JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 21, Number 2, Summer 2009

NEWTON'S METHOD FOR INVERSE OBSTACLE SCATTERING OF BURIED OBJECTS

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Communicated by Charles Groetsch

This paper is dedicated to Professor Rainer Kress on the occasion of his 65th birthday.

ABSTRACT. A Newton method to reconstruct the shape of a buried sound soft object through the measured far-field scattering data is given. The scattered field is represented as a single-layer potential which leads to an ill-posed integral equation of the first kind that is solved via Tikhonov regularization. The presented Newton based method combines ideas of both the iterative and decomposition methods and inherits the advantages of each of them, such as getting good reconstructions and not requiring a forward solver at each step. The numerical results show that the method yields good reconstruction.

1. Introduction. Inverse scattering of waves is a fundamental principle of applications such as radar and sonar techniques, nondestructive evaluation, geophysical exploration and medical imaging. In principle, in these applications, the effects of scattering objects on the propagation of the waves are exploited to obtain some information about the unknown object. As opposed to classical techniques of imaging such as computerized tomography, which are based on the fact that X rays travel along straight lines, inverse scattering problems take into account that the propagation of acoustic, electromagnetic and elastic waves has to be modeled by a wave equation. This means that inverse scattering requires a nonlinear model, whereas inverse tomography does linear.

The detection and identification of buried objects using electromagnetic waves are the areas of current importance for applications in remote sensing. The considered problem has practical applications such as detection of underground mines, pipes and cables. Most papers con-

Received by the editors on April 17, 2007, and in revised form on April 15, 2008.
DOI:10.1216/JIE-2009-21-2-317 Copyright ©2009 Rocky Mountain Mathematics Consortium

cerning the inverse scattering calculation are in free-space background. Colton and Monk [**1-3**] have carried out a series of acoustic wave inverse scattering calculations for two- and three-dimensional sound-soft impenetrable targets of several shapes. Ochs [**4**] has also discussed the limited-aperture problem of inverse scattering by applying the same method. Inverse obstacle reconstruction problem in free space was treated by Kirsch et al. [**5**], Jones and Mao [**6**], and Zinn [**7**] using different inversion techniques. Based on the Newton-Kantorovitch method, Roger [**8**], Kristensson and Vogel [**9**], Murch et al. [**10**], and Tobocman [**11**] have also solved two-dimensional inverse scattering problems of this type.

Because of the difficulties in obtaining the fundamental solution by numerical methods, the problem of inverse scattering in a half-space has rarely been attempted. Hohmann [**12**] has calculated the fundamental solution, but his calculations are restricted to the case of media with high conductivity at low frequency. Moreover, Chammeloux et al. [**13**] have applied the technique of computer tomography to process the images of buried cylindrical inhomogeneities. This method can avoid the calculation of the Green function, but they only obtained an approximate image of induced current, instead of the image of dielectric constant. Chiu and Kiang [**14**] solved buried inverse obstacle problem by Newton-Kantorovitch method. However, this method needs forward solver at each iteration step that needs high computational time especially for this problem due to the numerical evaluation of the fundamental solution of layered media. Recently, Kress et al. [**15**] reconstructed inhomogeneous surface impedance of a scatterer located inside a dielectric object by using the fundamental solution of space containing dielectric object . This problem was solved in free space by Kress and Akduman [**16**].

In this study, we are interested in shape reconstruction of a completely buried sound soft obstacle from measurements of the far field pattern on upper half space as depicted Fig. 1. Major advantage of the proposed method is that method does not need forward solver at each iteration step. Although obstacle is assumed to be C^2 smooth, the analysis can be extended for the domains with corners [**17.18**], and to cracks [**19**]. The presented method can also be applied for the reconstruction of both shape and boundary impedance of the obstacle [**20**]. The proposed Newton based method, which is sometimes called as Hybrid method,

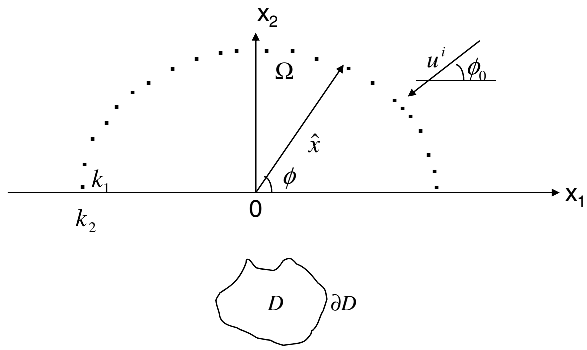

Figure 1. Geometry of the considered problem.

has been applied for shape reconstruction of the sound hard object [**21**] and cracks [**22**] in free space.

Given an open-bounded obstacle $D \subset R^2$ with an unbounded and connected complement and an incident field u^i, the direct scattering problem consists of finding the total field $u = u^i + u^s$ as the sum of the known incident field u^i and the scattered field u^s such that both the Helmholtz equation

$$\begin{aligned} &\Delta u + k_1^2 u = 0, \quad x_2 \geq 0 \\ &\Delta u + k_1^2 u = 0, \quad x_2 \leq 0 \quad \in R^2/\overline{D} \end{aligned} \tag{1.1}$$

with positive wave number k_1 and k_2, total field satisfies Dirichlet boundary condition as

$$u(x) = 0, \quad x \in \partial D \tag{1.2}$$

and the scattered wave u^s fulfills the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - ik_1 u^s\right) = 0, \quad r = |x| \tag{1.3}$$

uniformly with respect to all directions on upper unit semi circle Ω and the scattered wave has a asymptotic behavior as

$$u^s(x) = \frac{e^{ik_1|x|}}{\sqrt{|x|}}\left\{u_\infty(\hat{x}) + O\left(\frac{1}{|x|}\right)\right\}, \quad |x| \to \infty, \quad \hat{x} \in \Omega \tag{1.4}$$

with $\hat{x} = \frac{x}{|x|} = (\cos\phi, \sin\phi), \phi \in (0, \pi)$, uniformly in all directions on upper semi circle, with the far field pattern u_∞ defined on the unit upper semi circle Ω.

The inverse obstacle problem is the determination of boundary ∂D by means of the given far field pattern $u_\infty(\hat{x}), \hat{x} \in \Omega$ for the given excitation such as plane wave. The direct scattering problem for any illumination can be represented by an operator as $F : \partial D \to u_\infty$ that maps the boundary ∂D onto the far field pattern u_∞. By using this operator, given a far field pattern u_∞, the inverse problem is expressed as the solution of nonlinear and ill-posed operator equation,

$$F(\partial D) = u_\infty \tag{1.5}$$

for the unknown surface ∂D. Because of the fact that direct scattering problem depends nonlinearly on the boundary, Eq. (1.5) is nonlinear. Also, it is ill posed since the far field pattern is an analytic function on the unit semi circle.

2. Regularized Newton Iteration.

Field u^0 which would be observed when the scatterer was absent can be expressed straight as [**19**],

$$u^0(x) = \begin{cases} e^{-ik_1(x_1\cos\phi_0 + x_2\sin\phi_0)} \\ \qquad +\mathrm{Re}^{-ik_1(x_1\cos\phi_0 - x_2\sin\phi_0)}, & x_2 \geq 0 \\ Te^{-ik_1(x_1\cos\phi_1 + x_2\sin\phi_1)}, & x_2 < 0 \end{cases} \tag{2.1}$$

where ϕ_0, incident angle using the continuity of field and its normal derivative on $x_2 = 0, \phi_1, R, T$ can be obtained straightforwardly as [**23**],

$$\phi_1 = \arccos\left(\frac{k_1}{k_2}\cos(\phi_0)\right) \tag{2.2}$$

$$R = \frac{k_1\sin\phi_0 - k_2\sin\phi_1}{k_1\sin\phi_0 + k_2\sin\phi_1} \tag{2.3}$$

$$T = \frac{2k_1\sin\phi_0}{k_1\sin\phi_0 + k_2\sin\phi_1} \tag{2.4}$$

The presented Newton iteration starts with an initial estimate Γ_0. The scattered field in the closed exterior of Γ_0 can be expressed by using single layer potential [**24**]

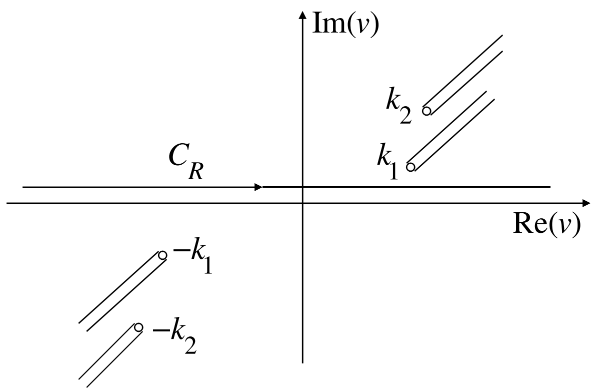

Figure 2. The regularity line C_R and cut of the complex v plane.

$$(P_{\Gamma_0}\varphi)(x) = \int_{\Gamma_0} \phi(x,y)\varphi(y)ds(y), \quad x \in R^2/\Gamma_0 \tag{2.5}$$

with density $\varphi \in L^2(\Gamma_0)$ in the exterior of the surface Γ_0, where

$$\phi(x,y) = \begin{cases} \phi_t(x,y), & x_2 > 0,\ y_2 < 0 \\ \frac{i}{4}H_0^{(1)}(k_2|x-y|) + \phi_r(x,y), & x_2 < 0,\ y_2 < 0 \end{cases} \tag{2.6}$$

represents the fundamental solution of Helmoltz equation in layered medium. Where $H_0^{(1)}(.)$ denotes the zero order Hankel function of the first kind. In (2.6), the terms $\phi_t(x,y)$ and $\phi_r(x,y)$ are smooth part of fundamental solution and given as [23]

$$\phi_t(x,y) = \frac{1}{2\pi}\int_{C_R} \frac{1}{\gamma_2+\gamma_1} e^{\gamma_2 y_2 - \gamma_1 x_2} e^{iv(x_1-y_1)} dv \tag{2.7}$$

$$\phi_r(x,y) = \frac{1}{2\pi}\int_{C_R} \frac{1}{2\gamma_2}\frac{\gamma_2-\gamma_1}{\gamma_2+\gamma_1} e^{\gamma_2(x_2+y_2)} e^{iv(x_1-y_1)} dv \tag{2.8}$$

where C_R is regularity line as depicted in Fig.2, γ_1 and γ_2 are the square root functions

$$\gamma_1(v) = \sqrt{v^2-k_1^2}, \qquad \gamma_2(v) = \sqrt{v^2-k_2^2} \tag{2.9}$$

which are defined in the complex v plane cut as depicted in Fig. 2 with the conditions

$$\gamma_j(0) = -ik_j, \qquad j = 1,2 \tag{2.10}$$

The far field pattern of the potential (2.5) for the scattered direction $\hat{x} = (\cos\phi, \sin\phi), \phi \in (0, \pi)$ as shown in Fig.1, denoted by $(P_{\Gamma_0,\infty}\varphi)(\hat{x})$ can be derived from $\phi_t(x, y)$ given in (2.7) while $|x| \to \infty$ as [**23**]

$$(P_{\Gamma_0,\infty}\varphi)(\hat{x}) = \frac{e^{i\pi/4}}{\sqrt{2\pi k_1}} \frac{2k_1 \sin\phi}{k_1 \sin\phi + \sqrt{k_2^2 - k_1^2 \cos^2\phi}} \int_{\Gamma_0} e^{-i\left\{k_1 y_1 \cos\phi + y_2 \sqrt{k_2^2 - k_1^2 \cos^2\phi}\right\}} \varphi(y) ds(y) \tag{2.11}$$

Because of the fact that the scattered field u^s is uniquely determined by its far field pattern u_∞, the density φ can be seen to be the unique solution of the ill posed integral equation

$$(P_{\Gamma_0,\infty}\varphi)(\hat{x}) = u_\infty(\hat{x}) \tag{2.12}$$

Due to its analytic kernel, integral equation in (2.12) is severely ill posed [**20**]. However, the operator $P_{\infty,\Gamma_0} : L^2(\Gamma_0) \to L^2(\Omega)$ in (2.11) is known to be injective and has dense range. Therefore, Tikhonov regularization can be applied for a stable approximate solution of (2.13), that is, the ill-posed equation (2.12) is replaced by

$$\alpha\varphi + P^*_{\Gamma_0,\infty} P_{\Gamma_0,\infty}\varphi = P^*_{\Gamma_0,\infty} u_\infty \tag{2.13}$$

with some positive regularization parameter α and the adjoint $P^*_{\Gamma_0,\infty}$ of $P_{\Gamma_0,\infty}$.

For the further description of the reconstruction scheme we represent the curve Γ_0 by a regular parameterization

$$\Gamma_0 = \{z_0(t) : t \in [0, 2\pi)\} \tag{2.14}$$

with a 2π periodic function $z_0 : R \to R^2$. Searching the location where the boundary condition (1.2) is satisfied, we approximate the total field u by the Taylor formula of order one with respect to the normal direction at Γ_0. For this purpose, we try to update in the form

$$\Gamma_1 = \{z_1(t) = z_0(t) + h(t)v_0(t) : t \in [0, 2\pi)\} \tag{2.15}$$

where v_0 denotes the outward normal vector to Γ_0 and $h : IR \to IR$ is a sufficiently small 2π periodic function. The normal vector can be expressed through the parameterization (2.14) via

$$v_0(t) = \frac{(z_0'(t))^\perp}{|z_0(t)|}, \qquad t \in [0, 2\pi) \tag{2.16}$$

where for any vector $a = (a_1, a_2)$, we set $a^\perp = (a_2, -a_1)$. Then the first order Taylor formula requires the update function h to satisfy

$$u + \frac{\partial u}{\partial v_0} h = 0 \tag{2.17}$$

Once the single layer density φ is known from (2.13), the values u and normal derivative $\partial u / \partial v_0$ of the total field on Γ_0 can be obtained through the jump relations for the single-layer potential [**24**], that is, by

$$u(x) = u^0(x) + \int_{\Gamma_0} \phi(x, y)\varphi(y) ds(y), \qquad x \in \Gamma_0 \tag{2.18}$$

$$\frac{\partial u}{\partial v_0}(x) = \frac{\partial u^0}{\partial v_0}(x) + \int_{\Gamma_0} \frac{\partial \phi(x, y)}{\partial v_0(x)} \varphi(y) ds(y) - \frac{1}{2}\varphi(x), \quad x \in \Gamma_0 \tag{2.19}$$

The integrals in (2.18) and (2.19) can be accurately evaluated by the quadrature rules as described in [**24**].

Since the solution of (2.17) is sensitive to errors in the normal derivative of u in the vicinity of zeros, Eq. (2.17) is solved in a stable way by the least squares method. For this purpose, we express h_0 in terms of the basis functions $\omega_1, \omega_2, \dots, \omega_j$ by

$$h(t) = \sum_{j=1}^{J} a_j \omega_j(t), \quad t \in [0, 2\pi) \tag{2.20}$$

with possible choices of the basis functions given by splines or trigonometric polynomials. Then, we satisfy (2.17) in a penalized least squares sense, that is, the coefficients $a_1, a_2, \dots, a_j$ in (2.20) are chosen such that for a set of collocation points $t_1, t_2, \dots, t_N$ in $[0, 2\pi)$ the penalized least squares sum

$$\sum_{n=1}^{N} \left| u(z_0(t_n)) + \frac{\partial u}{\partial v_0}(z_0(t_n)) \sum_{j=1}^{J} a_j \omega_j(t_n) \right|^2 + \beta \sum_{j=1}^{J} a_j^2 \tag{2.21}$$

with some regularization parameter $\beta > 0$ is minimized. After the solution of (2.21), updated boundary Γ_1 is obtained by means of (2.15) and the same procedure applied for Γ_1.

3. Numerical Examples. In the following examples, the degree of the trigonometric polynomial used for the approximation of the boundary is denoted by J. The Tikhonov regularization parameters for (2.13) is denoted by α , the number of Newton steps is denoted by T , the penalty factor in (2.21) by β, and the incidence angle by ϕ_0. In all examples, we used $N = 50$ collocation points. In order to avoid an inverse crime, the synthetic data were obtained by solving the combined single- and double-layer boundary integral equation for the direct scattering problem by the Nyström method as described in [**24**] with 100 quadrature points. The wave number of upper medium is chosen as $k_1 = 1$, and the iterations were started with circle of radius $1\,m$ buried same distance as object. The parameterizations of boundaries are given with respect to $0x_1x_2$ coordinate system.

In the first example, we consider the identification of a peanut-shaped boundary object $2\,m$ buried in the medium whose wave number is $k_2 = 1.5 + 0.01i$. The parameterization of the boundary given as

(3.1)
$$\partial D = \left\{ \left(\sqrt{\cos^2 t + 0.25 \sin^2 t} \right) (\cos t, \sin t - 2), \quad t \in [0, 2\pi) \right\} [m]$$

As parameters, we choose $\alpha = 10^{-7}$ and $\alpha = 10^{-4}$ for exact and noisy data respectively, $\beta = 0.001$, $J = 6$, and $T = 7$.

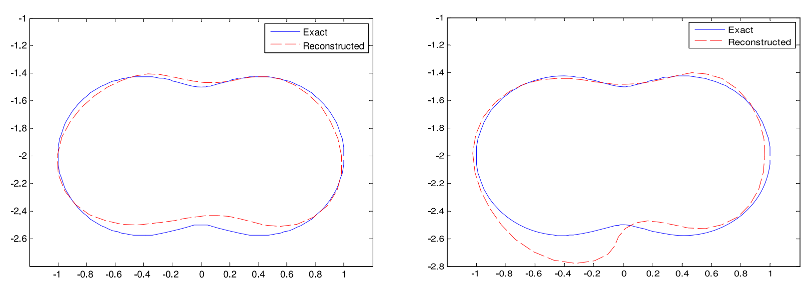

Figure 3. The reconstruction of the peanut for $\phi_0 = \pi/2$ without noise (left) and with %2 noise (right).

In the second example, the reconstruction of a kite-shaped boundary object $3\,m$ buried in the medium is considered. The wave number of the medium where the object is buried is $k_2 = 2$. The parameterization of

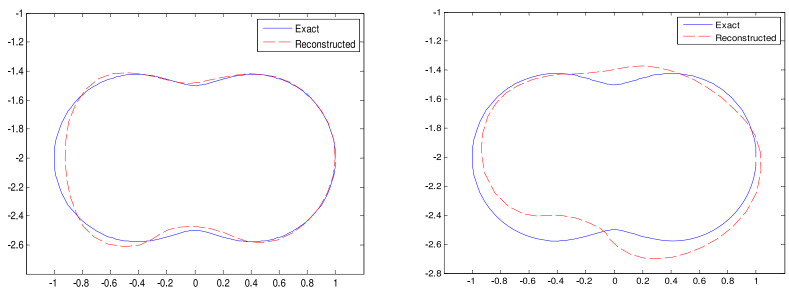

Figure 4. The reconstruction of the peanut for $\phi_0 = \pi/4$ without noise (left) and with %2 noise (right).

kite-shaped boundary is given by

$$\partial D = \{(\cos t + 0.65 \cos 2t - 0.65,\ 1.5 \sin t - 3), \quad t \in [0, 2\pi)\}[m]$$

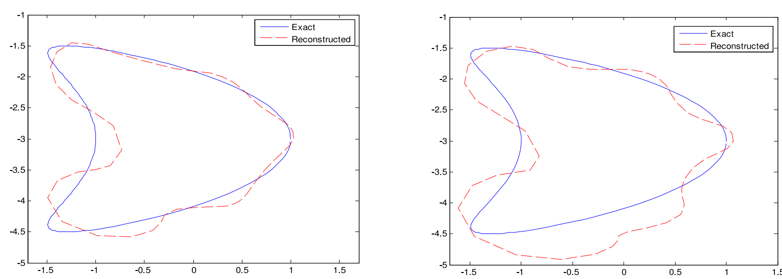

Figure 5. The reconstruction of the kite for $\phi_0 = \pi/2$ without noise (left) and with %2 noise (right).

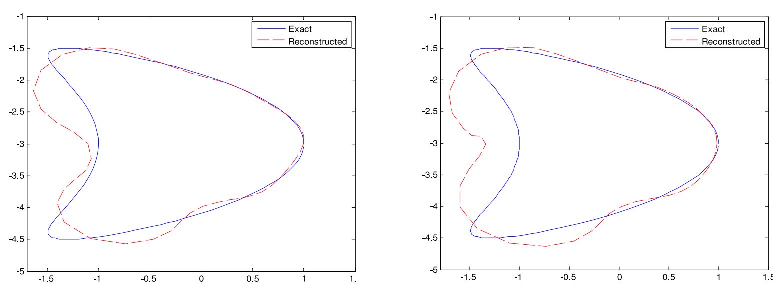

Figure 6. The reconstruction of the kite for $\phi_0 = \pi/4$ without noise (left) and with %2 noise (right).

4. Conclusions. In this study, the shape reconstruction of a completely buried sound soft object through the measured far-field scattering data is given. The scattered field is represented as a single-layer potential which leads to an ill-posed integral equation of the first kind that is solved via Tikhonov regularization. The field and its normal derivative are obtained by the single layer potential for an initial estimate. By applying the Newton method, a new estimate of boundary is obtained in the sense of least square. The main advantage of the proposed iterative method is that this method does not need forward solver for each iteration step that needs high computational time especially in the evaluation of the fundamental solution of a layered medium. As seen from the figures obtained, the presented method gives good reconstruction and stability against noisy data. We observed that if noise level exceeds %3, reconstructions start to deteriorate. It is also observed that we have a better reconstruction in the illuminated region than in shadow region as expected. Because of the nature of the considered problem, one can obtain scattered data only semi upper unit circle contrary to the same problem considered in free space. In addition, in principle, there is a straight forward extension to the case of limited angle data by modifying the far field equation (2.11) appropriately, of course, at the cost of increasing the degree of ill-posedness. Furthermore, also the case of near field data can be accommodated through again modifying the data equation (2.11) accordingly.

Acknowledgments. This research was carried out while Necmi Serkan Tezel was visiting the University of Göttingen as a fellow of the Scientific and Technical Research Council of Turkey (TUBITAK). The hospitality of the University of Göttingen and the support of the TUBITAK are gratefully acknowledged.

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