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INTEGRO-DIFFERENTIAL EQUATIONS OF FIRST ORDER WITH AUTOCONVOLUTION INTEGRAL

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ABSTRACT. In the paper two classes of first order integro-differential equations with autoconvolution integral are studied generalizing an equation of J. M. Burgers from the turbulence theory. General existence and stability theorems in a finite interval are proved and the asymptotic behavior of the solutions at infinity is discussed.

1. Introduction. In his theory of turbulence J. M. Burgers [**3**] (for Burgers' turbulence see also [**6, 7, 12**]) studied an integro-differential equation which can be reduced to the equation

$$y'(x) + \left(\frac{1}{2x} - \frac{1}{16}x^2\right) y(x) = \int_0^x y(\xi)y(x-\xi)d\xi\,, \quad x > 0 \tag{1.1}$$

with autoconvolution integral $I(y) = \int_0^x y(\xi)y(x-\xi)d\xi$ and derived a solution of this equation by series expansions in powers and exponentials.

In this paper we deal with a general first order integro-differential equation of the form

$$\begin{aligned} y'(x) + k(x)y(x) = &\int_0^x a(x,\xi)y(\xi)y(x-\xi)d\xi \\ &+ \int_0^x b(x,\xi)y(\xi)d\xi + g(x), \quad x \in (0,T) \end{aligned} \tag{1.2}$$

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with given numbers $T \in (0, \infty)$ and given functions k, a, b and g. Equation (1.2) comprises equations with singular coefficients of y at $x = 0$ like (1.1) (Type I) as well as related equations with singular coefficients of y and $I(y)$ (Type II). For both types of equations we prove general existence and stability theorems applying the iteration method with weighted norms in the form of our paper [**10**]. For convenience of the reader we state the corresponding theorem from [**10**] below in this Introduction. The theorem was formerly used in papers [**2, 9, 11, 14**] to study integral equations of the third kind with autoconvolution integral. In this way we obtain a nearly complete picture about the solvability of the integro-differential equations on a finite interval $[0, T]$.

Further, following Burgers, we derive some solutions by power and exponential series expansions as a basis for a discussion of the asymptotics of the solutions at infinity and state basic asymptotic solutions of generalized Burgers' equation and a related equation of type II.

The existence proofs in the paper are based on an existence theorem from [**10**] for operator equations of the form

$$y = f + G[y] + L[y, y] \tag{1.3}$$

with a linear operator G and a bilinear operator L in a Banach space X endowed with the scale of norms $\|z\|_\sigma$, $\sigma \geq 0$ satisfying the condition

$$\lambda(\sigma)\|z\|_0 \leq \|z\|_\sigma \leq \|z\|_0 \quad \text{for any} \quad z \in X \quad \text{and} \quad \sigma \geq \sigma_0 \geq 0 \tag{1.4}$$

where $\lambda \in C(\mathbb{R}_+ \to \mathbb{R}_+)$, $\lambda > 0$, which we cite here as Lemma 1.

Lemma 1. *Let the linear operator $G : X \to X$ and the bilinear operator $L : X \times X \to X$ fulfill the inequalities*

$$\|G[z]\|_\sigma \leq M(\sigma)\|z\|_\sigma\,, \quad \sigma \geq \sigma_0 \tag{1.5}$$

for any $z \in X$ with a continuous function M satisfying $M(\sigma) \to 0$ as $\sigma \to \infty$, and

$$\|L[z_1, z_2]\|_\sigma \leq N\|z_1\|_\sigma\|z_2\|_\sigma\,, \quad \sigma \geq \sigma_0 \tag{1.6}$$

with a constant N and

$$\|L[z_1, z_2]\|_\sigma \leq \begin{cases} \nu_1(\sigma)\|z_1\|\,\|z_2\|_\sigma \\ \nu_2(\sigma)\|z_1\|_\sigma\|z_2\| \end{cases} \tag{1.7}$$

with continuous functions ν_k, $k = 1, 2$, satisfying $\nu_k(\sigma) \to 0$ as $\sigma \to \infty$ for any pair $z_1, z_2 \in X$.

Then equation (1.3) has a uniquely determined solution $y \in X$. Moreover, for solutions y_1 and y_2, corresponding to functions $f = f_1$ and $f = f_2$, respectively, the stability estimate

$$\|y_1 - y_2\| \le \Lambda(Q_1, Q_2)\|f_1 - f_2\| \tag{1.8}$$

holds, where $Q_k = (\|f_k\|, \|G[f_k]\|)$, $k = 1, 2$, and $\Lambda \in C\left(\mathbb{R}^4_+ \to \mathbb{R}\right)$, $\Lambda > 0$ and $\Lambda(x_1, \dots, x_4)$ is increasing in $x_1, \dots, x_4$.

The plan for the paper is as follows. After formulating the problem in Section 2 we prove the existence theorems in Section 3 and 4, respectively. In Section 5 we discuss the asymptotics of the solutions at infinity whereas the asymptotic solutions are dealt with in Section 6. In Appendix we perform Burgers' straightforward calculation of some sums with roots of the Bessel function which are needed for the solution to equation (1.1) in form of an exponential series.

2. Problem formulation. Reduction to a family of integral equations. We are going to study the equation (1.2) in several weighted functional spaces where the weight is defined by the main part of the coefficient k. In the sequel we will always assume that

$$k = \varkappa + B \quad \text{with} \quad \varkappa \in \mathcal{I}\,, \quad B \in L^1(0, T) \tag{2.1}$$

where

$$\mathcal{I} = \bigcap_{\epsilon \in (0,T)} L^1(\epsilon, T)\,. \tag{2.2}$$

Important examples are $\varkappa(x) = \gamma x^{-\alpha}$, $\alpha > 0$, $\gamma \in \mathbb{R}$ and (with $T < 1$) $\varkappa(x) = \frac{\gamma}{x|\ln x|}$ and $\varkappa(x) = \frac{\gamma|\ln x|}{x}$, $\gamma \in \mathbb{R}$. For Burger's equation (1.1) we have $\varkappa(x) = \frac{1}{2x}$ and $B(x) = -\frac{1}{16}x^2$, $a \equiv 1$, $b \equiv 0$.

We remark that instead of $B \in L^1(0, T)$ we also can assume that B is (improperly) Riemann integrable with finite integral $\int_0^T B(x)dx$.

Let us define the following basic functional spaces related to the coefficient $\varkappa \in \mathcal{I}$:

$$(2.3)\quad L^p_\varkappa := \left\{ u \,:\, e^{-\int_\cdot^T \varkappa(\xi)d\xi} u \in L^p(0,T) \right\} \quad \text{with the norm } \|u\|_{L^p_\varkappa} = \left\| e^{-\int_\cdot^T \varkappa(\xi)d\xi} u \right\|_{L^p(0,T)},$$

$$(2.4)\quad C_\varkappa := \left\{ u \,:\, e^{-\int_\cdot^T \varkappa(\xi)d\xi} u \in C[0,T] \right\} \quad \text{with the norm } \|u\|_{C_\varkappa} = \left\| e^{-\int_\cdot^T \varkappa(\xi)d\xi} u \right\|_{C[0,T]},$$

$$(2.5)\qquad W^1_\varkappa := \{ y \,:\, y \in C_\varkappa,\, y' \in \mathcal{I} \}\,.$$

Note that

$$(2.6)\qquad W^1_0 = \{ y \,:\, y \in C[0,T],\, y' \in \mathcal{I} \}.$$

In this paper we will treat the case when equation (1.2) can be reduced to a family of integral equations of the second kind by means of solving it with respect to the left-hand side. For such a reduction we need the following lemmas.

Lemma 2. *If $\varphi \in L^1_\varkappa$, then the family of solutions of the equation $y'(x) + k(x)y(x) = \varphi(x)$, $x \in (0,T)$, is in the space*

$$(2.7)\qquad \mathcal{W} := \{ y \,:\, y \in C(0,T],\, y' \in \mathcal{I} \}$$

given by the formula

$$(2.8)\quad y(x) = K e^{\int_x^T k(\xi)d\xi} + \int_0^x e^{-\int_\xi^x k(\eta)d\eta} \varphi(\xi)d\xi\,, \qquad x \in (0,T)\,,\ K \in \mathbb{R}.$$

Proof. uses well-known arguments from the theory of linear ordinary differential equations. □

Lemma 3. *If $g \in L^1_\varkappa$, then the equation (1.2) is in the space*

$$\mathcal{S}_{\varkappa,a,b} = \left\{ y \,:\, y \in \mathcal{W},\ \int_0^{\cdot} a(\cdot,\tau)y(\cdot-\tau)y(\tau)d\tau \in L^1_\varkappa, \right.$$
$$\left. \int_0^{\cdot} b(\cdot,\tau)y(\tau)d\tau \in L^1_\varkappa \right\} \tag{2.9}$$

equivalent to the following family of integral equations

$$\begin{aligned} y(x) \,=\, & K e^{\int_x^T k(\xi)d\xi} + \int_0^x e^{-\int_\xi^x k(\eta)d\eta} g(\xi)d\xi \\ & + \int_0^x e^{-\int_\xi^x k(\eta)d\eta} \int_0^\xi a(\xi,\tau)y(\tau)y(\xi-\tau)d\tau d\xi \\ & + \int_0^x e^{-\int_\xi^x k(\eta)d\eta} \int_0^\xi b(\xi,\tau)y(\tau)d\tau d\xi\,, \quad x \in (0,T), \end{aligned} \tag{2.10}$$

where $K \in \mathbb{R}$ is an arbitrary parameter.

Proof. Denoting

$$\varphi[y](x) \,:=\, g(x) + \int_0^x a(x,\tau)y(x-\tau)y(\tau)d\tau + \int_0^x b(x,\tau)y(\tau)d\tau,$$

the equation (1.2) can be rewritten in the form $y'(x) + k(x)y(x) = \varphi[y](x)$, $x \in (0,T)$. Further, by the assumptions of Lemma 3 the function $\varphi[y]$ belongs to $L^1_\varkappa$ for any $y \in \mathcal{S}_{\varkappa,a,b}$. Now we observe that the assertion of Lemma 3 immediately follows from Lemma 2. □

Remark 1. Any solution of (2.10) belongs to the space $\mathcal{S}_{\varkappa,a,b}$. Indeed, let y solve (2.10). It is easy to see that the right-hand side of (2.10) is continuous and differentiable for any $x \in (0,T]$. Therefore, $y \in \mathcal{W}$. Moreover, the necessary conditions for the second and third addend in the right-hand side of (2.10) to exist for any $x \in (0,T)$, are

$$\int_0^{\cdot} a(\cdot,\tau)y(\cdot-\tau)y(\tau)d\tau \in L^1_\varkappa \quad \text{and} \quad \int_0^{\cdot} b(\cdot,\tau)y(\tau)d\tau \in L^1_\varkappa.$$

These conditions must hold for any solution y. Summing up, $y \in \mathcal{S}_{\varkappa,a,b}$ for any solution y.

We can establish the behavior of the solution of (2.10) at $x \to 0^+$, as well. Namely, the following lemma is valid:

Lemma 4. *Let $g \in L^1_\varkappa$, K be some real number and $y \in \mathcal{S}_{\varkappa,a,b}$ solve (2.10). Then $y \in W^1_\varkappa$. Moreover, y has the property*

$$\lim_{x\to 0^+} e^{-\int_x^T \varkappa(\xi)d\xi} y(x) = A := K e^{\int_0^T B(\xi)d\xi}. \tag{2.11}$$

Proof. Multiplying (2.10) by $e^{-\int_x^T \varkappa(\xi)d\xi}$ and observing that $k = \varkappa + B$ we obtain

$$\begin{aligned} (2.12)\qquad & e^{-\int_x^T \varkappa(\eta)d\eta} y(x) \\ &= K e^{\int_x^T B(\xi)d\xi} + \int_0^x e^{-\int_\xi^x B(\xi)d\xi} e^{-\int_\xi^T \varkappa(\eta)d\eta} g(\xi)d\xi \\ &\quad + \int_0^x e^{-\int_\xi^x B(\xi)d\xi} e^{-\int_\xi^T \varkappa(\eta)d\eta} \int_0^\xi a(\xi,\tau)y(\tau)y(\xi-\tau)d\tau d\xi \\ &\quad + \int_0^x e^{-\int_\xi^x B(\xi)d\xi} e^{-\int_\xi^T \varkappa(\eta)d\eta} \int_0^\xi b(\xi,\tau)y(\tau)d\tau d\xi\,, \quad x \in (0,T). \end{aligned}$$

Observing that $B \in L^1(0,T)$, $g \in L^1_\varkappa$, $\int_0^\cdot a(\cdot,\tau)y(\cdot-\tau)y(\tau)d\tau \in L^1_\varkappa$, $\int_0^\cdot b(\cdot,\tau)y(\tau)d\tau \in L^1_\varkappa$ (cf. definition of $\mathcal{S}_{\varkappa,a,b}$), and the definition of $L^1_\varkappa$ we see that the right-hand side of (2.12) belongs to $C[0,T]$. Thus, $y \in C_\varkappa$. Further, since $y' \in \mathcal{I}$ for $y \in \mathcal{S}_{\varkappa,a}$, we obtain $y \in W^1_\varkappa$. Finally, taking the limit $x \to 0^+$ in (2.12) we deduce (2.11). □

Remark 2. According to Remark 1 and Lemma 4 under the assumption $g \in L^1_\varkappa$ any solution of (2.10) belongs to $W^1_\varkappa$.

We note that in case of positive $\varkappa(x)$ the solution space $W^1_\varkappa$ may contain functions that are singular at $x = 0$. However, this singularity may be only integrable provided either a or b is bounded away from zero. This follows form the next lemma.

Lemma 5. *Let there exist $\delta > 0$ such that either the inequality*

$$|a(x,\xi)| \geq \delta \quad \textit{for a.e.} \quad 0 < \xi < x < T \tag{2.13}$$

or the inequality

$$|b(x,\xi)| \geq \delta \quad \textit{for a.e.} \quad 0 < \xi < x < T \tag{2.14}$$

is fulfilled. Then any solution $y \in W^1_\varkappa$ *of* (2.10) *belongs to the space* $L^1(0,T)$.

Proof. Let $y \in W^1_\varkappa$ solve (2.10). If $y \equiv 0$ then $y \in L^1(0,T)$ trivially. Thus, let $y \not\equiv 0$. This in view of $W^1_\varkappa \subset C(0,T]$ implies that there exist $x_0 \in (0,T)$, $s \in (0,x_0)$ and $q > 0$ such that

$$|y(x_0 - \xi)| \geq q \quad \text{for any} \quad \xi \in (0,s). \tag{2.15}$$

Let us prove the assertion of lemma in the case (2.13). From (2.10) we see that $a(\xi,\cdot)y(\cdot)y(\xi-\cdot) \in L^1(0,T)$ for a.e. $\xi \in (0,T)$. Evidently, we can choose x_0 so that $a(x_0,\cdot)y(x_0-\cdot)y(\cdot) \in L^1(0,s)$. Due to (2.13) and (2.15) we have $|a(x_0,\xi)y(x_0-\xi)| \geq \delta q > 0$ for any $\xi \in (0,s)$. Therefore, $y \in L^1(0,s)$. This with $y \in C(0,T]$ implies $y \in L^1(0,T)$. In case (2.14) the proof is similar. □

In Sections 2 and 3 we will study the solvability of the family of equations (2.10) or, equivalently, the equation (1.2) mainly in the largest possible solution space $W^1_\varkappa$.

We are going to deal with two main types of the integro-differential equation (1.2):

Type I. The kernels $a(x,\xi)$ and $b(x,\xi)$ are integrable with respect to x and ξ.

Type II. The kernels a and b are representable in the form

$$a(x,\xi) = \varkappa(x)a_0(x,\xi)\,, \quad b(x,\xi) = \varkappa(x)b_0(x,\xi)\,, \tag{2.16}$$

where $a_0(x,\xi)$ and $b_0(x,\xi)$ are integrable with respect to x and ξ. We remark that equation (1.2) with a and b of the form (2.16) can be obtained from the integro-differential equation of the third kind

$$\begin{aligned}\nu(x)y'(x) + (1+B_0(x))y(x) &= \int_0^x a_0(x,\xi)y(\xi)y(x-\xi)d\xi \\ &\quad + \int_0^x b_0(x,\xi)y(\xi)d\xi + h(x)\,, \quad x \in (0,T)\,,\end{aligned} \tag{2.17}$$

where $\nu \in C[0,T]$ with $\nu(0) = 0$, $\nu(x) \neq 0$ for $0 < x \leq T$ if we set $\varkappa(x) = \frac{1}{\nu(x)}$, $B(x) = \frac{B_0(x)}{\nu(x)}$, $g(x) = \frac{h(x)}{\nu(x)}$ and assume $\frac{B_0}{\nu} \in L^1(0,T)$.

3. Integro-differential equation of type I.

3.1. *The cases of non-positive and integrable $\varkappa$.* Here we will study the equation (1.2) of type I in the cases when either $\varkappa$ is non-positive having possibly non-integrable singularity at $x = 0$ or $\varkappa \in L^1(0,T)$.

We start by proving a technical lemma.

Lemma 6. *Let $l \in L^1(0,T)$, $l(x) \geq 0$ and $u_\sigma(x) = \int_0^x e^{-\sigma(x-\xi)} l(\xi)d\xi$. Then $u_\sigma \to 0$ in $C[0,T]$ as $\sigma \to \infty$.*

Proof. To prove Lemma 6, we make use of the following general result (see [**8**] p. 43):

Lemma 6a. *Let u_σ, $\sigma \geq 0$, be an equicontinuous family of functions in $C[0,T]$ such that $u_\sigma(x) \to u(x)$ as $\sigma \to \infty$ for any $x \in [0,T]$ where $u \in C[0,T]$. Then $u_\sigma \to u$ in $C[0,T]$ as $\sigma \to \infty$.*

Let ω be the modulus of continuity of the continuous function $v(x) = \int_0^x l(\xi)d\xi$. Then, for any $\sigma \geq 0$ and $x_1 \leq x_2$ from $[0,T]$ we have

$$\begin{aligned} |u_\sigma(x_1) - u_\sigma(x_2)| = &\int_{x_1}^{x_2} e^{-\sigma(x_2-\xi)} l(\xi)d\xi \\ &+ \int_0^{x_1} \Big(e^{-\sigma(x_2-\xi)} - e^{-\sigma(x_1-\xi)}\Big) l(\xi)d\xi \\ \leq &\int_{x_1}^{x_2} l(\xi)d\xi \; = \; \omega(|x_1 - x_2|) \end{aligned}$$

because $l(\xi) \geq 0$ and $e^{-\sigma(x_2-\xi)} \leq 1$ for $0 \leq \xi \leq x_2$ and $e^{-\sigma(x_2-\xi)} - e^{-\sigma(x_1-\xi)} \leq 0$ for $0 \leq \xi \leq x_1 \leq x_2$. This implies that the family u_σ, $\sigma \geq 0$, is equicontinuous. Furthermore, since $e^{-\sigma(x-\xi)} l(\xi) \to 0$ as $\sigma \to \infty$ a.e. $\xi \in (0,x)$ for any $x \in [0,T]$ we have $u_\sigma(x) \to 0$ as $\sigma \to \infty$ for any $x \in [0,T]$. Consequently, by Lemma 6a we obtain $u_\sigma \to 0$ in $C[0,T]$ as $\sigma \to \infty$. $\square$

Now we prove a theorem concerning the equation (1.2) in the case of non-positive $\varkappa$.

Theorem 1. *Let $g \in L^1_\varkappa$ and $\varkappa(x) \le 0$, $x \in (0,T)$. Assume that*

$$\int_0^{\cdot} |a(\cdot,\xi)|d\xi\,,\quad \int_0^{\cdot} |b(\cdot,\xi)|d\xi \in L^1(0,T). \tag{3.1}$$

Then the equation (1.2) *has a one-parametric family of solutions in the space $W^1_\varkappa$ with the parameter*

$$A = \lim_{x\to 0^+} e^{-\int_x^T \varkappa(\xi)d\xi} y(x) \in \mathbb{R}. \tag{3.2}$$

Any solution of (1.2) *belongs to this family. Moreover, for solutions y_1 and y_2, corresponding to the functions $g = g_1$ and $g = g_2$, respectively, and satisfying the initial condition $A = \lim_{x\to 0^+} e^{-\int_x^T \varkappa(\xi)d\xi} y_1(x) = \lim_{x\to 0^+} e^{-\int_x^T \varkappa(\xi)d\xi} y_2(x)$ the stability estimate*

$$\|y_1 - y_2\|_{C_\varkappa} \le \Lambda(|A|, \|f_1\|_{C_\varkappa}, \|f_2\|_{C_\varkappa})\|f_1 - f_2\|_{C_\varkappa} \tag{3.3}$$

holds where

$$f_k(x) = \int_0^x e^{-\int_\xi^x k(\eta)d\eta} g_k(\xi)d\xi\,,\quad k = 1,2, \tag{3.4}$$

and

$$\Lambda \in C\left(\mathbb{R}^3_+ \to \mathbb{R}\right)\,,\quad \Lambda > 0\,,\quad \Lambda(x_1,x_2,x_3) \quad - \textit{increasing in } x_1,x_2,x_3. \tag{3.5}$$

Proof. Let us fix $K \in \mathbb{R}$ and rewrite the equation (2.10) in the form $y = f + G[y] + L[y,y]$, where

$$f(x) = Ke^{\int_x^T k(\xi)d\xi} + \int_0^x e^{-\int_\xi^x k(\eta)d\eta} g(\xi)d\xi\,, \tag{3.6}$$

$$G[y](x) = \int_0^x e^{-\int_\xi^x k(\eta)d\eta} \int_0^\xi b(\xi,\tau)y(\tau)d\tau d\xi\,, \tag{3.7}$$

$$L[y,z](x) = \int_0^x e^{-\int_\xi^x k(\eta)d\eta} \int_0^\xi a(\xi,\tau)y(\tau)z(\xi-\tau)d\tau d\xi. \tag{3.8}$$

Observing the decomposition $k = \varkappa + B$ from (3.6) we have

$$e^{-\int_x^T \varkappa(\xi)d\xi} f(x) = K e^{\int_x^T B(\xi)d\xi} + \int_0^x e^{-\int_\xi^x B(\eta)d\eta} e^{-\int_\xi^T \varkappa(\eta)d\eta} g(\xi)d\xi. \tag{3.9}$$

Note that

$$e^{\int_\cdot^T B(\xi)d\xi} \in C[0,T]\,, \quad e^{-\int_\xi^x B(\eta)d\eta} \in C(\Delta_T) \quad \text{where} \quad \Delta_T = \{(x,\xi)\,:\, 0 \le x \le T,\, 0 \le \xi \le x\} \tag{3.10}$$

because $B \in L^1(0,T)$. Moreover, $e^{-\int_\cdot^T \varkappa(\eta)d\eta} g \in L^1(0,T)$ due to the assumption $g \in L^1_\varkappa$. Consequently, from (3.9) we see that $e^{-\int_\cdot^T \varkappa(\eta)d\eta} f \in C[0,T]$, hence

$$f \in C_\varkappa. \tag{3.11}$$

Further, we introduce the scale of norms

$$\|u\|_\sigma = \left\|e^{-\sigma\cdot} u\right\|_{C_\varkappa} = \left\|e^{-\sigma\cdot} e^{-\int_\cdot^T \varkappa(\eta)d\eta} u\right\|_{C[0,T]}, \quad \sigma \ge 0, \tag{3.12}$$

in the space $C_\varkappa$. This scale satisfies the condition (1.4) with $\lambda(\sigma) = e^{-\sigma T}$. Using the relation $k = \varkappa + B$ we compute

$$\begin{aligned} &e^{-\sigma x} e^{-\int_x^T \varkappa(\eta)d\eta} L[z_1, z_2](x) \\ &= e^{-\sigma x} e^{-\int_x^T \varkappa(\eta)d\eta} \int_0^x e^{-\int_\xi^x k(\eta)d\eta} \int_0^\xi a(\xi,\tau) z_1(\tau) z_2(\xi-\tau)d\tau d\xi \\ &= \int_0^x e^{-\sigma(x-\xi)} e^{-\int_\xi^x B(\eta)d\eta} \Psi(\xi)d\xi \end{aligned} \tag{3.13}$$

where

$$\begin{aligned} \Psi(\xi) = \int_0^\xi & a(\xi,\tau) e^{\int_\tau^\xi \varkappa(\eta)d\eta} e^{\int_{\xi-\tau}^T \varkappa(\eta)d\eta} \\ &\times e^{-\sigma\tau} e^{-\int_\tau^T \varkappa(\eta)d\eta} z_1(\tau) e^{-\sigma(\xi-\tau)} e^{-\int_{\xi-\tau}^T \varkappa(\eta)d\eta} z_2(\xi-\tau)d\tau. \end{aligned} \tag{3.14}$$

Due to the assumption $\varkappa(\eta) \leq 0$, $\eta \in (0,T)$, the functions $e^{\int_\tau^\xi \varkappa(\eta)d\eta}$ and $e^{\int_{\xi-\tau}^T \varkappa(\eta)d\eta}$ are bounded by 1 for $0 < \tau < \xi < T$. This due to (3.1) implies

$$(3.15)\quad |\Psi(\xi)| \leq l_1(\xi)\,\|z_1\|_\sigma \|z_2\|_\sigma\,, \quad l_1(\xi) = \int_0^\xi |a(\xi,\tau)|d\tau \in L^1(0,T).$$

The relation (3.15) with (3.10) implies that the second row of (3.13) belongs to $C[0,T]$ provided $z_1, z_2 \in C_\varkappa$. Thus,

$$(3.16)\qquad L[z_1,z_2] \in C_\varkappa \quad \text{for any} \quad z_1, z_2 \in C_\varkappa.$$

Similarly, from (3.7) we have

$$(3.17)\qquad e^{-\sigma x} e^{-\int_x^T \varkappa(\eta)d\eta} G[z](x) = \int_0^x e^{-\sigma(x-\xi)} e^{-\int_\xi^x B(\eta)d\eta} \Phi(\xi) d\xi$$

where

$$(3.18)\quad \Phi(\xi) = \int_0^\xi b(\xi,\tau) e^{\int_\tau^\xi \varkappa(\eta)d\eta} e^{-\sigma(\xi-\tau)} e^{-\sigma\tau} e^{-\int_\tau^T \varkappa(\eta)d\eta} z(\tau) d\tau.$$

Due to (3.1) and the boundedness of $e^{\int_\tau^\xi \varkappa(\eta)d\eta}$ we immediately obtain

$$(3.19)\qquad |\Phi(\xi)| \leq l_2(\xi)\,\|z\|_\sigma\,, \quad l_2(\xi) = \int_0^\xi |b(\xi,\tau)|d\tau \in L^1(0,T).$$

The relation (3.19) with (3.10) implies that (3.17) belongs to $C[0,T]$ provided $y \in C_\varkappa$. Thus,

$$(3.20)\qquad G[z] \in C_\varkappa \quad \text{for any} \quad z \in C_\varkappa.$$

The relations (3.11), (3.16) and (3.20) show that the equation $y = f + G[y] + L[y,y]$ is well-defined in the space $C_\varkappa$.

Taking in (3.13) and (3.17) maximum over $x \in [0,T]$ and observing (3.15), (3.19) we obtain

$$(3.21)\quad \|L[z_1,z_2]\|_\sigma \leq \nu(\sigma)\,\|z_1\|_\sigma \|z_2\|_\sigma\,, \quad \|G[z]\|_\sigma \leq \nu(\sigma)\,\|z\|_\sigma, \ \ \sigma \geq 0$$

with

$$\nu(\sigma) = \max_{0\le\xi\le x\le T} \left| e^{-\int_\xi^x B(\eta)d\eta} \right| \|u_\sigma\|_{C[0,T]} \quad \text{and} \quad u_\sigma(x) = \int_0^x e^{-\sigma(x-\xi)} l(\xi)d\xi \tag{3.22}$$

where $l = \max\{l_1, l_2\} \in L^1(0,T)$. Lemma 6 yields

$$\nu(\sigma) \to 0 \quad \text{as} \quad \sigma \to \infty.$$

Taking this relation and (3.21) into account and using Lemma 1 we come to the conclusion that the equation $y = f + G[y] + L[y,y]$ or, equivalently, (2.10) with given K has a unique solution in the space $C_\varkappa$. This solution is differentiable for any $x \in (0,T)$ (cf. Remark 1). Thus, $y \in W^1_\varkappa$. Due to the assertion (2.11) of Lemma 4 the solution y corresponding to $K \in \mathbb{R}$ satisfies the condition (3.2) with $A = Ke^{\int_0^T B(\xi)d\xi}$.

Summing up, we have shown the existence of a one-parametric family of solutions of (2.10) in $W^1_\varkappa$ with the parameter (3.2). From the uniqueness of the solution for fixed $K \in \mathbb{R}$ and Lemma 4 we deduce that any solution of (2.10) belongs to the constructed family. Since by Lemma 3 and Remarks 1,2 equation (1.2) and the family of equations (2.10) are equivalent in $W^1_\varkappa$ all these statements remain valid for the equation (1.2), too. This proves the solvability assertions of Theorem 1.

Finally, the stability estimate (3.3) follows from the estimate (1.8) of Lemma 1 in view of the relation (3.6) for the function f in the equation $y = f + G[y] + L[y,y]$ and (2.11). Theorem is completely proved. $\square$

At the end of this subsection we deal with the case $\varkappa \in L^1(0,T)$ without any assumptions about the sign of $\varkappa$. Note that in this case according to the definitions (2.3), (2.5) and (2.6) we have $L^1_\varkappa = L^1(0,T)$ and $W^1_\varkappa = W^1_0 = \{y : y \in C[0,T],\, y' \in \mathcal{I}\}$.

Corollary 1. *Let $g, \varkappa \in L^1(0,T)$. Assume that* (3.1) *holds. Then* (1.2) *has a one-parametric family of solutions in the space W^1_0 with the parameter $A = y(0) \in \mathbb{R}$. Any solution of* (1.2) *belongs to this family* *and its derivative satisfies* $y' \in L^1(0,T)$. *Moreover, for solutions* y_1 *and* y_2, *corresponding to the functions* $g = g_1$ *and* $g = g_2$, *respectively, and satisfying the initial condition* $A = y_1(0) = y_2(0)$ *the stability estimate*

$$\|y_1 - y_2\|_{C[0,T]} \le \Lambda(|A|, \|f_1\|_{C[0,T]}, \|f_2\|_{C[0,T]})\|f_1 - f_2\|_{C[0,T]} \tag{3.23}$$

holds, where f_k *are given in terms* g_k *by* (3.4) *and* Λ *is a function with properties* (3.5).

Proof. Since $\varkappa \in L^1(0,T)$ we can decompose $k = \varkappa_1 + B_1$ with $\varkappa_1 \equiv 0$, $B_1 = \varkappa + B \in L^1(0,T)$ and apply Theorem 1 with $\varkappa_1$ instead of $\varkappa$. This yields that (1.2) has a one-parametric family of solutions in the space $W_0^1 = \{y : y \in C[0,T], y' \in \mathcal{I}\}$ with the parameter $A = y(0) \in \mathbb{R}$ and that any solution of (1.2) belongs to this family. Further, if y is the solution of (1.2) then due to the relations $g, k \in L^1(0,T)$, $\int_0^{\cdot} |a(\cdot,\xi)|d\xi, \int_0^{\cdot} |b(\cdot,\xi)|d\xi \in L^1(0,T)$ and $y \in C[0,T]$ all terms except for y' in (1.2) belong to $L^1(0,T)$. Thus, we have $y' \in L^1(0,T)$, too. Finally, the estimate (3.23) follows from (3.3). □

3.2. *The case of positive* $\varkappa$. Here we will study the equation (1.2) of type I provided $\varkappa$ is positive.

In case $e^{\int_x^T \varkappa(\eta)d\eta}$ is integrable at $x = 0$, we can prove a result that is similar to Theorem 1.

Theorem 2. *Let* $g \in L^1_\varkappa$ *and* $\varkappa(x) > 0$, $x \in (0,T)$. *Assume that*

$$e^{\int_\cdot^T \varkappa(\eta)d\eta} \in L^1(0,T) \tag{3.24}$$

and

$$\sup_{\xi\in(0,\cdot)} |a(\cdot,\xi)| \in L^\infty_\varkappa, \quad \sup_{\xi\in(0,\cdot)} |b(\cdot,\xi)| \in L^1_\varkappa. \tag{3.25}$$

Then the assertions of Theorem 1 *are valid.*

Proof. of this theorem repeats the proof of Theorem 1. The only difference is the way of deriving the relations (3.16), (3.20) and (3.21) for the operators L and G.

Let us start by deducing (3.16) for the operator L. We have the formula (3.13) with the function Ψ that we rewrite in the following form:

$$\begin{aligned}(3.26)\quad \Psi(\xi) \,=\, & e^{-\int_\xi^T \varkappa(\eta)d\eta}\int_0^\xi a(\xi,\tau)e^{\int_\tau^T \varkappa(\eta)d\eta}e^{\int_{\xi-\tau}^T \varkappa(\eta)d\eta} \\ & \times e^{-\sigma\tau}e^{-\int_\tau^T \varkappa(\eta)d\eta}z_1(\tau)e^{-\sigma(\xi-\tau)}e^{-\int_{\xi-\tau}^T \varkappa(\eta)d\eta}z_2(\xi-\tau)d\tau.\end{aligned}$$

By virtue of the assumptions (3.24) and (3.25), the definition of $L^\infty_\varkappa$ and the well-known relation for convolutions

$$(3.27)\qquad \varphi_1,\varphi_2\in L^1(0,T)\Longrightarrow \int_0^{\cdot}\varphi_1(\tau)\varphi_2(\cdot-\tau)d\tau\in L^1(0,T)$$

we obtain

$$\begin{aligned}(3.28)\quad & |\Psi(\xi)|\le l_3(\xi)\,\|z_1\|_\sigma\|z_2\|_\sigma\,, l_3(\xi) \\ & = \Big\|\sup_{\tau\in(0,\cdot)}|a(\cdot,\tau)|\Big\|_{L^\infty_\varkappa}\int_0^\xi e^{\int_\tau^T \varkappa(\eta)d\eta}e^{\int_{\xi-\tau}^T \varkappa(\eta)d\eta}d\tau\in L^1(0,T).\end{aligned}$$

In view of this relation and (3.10) the right hand side of (3.13) belongs to $C[0,T]$ provided $z_1,z_2\in C_\varkappa$. Thus, we obtain (3.16).

Next we show (3.20) for the quantity $G[z]$. We make use of the formula (3.17) that holds with the function Φ of the form

$$\begin{aligned}(3.29)\quad & \Phi(\xi) \\ & = e^{-\int_\xi^T \varkappa(\eta)d\eta}\int_0^\xi b(\xi,\tau)e^{\int_\tau^T \varkappa(\eta)d\eta}e^{-\sigma(\xi-\tau)}e^{-\sigma\tau}e^{-\int_\tau^T \varkappa(\eta)d\eta}z(\tau)d\tau.\end{aligned}$$

Due to the assumptions (3.24), (3.25) and the definition of $L^1_\varkappa$ we obtain

$$\begin{aligned}(3.30)\quad & |\Phi(\xi)|\le l_4(\xi)\,\|z\|_\sigma\,, \\ & l_4(\xi)=\int_0^T e^{\int_\tau^T \varkappa(\eta)d\eta}d\tau\, e^{-\int_\xi^T \varkappa(\eta)d\eta}\sup_{\tau\in(0,\xi)}|b(\xi,\tau)|\in L^1(0,T).\end{aligned}$$

In view of this relation and (3.10) the right hand side of (3.17) belongs to $C[0,T]$ provided $z\in C_\varkappa$. Consequently, we get (3.20).

Finally, taking in (3.13) and (3.17) maximum over $x \in [0,T]$ and observing (3.28), (3.30) we obtain the estimates (3.21) with (3.22) where $l = \max\{l_3, l_4\} \in L^1(0,T)$ and $\nu(\sigma) \to 0$ as $\sigma \to \infty$ due to Lemma 6. $\square$

Before we continue investigating the set of solutions of equation (1.2) in $W^1_\varkappa$ in case $e^{\int_\cdot^T \varkappa(\eta)d\eta} \notin L^1(0,T)$, we study this equation in the space W^1_0 that is a subspace of $W^1_\varkappa$ provided $\varkappa(x) > 0$. (Indeed, according to the definition of $C_\varkappa$, $W^1_\varkappa$ and $\varkappa(x) > 0$ we have $\{u : u \in C_0 = C[0,T],\, u' \in \mathcal{I}\} = W^1_0 \subseteq W^1_\varkappa = \{u : u \in C_\varkappa,\, u' \in \mathcal{I}\}$.) Since the case $\varkappa \in L^1(0,T)$ was completely covered by Corollary 1, we treat only the case $\varkappa \notin L^1(0,T)$.

Theorem 3. *Let $g \in L^1(0,T)$, $\varkappa(x) > 0$, $x \in (0,T)$ and $\varkappa \notin L^1(0,T)$. Assume that a and b satisfy (3.1). Then the equation (1.2) has a unique solution in the space W^1_0. This solution has the initial value $y(0) = 0$. Moreover, for solutions y_1 and y_2, corresponding to the functions $g = g_1$ and $g = g_2$, respectively, the stability estimate*

$$\|y_1 - y_2\|_{C[0,T]} \le \Lambda(\|f_1\|_{C[0,T]}, \|f_2\|_{C[0,T]})\|f_1 - f_2\|_{C[0,T]} \tag{3.31}$$

holds where f_k are given in terms g_k by (3.4) and

$$\Lambda \in C\left(\mathbb{R}^2_+ \to \mathbb{R}\right), \quad \Lambda > 0, \quad \Lambda(x_1,x_2) - \textit{increasing in } x_1, x_2. \tag{3.32}$$

Proof. Again, we rewrite the equation (2.10) in the form $y = f + G[y] + L[y,y]$, where f, G and L are given by (3.6 - 3.8). Observing (2.1), (3.10) and the assumptions of Theorem 3 we have

$$e^{-\int_\xi^x k(\eta)d\eta} \in C(\Delta_T \setminus \{(0,0)\}) \cap L^\infty(\Delta_T), \tag{3.33}$$

$$e^{\int_x^T k(\xi)d\xi} \to \infty \text{ as } x \to 0^+. \tag{3.34}$$

Thus, in view of $g \in L^1(0,T)$ from (3.6) we get

$$f \in C[0,T] \quad \text{in case} \quad K = 0, \qquad f \notin C[0,T] \quad \text{in case} \quad K \neq 0. \tag{3.35}$$

Introduce the scale of norms

$$\|u\|_\sigma \;=\; \left\|e^{-\sigma\cdot}u\right\|_{C[0,T]}\,, \quad \sigma \ge 0, \tag{3.36}$$

in the space $C[0,T]$. This scale satisfies (1.4) with $\lambda(\sigma) = e^{-\sigma T}$. From (3.8) we obtain

$$e^{-\sigma x}L[z_1,z_2](x) = \int_0^x e^{-\sigma(x-\xi)}e^{-\int_\xi^x k(\eta)d\eta}\Psi(\xi)d\xi \tag{3.37}$$

where

$$\Psi(\xi) \;=\; \int_0^\xi a(\xi,\tau)e^{-\sigma\tau}z_1(\tau)e^{-\sigma(\xi-\tau)}z_2(\xi-\tau)d\tau. \tag{3.38}$$

Observing the assumption (3.1) we have

$$|\Psi(\xi)| \;\le\; l_5(\xi)\|z_1\|_\sigma\|z_2\|_\sigma\,, \quad l_5(\xi) = \int_0^\xi |a(\xi,\tau)|d\tau \in L^1(0,T). \tag{3.39}$$

In view of this relation and (3.33) the right hand side of (3.37) belongs to $C[0,T]$ provided $z_1, z_2 \in C[0,T]$. Thus,

$$L[z_1,z_2] \in C[0,T] \quad \text{for any} \quad z_1,z_2 \in C[0,T]. \tag{3.40}$$

Similarly for the quantity $G[z]$ from (3.7) we have

$$e^{-\sigma x}G[z](x) = \int_0^x e^{-\sigma(x-\xi)}e^{-\int_\xi^x k(\eta)d\eta}\Phi(\xi)d\xi \tag{3.41}$$

where

$$\Phi(\xi) \;=\; \int_0^\xi b(\xi,\tau)e^{-\sigma(\xi-\tau)}e^{-\sigma\tau}z(\tau)d\tau. \tag{3.42}$$

Due to the assumption (3.1) we obtain

$$|\Phi(\xi)| \;\le\; l_6(\xi)\|z\|_\sigma\,, \quad l_6(\xi) = \int_0^\xi |b(\xi,\tau)|d\tau \in L^1(0,T). \tag{3.43}$$

Again, due to this relation and (3.33) the right hand side of (3.41) belongs to $C[0,T]$ provided $z \in C[0,T]$. Therefore,

$$G[z] \in C[0,T] \quad \text{for any} \quad z \in C[0,T]. \tag{3.44}$$

Summing up, the relations (3.35), (3.40) and (3.44) show that the equation $y = f + G[y] + L[y,y]$ is well-defined in the space $C[0,T]$ in case $K = 0$ and has no solution in the space $C[0,T]$ in case $K \neq 0$. Therefore, we continue studying this equation in case $K = 0$.

Taking in (3.37) and (3.41) maximum over $x \in [0,T]$ and observing (3.39), (3.43) we obtain the estimates (3.21) with (3.22) where $l = \max\{l_5, l_6\} \in L^1(0,T)$. Lemma 6 yields

$$\nu(\sigma) \to 0 \quad \text{as} \quad \sigma \to \infty.$$

Thus, by Lemma 1 the equation $y = f + G[y] + L[y,y]$ or, equivalently, (2.10) with $K = 0$ has a unique solution y in the space $C[0,T]$. This solution is differentiable for any $x \in (0,T)$ (see Remark 1), which implies $y \in W_0^1$. By Lemma 3 and Remarks 1,2 also y is the unique solution of (1.2) in W_0^1. The property $y(0) = 0$ follows from the equality $y(t) = f(t) + G[y](t) + L[y,y](t)$, because its right-hand side equals 0 at $t = 0$ (cp. (3.6) with $K = 0$, (3.7), (3.8)) . Finally, the stability estimate (1.8) follows from the estimate (1.8) of Lemma 1 in view of the relation (3.6) for the function f in the equation $y = f + G[y] + L[y,y]$ and $K = 0$. Theorem is proved. □

Now we return to the study of (2.10) in $W_\varkappa^1$.

Lemma 7. *Let $g \in L^1(0,T)$ and $\varkappa(x) > 0$, $x \in (0,T)$ Assume that*

$$e^{\int_\cdot^T \varkappa(\eta)d\eta} \notin L^1(0,T) \tag{3.45}$$

and a, b satisfy the conditions

$$\sup_{\xi\in(0,\cdot)} |a(\cdot,\xi)| \in L^\infty(0,T)\,, \qquad \sup_{\xi\in(0,\cdot)} |b(\cdot,\xi)| \in L^1(0,T)\,. \tag{3.46}$$

Moreover, let either (2.13) or (2.14) holds with some $\delta > 0$. Then any solution $y \in W_\varkappa^1$ of (1.2) is a solution of (2.10) with $K = 0$ and belongs to $C[0,T]$.

Proof. As in the proof of Theorem 3 we have the relation (3.33). Furthermore, Lemma 5 implies $y \in L^1(0,T)$. This means that the right-hand side of (2.10) must belong to $L^1(0,T)$. The relation $y \in L^1(0,T)$ with the assumptions (3.46) yields $\int_0^x a(x,\xi)y(\xi)y(x-\xi)d\xi$, $\int_0^x b(x,\xi)y(\xi)d\xi \in L^1(0,T)$. In view of these relations, the assumption $g \in L^1(0,T)$ and (3.33) the right-hand side of (2.10) besides the term $Ke^{\int_\cdot^T \varkappa(\eta)d\eta}$ belongs to $C[0,T]$. Due to the assumption (3.45) the term $Ke^{\int_\cdot^T \varkappa(\eta)d\eta} \in L^1(0,T)$ if and only if $K=0$. Consequently, we must have $K=0$. But then the whole right-hand side of (2.10) belongs to $C[0,T]$. This proves $y \in C[0,T]$. □

As a corollary of Theorem 3 and Lemma 7 we can formulate the following theorem.

Theorem 4. *Let* $g \in L^1(0,T)$, $\varkappa(x) > 0$, $x \in (0,T)$ *and* (3.45) *hold. Moreover, assume that* a, b *satisfy* (3.46) *and either* (2.13) *or* (2.14) *holds with some* $\delta > 0$. *Then the equation* (1.2) *has a unique solution in the space* $W^1_\varkappa$. *This solution belongs to* W^1_0 *and has the initial value* $y(0) = 0$. *Moreover, for solutions* y_1 *and* y_2, *corresponding to the functions* $g = g_1$ *and* $g = g_2$, *respectively, the stability estimate* (3.31) *is valid where* f_k *are given in terms* g_k *by* (3.4) *and* Λ *is a function with the properties* (3.32).

3.3. *Examples.* Here we apply results of previous subsections to the equation (1.2) with the function $k = \varkappa + B$ where $B \in L^1(0,T)$ and

$$\varkappa(x) = \frac{\gamma}{x^\alpha |\ln x|^\beta} \quad in \quad (0,T] \tag{3.47}$$

where $0 < T < 1$ and $\gamma \neq 0$, $\alpha \geq 0$, $\beta \in \mathbb{R}$ are constants. Using results of this section we can formulate the following statements concerning this equation.

1. *Assume that* $\gamma < 0$, $\alpha \geq 0$, $\beta \in \mathbb{R}$. *Moreover, let* $g \in L^1_\varkappa$ *and* a, b *satisfy* (3.1). *Then Theorem* 1 *and in case* $\varkappa \in L^1(0,T)$ *Corollary* 1 *hold.*

2. *Assume that either* $\gamma > 0$, $0 \leq \alpha < 1$, $\beta \in \mathbb{R}$ *or* $\gamma > 0$, $\alpha = 1$, $\beta > 0$ *or* $0 < \gamma < 1$, $\alpha = 1$, $\beta = 0$. *Moreover, let again* $g \in L^1_\varkappa$ *and* a, b

satisfy (3.25). *Then Theorem* 2 *and in case* $\varkappa \in L^1(0,T)$ *Corollary* 1 *hold.*

3. *Assume that either* $\gamma > 0,\ \alpha = 1,\ \beta \le 1$ *or* $\gamma > 0,\ \alpha > 1,\ \beta \in \mathbb{R}$. *Moreover, let* $g \in L^1(0,T)$ *and* a, b *satisfy* (3.1). *Then Theorem* 3 *holds.*

4. *Assume that either* $\gamma > 0,\ \alpha > 1,\ \beta \in \mathbb{R}$ *or* $\gamma \ge 1,\ \alpha = 1,\ \beta \le 0$ *or* $0 < \gamma < 1,\ \alpha = 1,\ \beta < 0$. *Moreover, let* $g \in L^1(0,T)$ *and* a, b *satisfy* (3.46) and either (2.13) or (2.14) with some $\delta > 0$. *Then Theorem* 4 *holds.*

Let us describe more precisely the solution sets in particular cases under suitable assumptions on g, a and b.

If either $0 \le \alpha < 1$, $\beta \in \mathbb{R}$ or $\alpha = 1$, $\beta > 1$ then $\varkappa \in L^1(0,T)$, hence (1.2) has a one-parametric family of solutions in the space W_0^1 with the parameter $A = y(0) \in \mathbb{R}$, if $\alpha = \beta = 1$ then (1.2) has a one-parametric family of solutions in the space $\{y : |\ln x|^{-\gamma} y(x) \in C[0,T], y' \in \mathcal{I}\}$ with the parameter $A = \lim_{x\to 0^+} |\ln x|^{-\gamma} y(x) \in \mathbb{R}$, if either $\gamma < 0$, $\alpha = 1$, $\beta < 1$ or $\gamma > 0$, $\alpha = 1$, $0 < \beta < 1$ or $0 < \gamma < 1$, $\alpha = 1$, $\beta = 0$ then (1.2) has a one-parametric family of solutions in the space

$$\left\{y : x^{\overline{\frac{\gamma}{(1-\beta)\,|\ln x|^{\beta}}}} y(x) \in C[0,T], y' \in \mathcal{I}\right\}$$

with the parameter $A = \lim_{x\to 0^+} x^{\gamma/(1-\beta)\,|\ln x|^{\beta}} y(x) \in \mathbb{R}$ (in particular, if $\gamma < 1$, $\alpha = 1$, $\beta = 0$ then (1.2) has a one-parametric family of solutions in the space $\{y : x^{\gamma} y(x) \in C[0,T], y' \in \mathcal{I}\}$ with the parameter $A = \lim_{x\to 0^+} x^{\gamma} y(x) \in \mathbb{R}$), if $\gamma < 0$, $\alpha > 1$, $\beta \in \mathbb{R}$ then (1.2) has a one-parametric family of solutions in the space

$$\left\{y : e^{-\gamma(1-\alpha)^{\beta-1}\Gamma(1-\beta,(1-\alpha)|\ln x|)} y(x) \in C[0,T], y' \in \mathcal{I}\right\}$$

with the parameter $A = \lim_{x\to 0^+} e^{-\gamma(1-\alpha)^{\beta-1}\Gamma(1-\beta,(1-\alpha)|\ln x|)} y(x) \in \mathbb{R}$ (in particular, in case $\gamma < 0$, $\alpha > 1$, $\beta = 0$ equation (1.2) has a one-parametric family of solutions in the space

$$\left\{y : e^{-\overline{\frac{\gamma}{(\alpha-1)x^{\alpha-1}}}} y(x) \in C[0,T], y' \in \mathcal{I}\right\}$$

with the parameter $A = \lim_{x\to 0^+} e^{-\gamma/(\alpha-1)x^{\alpha-1}} y(x) \in \mathbb{R}$). Here $\Gamma(a,x) = \int_x^\infty t^{a-1}e^{-t}dt$ is the (complementary) incomplete Gamma function (see [**4**] Vol. II, Chap. IX).

4. Integro-differential equation of type II.

4.1. *The cases of non-positive and integrable $\varkappa$.* In this subsection we deal with the equation (1.2) of type II in the cases of non-positive and integrable $\varkappa$. Let us start with the former case.

Theorem 5. *Let $g \in L^1_\varkappa$ and $\varkappa(x) \le 0$, $x \in (0,T)$. Moreover, let*

$$\varkappa(\tau) \le \varkappa(\xi) \quad \textit{for} \quad 0 < \tau < \xi < T. \tag{4.1}$$

Assume that a and b have the form (2.16) *where a_0 and b_0 satisfy*

$$\sup_{\xi\in(0,\cdot)} |a_0(\cdot,\xi)| \in L^1(0,T)\,, \quad \sup_{\xi\in(0,\cdot)} |b_0(\cdot,\xi)| \in L^1(0,T). \tag{4.2}$$

Then assertions of Theorem 1 *are valid.*

Proof. repeats the proof of Theorem 1. Only it is necessary to deduce again the relations (3.16), (3.20) and (3.21) for the operators L and G under the assumptions of Theorem 5.

We start with the relation (3.13) for L with the function Ψ given by (3.14). Due to the assumptions $\varkappa(x) \le 0$ and (4.1) we have $|\varkappa(\xi)| = -\varkappa(\xi) \le -\varkappa(\tau)$ for $0 < \tau \le \xi$. Thus,

$$|\varkappa(\xi)| e^{\int_\tau^\xi \varkappa(\eta)d\eta} \le \frac{d}{d\tau} e^{\int_\tau^\xi \varkappa(\eta)d\eta} \tag{4.3}$$

and from (3.14) in view of (2.16), (4.2) and the inequality $e^{\int_{\xi-\tau}^T \varkappa(\eta)d\eta} \le 1$ following from the assumption $\varkappa(x) \le 0$ we obtain

$$
\begin{aligned}
(4.4)\quad |\Psi(\xi)| &\le \int_0^\xi |\varkappa(\xi)||a_0(\xi,\tau)| e^{\int_\tau^\xi \varkappa(\eta)d\eta} e^{\int_{\xi-\tau}^T \varkappa(\eta)d\eta} d\tau \, \|z_1\|_\sigma \|z_2\|_\sigma \\
&\le \sup_{\tau\in(0,\xi)} |a_0(\xi,\tau)| \int_0^\xi \frac{d}{d\tau} e^{\int_\tau^\xi \varkappa(\eta)d\eta} d\tau \, \|z_1\|_\sigma \|z_2\|_\sigma \\
&= \sup_{\tau\in(0,\xi)} |a_0(\xi,\tau)| \Big(1 - \lim_{\tau\to 0^+} e^{\int_\tau^\xi \varkappa(\eta)d\eta}\Big) \|z_1\|_\sigma \|z_2\|_\sigma \\
&\le l_7(\xi)\, \|z_1\|_\sigma \|z_2\|_\sigma \,, \quad l_7(\xi) = \sup_{\tau\in(0,\xi)} |a_0(\xi,\tau)| \in L^1(0,T).
\end{aligned}
$$

The relation (4.4) with $B \in L^1(0,T)$ implies that the second row of (3.13) belongs to $C[0,T]$ provided $z_1, z_2 \in C_\varkappa$. Consequently, we obtain the relation (3.16).

Next we proceed to the formula (3.17) with $\Phi(x)$ given by (3.18). From (3.18) due to (2.16), (4.2), (4.3) and the inequality $e^{\int_{\xi-\tau}^T \varkappa(\eta)d\eta} \le 1$ we deduce

$$
\begin{aligned}
(4.5)\qquad |\Phi(\xi)| &\le \sup_{\tau\in(0,\xi)} |b_0(\xi,\tau)| \int_0^\xi \frac{d}{d\tau} e^{\int_\tau^\xi \varkappa(\eta)d\eta} d\tau \, \|z\|_\sigma \\
&\le l_8(\xi)\, \|z\|_\sigma \,, \quad l_8(\xi) = \sup_{\tau\in(0,\xi)} |b_0(\xi,\tau)| \in L^1(0,T).
\end{aligned}
$$

The relation (4.5) with $B \in L^1(0,T)$ implies that the right-hand side of (3.17) belongs to $C[0,T]$ provided $y \in C_\varkappa$. Thus, we get (3.20).

Finally, taking in (3.13) and (3.17) maximum over $x \in [0,T]$ and observing (4.4), (4.5) we obtain the estimates (3.21) with (3.22) where $l = \max\{l_7, l_8\} \in L^1(0,T)$ and $\nu(\sigma) \to 0$ as $\sigma \to \infty$. ◻

Next we state a result concerning the case of integrable $\varkappa$.

Corollary 2. *Let $g, \varkappa \in L^1(0,T)$. Assume that* (2.16) *and* (4.2) *hold. Then assertions of Corollary* 1 *are valid.*

Proof. is analogous to the proof of Corollary 1. We set $k = \varkappa_1 + B_1$ where $\varkappa_1 = 0$ and $B_1 = \varkappa + B$. Then $\varkappa_1$ satisfies the conditions $\varkappa_1(x) \le 0$, (4.1) and the function B_1 belongs to $L^1(0,T)$, hence we can apply Theorem 4. ◻

4.2. *The case of positive $\varkappa$.* In this subsection we deal with the equation (1.2) of type II in the case of positive $\varkappa$. Here we didn't succeed to prove the solvability in $W^1_\varkappa$ in case of arbitrary integrable $e^{\int_x^T \varkappa(\eta)d\eta}$ as in the case of type I. We were able to prove such a result only in the particular case when $\varkappa$ satisfies the condition

$$0 < \varkappa(x) \le \frac{\gamma}{x}\,, \quad x \in (0,T) \quad \text{with some} \quad \gamma \in (0,1). \tag{4.6}$$

Note that in this case the function $e^{\int_x^T \varkappa(\eta)d\eta}$ can have maximally a power-type integrable singularity at $t = 0$, i.e.

$$0 < e^{\int_x^T \varkappa(\eta)d\eta} \le \frac{T^\gamma}{x^\gamma}\,, \quad x \in (0,T).$$

Theorem 6. *Let $g \in L^1_\varkappa$ and $\varkappa$ satisfies* (4.5). *Assume that a and b have the form* (2.16) *where a_0 and b_0 satisfy*

$$x^{-\gamma} \sup_{\xi \in (0,x)} |a_0(x,\xi)| \in L^1(0,T)\,, \qquad \sup_{\xi \in (0,\cdot)} |b_0(\cdot,\xi)| \in L^1(0,T). \tag{4.7}$$

Then assertions of Theorem 1 *are valid.*

Proof. Again, the proof repeats proof of Theorem 1. The only difference is the deduction of (3.16), (3.20) and (3.21).

To derive (3.16) we follow the equality (3.13) and rewrite the involved function Ψ in the form

$$\begin{aligned} \Psi(\xi) \;=\; \varkappa(\xi) \int_0^\xi a_0(\xi,\tau) e^{\int_\tau^\xi \varkappa(\eta)d\eta} e^{\int_{\xi-\tau}^T \varkappa(\eta)d\eta} \\ \times\, e^{-\sigma\tau} e^{-\int_\tau^T \varkappa(\eta)d\eta} z_1(\tau) e^{-\sigma(\xi-\tau)} e^{-\int_{\xi-\tau}^T \varkappa(\eta)d\eta} z_2(\xi-\tau)d\tau. \end{aligned} \tag{4.8}$$

Owing to the assumption (4.6) we have

$$\begin{aligned} \int_0^\xi e^{\int_\tau^\xi \varkappa(\eta)d\eta} e^{\int_{\xi-\tau}^T \varkappa(\eta)d\eta} d\tau \le \xi^\gamma T^\gamma \int_0^\xi \tau^{-\gamma}(\xi-\tau)^{-\gamma} d\tau \\ = \mathrm{B}(1-\gamma, 1-\gamma) T^\gamma \xi^{1-\gamma} \end{aligned}$$

where B is the Beta function (see [4] Vol I, Sect. 1.5). Using this estimate and $\varkappa(\xi) \leq \frac{\gamma}{\xi}$ as well as the assumption (4.7) in (4.8) we obtain

$$\begin{aligned} (4.9) \quad & |\Psi(\xi)| \leq l_9(\xi)\, \|z_1\|_\sigma \|z_2\|_\sigma\,, \\ & l_9(\xi) = \gamma T^\gamma\, \mathrm{B}(1-\gamma, 1-\gamma)\; \xi^{-\gamma} \sup_{\tau\in(0,\xi)} |a_0(\xi,\tau)| \in L^1(0,T). \end{aligned}$$

In view of this relation and (3.10) the right hand side of (3.13) belongs to $C[0,T]$ provided $z_1, z_2 \in C_\varkappa$. Thus, we have deduced (3.16).

Next we consider (3.17) where we rewrite Φ as follows.

$$(4.10) \quad \Phi(\xi) = \varkappa(\xi) \int_0^\xi b_0(\xi,\tau) e^{\int_\tau^\xi \varkappa(\eta)d\eta} e^{-\sigma(\xi-\tau)} e^{-\sigma\tau} e^{-\int_\tau^T \varkappa(\eta)d\eta} z(\tau) d\tau.$$

Due to the assumption (4.6) we have $\int_0^\xi e^{\int_\tau^\xi \varkappa(\eta)d\eta} d\tau \leq \xi/(1-\gamma)$. By this relation, $\varkappa(\xi) \leq \frac{\gamma}{\xi}$ and (4.7) from (4.10) we deduce

$$(4.11) \quad |\Phi(\xi)| \leq l_{10}(\xi)\, \|z\|_\sigma\,, \quad l_{10}(\xi) = \frac{\gamma}{1-\gamma} \sup_{\tau\in(0,\xi)} |b_0(\xi,\tau)| \in L^1(0,T).$$

In view of this relation and (3.10) the right-hand side of (3.17) belongs to $C[0,T]$ provided $z \in C_\varkappa$. This means that we have proved (3.20).

Finally, from (3.13) and (3.17) with the help of (4.9) and (4.11) we deduce (3.21) with (3.22) where $l = \max\{l_9, l_{10}\} \in L^1(0,T)$ and $\nu(\sigma) \to 0$ as $\sigma \to \infty$. □

The analogue of Theorem 3 in the case of equation of type II is as follows.

Theorem 7. *Let $g \in L^1(0,T)$, $\varkappa(x) > 0$, $x \in (0,T)$ and $\varkappa \notin L^1(0,T)$. Assume that a and b have the form* (2.16) *where a_0 and b_0 satisfy the conditions*

$$(4.12) \quad \sup_{x\in(\cdot,T)} |a_0(x,\cdot)| \in L^1(0,T)\,, \quad \sup_{x\in(\cdot,T)} |b_0(x,\cdot)| \in L^1(0,T).$$

Then the assertions of Theorem 3 *are valid.*

Proof. is partially similar to proof of Theorem 3. We write the equation (2.10) in the form $y = f + G[y] + L[y,y]$, where f, G and L are given by (3.6 - 3.8). Then have the relations (3.33), (3.34) and the function f given by (3.6) satisfies (3.35). To analyze the equation in the space $C[0,T]$ we make use of the scale of norms (3.36).

From (3.8) using the relations $k = \varkappa + B$ and $a(\xi,\eta) = \varkappa(\xi)a_0(\xi,\eta)$ we obtain

$$\begin{aligned}(4.13)\quad e^{-\sigma x}L[z_1,z_2](x) = \int_0^x & e^{-\sigma(x-\xi)}e^{-\int_\xi^x B(\eta)d\eta}\varkappa(\xi)e^{-\int_\xi^x \varkappa(\eta)d\eta} \\ & \int_0^\xi a_0(\xi,\tau)e^{-\sigma\tau}z_1(\tau)e^{-\sigma(\xi-\tau)}z_2(\xi-\tau)d\tau d\xi.\end{aligned}$$

Observing the equality $\varkappa(\xi)e^{-\int_\xi^x \varkappa(\eta)d\eta} = \frac{d}{d\xi}e^{-\int_\xi^x \varkappa(\eta)d\eta}$, the assumption (4.12), the positivity of $\varkappa$ and the equality $\lim\limits_{\xi\to 0^+} e^{-\int_\xi^x \varkappa(\eta)d\eta} = 0$ (that holds in view of (2.1) and $\varkappa \notin L^1(0,T)$) from (4.13) we deduce the estimate

$$\begin{aligned}(4.14)\qquad \left|e^{-\sigma x}L[z_1,z_2](x)\right| &\le C_B\int_0^x \sup_{\xi\in(\tau,T)}|a_0(\xi,\tau)|d\tau \\ &\qquad \int_0^x d\left(e^{-\int_\xi^x \varkappa(\eta)d\eta}\right)\ \|z_1\|_\sigma\|z_2\|_\sigma \\ &= C_B\int_0^x \sup_{\xi\in(\tau,T)}|a_0(\xi,\tau)|d\tau\ \|z_1\|_\sigma\|z_2\|_\sigma\end{aligned}$$

where $C_B = \max\limits_{0\le\xi\le x\le T} e^{-\int_\xi^x B(\eta)d\eta}$. This implies

$$(4.15)\qquad \|L[z_1,z_2]\|_\sigma \ \le\ N\|z_1\|_\sigma\|z_2\|_\sigma\,,\quad \sigma\ge 0,$$

with the positive constant $N = C_B\int_0^T \sup\limits_{\xi\in(\tau,T)}|a_0(\xi,\tau)|d\tau$. Moreover, (4.14) yields the continuity of $L[z_1,z_2](x)$ at $x=0$ and the equality $L[z_1,z_2](0) = 0$. The continuity of $L[z_1,z_2](x)$ in $(0,T]$ follows from the continuity in the domain $\Delta_T\setminus\{(0,0)\}$ of the function of arguments (x,ξ) under the integral $\int_0^x$ in the right-hand side of the formula (4.13) (cp. (3.33)) . Thus,

$$(4.16)\qquad L[z_1,z_2]\in C[0,T]\quad\text{for any}\quad z_1,z_2\in C[0,T].$$

Analogously to (4.14) from (4.13) we derive the estimates

$$\|L[z_1,z_2]\|_\sigma \le \begin{cases} \nu_1(\sigma)\|z_1\|_0\,\|z_2\|_\sigma \\ \nu_2(\sigma)\|z_1\|_\sigma\|z_2\|_0 \end{cases}, \quad \sigma \ge 0, \tag{4.17}$$

where

$$\begin{aligned}\nu_1(\sigma) &= C_B \max_{x\in[0,T]} \int_0^x e^{-\sigma(x-\tau)} \sup_{\xi\in(\tau,T)} |a_0(\xi,\tau)|d\tau\,, \quad \nu_2(\sigma)\\ &= C_B \int_0^T e^{-\sigma\tau} \sup_{\xi\in(\tau,T)} |a_0(\xi,\tau)|d\tau.\end{aligned}$$

Due to the assumption (4.12) and Lemma 6 we have

$$\nu_k(\sigma) \to 0 \quad \text{as} \quad \sigma\to\infty\,, \quad k=1,2. \tag{4.18}$$

For the quantity $G[z]$ in view of (2.16) from (3.7) we get

$$\begin{aligned}e^{-\sigma x}G[z](x) = \int_0^x e^{-\sigma(x-\xi)} e^{-\int_\xi^x B(\eta)d\eta} \varkappa(\xi) e^{-\int_\xi^x \varkappa(\eta)d\eta}\\ \int_0^\xi b_0(\xi,\tau)e^{-\sigma(\xi-\tau)}e^{-\sigma\tau}z(\tau)d\tau d\xi.\end{aligned} \tag{4.19}$$

Similarly as above we deduce the estimate

$$\left|e^{-\sigma x}G[z](x)\right| \le C_B \max_{0\le y\le x} \int_0^y e^{-\sigma(y-\tau)} \sup_{\xi\in(\tau,T)} |b_0(\xi,\tau)|d\tau\ \|z\|_\sigma. \tag{4.20}$$

Thus, we obtain

$$\|G[z]\|_\sigma \le M(\sigma)\|z\|_\sigma\,, \quad \sigma\ge 0, \tag{4.21}$$

with

$$M(\sigma) = C_B \max_{0\le y\le T} \int_0^y e^{-\sigma(y-\tau)} \sup_{\xi\in(\tau,T)} |b_0(\xi,\tau)|d\tau.$$

Due to Lemma 6 we have

$$M(\sigma)\to 0 \quad \text{as} \quad \sigma\to\infty. \tag{4.22}$$

Moreover, (4.20) implies that $G[z](x)$ is continuous at $x = 0$ and $G[z](0) = 0$. The continuity of $G[z](x)$ in $(0, T]$ follows from the continuity in the domain $\Delta_T \setminus \{(0,0)\}$ of the function of arguments (x, ξ) under the integral $\int_0^x$ in the right-hand side of the formula (4.19). Summing up,

$$G[z] \in C[0,T] \quad \text{for any} \quad z \in C[0,T]. \tag{4.23}$$

The relations (3.35, 4.16, 4.23) show that the equation $y = f + G[y] + L[y,y]$ is well-defined in the space $C[0,T]$ in case $K = 0$ and has no solution in the space $C[0,T]$ in case $K \neq 0$. Therefore, we take into consideration only the case $K = 0$. Observing (4.15, 4.17, 4.18, 4.21, 4.22, and Lemma 1) we see that the equation $y = f + G[y] + L[y,y]$ or, equivalently, (2.10) with $K = 0$ has a unique solution in the space $C[0,T]$. The rest of the proof is identical to the proof of Theorem 3. □

We emphasize that Theorem 4 cannot be extended to the equation of type II. This means that in case $e^{\int_x^T \varkappa(\eta)d\eta} \notin L^1(0,T)$ solutions may exist in $W^1_\varkappa \setminus W^1_0$, too. In next subsection we provide a corresponding example.

4.3. *Examples and special cases.* Firstly, let us analyse the equation (1.2) of type II with $\varkappa$ of the form (3.47). For the sake of shortness we consider only the case $\beta = 0$, i.e.

$$\varkappa(x) = \frac{\gamma}{x^\alpha} \quad in \quad (0,T] \tag{4.24}$$

with $\gamma \neq 0$ and $\alpha \geq 0$. We can formulate the following statements for this equation.

1. *Assume that $\gamma < 0$, $\alpha \geq 0$. Moreover, let $g \in L^1_\varkappa$ and a_0, b_0 satisfy* (4.2). *Then Theorem* 5 *and in case $\varkappa \in L^1(0,T)$ Corollary* 2 *hold.*

2. *Assume that either $\gamma > 0$, $0 \leq \alpha < 1$ or $0 < \gamma < 1$, $\alpha = 1$. Moreover, let $g \in L^1_\varkappa$ and a_0, b_0 satisfy* (4.7). *Then Theorem* 6 *and in case $\varkappa \in L^1(0,T)$ Corollary* 2 *hold.*

3. *Assume that $\gamma > 0$, $\alpha \geq 1$. Moreover, let $g \in L^1(0,T)$ and a_0, b_0 satisfy* (4.12). *Then Theorem* 7 *holds.*

More precisely, the solution sets in particular cases under suitable assumptions on g, a_0 and b_0 are as follows.

If $0 \le \alpha < 1$ then $\varkappa \in L^1(0,T)$, hence (1.2) has a one-parametric family of solutions in the space W_0^1 with the parameter $A = y(0) \in \mathbb{R}$,

if $\gamma < 1$, $\alpha = 1$ then (1.2) has a one-parametric family of solutions in the space $\{y : x^\gamma y(x) \in C[0,T], y' \in \mathcal{I}\}$ with the parameter $A = \lim\limits_{x\to 0^+} x^\gamma y(x) \in \mathbb{R}$,

if $\gamma < 0$, $\alpha > 1$ then (1.2) has a one-parametric family of solutions in the space

$$\left\{y \,:\, e^{-\frac{\gamma}{(\alpha-1)x^{\alpha-1}}} y(x) \in C[0,T], y' \in \mathcal{I}\right\}$$

with the parameter $A = \lim\limits_{x\to 0^+} e^{-\gamma/(\alpha-1)x^{\alpha-1}} y(x) \in \mathbb{R}$.

Further, let us analyse the equation (1.2) of type II with $\varkappa$ of the form (4.24) more closely in the case $\gamma = \alpha = 1$. Then $e^{\int_x^T \varkappa(\eta)d\eta} = \frac{T}{x} \notin L^1(0,T)$. We show that there exist non-continuous solutions to this equation in $W_\varkappa^1 \setminus W_0^1$.

The basic equation of this form is

$$y'(x) + \frac{y(x)}{x} = \frac{1}{x}\int_0^x y(x-\xi)y(\xi)d\xi\,, \quad x > 0. \tag{4.25}$$

It is easy to check (using Laplace transform) that this equation has besides the solution $y_0 \equiv 0 \in W_0^1$ the one-parametric family of solutions

$$y_K(x) = \frac{1}{K}\nu'\left(\frac{x}{K}\right)\,, \quad K > 0 \tag{4.26}$$

with the Volterra's function (see [**4**] Vol. III, Sect. 18.3)

$$\nu(x) = \int_0^\infty \frac{x^t}{\Gamma(t+1)}dt\,.$$

The function ν and its derivatives ν', ν'' have the asymptotic expansions

$$\nu(x) \sim -\frac{1}{\ln x} + \frac{C}{\ln^2 x} \quad \text{as} \quad x \to 0^+$$

where $C = -\Gamma'(1)$ is the Euler constant and

$$\nu'(x) \sim \frac{1}{x\ln^2 x} - \frac{2C}{x\ln^3 x} \qquad \text{as} \quad x \to 0^+\,, \tag{4.27}$$

$$\nu''(x) \sim -\frac{1}{x^2\ln^2 x} - \frac{2(1-C)}{x^2\ln^3 x} \quad \text{as} \quad x \to 0^+\,. \tag{4.28}$$

The sign $\sim$ denotes the asymptotic equality (see [**1**] Sect. 2). In particular, for the solutions (4.26) the asymptotic expansion

$$y_K(x) \sim \frac{1}{x\ln^2 x} + \frac{2(\ln K - C)}{x\ln^3 x} \quad \text{as} \quad x \to 0^+ \tag{4.29}$$

holds. Thus, $y_K \in W^1_\varkappa \setminus W^1_0$ for any $K > 0$ in every finite interval $(0,T)$.

Finally, we study a more general equation

$$\begin{aligned} y'(x) + \left(\frac{1}{x} + B(x)\right) y(x) \\ = \frac{1}{x}\int_0^x a_0(x,\xi)y(x-\xi)y(\xi)d\xi + g(x), \quad x \in (0,T)\,, \end{aligned} \tag{4.30}$$

where $0 < T < 1$ and

$$\begin{aligned} B(x) \in L^1(0,T)\,,\ \ |\ln x|^\delta \sup_{\xi\in(0,x)} |a_0(x,\xi)| \in L^\infty(0,T), \\ x\ln^2 x\, g(x) \in L^1(0,T) \end{aligned} \tag{4.31}$$

with some $\delta > 0$. We seek a solution in the form $y(x) = \rho(x)w(x)$ where $\rho(x) = \nu'(x)$ is the above solution $y_1(x)$ of equation (4.25) and $w \in W^1_0$ is the unknown function. The function w obeys the integro-differential equation

$$\begin{aligned} w'(x) + k_1(x)w(x) \ = \int_0^x a_1(x,\xi)w(x-\xi)w(\xi)d\xi + g_1(x)\,, \\ x \in (0,T)\,, \end{aligned} \tag{4.32}$$

where

$$k_1(x) \ = \ \varkappa_1(x) + B(x) \quad \text{with} \quad \varkappa_1(x) \ = \ \frac{1}{x} + \frac{\nu''(x)}{\nu'(x)}\,,$$

$$a_1(x,\xi) \ = \ \frac{1}{x\nu'(x)} a_0(x,\xi)\nu'(\xi)\nu'(x-\xi) \quad \text{and} \quad g_1(x) \ = \ \frac{g(x)}{\nu'(x)}\,.$$

We are going to show that under the assumptions (4.31) the conditions of Theorem 3 for equation (4.32) are fulfilled. By (4.27) we have $g_1 \in L^1(0,T)$. Further, by (4.27) and (4.28) the asymptotic relation

$$\varkappa_1(x) \sim \frac{1}{x} - \frac{\frac{1}{x^2\ln^2 x} + \frac{2(1-C)}{x^2\ln^3 x}}{\frac{1}{x\ln^2 x} - \frac{2C}{x\ln^3 x}} \sim -\frac{2}{x\ln x} \quad \text{as} \quad x \to 0^+$$

holds implying $\varkappa_1 \notin L^1(0,T)$ together with $B \in L^1(0,T)$. Finally, observing the solution $y_1 = \nu'$ of (4.25) and using its positivity we again have

$$\begin{aligned}\int_0^x |a_1(x,\xi)|d\xi &\leq \frac{C_{a_1}}{x\nu'(x)|\ln x|^\delta}\int_0^x \nu'(\xi)\nu'(x-\xi)d\xi \\ &= \frac{C_{a_1}}{x\nu'(x)|\ln x|^\delta}\,[x\nu''(x)+\nu'(x)] = C_{a_1}\frac{\varkappa_1(x)}{|\ln x|^\delta} \in L^1(0,T)\end{aligned}$$

with the constant $C_{a_1} = \sup\limits_{0<\xi<x<T} |\ln x|^\delta\,|a_0(x,\xi)|$. Hence, applying Theorem 3, we prove

Theorem 8. *Let the assumptions* (4.31) *hold. Then equation* (4.30) *has a solution of the form $y(x) = \nu'(x)w(x)$ where $w \in W_0^1$ with $w(0) = 0$.*

5. Series expansion of solutions and asymptotics at infinity.

5.1. *Exponential series. Burgers' equation*

$$y'(x) + \left(\frac{1}{2x} - \beta_0 x^2\right)y(x) = \int_0^x y(\xi)y(x-\xi)d\xi \quad (\beta_0 > 0) \tag{5.1}$$

has a unique solution $y = y_A$ in $(0,\infty)$ which fulfills the initial condition

$$x^{1/2}y(x)|_{x=0} = A \tag{5.2}$$

for prescribed A. With Burgers we are looking for a solution in form of the (for $x > 0$ convergent) series

$$y(x) = Kx\sum_{n=1}^{\infty} e^{-\alpha_n x} \tag{5.3}$$

with $K \in \mathbb{R}$ and $0 < \alpha_1 < \alpha_2 < \cdots$ for some $A \in \mathbb{R}$ in (5.2).

Inserting the ansatz (5.3) into equation (5.1), we get the equation for K and α_n, $n = 1, 2, \ldots$

$$\begin{aligned}\frac{3}{2}K\sum_{n=1}^{\infty} e^{-\alpha_n x} - Kx\sum_{n=1}^{\infty}\alpha_n e^{-\alpha_n x} - K\beta_0 x^3 \sum_{n=1}^{\infty} e^{-\alpha_n x} \\ = \frac{K^2}{6}x^3\sum_{n=1}^{\infty} e^{-\alpha_n x} + 2K^2 x\sum_{n=1}^{\infty}\Big(\sum_{m\neq n}\frac{1}{(\alpha_n-\alpha_m)^2}\Big)e^{-\alpha_n x} \\ + 4K^2\sum_{n=1}^{\infty}\Big(\sum_{m\neq n}\frac{1}{(\alpha_n-\alpha_m)^3}\Big)e^{-\alpha_n x}\end{aligned}$$

which is satisfied if $K = -6\beta_0$ and

$$\sum_{m\neq n}\frac{1}{(\alpha_m-\alpha_n)^2} = \frac{\alpha_n}{12\beta_0}\,, \quad \sum_{m\neq n}\frac{1}{(\alpha_m-\alpha_n)^3} = \frac{1}{16\beta_0}\,. \tag{5.4}$$

Comparison with (A.8) shows that the relations (5.4) are fulfilled if $\lambda = 1/(3\sqrt{\beta_0})$ in (A.8) i.e. if $-\alpha_n$ are chosen as the zeros of the entire function $q_0(z) = z^{1/2}K_{1/3}\left(\frac{1}{3\sqrt{\beta_0}}z^{3/2}\right)$.

It remains to calculate the initial condition (5.2) for this solution (5.3) of equation (5.1). In view of (A.2) we have

$$\varphi(x) = \sum_{n=1}^{\infty} e^{-\alpha_n x} \sim \varphi_0(x) = \sum_{n=1}^{\infty}\exp\left[-\left(\frac{\pi}{\lambda}\right)^{2/3} x n^{2/3}\right]$$

and by [**5**] pp. 36-39

$$\varphi_0(x) \sim \frac{3}{4}\frac{\lambda}{\sqrt{\pi}}x^{-3/2} \quad \text{as} \quad x \to 0^+\,,$$

hence

$$y_A(x) \sim Ax^{-1/2}\,, \quad A = \frac{3}{4}\lambda\frac{K}{\sqrt{\pi}} = -\frac{3}{2}\sqrt{\frac{\beta_0}{\pi}}\,. \tag{5.5}$$

5.2. *Power series.* The *generalized Burgers' equation*

$$y'(x) + \left(\frac{1}{2x} + \omega x^{1/2} - \beta_0 x^2\right) y(x) = \int_0^x y(\xi)y(x-\xi)d\xi \tag{5.6}$$

where $\omega, \beta_0 \in \mathbb{R}$ also has a unique continuous solution $y = y_A$ on $(0,\infty)$ satisfying the initial condition (5.2) for prescribed $A \in \mathbb{R}$. The solution is of the form

$$y(x) = Ax^{-1/2} + x^{-1/2}z(x)$$

with a continuous function z on $[0,\infty)$ where $z(0) = 0$. Like Burgers ([**3**] Chap. IV, Sect. 21) for $\omega = 0$ we are looking for the solution in form of a power series

$$y(x) = \sum_{n=0}^{\infty} a_n x^{-1/2+3/2n}\,, \quad a_0 = A\,. \tag{5.7}$$

Inserting the ansatz (5.7) into equation (5.6), we obtain $a_1 = \frac{2}{3}(\pi A^2 - \omega A)$ and the recurrence system for $a_2, a_3, \ldots$

$$\frac{3}{2}(n+1)a_{n+1} + \omega a_n - \beta_0 a_{n-1} = \sum_{m=0}^{n} a_m a_{n-m} B_{m,n-m}\,, \quad n = 1, 2, \ldots$$

where $B_{m,k} = \mathrm{B}\left(\frac{3}{2}m + \frac{1}{2}, \frac{3}{2}k + \frac{1}{2}\right)$ with the Euler Beta function $\mathrm{B}(p,q)$.

We ask for a solution of the simple form

$$y(x) = Ax^{-1/2} + a_1 x\,, \quad a_1 = \frac{2}{3}(\pi A^2 - \omega A). \tag{5.8}$$

Such a solution exists if

$$\text{either} \quad \omega = \pi A\,, \ \beta_0 = 0 \quad \text{or} \quad \omega = \frac{5}{2}A\,, \ \beta_0 = -\frac{1}{9}\left(\pi - \frac{5}{2}\right)A^2 \tag{5.9}$$

where $a_1 = 0$ or $a_1 = \frac{2}{3}\left(\pi - \frac{5}{2}\right)A^2$, respectively.

5.3. *Asymptotics at infinity.* The representation (5.3) of the solution y_A to equation (5.1) with (5.2) for $A = -\frac{3}{2}\sqrt{\frac{\beta_0}{\pi}}$ is an asymptotic expansion for y_A as $x \to +\infty$ and yields the asymptotic representation

$$y_A(x) \sim Kxe^{-\alpha_1 x} \quad \text{as} \quad x \to +\infty. \tag{5.10}$$

Formula (5.10) shows that y_A tends (exponentially) to zero at infinity.

The function $\widetilde{y}(x) = e^{\epsilon x} y_A(x)$, $\epsilon \in \mathbb{R}$, satisfies the integro-differential equation

$$\widetilde{y}\,'(x) + \left(\frac{1}{2x} - \beta_0 x^2 - \epsilon\right)\widetilde{y}(x) = \int_0^x \widetilde{y}(\xi)\widetilde{y}(x-\xi)d\xi \tag{5.11}$$

and the same initial condition (5.2) as y_A. But although the coefficient of $\widetilde{y}$ in (5.11) has analogous asymptotic behavior like the coefficient of y_A in (5.1) as $x \to +\infty$ we have a different asymptotic behavior of $\widetilde{y}$ and y_A as $x \to +\infty$ for $\epsilon \neq 0$. In particular, for $\epsilon > \alpha_1$ the function $\widetilde{y}$ tends (exponentially) to infinity as $x \to +\infty$.

Further, the solution (5.8) of equation (5.6) under the conditions (5.9) with $\beta_0 < 0$ behaves like $a_1 x$ as $x \to +\infty$, i.e. also tends to infinity and the solutions (4.26) of equation (4.25) are asymptotically equal to $\frac{1}{K} e^{x/K}$ as $x \to +\infty$. Since it seems difficult to obtain a more or less complete picture about the asymptotic behavior of the exact solutions as $x \to +\infty$ we shall study *asymptotic solutions* ([**1**] Chap. II) for two classes of integro-differential equations in the next section. This means the left-hand side of equation (1.2) is asymptotically equal to the right-hand side but not necessarily it is the asymptotic representation of an exact solution.

6. Asymptotic solutions.

6.1. *Integro-differential equations of type I.* We consider the class of equations

$$y'(x) + k(x)y(x) = \int_0^x y(\xi)y(x-\xi)d\xi \tag{6.1}$$

where

$$k(x) = \frac{1}{x^\alpha}\left[\gamma + \delta x^\beta + B(x)\right] \tag{6.2}$$

with $0 < \alpha < \beta$, $\gamma \neq 0$, $\delta \neq 0$ and $B(x) = o\big(x^\beta\big)$ as $x \to +\infty$, $B(x) = o(1)$ as $x \to 0^+$. Equation (6.1) with (6.2) has the asymptotic solutions as $x \to +\infty$

$$\overline{y}(x) = \lambda x^\mu e^{\nu x} \tag{6.3}$$

where $\lambda \mathrm{B}(\beta-\alpha, \beta-\alpha) = \delta$, $\mu = \beta - \alpha - 1 > -1$ and arbitrary ν. The asymptotic solutions (6.3) are the solutions of the approximate equation

$$\delta x^{\beta-\alpha}\overline{y}(x) \;=\; \int_0^x \overline{y}(\xi)\overline{y}(x-\xi)d\xi\,.$$

To fix the parameter ν the equation (6.1) with $\overline{k}(x) = \gamma x^{-\alpha} + \delta x^{\beta-\alpha}$ and the corresponding initial condition

$$x^{\gamma}y(x)|_{x=0} \;=\; A \tag{6.4}$$

has to be taken into account. An approximate value for ν in case of continuous B can be obtained by the simple approximation

$$\widehat{y}(x) \;=\; \begin{cases} Ax^{-\gamma} & \text{in} \quad (0,\rho) \\ \lambda x^{\mu}e^{\nu x} & \text{in} \quad (\rho,\infty) \end{cases}$$

for the exact y with some $\rho \in (0,\infty)$. The values of ν and ρ are determined by the continuity of $\widehat{y}$ and $\widehat{y}\,'$ at $x = \rho$. In the case of Burgers' equation (5.1) with $\beta_0 = \frac{1}{16}$ where $\alpha = 1$, $\beta = 3$, $\gamma = \frac{1}{2}$, $\delta = -\frac{1}{16}$ and $B(x) = 0$ we get $\rho \approx 0.5388$ and $\nu \approx -2.784$ in this way where the exact value is $\nu = -2.920$ (cf. [**3**] Chap. V). Further, for non-vanishing B the shifting of ν by the substitution $\widetilde{y} = e^{\epsilon x}y$ (cf. Section 5.3) should be observed.

We remark that in the case $\alpha = 1$ with $\gamma < 1$ if $k(x) \sim \delta x^{\beta-1}\ln x$ as $x \to +\infty$ asymptotic solutions of the form

$$\overline{y}(x) \;=\; \lambda x^{\mu}e^{\nu x}\ln x\,, \quad \mu = -\gamma > -1$$

exist.

6.2. *Integro-differential equations of type II.* Finally, the class of equations

$$x^{\alpha}y'(x) + k_0(x)y(x) \;=\; \int_0^x y(\xi)y(x-\xi)d\xi \tag{6.5}$$

where

$$k_0(x) \;=\; \gamma + \delta x^{\beta} + B_0(x) \tag{6.6}$$

with $\alpha > 0$, $\beta > 0$, $\gamma \neq 0$, $\delta \neq 0$ and $B_0(x) = o\bigl(x^\beta\bigr)$ as $x \to +\infty$, $B(x) = o(1)$ as $x \to 0^+$ is dealt with. Equation (6.5) with (6.6) has the asymptotic solutions (6.3) where for arbitrary $\nu \neq 0$

$$\mu = \mu_0 - 1\,,\ \mu_0 = \max\{\alpha, \beta\} > 0 \quad \text{and} \quad \lambda \mathrm{B}(\mu_0, \mu_0) = \begin{cases} \delta & \text{if } \ \beta > \alpha \\ \delta + \nu & \text{if } \ \beta = \alpha \\ \nu & \text{if } \ \beta < \alpha \end{cases}$$

and for $\nu = 0$

$$\mu = \mu_1 - 1\,,\ \mu_1 = \max\{\alpha - 1, \beta\} > 0$$
$$\text{and} \quad \lambda \mathrm{B}(\mu_1, \mu_1) \ = \ \begin{cases} \delta & \text{if } \ \beta > \alpha - 1 \\ \delta + \alpha - 2 & \text{if } \ \beta = \alpha - 1 \\ \alpha - 2 & \text{if } \ \beta < \alpha - 1\,. \end{cases}$$

Appendix. Sums with roots of a Bessel function.

Following Burgers ([**3**] Chap. V) we consider the entire function

$$(A.1) \quad q_0(z) \ = \ z^{1/2} K_{1/3}(\lambda z^{3/2}) \ = \ \frac{\pi}{\sqrt{3}} \left[\sum_{k=0}^{\infty} \frac{(\lambda/2)^{2k-1/3}}{k!\,\Gamma(k+2/3)} z^{3k} - \sum_{k=0}^{\infty} \frac{(\lambda/2)^{2k+1/3}}{k!\,\Gamma(k+4/3)} z^{3k+1} \right]$$

where K_ν, $\nu > 0$, denotes the modified Bessel function of the third kind (Macdonald function) (see [**4**] Vol. III, Chap. VII) and $\lambda > 0$. For $-\pi < \arg z < \pi$ there holds the asymptotic relation (cf. [**4**] Vol. II, Sect. 7.4.1)

$$q_0(z) \ \sim \ \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{\lambda}} z^{-1/4} e^{-\lambda z^{3/2}} \quad \text{as} \quad |z| \to \infty$$

and $q_0(z)$ has only zeros on the negative real axis. For $z = -x < 0$ by (A.1) we have

$$q_0(-x) \ = \ \frac{\pi}{\sqrt{3}}\, x^{1/2} \left[J_{1/3}(\lambda x^{3/2}) + J_{-1/3}(\lambda x^{3/2}) \right]$$

with the Bessel function of the first kind J_ν implying the asymptotic relation (cf. [**4**] Sect. 7.13.1)

$$q_0(-x) \sim \sqrt{\frac{2\pi}{\lambda}} x^{-1/4} \cos\left(\lambda x^{3/2} - \frac{\pi}{4}\right) + O\left(x^{-7/4}\right)$$

as $x \to \infty$. The function $q_0(x)$ has the simple zeros $z_n = -\alpha_n$, $n = 1, 2, \ldots$, with $0 < \alpha_1 < \alpha_2 < \cdots$ and

$$\alpha_n = \left(\frac{\pi}{\lambda}\right)^{2/3} n^{2/3} + O\left(n^{-1/3}\right) \tag{A.2}$$

as $n \to \infty$.

The theorem of residues applied to a contour integral of the function $\frac{x}{z(x-z)} \frac{q_n'(z)}{q_n(z)}$ where $q_n(z) = q_0(z - \alpha_n)$, $x \in \mathbb{R}$, yields the relations (cf. [**3**] Chap. V, [**13**] pp. 497-498)

$$\sum_{\substack{m=1,2,\ldots\infty \\ m\neq n}} \frac{x}{(\alpha_n - \alpha_m)(\alpha_n - \alpha_m - x)} = \frac{1}{x} - \frac{q_n'(x)}{q_n(x)}, \quad n = 1, 2, \ldots . \tag{$A.3$}$$

In view of (A.2) the series in (A.3) is absolutely convergent in a neighborhood of $x = 0$. We expand the right-hand side of (A.3) at $x = 0$. It holds

$$\begin{aligned} \frac{q_n'(x)}{q_n(x)} = \frac{1}{x} \Bigg\{ 1 + \frac{q_0''}{2q_0'} x + \left(\frac{q_0'''}{3q_0'} - \frac{{q_0''}^2}{4q_0'} \right) x^2 \\ + \left(\frac{q_0^{IV}}{8q_0'} - \frac{q_0'' q_0'''}{4{q_0'}^2} + \frac{1}{8} \left[\frac{q_0''}{q_0'} \right]^3 \right) x^3 \Bigg\} + O(x^3) \end{aligned} \tag{$A.4$}$$

as $x \to 0$ where the derivatives of q_0 are calculated at $-\alpha_n$. The function $q_0(-x)$ (like $q_0(x)$) is an Airy integral (cf. [**4**] Vol. II, Sect. 7.3.7) satisfying Airy's differential equation (cf. [**13**] Sect. 6.4) from which the equalities $q_0''(-\alpha_n) = 0$, $n = 1, 2, \ldots$ follow. Hence, (A.4) simplifies to

$$\frac{1}{x} - \frac{q_n'(x)}{q_n(x)} = -\frac{q_0'''}{3q_0'} x - \frac{q_0^{IV}}{8q_0'} x^2 + O(x^3) \; . \tag{A.5}$$

Further, by differentiating twice the Airy's differential equation, we obtain the relations

$$\text{(A.6)} \qquad \frac{1}{3}\frac{q_0'''(-\alpha_n)}{q_0'(-\alpha_n)} = -\frac{3}{4}\lambda^2\alpha_n\,, \quad \frac{1}{8}\frac{q_0^{IV}(-\alpha_n)}{q_0'(-\alpha_n)} = \frac{9}{16}\lambda^2\,.$$

The left-hand side of (A.3) has the expansion at $x = 0$

$$(A.7) \quad \sum_{m\neq n}\frac{x}{(\alpha_n-\alpha_m)(\alpha_n-\alpha_m-x)} = \sum_{m\neq n}\frac{x}{(\alpha_n-\alpha_m)^2\left[1-\frac{x}{\alpha_n-\alpha_m}\right]}$$
$$= x\sum_{m\neq n}\frac{1}{(\alpha_m-\alpha_n)^2} - x^2\sum_{m\neq n}\frac{1}{(\alpha_m-\alpha_n)^3} + O\big(x^3\big)\,.$$

Now comparing the coefficients of x and x^2 in both sides of (A.3) and observing (A.7) as well as (A.5) with (A.6), we obtain the sums

$$\text{(A.8)} \qquad \sum_{m\neq n}\frac{1}{(\alpha_m-\alpha_n)^2} = \frac{3}{4}\lambda^2\alpha_n\,, \quad \sum_{m\neq n}\frac{1}{(\alpha_m-\alpha_n)^3} = \frac{9}{16}\lambda^2\,.$$

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REFERENCES

1. L. Berg, *Asymptotische Darstellungen und Entwicklungen*, Deutscher Verlag der Wissenschaften, Berlin, (1968).

2. L. Berg and L. v. Wolfersdorf, *On a class of generalized autoconvolution equations of the third kind*, Z. Anal. Anw. (J. Anal. Appl.) **24** (2005), 217–250.

3. J. M. Burgers, *The nonlinear diffusion equation*, Reidel Publ. Co., Dordrecht-Holland, (1974).

4. A. Erdélyi (ed.), *Higher transcendental functions*, Vo. I - III, McGraw Hill, New York, (1953, 1955).

5. M. A. Evgrafov, *Asymptotic estimates and entire functions* (in Russ.), Gos. Izd. Tekh.-Teor. Lit., Moscow, (1957).

6. U. Frisch, *Turbulence: the legacy of A. N. Kolmogorov*, Cambridge University Press, Cambridge, (1995).

7. S. N. Gurbatov, A. I. Saichev and I. G. Yakushkin, *Nonlinear waves and one-dimensional turbulence in nondispersive media*, Sov. Phys. Usp. **26** (1983), 857–876.

8. F. Hirsch and G. Lacombe, *Elements of functional analysis*, Springer, New York, (1999).

9. B. Hofmann and L. v. Wolfersdorf, *On the determination of a density function by its autoconvolution coefficient*, Numer. Funct. Anal. Optim. **27** (2006), 357–375.

10. J. Janno, *Nonlinear equations with operators satisfying generalized Lipschitz-conditions in scales*, Z. Anal. Anw. (J. Anal. Appl.) **18** (1999), 287–295.

11. J. Janno and L. v. Wolfersdorf, *A general class of autoconvolution equations of the third kind*, Z. Anal. Anw. (J. Anal. Appl.) **24** (2005), 523–543.

12. J. Qian, *Numerical experiments on one-dimensional model of turbulence*, Phys. of Fluids **27** (1984), 1957–1965.

13. G. N. Watson, *A treatise on the theory of Bessel functions*, Cambridge University Press, Cambridge, (1996).

14. L. v. Wolfersdorf, *On the theory of convolution equations of the third kind*, J. Math. Anal. Appl. **331** (2007), 1314–1336.

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