin{aligned} (3.11) \quad & \|\xi w_0\|^2_{H^s(S_{\kappa_0})} \\ & = \int_0^\infty t^{-2s} \inf_{\xi w_0 = w_1 + w_2} \left(\|w_1\|^2_{L_2(S_{\kappa_0})} + t^2 \|w_2\|^2_{H^1(S_{\kappa_0})} \right) \frac{dt}{t}. \end{aligned}$$

For any $t \in (0, \Delta)$ we define

$$\omega_t(r) = \omega\left(\tfrac{r}{t}\right), \quad \tilde{\omega}_t(r) = 1 - \omega_t(r), \quad r \geq 0,$$

where ω is as in (3.5). Then by (3.11) we have

$$\begin{aligned} (3.12)\ \|\xi w_0\|^2_{H^s(S_{\kappa_0})} \leq & \int_0^\Delta t^{-2s-1} \left(\|\xi w_0 \tilde{\omega}_t\|^2_{L_2(S_{\kappa_0})} + t^2 \|\xi w_0 \omega_t\|^2_{H^1(S_{\kappa_0})} \right) dt \\ & + \int_\Delta^\infty t^{-2s-1} \|\xi w_0\|^2_{L_2(S_{\kappa_0})} dt. \end{aligned}$$

Now we estimate the norms on the right-hand side of (3.12). Since $\tilde{\omega}_t(r) = 0$ for $r \geq 2t$, we obtain similarly to (3.10)

$$(3.13) \qquad \|\xi w_0 \tilde{\omega}_t\|^2_{L_2(S_{\kappa_0})} \leq C \int_0^{2t} r^{2\lambda+1} |\log r|^{2\beta}\, dr \leq C t^{2\lambda+2} |\log t|^{2\beta}.$$

To estimate the norm $\|\xi w_0 \omega_t\|_{H^1(S_{\kappa_0})}$ we evaluate the derivatives of $\xi w_0 \omega_t$. We will use the following inequalities

$$\left|\frac{\partial r}{\partial x_i}\right| \leq 1, \quad \left|\frac{\partial \theta}{\partial x_i}\right| \leq \frac{1}{x_i} \leq C(\kappa)\frac{1}{r}, \quad x \in S_\kappa, \quad i = 1, 2;$$

$$\begin{aligned} \left|\frac{d\omega^\Delta(r)}{dr}\right| = \left|\frac{d\tilde{\omega}^\Delta(r)}{dr}\right| & = \begin{cases} 0 & \text{for } 0 < r < \Delta \text{ or } r > 2\Delta, \\ \left|\omega'\left(\frac{r}{\Delta}\right)\right| \frac{1}{\Delta} & \text{for } \ \Delta \leq r \leq 2\Delta \end{cases} \\ & \leq C\, r^{-1} \qquad \text{for } \ r > 0, \end{aligned}$$

and a similar estimate for $\left|\frac{d\omega_t(r)}{dr}\right|$. Hence, for any $x \in S_\kappa$ we find by simple calculations

$$\begin{aligned}
(3.14) \qquad \left|\frac{\partial}{\partial x_i}(\xi w_0 \omega_t)\right| &= \left|\frac{\partial}{\partial x_i}\left(r^\lambda |\log r|^\beta \chi(r)\tilde{\chi}(\theta) w(\theta) \tilde{\omega}^\Delta(r) \omega_t(r)\right)\right| \\
&\le C\left[\left|\frac{\partial}{\partial x_i}(r^\lambda |\log r|^\beta)\right| + r^\lambda |\log r|^\beta \left(\left|\frac{d\chi}{dr}\right|\right.\right. \\
&\quad \left.\left. + \left|\frac{d\tilde{\chi}}{d\theta}\right| \frac{1}{r} + \left|\frac{dw}{d\theta}\right| \frac{1}{r} + \left|\frac{d\tilde{\omega}^\Delta}{dr}\right| + \left|\frac{d\omega_t}{dr}\right|\right)\right] \\
&\le C r^{\lambda - 1} |\log r|^\beta, \qquad i = 1, 2.
\end{aligned}$$

Since $\xi w_0 \omega_t$ vanishes on ∂S_κ and outside the domain $K^1_\Delta = \{x \in S_\kappa;\ t < r < 2\Delta\}$, we deduce from (3.14) that

$$\begin{aligned}
(3.15) \qquad \|\xi w_0 \omega_t\|^2_{H^1(S_{\kappa_0})} &\le C |\xi w_0 \omega_t|^2_{H^1(K^1_\Delta)} \\
&\le C \int_t^{2\Delta} \int_{\theta_1}^{\theta_2} r^{2\lambda - 2} |\log r|^{2\beta}\, r\, d\theta dr \\
&\le C \int_t^{2\Delta} r^{2\lambda - 1} |\log r|^{2\beta} dr \\
&\le C \begin{cases} t^{2\lambda} |\log t|^{2\beta} & \text{if} \quad \lambda < 0, \\ \Delta^{2\lambda} |\log \Delta|^{2\beta} & \text{if} \quad \lambda > 0. \end{cases}
\end{aligned}$$

If $\lambda = 0$, we introduce a small $\varepsilon \in (0, 2 - 2s)$ and estimate the norm $\|\xi w_0 \omega_t\|^2_{H^1(S_{\kappa_0})}$ as follows

$$\begin{aligned}
(3.16) \qquad \|\xi w_0 \omega_t\|^2_{H^1(S_{\kappa_0})} &\le C \int_t^{2\Delta} r^{-1} |\log r|^{2\beta} dr \\
&\le C |\log t|^{2\beta} \int_t^{2\Delta} r^{-1-\varepsilon} r^\varepsilon dr \\
&\le C \Delta^\varepsilon t^{-\varepsilon} |\log t|^{2\beta}.
\end{aligned}$$

Using estimates (3.10, 3.13, 3.15) for the norms on the right-hand side of (3.12) we obtain for $0 < s < \min\{1, \lambda+1\}$

$$\begin{aligned} \|\xi w_0\|^2_{H^s(S_{\kappa_0})} &\le C \int_0^{\Delta} t^{2\lambda+1-2s} |\log t|^{2\beta} dt \\ &\quad + C\Delta^{2\lambda+2} |\log \Delta|^{2\beta} \int_{\Delta}^{\infty} t^{-2s-1} dt \\ &\le C\Delta^{2(\lambda+1-s)} |\log \Delta|^{2\beta} \qquad \text{if } \ -1 < \lambda < 0 \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} \|\xi w_0\|^2_{H^s(S_{\kappa_0})} &\le C \int_0^{\Delta} t^{2\lambda+1-2s} |\log t|^{2\beta} dt \\ &\quad + C\Delta^{2\lambda} |\log \Delta|^{2\beta} \int_0^{\Delta} t^{-2s+1} dt \\ &\quad + C\Delta^{2\lambda+2} |\log \Delta|^{2\beta} \int_{\Delta}^{\infty} t^{-2s-1} dt \\ &\le C\Delta^{2(\lambda+1-s)} |\log \Delta|^{2\beta} \qquad \text{if } \ \lambda > 0. \end{aligned} \tag{3.18}$$

In the case when $\lambda = 0$ we proceed similarly and use (3.16) instead of (3.15). Then recalling that $0 < \varepsilon < 2 - 2s$ we have for $0 < s < 1$

$$\begin{aligned} \|\xi w_0\|^2_{H^s(S_{\kappa_0})} &\le C \int_0^{\Delta} t^{-2s-1} \Big(t^2 + t^{2-\varepsilon}\Delta^{\varepsilon}\Big) |\log t|^{2\beta} dt \\ &\quad + C\Delta^{2} |\log \Delta|^{2\beta} \int_{\Delta}^{\infty} t^{-2s-1} dt \\ &\le C\Delta^{2-2s} |\log \Delta|^{2\beta} + C\Delta^{\varepsilon} \int_0^{\Delta} t^{-2s+1-\varepsilon} |\log t|^{2\beta} dt \\ &\le C\Delta^{2(1-s)} |\log \Delta|^{2\beta} \qquad \text{if } \ \lambda = 0. \end{aligned} \tag{3.19}$$

Taking $\Delta = p^{-2}$ and using estimates (3.10, 3.17 - 3.19) we prove (3.8).

Let ℓ be the line $x_1 = \tilde{\kappa} x_2$, where $\kappa_0^{-1} \le \tilde{\kappa} \le \kappa_0$. Then, recalling that $\operatorname{supp} w_0 \subset \bar{K}_\Delta$, we find by simple calculations

$$\begin{aligned} \|\xi w_0\|^2_{L_2(\ell \cap \bar{S}_{\kappa_0})} &\le C(\lambda, \beta, \tilde{\kappa}) \int\limits_0^{2\Delta(1+\tilde{\kappa}^2)^{-1/2}} z^{2\lambda} |\log z|^{2\beta} dz \\ &\le C \Delta^{2\lambda+1} |\log \Delta|^{2\beta}. \end{aligned}$$

Setting $\Delta = p^{-2}$ we obtain (3.9). □

Proof of Theorem 3.1. The desired statement follows from Lemmas 3.1 and 3.2 making use of decomposition (3.6). □

Now we consider an element (triangle or parallelogram) $K \subset \mathbf{R}^2$ of the fixed size (i.e., we assume that $\operatorname{diam} K \simeq \rho_K \simeq 1$).

Theorem 3.2. *Let $K \subset \mathbf{R}^2$ and suppose that $O = (0,0)$ is a vertex of K. Let u be given by (3.1) on K. Then there exists a sequence $u_p \in \mathcal{P}_p(K)$, $p = 1, 2, \ldots$ such that for $0 \le s < \min\{1, \lambda + 1\}$*

$$\|u - u_p\|_{H^s(K)} \le C\, p^{-2(\lambda+1-s)} (1 + \log p)^\beta.$$

Moreover, $u_p(0,0) = 0$, $u_p = 0$ on the sides $l_i \subset \partial K$, $\bar{l}_i \not\ni O$, and

$$\|u - u_p\|_{L_2(l_k)} \le C\, p^{-2(\lambda+1/2)} (1 + \log p)^\beta \text{ for each side } l_k \subset \partial K,\ O \in \bar{l}_k.$$

The proof is based on Theorem 3.1 and repeats exactly the arguments in [**15**, Theorem 8.2].

4. ***hp*-approximation on quasi-uniform meshes.** In this section we prove the result on the approximation of vertex singularities by piecewise polynomials defined on the quasi-uniform mesh Δ_h discretising polyhedral surface Γ. Let us fix a vertex v of Γ. We will consider the vertex singularity u given by (3.1), where (r, θ) now refers to local polar coordinates (with origin at v) on each face of Γ containing v.

Theorem 4.1. *Let u be given by* (3.1) *with $\lambda > -\frac{1}{2}$ and an integer $\beta \geq 0$. Then there exists $u_{hp} \in S^{hp}(\Gamma)$ with $p \geq \lambda$ such that for $0 \leq s < \min\{1, \lambda+1\}$*

$$\|u - u_{hp}\|_{H^s(\Gamma)} \leq C\, h^{\lambda+1-s}\, p^{-2(\lambda+1-s)}\, (1 + \log(p/h))^{\beta+\nu}, \tag{4.1}$$

where $\nu = \frac{1}{2}$ if $p = \lambda$, and $\nu = 0$ otherwise.

If $1 \leq p < \lambda$, then there exists $u_{hp} \in S^{hp}(\Gamma)$ satisfying for $s \in [0,1]$

$$\|u - u_{hp}\|_{H^s(\Gamma)} \leq C\, h^{p+1-s}. \tag{4.2}$$

Proof. Note that assumption $1 \leq p < \lambda$ implies $\lambda > 1$. This case was considered in [**6**, Theorem 6.1], where estimate (4.2) was proved.

To prove (4.1) we decompose u as $u = \varphi_1 + \varphi_2$, where

$$\varphi_1 := u\chi(r/h_0), \quad \varphi_2 := u(1 - \chi(r/h_0)), \quad h_0 = (\sigma_1\sigma_2)^{-1}h, \tag{4.3}$$

χ is the cut-off function in (3.1), and $\sigma_1,\ \sigma_2$ are the same as in (2.1).

The singular function φ_1 has small support, $\operatorname{supp}\varphi_1 \subset \bar{A}_v := \cup\{\bar{\Gamma}_j;\ v \in \bar{\Gamma}_j\}$. Let $K^h = \Gamma_j \subset A_v$ and let $K \subset \mathbf{R}^2$ be a triangle or parallelogram such that $K^h = M_h(K)$ under the affine mapping $M_h : x_i = h\hat{x}_i,\ i = 1,2,\ x \in K^h,\ \hat{x} \in K$. Then $O = (0,0)$ is a vertex of K and for $h < \frac{1}{2}$ we have

$$\hat{\varphi}_1(\hat{x}) = \varphi_1(h\hat{x}_1, h\hat{x}_2) = h^\lambda \hat{r}^\lambda \sum_{k=0}^{\beta} \binom{\beta}{k} |\log h|^k |\log \hat{r}|^{\beta-k} \chi(\sigma_1\sigma_2\hat{r}) w(\hat{\theta}),$$

where $\hat{r} = (\hat{x}_1^2 + \hat{x}_2^2)^{1/2},\ \hat{\theta} = \arctan(\hat{x}_2/\hat{x}_1)$.

Let $\mathcal{A} = \{l_i\}$ contain those sides $l_i \subset \partial K$ for which $O \in \bar{l}_i$, and let $\mathcal{B}$ be the union of the other sides of K. Then applying Theorem 3.2 to each function $\hat{r}^\lambda |\log \hat{r}|^k \chi(\sigma_1\sigma_2\hat{r}) w(\hat{\theta}),\ k = 0, \ldots, \beta$, we find a polynomial $\hat{\phi} \in \mathcal{P}_p(K)$ such that $\hat{\phi}(0,0) = 0$, $\hat{\phi} = 0$ on $\mathcal{B}$,

$$\|\hat{\varphi}_1 - \hat{\phi}\|_{H^s(K)} \leq C(\beta)\, h^\lambda\, p^{-2(\lambda+1-s)}\, (1 + \log(p/h))^\beta, \quad 0 \leq s < \min\{1, \lambda+1\}, \tag{4.4}$$

$$\|\hat{\varphi}_1 - \hat{\phi}\|_{L_2(l)} \leq C(\beta)\, h^\lambda\, p^{-2(\lambda+1/2)}\, (1 + \log(p/h))^\beta \quad \text{for every } l \in \mathcal{A}. \tag{4.5}$$

Let us define $\phi_j := \hat{\phi} \circ M_h^{-1}$. Then $\phi_j \in \mathcal{P}_p(\Gamma_j)$, $\phi_j = 0$ at the vertex v and on the sides $l_i^h \in \mathcal{B}_j = M_h(\mathcal{B})$. Furthermore, making use of Lemma 2.1, we obtain by (4.4, 4.5)

$$\|\varphi_1 - \phi_j\|_{H^s(\Gamma_j)} \le C\, h^{\lambda+1-s}\, p^{-2(\lambda+1-s)}\, (1 + \log(p/h))^\beta, \quad 0 \le s < \min\{1, \lambda+1\}, \tag{4.6}$$

$$\|\varphi_1 - \phi_j\|_{L_2(l^h)} \le C\, h^{\lambda+1/2}\, p^{-2(\lambda+1/2)}\, (1 + \log(p/h))^\beta \quad \text{for every } l^h \in \mathcal{A}_j = M_h(\mathcal{A}). \tag{4.7}$$

Suppose that $\Gamma_i, \Gamma_j \subset A_v$ are two elements having the common edge $l^h = \bar{\Gamma}_i \cap \bar{\Gamma}_j$ (these elements may lie on different faces of Γ). Let $\phi_i \in \mathcal{P}_p(\Gamma_i)$ and $\phi_j \in \mathcal{P}_p(\Gamma_j)$ be the approximations of φ_1 constructed above and satisfying estimates (4.6 - 4.7). Then the jump $g = (\phi_j - \phi_i)|_{l^h}$ vanishes at the end points of l^h and

$$\|g\|_{L_2(l^h)} \le C\, h^{\lambda+1/2}\, p^{-2(\lambda+1/2)}\, (1 + \log(p/h))^\beta.$$

If Γ_i is a parallelogram, we use Lemma 2.2 to find a polynomial $z \in \mathcal{P}_p(\Gamma_i)$ such that

$$z = g \ \text{ on } \ l^h, \qquad z = 0 \ \text{ on } \ \partial\Gamma_i \backslash l^h, \tag{4.8}$$

and for $0 \le s \le 1$

$$\|z\|_{H^s(\Gamma_i)} \le C\, h^{\lambda+1-s}\, p^{-2(\lambda+1-s)}\, (1 + \log(p/h))^\beta. \tag{4.9}$$

In the case that Γ_i is a triangle, we note that (4.6) and (4.7) also hold for a polynomial ψ_j of degree $\left[\frac{p-1}{2}\right]$ (with different constants C for the upper bounds in (4.6) and (4.7)). Then Lemma 2.2 yields a polynomial $z \in P_p(\Gamma_i)$ which satisfies (4.8, 4.9) for Γ_i being a triangle.

Setting $\tilde{\phi} = \phi_i + z$ on Γ_i and $\tilde{\phi} = \phi_j$ on Γ_j we find a continuous piecewise polynomial $\tilde{\phi}$ such that the norms $\|\varphi_1 - \tilde{\phi}\|_{H^s(\Gamma_i)}$ and $\|\varphi_1 - \tilde{\phi}\|_{H^s(\Gamma_j)}$ are bounded as in (4.6) for $0 \le s < \min\{1, \lambda+1\}$.

Repeating the above procedure we construct a continuous function $\psi_1 \in C^0(\bar{A}_v)$ such that $\psi_1 = 0$ on ∂A_v, $\psi_1 \in \mathcal{P}_p(\Gamma_j)$ for each $\Gamma_j \subset A_v$, and

$$\|\varphi_1 - \psi_1\|_{H^s(\Gamma_j)} \le C\, h^{\lambda+1-s}\, p^{-2(\lambda+1-s)}\, (1 + \log(p/h))^\beta, \quad 0 \le s < \min\{1, \lambda+1\}. \tag{4.10}$$

Now we extend ψ_1 by zero onto $\Gamma \backslash A_v$ (keeping the notation ψ_1 for the extension). Then $\psi_1 \in S^{hp}(\Gamma)$ and there holds for $0 \le s < \min\{1, \lambda+1\}$

$$\|\varphi_1 - \psi_1\|_{H^s(\Gamma)} \le C\, h^{\lambda+1-s}\, p^{-2(\lambda+1-s)}\, (1 + \log(p/h))^{\beta}. \tag{4.11}$$

In fact, for $s = 0$ estimate (4.11) on Γ immediately follows from inequalities (4.10) on individual elements. If $1/2 < s < \min\{1, \lambda+1\}$, then we use Lemma 2.3:

$$\begin{aligned}\|\varphi_1 - \psi_1\|^2_{H^s(\Gamma)} &\le C\biggl(h^{-2s}\|\varphi_1 - \psi_1\|^2_{L_2(\Gamma)} + \sum_{j:\Gamma_j \subset \Gamma} |\varphi_1 - \psi_1|^2_{H^s(\Gamma_j)}\biggr)\\ &\le C\biggl(h^{-2s}\|\varphi_1 - \psi_1\|^2_{L_2(A_v)} + \sum_{j:\Gamma_j \subset A_v} \|\varphi_1 - \psi_1\|^2_{H^s(\Gamma_j)}\biggr),\end{aligned}$$

and (4.11) follows again from (4.10), because the number ν_v of elements in A_v is independent of h ($\nu_v \le \omega_v/\theta_0$, where ω_v is the total length of the closed piecewise smooth arc cut out in the unit sphere $\mathbf{S}^2$ by the edges of Γ having v as an endpoint, θ_0 is the minimal angle of elements in the mesh).

Finally, for $0 < s \le 1/2$, estimate (4.11) follows via interpolation between $H^0(\Gamma)$ and $H^{s'}(\Gamma)$ for some $s' \in (\frac{1}{2}, \min\{1, \lambda+1\})$.

Let $\Gamma^{(i)}$, $i = 1, \dots, \mathcal{I}_v$ be the faces of Γ at the vertex v. For the function φ_2 (see (4.3)) one has

$$\varphi_2 = r^{\lambda} |\log r|^{\beta} \chi(r)(1 - \chi(r/h_0)) w(\theta) \in H^m(\Gamma^{(i)}), \quad i = 1, \dots, \mathcal{I}_v,$$

where m depends on the regularity of $w(\theta)$, m is fixed and as large as required. Furthermore,

$$\operatorname{supp} \varphi_2 \subset \bar{R}^h, \quad \text{where } R^h = \Bigl\{x \in \bigcup_{i=1}^{\mathcal{I}_v} \bar{\Gamma}^{(i)};\ \tfrac{\delta}{2} h_0 < r(x) < \delta \text{ on each face } \Gamma^{(i)},\ i = 1, \dots, \mathcal{I}_v\Bigr\},$$

where δ is the same as in (3.2).

To bound the norm $\|\varphi_2\|_{H^k(\Gamma^{(i)})}$ we need the following inequalities:

$$\left|\frac{\partial^{l+n} r}{\partial x_1^l \partial x_2^n}\right| \le C r^{1-l-n}, \qquad \left|\frac{\partial^{l+n} \theta}{\partial x_1^l \partial x_2^n}\right| \le C r^{-l-n}$$

for any integer $l, n \geq 0$, and

$$\left|\frac{\partial^l}{\partial r^l}\Big(1-\chi(r/h_0)\Big)\right| = \begin{cases} 0 & \text{for } 0 < r < \frac{\delta}{2}h_0 \text{ and } r > \delta h_0, \\ |\chi^{(l)}|h_0^{-l} & \text{for } \ \frac{\delta}{2}h_0 \leq r \leq \delta h_0 \end{cases}$$
$$\leq C\, r^{-l} \qquad \text{for } \ r > 0$$

with any integer $l \geq 1$.

Hence, we find by simple calculations

$$\|\varphi_2\|^2_{H^k(\Gamma^{(i)})} \leq C(\log(1/h))^{2\beta} \int\limits_{\delta h_0/2}^{\delta} r^{2(\lambda-k)}\, r dr, \quad 0 \leq k \leq m. \tag{4.12}$$

Further, due to Theorem 2.1, there exists $\psi_2 \in S^{hp}(\Gamma)$ such that for $s \in [0,1]$

$$\|\varphi_2 - \psi_2\|^2_{H^s(\Gamma)} \leq C h^{2(\mu-s)} p^{-2(k-\tilde{s})} \sum_{i=1}^{\mathcal{I}_v} \|\varphi_2\|^2_{H^k(\Gamma^{(i)})}, \tag{4.13}$$

where $k \in (1, m]$ is integer, $\mu = \min\{k, p+1\}$, and $\tilde{s}$ is defined by (2.5).

If $\lambda + 1 \leq k \leq m$, then (4.12) and (4.13) yield

$$\|\varphi_2 - \psi_2\|_{H^s(\Gamma)} \leq C h^{\mu-s+\lambda-k+1} p^{-(k-\tilde{s})} \log^{\beta+\bar{\nu}}(1/h), \quad s \in [0,1], \tag{4.14}$$

where $\bar{\nu} = \frac{1}{2}$ if $k = \lambda+1$, and $\bar{\nu} = 0$ if $k > \lambda + 1$.

If $p > 2\lambda + \frac{3}{2}$, we select an integer k satisfying

$$2\lambda + \tfrac{5}{2} < k \leq p + 1.$$

Then $\mu = k > \frac{3}{2}$ and $p^{-(k-\tilde{s})} \leq p^{-2(\lambda+1-s)}$ for any $s \in [0,1]$.

If $\lambda < p \leq 2\lambda + \frac{3}{2}$ (i.e., p is bounded), we choose an integer k such that

$$\max\{1,\, \lambda+1\} < k \leq p + 1,$$

and if $p = \lambda$, then we take $k = \lambda + 1 = p + 1$. In both these cases $\mu = k > 1$ and $p^{-(k-\tilde{s})} \leq C(\lambda)\, p^{-2(\lambda+1-s)}$ for any $s \in [0,1]$.

Thus, for any $p \geq \lambda$, selecting k as indicated above we find by (4.14)

$$\|\varphi_2 - \psi_2\|_{H^s(\Gamma)} \leq C h^{\lambda+1-s} p^{-2(\lambda+1-s)} \log^{\beta+\nu}(1/h), \quad s \in [0,1], \tag{4.15}$$

where $\nu = \frac{1}{2}$ if $p = \lambda$ and $\nu = 0$ otherwise.

Now combination of (4.11) and (4.15) gives (4.1) with $u_{hp} := \psi_1 + \psi_2 \in S^{hp}(\Gamma)$. $\square$

Remark 4.1. If Γ is an open piecewise plane surface and the function u in (3.1) vanishes on $\partial\Gamma$, then $u \in \tilde{H}^s(\Gamma)$ for any $0 \leq s < \min\{1, \lambda+1\}$. In this case the same arguments as in the proof of Theorem 4.1 lead to even stronger result: if $p \geq \lambda$, then there exists $u_{hp} \in S_0^{hp}(\Gamma) := S^{hp}(\Gamma) \cap H_0^1(\Gamma)$ such that for $0 \leq s < \min\{1, \lambda+1\}$

$$\|u - u_{hp}\|_{\tilde{H}^s(\Gamma)} \leq C\, h^{\lambda+1-s}\, p^{-2(\lambda+1-s)}\, (1 + \log(p/h))^{\beta+\nu},$$

where ν is the same as in (4.1); if $1 \leq p < \lambda$, then there exists $u_{hp} \in S_0^{hp}(\Gamma)$ satisfying for $s \in [0,1]$

$$\|u - u_{hp}\|_{\tilde{H}^s(\Gamma)} \leq C\, h^{p+1-s}.$$

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Department of Mathematical Sciences, Brunel University, Uxbridge, West London UB8 3PH, UK.
Email address: albespalov@yahoo.com