JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 21, Number 1, Spring 2009

# THE COMPLEXITY OF FREDHOLM EQUATIONS OF THE SECOND KIND: NOISY INFORMATION ABOUT EVERYTHING

ARTHUR G. WERSCHULZ

Communicated by Ian Sloan

ABSTRACT. We study the complexity of Fredholm problems of the second kind $u - \int_\Omega k(\cdot, y)u(y)\,dy = f$. Previous work on the complexity of this problem has assumed that $\Omega$ was the unit cube $I^d$. In this paper, we allow $\Omega$ to be part of the data specifying an instance of the problem, along with $k$ and $f$. More precisely, we assume that $\Omega$ is the diffeomorphic image of the unit $d$-cube under a $C^{r_1}$ mapping $\rho\colon I^d \to I^l$. In addition, we assume that $k \in C^{r_2}(I^{2l})$ and $f \in C^{r_3}(I^l)$. Using a change of variables, we can reduce this problem to an integral equation over $I^d$. Our information about the problem data consists of function evaluations, contaminated by $\delta$-bounded noise. Error is measured by the max norm. We show that the problem is unsolvable if $r_1 = 1$ and $d < l$. Hence we assume that either $r_1 \geq 2$ or $d = l$ in what follows. We find that the $n$th minimal error is bounded from below by $\Theta(n^{-\mu_1} + \delta)$ and from above by $\Theta(n^{-\mu_2} + \delta)$, where

$$\mu_1 = \min\left\{\frac{r_1}{d}, \frac{r_2}{2d}, \frac{r_3}{d}\right\} \quad \text{and} \quad \mu_2 = \min\left\{\frac{r_1 - 1}{d}, \frac{r_2}{2d}, \frac{r_3}{d}\right\}.$$

The upper bound is attained by a noisy modified Galerkin method, which can be efficiently implemented by a two-grid algorithm. We thus find bounds on the $\varepsilon$-complexity of the problem, these bounds depending on the cost $\mathbf{c}(\delta)$ of calculating a $\delta$-noisy function value. As an example, if $\mathbf{c}(\delta) = \delta^{-b}$, we find that the $\varepsilon$-complexity is between $(1/\varepsilon)^{b+1/\mu_1}$ and $(1/\varepsilon)^{b+1/\mu_2}$.

## 1. Introduction.

We are interested in the worst case complexity of solving Fredholm problems of the second kind

$$u(s) - \int_\Omega k(s,t)u(t)\,dt = f(s) \qquad \forall\, s \in \Omega. \tag{1.1}$$

This research was supported in part by the National Science Foundation under Grant CCR-99-87858, as well as by a Fordham Univeristy Faculty Fellowship.
Received by the editors on September 16, 2006, and in revised form on March 13, 2007.
DOI:10.1216/JIE-2009-21-1-113 

Previous work on the complexity of this problem has dealt with the case where the domain $\Omega$ of the integral equation has been the unit cube $I^d$. Moreover, most of this work has either assumed that we have had complete information about $k$, or that $k$ and $f$ have had the same smoothness (see, e.g., [**8, 9, 11, 14, 20, 21**, Sec. 6.3], and the references contained therein). Furthermore, most of the work (with the exception of [**11**] and a few papers referenced therein) has assumed that the information was exact.

In [**22**], we studied the complexity of this problem under the assumption that we had noisy standard information about the kernel $k$ and the right-hand side $f$, with $k$ and $f$ having different smoothness. This lifted many of the restrictions in the previous studies of this problem. However, [**22**] still assumed that the problem was being solved over the unit cube.

Clearly, the assumption $\Omega = I^d$ is exceptionally restrictive. We need to be able to solve Fredholm problems over whatever domains they naturally arise. Examples include the following:

• The solution of Poisson's equation can be written in terms of integral equations involving single layer potentials, see (e.g.) [**6**, pg. 390] and [**10**, Chap. 8].

• The solution of the exterior Helmholtz problem (which arises in scattering theory) can be expressed in terms of the solution of a Fredholm problem, see [**2**].

Note that the integral equations arising in these examples need to be solved over whatever domain the particular problem is defined, and not merely (say) a cube. For problems defined over boundaries of regions (such as the examples given above), the domain in question is a $d$-dimensional subset of $\mathbb{R}^{d+1}$. This motivates our interest in solving Fredholm problems over general $d$-dimensional subsets of $\mathbb{R}^l$, where $d \le l$.

In this paper, we study the worst case complexity of Fredholm problems, assuming that we have noisy standard information about all the elements that prescribe our problem. Roughly speaking, this means the following:

1. The domain $\Omega$ is the image $\rho(I^d)$ of the closed unit $d$-cube under an injection $\rho \in C^{r_1}(I^d; I^l)$. Hence $\Omega$ is a subregion of $I^l$ when $d = l$, whereas $\Omega$ is a $d$-dimensional surface in $I^l$ if $d < l$.

2. The kernel $k$ belongs to a ball of $C^{r_2}(I^{2l})$. Moreover, the operator appearing on the left-hand side of (1.1) is invertible, with all such operators satisfying a "uniform invertibility" condition.

3. The right-hand side $f$ belongs to the unit ball of $C^{r_3}(I^l)$.

4. Only $\delta$-noisy standard information (i.e., noisy function values) is available about the functions $\rho$, $f \circ \rho$, and $k \circ \rho$ determining a particular problem instance.[1]

Under these conditions, we can use a change of variables to reduce the problem to a new integral equation that is defined over $I^d$. Having done so, we measure the error of an approximation as its worst case error in the $C(I^d)$-norm. The full details are given in §2.

We are able to determine bounds on the $n$th minimal radius $r(n, \delta)$ of $\delta$-noisy information, i.e., the minimal error when we use $n$ evaluations with a noise level of $\delta$. In §3, we establish the following lower bounds:[2]

1. Let $d < l$ and $r_1 = 1$. Then

$$r(n, \delta) \asymp 1.$$

2. Let $d = l$ or $r_1 \geq 2$. Then

$$r(n, \delta) \succcurlyeq \left(\frac{1}{n}\right)^{\mu_1} + \delta,$$

where

$$\mu_1 = \min\left\{\frac{r_1}{d}, \frac{r_2}{2d}, \frac{r_3}{d}\right\}.$$

Note that the problem is *unsolvable* if $d < l$ and $r_1 = 1$, i.e., we cannot make the error arbitrarily small using finitely many noisy evaluations, no matter how small the noise level nor how large the number of evaluations. Hence, the problem is solvable only if $d = l$ or if $r_1 > 1$.

[1] One might expect that we would use simpler $\delta$-noisy values of $f$ and $k$, rather than of composite noisy information about $f \circ \rho$ and $k \circ \rho$. Our choice simplifies the exposition somewhat. In addition, our choice involves no essential loss of generality; the simpler noisy information is also noisy composite information, albeit with a different value of $\delta$ that involves a Lipshcitz constant for $\rho$.

[2] In this paper, we use $\preccurlyeq$, $\succcurlyeq$, and $\asymp$ to denote $O$-, $\Omega$-, and $\Theta$-relations. Here, all proportionality factors are independent of $n$ and $\delta$.

Next, we seek upper bounds on the $n$th minimal noisy error. These bounds are given by a noisy Galerkin method, described in §4. This method uses two meshsizes $\bar{h}$ and $h$, for approximating the Fredholm kernel $k \circ \rho$ and the right-hand side $f \circ \rho$, respectively. In §5, we analyze the error of this method in terms of $h$, $\bar{h}$, and $\delta$. Then in §6, we show how to choose $h$ and $\bar{h}$ minimizing, for a given number $n$ of $\delta$-noisy function evaluations, the upper bound on the error of the noisy Galerkin method. We find that if $d = l$ or $r_1 \geq 1$, then

$$r(n, \delta) \preccurlyeq \left(\frac{1}{n}\right)^{\mu_2} + \delta,$$

where

$$\mu_2 = \min\left\{\frac{r_1 - 1}{d}, \frac{r_2}{2d}, \frac{r_3}{d}\right\}.$$

Note that we have tight bounds

$$r(n, \delta) \asymp \left(\frac{1}{n}\right)^{\mu} + \delta \qquad \text{with} \qquad \mu = \min\left\{\frac{r_2}{2d}, \frac{r_3}{d}\right\}$$

for the solvable case ($d = l$ or $r_1 \geq 2$) only when

$$\min\left\{\frac{r_2}{2d}, \frac{r_3}{d}\right\} \leq \frac{r_1 - 1}{d}. \tag{1.2}$$

When this inequality does not hold, there is a gap between our lower and upper bonuds. An especially appalling case occurs when $d = l$ and $r_1 = 1$. Then the upper bound on the minimal error does *not* converge to zero as $n \to \infty$, whereas the lower bound *does* converge to zero as $n \to \infty$, and so we don't even know whether the problem is solvable for this case. The task of determining tight bounds on the minimal error in the remaining cases is currently an open problem.

What can we say about the cost of the noisy Galerkin method? We use the model of computation found in [**15**, §2.9], under the proviso that the noise bound $\delta$ is the same for all observations. Let $\mathbf{c}(\delta)$ denote the cost of evaluating a function with a noise level $\delta$. Then the information cost of this algorithm is $\mathbf{c}(\delta)\, n$. However, since this algorithm involves the solution of a full linear system of equations, the combinatory cost is much worse than $\Theta(n)$. We can overcome this difficulty by using a two-grid implementation of the noisy Galerkin method. This algorithm has the same order of error as the original noisy Galerkin method, and its combinatory cost is $\Theta(n)$. (We omit the detailed description and the analysis of this algorithm, which are substantially the same as in [**22**].) Hence, we can calculate the two-grid approximation using $\Theta(n)$ arithmetic operations, which is optimal.

We can now use our bounds on the $n$th minimal radius to determine bounds on the $\varepsilon$-complexity of the Fredholm problem. First, suppose that $d < l$ and $r_1 = 1$. Since the $n$th minimal radius is bounded away from zero, there exists $\varepsilon_0 > 0$ such that $\mathrm{comp}(\varepsilon) = \infty$ for $0 \le \varepsilon \le \varepsilon_0$. So, we consider the case where $d = l$ or $r_1 \ge 2$. We can show that there exist positive constants $C_1$, $C_2$, and $C_3$, independent of $\varepsilon$, such that the problem complexity is bounded from below by

$$\mathrm{comp}(\varepsilon) \ge \inf_{0<\delta<C_1\varepsilon} \left\{ c(\delta) \left\lceil \left( \frac{1}{C_1\varepsilon - \delta} \right)^{1/\mu_1} \right\rceil \right\}$$

and from above by

$$\mathrm{comp}(\varepsilon) \le C_2 \inf_{0<\delta<C_3\varepsilon} \left\{ c(\delta) \left\lceil \left( \frac{1}{C_3\varepsilon - \delta} \right)^{1/\mu_2} \right\rceil \right\}.$$

Once again, the details are substantially the same as in [**22**]. These upper bounds are attained by two-grid implementations of the noisy modified Galerkin method, with $\delta$ chosen to minimize the right-hand sides of the upper bound.

In particular, suppose that $\mathbf{c}(\delta) = \delta^{-b}$ for some $b > 0$. We find that

$$\left(\frac{1}{\varepsilon}\right)^{b+1/\mu_1} \preccurlyeq \mathrm{comp}(\varepsilon) \preccurlyeq \left(\frac{1}{\varepsilon}\right)^{b+1/\mu_2},$$

Note that when $\mu_1 = \mu_2 = \mu$, which happens if (1.2) holds, we have tight bounds

$$\mathrm{comp}(\varepsilon) \asymp \left(\frac{1}{\varepsilon}\right)^{b+1/\mu}$$

on the $\varepsilon$-complexity.

Finally, we note that the results of this paper lead to several open questions. We address these questions in §7.

**2. Problem description.** In this section, we precisely describe the class of Fredholm problems whose solutions we wish to approximate.

For an ordered ring $\mathscr{X}$, we shall let $\mathscr{X}^+$ and $\mathscr{X}^{++}$ respectively denote the non-negative and positive elements of $\mathscr{X}$. Hence (for example), $\mathbb{Z}^+$ denotes the set of natural numbers (non-negative integers), whereas $\mathbb{Z}^{++}$ denotes the set of strictly positive integers. For a normed linear space $\mathscr{Y}$, we let $\mathscr{B}\mathscr{Y}$ denote the unit ball of $\mathscr{Y}$. We assume that the reader is familiar with the standard concepts and notations involving Hölder and Sobolev norms and spaces, as found in, e.g., [**4, 13**].

As in [**24**], we shall deal only with nondegenerate domains that are bijective images of $I^d$ (see Figure 1), the nondegeneracy meaning that the Jacobian associated with the domain never vanishes.

More precisely, let $\rho\colon I^d \to I^l$ be a continuously differentiable injection, so that $d \le l$ must necessarily hold. The *gradient* of $\rho$ at $x \in I^d$ is

$$(\nabla\rho)(x) = \left[\frac{\partial \rho_i}{\partial x_j}(x)\right]_{1\le i\le l,1\le j\le d} \in \mathbb{R}^{l\times d},$$

where $\rho_1, \ldots, \rho_l$ are the components of $\rho$. The *Jacobian* of $\rho(I^d)$ at $x \in I^d$ is defined to be

$$J(x;\rho) = \sqrt{\det A(x;\rho)},$$

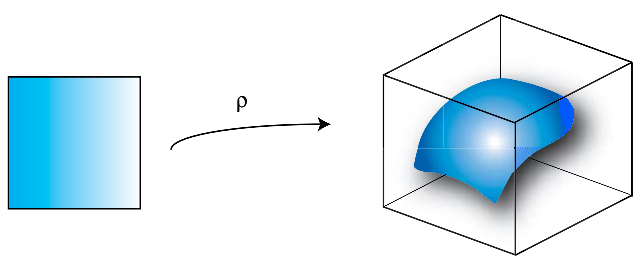


Figure 1: An admissible domain

where

$$A(x;\rho) = [(\nabla\rho)(x)]^{\mathsf{T}}[(\nabla\rho)(x)] \in \mathbb{R}^{d\times d}.$$

The region $\rho(I^d)$ is *nondegenerate* if $J(x;\rho) \neq 0$ for all $x \in I^d$. For a nondegenerate region, we have the change of variables formula

$$\int_{\Omega_\rho} v(t)\,dt = \int_{I^d} v\big(\rho(x)\big) J(x;\rho)\,dt, \tag{2.1}$$

and so $J(x;\rho)\,dx$ is the volume element (if $d = l$) or surface area element (if $d < l$) of $\Omega_\rho$ at $x \in I^d$. See [**7**, p. 334 ff.] for further discussion.

Given such $\rho$, let $\Omega_\rho = \rho(I^d)$, and suppose that $u$ is the solution of the Fredholm problem (1.1) over the domain $\Omega_\rho$, i.e.,

$$u(s) - \int_{\Omega_\rho} k(s,t)u(t)\,dt = f(s) \qquad \forall\, s \in \Omega_\rho.$$

Writing $s = \rho(x)$ and using the change of variables formula (2.1), this may be rewritten

$$u\big(\rho(x)\big) - \int_{I^d} k\big(\rho(x),\rho(y)\big)u\big(\rho(y)\big)J(y;\rho)\,dy = f\big(\rho(x)\big) \quad \forall\, x \in I^d \tag{2.2}$$

as a problem over $I^d$.

It will be convenient to write (2.2) as an operator equation. Define

$$k_\rho(x,y) = k\big(\rho(x),\rho(y)\big) \qquad \forall\, x,y \in I^d.$$

Moreover, for any $v\colon I^d \to \mathbb{R}$ and $g\colon I^{2d} \to \mathbb{R}$, let us write

$$T_{\rho,g}v = \int_{I^d} g(\cdot,y)v(y)J(y;\rho)\,dy$$

and

$$v_\rho = v \circ \rho.$$

Then we may rewrite (2.2) in the form

$$(I - T_{\rho,k_\rho})u_\rho = f_\rho. \tag{2.3}$$

We are now ready to describe the admissible problem elements $[\rho,k,f]$. We begin with the class $\mathscr{R}$ of functions $\rho\colon I^d \to I^l$ determining the domains $\Omega_\rho = \rho(I^d)$. Let positive numbers $c_1$ and $c_2$ be given, along with

$r_1 \geq 1$. Then $\mathscr{R}$ consists of the functions $\rho \in C^{r_1}(I^d; I^l)$ that satisfy the condition

$$\|\rho\|_{C^{r_1}(I^d;I^l)} \leq c_1,$$

where

$$\|\rho\|_{C^{r_1}(I^d;I^l)} = \max_{1 \leq i \leq d} \|\rho_i\|_{C^{r_1}(I^d)},$$

as well as the "uniform nondegeneracy condition"

$$\min_{x \in I^d} J(x) \geq c_2.$$

For simplicity, we shall assume that $c_2 < 1 \leq c_1$ in this paper.

*Remark.* The mapping $\mathrm{id}\colon I^d \to I^l$, defined as

$$\mathrm{id}(x) = (x, \underbrace{0, \ldots, 0}_{l-d \text{ zeros}}) \qquad \forall\, x \in I^d, \tag{2.4}$$

belongs to $\mathscr{R}$. □

*Remark.* Why do we require $\rho(I^d) \subseteq I^l$? Under this condition, any $k\colon I^{2l} \to \mathbb{R}$ will be defined on $\Omega_\rho \times \Omega_\rho$, and any $f\colon I^l \to \mathbb{R}$ will be defined on $\Omega_\rho$. Thus any such $k$ and $f$ will be allowable in our integral equation (1.1). Had we not imposed this condition on $\rho$, we would have needed to impose more complicated conditions on our $k$ and $f$ than those stated below. □

*Remark.* The conditions defining $\mathscr{R}$ imply that we have an a priori bound on the volume or surface area element of $\Omega_\rho$, which is independent of $\rho$, namely

$$J(\cdot;\rho) \leq \kappa_{d,l} = \begin{cases} 1 & \text{if } d = l, \\ \sqrt{d!\, l^d} c_1^d & \text{if } d < l. \end{cases} \qquad \forall\, \rho \in \mathscr{R}. \tag{2.5}$$

Indeed, the bound for the case $d = l$ follows from the fact that the codomain of $\rho$ is the unit cube, whereas a *very* rough calculation shows the bound for the case $d < l$. Hence for any $\rho \in \mathscr{R}$, the volume (or surface area) of $\Omega_\rho$, which is merely $\|J(\cdot;\rho)\|_{L_1(I^d)}$, is at most $\kappa_{d,l}$. □

Next, we describe our class $\mathscr{K}$ of kernels. Let $c_3 > 0$ and $c_4 > 1$ be given, along with $r_2 \geq 0$ and $p \in (1, \infty)$. Then $\mathscr{K}$ consists of the functions $k \in C^{r_2}(I^{2l})$ satisfying

$$\|k\|_{C^{r_2}(I^{2l})} \leq c_3,$$

for which the "uniform invertibility condition"

$$\|(I - T_{\rho,k_\rho})^{-1}\|_{\mathrm{Lin}[C(I^d)]} \leq c_4 \qquad \forall\, \rho \in \mathscr{R}$$

holds. Here, $\|\cdot\|_{\mathrm{Lin}[\mathscr{Y}]}$ is the usual operator norm.

Our class of right-hand sides will be $\mathscr{B}C^{r_3}(I^l)$. Hence our class of problem elements will be

$$\mathscr{F} = \mathscr{R} \times \mathscr{K} \times \mathscr{B}C^{r_3}(I^l).$$

Now we can define our solution operator $S\colon \mathscr{F} \to C(I^d)$ as

$$S([\rho, k, f]) = (I - T_{\rho,k_\rho})^{-1} f_\rho \qquad \forall\, [\rho, k, f] \in \mathscr{F}.$$

Hence $u_\rho = S([\rho, k, f])$ is the solution of the operator equation (2.3).

We wish to calculate approximate solutions to this problem, using noisy standard information. To be specific, we will be using uniformly sup-norm-bounded noise. Our notation and terminology is essentially that of [**15**], although we sometimes use modifications found in [**16**].

Let $\delta \in [0, 1]$ be a *noise level*. For $[\rho, k, f] \in \mathscr{F}$, we calculate $\delta$-*noisy information*

$$z = [z_1, \ldots, z_{n(z)}]$$

about $[\rho, k, f]$. Here, for each index $i \in \{1, \ldots, n(z)\}$, either

$$|z_i - \rho(x^{(i)})| \leq \delta \text{ for some } x^{(i)} \in I^d,$$

or

$$\left| z_i - k\big(\rho(x^{(i)}), \rho(y^{(i)})\big) \right| \leq \delta \text{ for some } (x^{(i)}, y^{(i)}) \in I^{2d}, \tag{2.6}$$

or

$$\left| z_i - f\big(\rho(x^{(i)})\big) \right| \leq \delta \text{ for some } x^{(i)} \in I^d. \tag{2.7}$$

The choice of whether to evaluate $\rho$, $k_\rho$ or $f_\rho$ at the $i$th sample point, as well as the choice of the $i$th sample point itself, may be determined either nonadaptively or adaptively. Moreover, the information is allowed to be of varying cardinality. Since we will be using the worst case setting, the *cardinality* of the information $\mathbb{N}_\delta$ is

$$\operatorname{card} \mathbb{N}_\delta = \sup_{z \in \mathscr{Z}(\mathbb{N}_\delta)} n(z).$$

Let $\mathbb{N}_\delta$ be noisy information of finite cardinality. For $[\rho, k, f] \in \mathscr{F}$, we let $\mathbb{N}_\delta([\rho, k, f])$ denote the set of all such $\delta$-noisy information $z$ about $[\rho, k, f]$, and we let

$$\mathscr{Z}(\mathbb{N}_\delta) = \bigcup_{[\rho,k,f] \in \mathscr{F}} \mathbb{N}_\delta([\rho, k, f])$$

denote the set of all possible noisy information values. Then an *algorithm* using the noisy information $\mathbb{N}_\delta$ is a mapping $\phi\colon \mathscr{Z}(\mathbb{N}_\delta) \to C(I^d)$, whose *error* is given as

$$e(\phi, \mathbb{N}_\delta) = \sup_{[\rho,k,f] \in \mathscr{F}} \sup_{z \in \mathbb{N}_\delta([\rho,k,f])} \|S([\rho, k, f]) - \phi(z)\|_{C(I^d)}.$$

The *radius of information*

$$r(\mathbb{N}_\delta) = \inf_{\phi \text{ using } \mathbb{N}_\delta} e(\phi, \mathbb{N}_\delta)$$

gives the minimal error achievable by algorithms using the given noisy information $\mathbb{N}_\delta$.

Finally, let $n \in \mathbb{Z}^+$ and $\delta \in [0, 1]$. The $n$th *minimal radius* of noisy information

$$r(n, \delta) = \inf\{\, r(\mathbb{N}_\delta) : \operatorname{card} \mathbb{N}_\delta \le n \,\},$$

is the minimal error among all algorithms using $\delta$-noisy information of cardinality at most $n$.

**3. Lower bounds.** In this section, we prove a lower bound on the $n$th minimal radius of $\delta$-noisy information.

**Theorem 3.1.**

1. *If $d < l$ and $r_1 = 1$, then*

$$r(n, \delta) \asymp 1.$$

2. *If $d = l$ or $r_1 \geq 2$, let*

$$\mu_1 = \min\left\{\frac{r_1}{d}, \frac{r_2}{2d}, \frac{r_3}{d}\right\}.$$

*There is a constant $M_0$, independent of $n$ and $\delta$, such that*

$$r(n, \delta) \geq M_0(n^{-\mu_1} + \delta)$$

*for all $n \in \mathbb{Z}^+$ and $\delta \in [0, 1]$.*

*Proof.* We first consider the case $d < l$ and $r_1 = 1$. Let

$$\rho^*(x) = (0, x_2, \dots, x_d, x_1, \underbrace{0, \dots, 0}_{l-d-1 \text{ zeros}}) \qquad \forall\, x = (x_1, \dots, x_d) \in I^d,$$

and define $k^* \equiv \frac{1}{2}$ and $f^* \equiv 1$. Since $J(\cdot, \rho^*) \equiv 1$, it follows that $[\rho^*, k^*, f^*] \in \mathscr{F}$. Moreover, $u^*_{\rho^*} = S([\rho^*, k^*, f^*])$ satisfies

$$u^*\big(\rho^*(x)\big) = \tfrac{1}{2} \int_{I^d} u^*\big(\rho^*(y)\big) J(y, \rho^*)\, dy + 1.$$

Since $\operatorname{area}(\Omega_{\rho^*}) = 1$, we have

$$u^*\big(\rho^*(x)\big) \equiv \frac{1}{1 - \frac{1}{2}\operatorname{area}(\Omega_{\rho^*})} = 2.$$

Let $N$ be noise-free information of cardinality at most $n$. Without loss of generality, assume that the $\rho$-evaluation points in $N([\rho^*, k^*, f^*])$ are $x^{(1)}, \dots, x^{(n')}$. As on [**23**, pg. 461], we can construct a function $z\colon I^d \to \mathbb{R}$ such that

$$z(x^{(1)}) = \cdots = z(x^{(n')}) = 0 \qquad \text{and} \qquad \|z\|_{C^{r_1}(I^d; I^l)} = 1,$$

and such that

$$\int_{I^d} \sum_{j=1}^{d} \left( \frac{\partial z}{\partial x_j} \right)^2 (x)\, dx \succcurlyeq 1. \tag{3.1}$$

Let

$$\rho^{**}(x) = \left[ \frac{z(x)}{2\sqrt{d}}, x_2, \ldots, x_d, x_1, \underbrace{0, \ldots, 0}_{l-d-1 \text{ zeros}} \right] \qquad \forall\, x = (x_1, \ldots, x_d) \in I^d.$$

We find

$$J(x, \rho^{**}) = \sqrt{1 + \frac{1}{4d} \sum_{j=1}^{d} \left( \frac{\partial z}{\partial x_j} \right)^2 (x)} \geq 1,$$

from which it follows that $\rho^{**} \in \mathscr{R}$. Hence $[\rho^{**}, k^*, f^*] \in \mathscr{F}$. Moreover, $u^{**}_{\rho^{**}}$ satisfies

$$u^{**}\big(\rho^{**}(x)\big) = \tfrac{1}{2} \int_{I^d} u^{**}\big(\rho^{**}(y)\big) J(y, \rho^{**})\big)\, dy + 1,$$

and so

$$u^{**}\big(\rho^{**}(x)\big) \equiv \frac{1}{1 - \frac{1}{2}\, \text{area}\, (\Omega_{\rho^{**}})}.$$

Using [**19**, pp. 45, 49], we see that

$$\begin{aligned} r(N) \geq \tfrac{1}{2} \|u^{**}_{\rho^{**}} - u^*_{\rho^*}\|_{C(I^d)} &= \tfrac{1}{2} \left| \frac{1}{1 - \frac{1}{2}\, \text{area}\, (\Omega_{\rho^{**}})} - \frac{1}{1 - \frac{1}{2}\, \text{area}\, (\Omega_{\rho^*})} \right| \\ &= \tfrac{1}{2}\, \frac{|\, \text{area}\, (\Omega_{\rho^{**}}) - \text{area}\, (\Omega_{\rho^*})|}{|1 - \frac{1}{2}\, \text{area}\, (\Omega_{\rho^{**}})|}\ . \end{aligned}$$

Now

$$\text{area}\, (\Omega_{\rho^{**}}) = \int_{I^d} J(x; \rho^{**})\, dx \geq 1,$$

and so

$$1 - \tfrac{1}{2}\, \text{area}\, (\Omega_{\rho^{**}}) \geq \tfrac{1}{2}.$$

Hence

$$r(N) \geq |\,\text{area}\,(\Omega_{\rho^{**}}) - \text{area}\,(\Omega_{\rho^*})| = \int_{I^d} \frac{\displaystyle\sum_{j=1}^{d} \left(\frac{\partial z}{\partial x_j}\right)^2 (x)}{\sqrt{1 + \dfrac{1}{4d} \displaystyle\sum_{j=1}^{d} \left(\frac{\partial z}{\partial x_j}\right)^2 (x) + 1}}\, dx$$

$$\geq \tfrac{1}{2} \int_{I^d} \sum_{j=1}^{d} \left(\frac{\partial z}{\partial x_j}\right)^2 (x)\, dx \succcurlyeq 1,$$

the latter by (3.1). Hence

$$r(n, \delta) \geq r(n, 0) \succcurlyeq 1.$$

To see the matching upper bound, let $\mathbb{N}_\delta$ be noisy information of cardinality at most $n$, and let $\phi^0$ be the zero algorithm

$$\phi^0(z) \equiv 0 \qquad \forall\, z \in \mathscr{Z}(\mathbb{N}_\delta).$$

It is easy to see that the error of $\phi^0$ is bounded, independent of $n$ and $\delta$, and so

$$r(n, \delta) \leq e(\phi^0, \mathbb{N}_\delta) \preccurlyeq 1.$$

Thus

$$r(n, \delta) \asymp 1,$$

as claimed.

We now treat the case where $d = l$ or $r_1 \geq 2$. First, note that if we choose $\rho = \text{id}$, where id is given by (2.4), we can follow the proof of [**22**, Thm. 2] to show that

$$r(n, \delta) \succcurlyeq n^{-\min\{r_2/2d, r_3/d\}} + \delta.$$

Hence, we only need to show that

$$r(n, 0) \succcurlyeq n^{-r_1/d}.$$

Let $\rho \in \mathscr{R}$ and define $f^* \in \mathscr{B}C^{r_3}(I^l)$ as

$$f^*(s) \equiv s \qquad \forall\, s \in I^l.$$

Since $T_{\rho,0} = 0$, we see that $S([\rho, 0, f^*]) = \rho_1$, the first component of $\rho$. Define a solution operator $\tilde{S}: \mathscr{R} \to C(I^d)$ as

$$\tilde{S}(\rho) = S([\rho, 0, f^*]) = \rho_1 \qquad \forall\, \rho \in \mathscr{R}.$$

Since the problem given by this solution operator is a special case of our Fredholm problem, we see that the $n$th minimal noise-free radius of $S$ is bounded from below by that for $\tilde{S}$, which we may write

$$r(n, 0; S) \geq r(n, 0; \tilde{S}).$$

Using [**24**, Lemma 3.4], we have

$$r(n, 0; \tilde{S}) \succcurlyeq n^{-r_1/d},$$

as required. □

**4. The noisy modified Galerkin method.** Having established a lower bound on the $n$th minimal radius for our problem, we now seek an upper bound. Since our problem is unsolvable when $d < l$ and $r_1 = 1$, we shall assume that $d = l$ or $r_1 \geq 2$ in the sequel. Our upper bound will be provided by a modified Galerkin method using noisy standard information. In this section, we describe the method; we analyze its error in the next section.

We first present a weak formulation of our problem. For $[\rho, k] \in \mathscr{R} \times \mathscr{K}$, define a bilinear form $B(\cdot, \cdot; \rho, k_\rho)$ on $L_\infty(I^d) \times L_1(I^d)$ as

$$B(v, w; \rho, k_\rho) = \langle (I - T_{\rho,k_\rho})v, w\rangle \qquad \forall\, v \in L_\infty(I^d), w \in L_1(I^d).$$

For $f \in \mathscr{B}C^{r_3}(I^l)$, we see that $u_\rho = S([\rho, k, f]) \in C(I^d)$ satisfies

$$B(u_\rho, w; \rho, k_\rho) = \langle f_\rho, w\rangle \qquad \forall\, w \in L_1(I^d),$$

where

$$\langle v, w\rangle = \int_{I^d} v(x)w(x)\,dx \qquad \forall\, v \in L_\infty(I^d), w \in L_1(I^d).$$

Next, we describe a class of useful spline spaces; for further details, see [**23**]. Let $m \in \mathbb{Z}^{++}$ (to be determined later) and $h > 0$. Then

$\mathscr{S}_h$ denotes a $d$-fold tensor product of one-dimensional $C^1$-splines of degree $m$, over a uniform grid of mesh-size $h$.

Let $n_h = \dim \mathscr{S}_h$, noting that $n_h \asymp h^{-d}$. Associated with $\mathscr{S}_h$ is a quasi-interpolation operator

$$(Q_h w)(x) = \sum_{j=1}^{n_h} \lambda_{j,h}(w) s_{j,h}(x) \qquad \forall\, x \in I^d, w \in C(I^d), \tag{4.1}$$

where each $s_{j,h}$ is a $d$-fold tensor product of one-dimensional splines and we can write

$$\lambda_{j,h}(w) = \lambda_j\bigl(\{w(x_{i,h})\}_i\bigr) \qquad \forall\, w \in C(I^d)$$

where each $\lambda_j(w)$ can be computed with cost independent of $h$, once the values $w(x_{1,h}), \ldots\, w(x_{n_h,h})$ have been computed. For any $h$ and any $q \in [1, \infty]$, there is a projection operator $\mathscr{P}_h\colon L_q(I^d) \to \mathscr{S}_h$, defined by

$$\langle \mathscr{P}_h v, w \rangle = \langle v, w \rangle \qquad \forall\, v \in L_p(I^d), w \in \mathscr{S}_h.$$

Not only is the projection operator well-defined, but we also have

**Lemma 4.1.** *For $q \in [1, \infty]$,*

$$\pi_q = \sup_{0<h\le 1} \|\mathscr{P}_h\|_{\mathrm{Lin}\,[L_q(I^d)]}$$

*is finite.*

*Proof.* If $q = 2$, the result clearly holds, with $\pi_2 = 1$. Shadrin's remarkable proof [**18**] of the de Boor conjecture establishes the result for the case $q = \infty$ and $d = 1$; the case $q = \infty$ for arbitrary $d$ easily follows from the case with $d = 1$, as in [**17**]. By duality, it follows that $\pi_1 = \pi_\infty$. Finally, the result for arbitrary $q \in (1, 2)$ may be obtained by interpolating the results for $q = 1$ and $q = 2$, and the result for $q \in (2, \infty)$ may be obtained by interpolating the results for $q = 2$ and $q = \infty$. □

We will also have need of a $2d$-variate spline space $\mathscr{S}_{\bar h} \otimes \mathscr{S}_{\bar h}$ involving a (possibly) different mesh-size $\bar h$. The quasi-interpolation operator $Q_{\bar h \otimes \bar h}$

of $\mathscr{S}_{\bar h} \otimes \mathscr{S}_{\bar h}$ takes the form

$$(4.2) \quad (Q_{\bar h \otimes \bar h} w)(x,y) = \sum_{i,j=1}^{n_{\bar h}} \lambda_{i,j,\bar h}\big(\{w(x_{i',\bar h}, x_{j',\bar h})\}_{i',j'}\big) s_{j,\bar h}(y) s_{i,\bar h}(x) \qquad \forall\, x,y \in I^d, w \in C(I^{2d}).$$

*Remark.* Note since the maximum continuous differentiability of a degree-$m$ spline is $m-2$, we must have $m \ge 3$ to guarantee that $\mathscr{S}_h$ and $\mathscr{S}_{\bar h} \otimes \mathscr{S}_{\bar h}$ are globally $C^1$. We also note that $\mathscr{S}_h$ and $\mathscr{S}_{\bar h} \otimes \mathscr{S}_{\bar h}$ are (respectively) subspaces of $W^{2,\infty}(I^d)$ and $W^{2,\infty}(I^{2d})$, since $\mathscr{S}_h$ is piecewise polynomial and globally $C^1$; this follows from the $L_\infty$ version of [**4**, Thm. 2.1.1]. $\square$

Now that we have a bilinear form and a family of spline spaces, we can define a "pure" Galerkin method. Let $[\rho, k, f] \in \mathscr{F}$ and let $h > 0$. Then the *pure Galerkin method* consists of finding $u_h \in \mathscr{S}_h$ such that

$$B(u_h, w; \rho, k_\rho) = \langle f_\rho, w\rangle \qquad \forall\, w \in \mathscr{S}_h.$$

Alternatively, we seek $u_h \in \mathscr{S}_h$ satisfying

$$(I - \mathscr{P}_h T_{\rho,k_\rho}) u_h = \mathscr{P}_h f_\rho,$$

where $\mathscr{P}_h$ is the projection operator mentioned above. Note that $u_h$ is an approximation of $u_\rho$, and not of $u$.

Expanding $u_h$ in terms of the basis functions $s_{1,h}, \dots, s_{n_h,h}$, we see that the pure Galerkin method requires the calculation of weighted integrals involving $\rho$, $k_\rho$ and $f_\rho$. Since we are only using (noisy) standard information, the pure Galerkin method is not admissible for us. Instead, we shall replace $\rho$, $k_\rho$, and $f_\rho$ by their noisy quasi-interpolants (defined below); this will give us an algorithm using permissible information.

Let $h, \bar h, \delta > 0$, and let $[\rho, k, f] \in \mathscr{F}$. For $j \in \{1, \dots, n_h\}$, calculate $\tilde\rho_{j;h,\delta}$ satisfying

$$|\tilde\rho_{j,h,\delta} - \rho(x_{j,h})| \le \delta$$

and $\tilde{f}_{j,h,\delta}$ satisfying

$$\left|\tilde{f}_{j,h,\delta} - f\big(\rho(x_{j,h})\big)\right| \le \delta.$$

For $i, j \in \{1, \dots, n_{\bar{h}}\}$, calculate $\tilde{k}_{i,j,\delta}$ satisfying

$$\left|\tilde{k}_{i,j,\delta} - k\big(\rho(x_{i,\bar{h}}), \rho(x_{j,\bar{h}})\big)\right| \le \delta.$$

Define noisy quasi-interpolants of $\rho$, $f_\rho$, and $k_\rho$ by using the quasi-interpolants (4.1) and (4.2), but using noisy function values instead of exact function values. Thus

$$(Q_{h,\delta}\rho)(x) = \sum_{j=1}^{n_h} \lambda_{j,h}\big(\{\tilde{\rho}_{i,h,\delta}\}_i\big) s_{j,h}(x),$$

$$(Q_{h,\delta}f_\rho)(x) = \sum_{j=1}^{n_h} \lambda_{j,h}\big(\{\tilde{f}_{i,h,\delta}\}_i\big) s_{j,h}(x),$$

$$(Q_{h,\bar{h},\delta}k_\rho)(x,y) = \sum_{i,j=1}^{n_{\bar{h}}} \lambda_{i,j,\bar{h}}\big(\{\tilde{k}_{i',j',h,\bar{h},\delta}\}_{i',j'}\big) s_{j,\bar{h}}(y) s_{i,\bar{h}}(x).$$

For $[\rho, k] \in \mathscr{R} \times \mathscr{K}$, we define a new bilinear form $B_{h,\bar{h},\delta}(\cdot,\cdot;\rho,k_\rho)$ on $C(I^d) \times L_1(I^d)$ as

$$B_{h,\bar{h},\delta}(v,w;\rho,k_\rho) = B(v,w;Q_{h,\delta}\rho, Q_{h,\bar{h},\delta}k_\rho) \quad \forall\, v \in C(I^d), w \in L_1(I^d)$$

and define a new linear functional $f(\cdot,\rho)$ on $L_1(I^d)$ as

$$f_{h,\delta}(w,\rho) = \langle Q_{h,\delta}f_\rho, w\rangle \qquad \forall\, w \in L_1(I^d).$$

It would be reasonable to seek $u_{h,\bar{h},\delta} \in \mathscr{S}_h$ satisfying

$$B_{h,\bar{h},\delta}(u_{h,\bar{h},\delta}, w;\rho,k_\rho) = f_{h,\delta}(w,\rho) \qquad \forall\, w \in \mathscr{S}_h.$$

However when $d < l$, this formulation leads to a linear system whose coefficient matrix contains entries that may not be computable. To see why, let us write

$$u_{h,\bar{h},\delta}(x) = \sum_{j=1}^{n_h} \upsilon_j s_{j,h}(x) \qquad \forall\, x \in I^d,$$

so that $\mathbf{u} = [v_1, \dots, v_{n_h}]$ satisfies the linear system

$$(\mathbf{A} - \mathbf{B})\mathbf{u} = \mathbf{f},$$

where

$$\mathbf{f} = [f_{h,\delta}(s_{1,h}, \rho) \dots f_{h,\delta}(s_{n_h,h}, \rho)]^{\mathsf{T}}$$

and, for $1 \le i, j \le n_h$, we have

$$a_{i.j} = \langle s_{j,h}, s_{i,h} \rangle \qquad \text{and} \qquad b_{i,j} = \langle T_{\rho,k_\rho;h,\bar{h},\delta} s_{j,h}, s_{i,h} \rangle,$$

where

$$T_{\rho,k_\rho;h,\bar{h},\delta} v = \int_{I^d} (Q_{h,\bar{h},\delta} k_\rho)(\cdot, y) v(y) J(y; Q_{h,\delta}\rho)\, dy.$$

Hence

$$b_{i,j} = \sum_{i',j'=1}^{n_{\bar{h}}} \tilde{k}_{i,j,\delta} \left[ \int_{I^d} s_{j',\bar{h}}(x) s_{j,h}(x)\, dx \right] \left[ \int_{I^d} s_{i',\bar{h}}(y) s_{i,h}(y) J(y; Q_{h,\delta}\rho)\, dy \right].$$

If $d < l$, the integrands $s_{i',\bar{h}}(y) s_{i,h}(y) J(y; Q_{h,\delta}\rho)$ involve the square roots of piecewise polynomials. Hence these integrands may not have closed form antiderivatives. Thus the entries of $\mathbf{B}$ may not be computable, as claimed.

To deal with this problem, we use an approach found in [**23**, pg. 458] (and given in more detail in [**24**]), namely, replacing the square root appearing above by its Taylor expansion. For $\eta \in \mathbb{R}^{++}$ and any integer $q$, let $R_q(\cdot, \eta)$ denote the Taylor series of degree $q-1$ for the square root at the point $\eta$, i.e.,

$$R_q(\xi, \eta) = \sqrt{\eta} + \sum_{i=1}^{q-1} \beta_j(\eta)(\xi - \eta)^j \qquad \forall\, \xi \in (\eta - 1, \eta + 1), \tag{4.3}$$

where

$$\beta_j(\eta) = \frac{1}{j!} \left( \frac{d}{d\xi} \right)^j \sqrt{\xi} \,\bigg|_{\xi=\eta} = \frac{1}{\eta^{(2j-1)/2}} \binom{j - \frac{3}{2}}{j}.$$

Then

$$\left| \sqrt{\xi} - R_q(\xi, \eta) \right| \le |\beta_q|\, |\xi - \eta|^q \qquad \forall\, \xi \in (\eta - 1, \eta + 1). \tag{4.4}$$

We now define a modification $\tilde{T}_{\rho,k_\rho;h,\bar{h},\delta}$ of our operator $T_{\rho,k_\rho;h,\bar{h},\delta}$. First of all, if $d = l$, we simply take $\tilde{T}_{\rho,k_\rho;h,\bar{h},\delta} = T_{\rho,k_\rho;h,\bar{h},\delta}$. Now suppose that $d < l$. Let $\mathscr{Q}_h$ denote the set of $h^{-d}$ cubes of side $h$ into which $I^d$ is partitioned when constructing $\mathscr{S}_h$. Then for $v \in C(I^d)$, we let

$$\tilde{T}_{\rho,k_\rho;h,\bar{h},\delta} v \Big|_K = \tilde{T}_{\rho,k_\rho;h,\bar{h},\delta;K} v \qquad \forall\, K \in \mathscr{Q}_h, \tag{4.5}$$

where

$$\tilde{T}_{\rho,k_\rho;h,\bar{h},\delta;K} v = \int_K (Q_{h,\bar{h},\delta} k_\rho)(\cdot, y)\, v(y)\, R_q\big(A(y; Q_{h,\delta}\rho), A(y^{(K)}; Q_{h,\delta}\rho)\big)\, dy \quad \forall\, K \in \mathscr{Q}_h, \tag{4.6}$$

with $y^{(K)}$ a fixed evaluation point in $K$ (such as the center or a specific corner) for each $K \in \mathscr{Q}_h$.

We are now ready to define our noisy modified Galerkin method. For $[\rho, k] \in \mathscr{R} \times \mathscr{K}$, we define a new bilinear form $\tilde{B}_{h,\bar{h},\delta}\big(\cdot, \cdot; \rho, k_\rho\big)$ on $C(I^d) \times L_1(I^d)$ as

$$\tilde{B}_{h,\bar{h},\delta}(v, w; \rho, k_\rho) = \langle (I - \tilde{T}_{\rho,k_\rho;h,\bar{h},\delta}) v, w \rangle \qquad \forall\, v \in C(I^d), w \in L_1(I^d).$$

Then the *noisy modified Galerkin method* consists of finding $u_{h,\bar{h},\delta} \in \mathscr{S}_h$ satisfying

$$\tilde{B}_{h,\bar{h},\delta}(u_{h,\bar{h},\delta}, w; \rho, k_\rho) = f_{h,\delta}(w, \rho) \qquad \forall\, w \in \mathscr{S}_h.$$

If we write

$$u_{h,\bar{h},\delta}(x) = \sum_{j=1}^{n_h} v_j s_{j,h}(x) \qquad \forall\, x \in I^d,$$

then $\mathbf{u} = [v_1, \dots, v_{n_h}]$ satisfies the linear system

$$(\mathbf{A} - \mathbf{B})\mathbf{u} = \mathbf{f},$$

where

$$\mathbf{f} = [f_{h,\delta}(s_{1,h}, \rho) \cdots f_{h,\delta}(s_{n_h,h}, \rho)]^{\mathsf{T}}$$

and, for $1 \le i, j \le n_h$, we have

$$a_{i.j} = \langle s_{j,h}, s_{i,h} \rangle \qquad \text{and} \qquad b_{i,j} = \langle \tilde{T}_{\rho,k_\rho;h,\bar{h},\delta} s_{j,h}, s_{i,h} \rangle.$$

Note that the integrand appearing in each $b_{i,j}$ is piecewise polynomial. Hence the entries of $\mathbf{B}$ are computable, as required.

Let

$$\mathbb{N}_{h,\bar{h},\delta}([\rho, k, f]) = [\mathbb{N}_{h,\delta}(\rho), \mathbb{N}_{h,\delta}(f_\rho), \overline{\mathbb{N}}_{\bar{h},\delta}(k_\rho)],$$

where

$$\begin{aligned} \mathbb{N}_{h,\delta}(\rho) &= [\tilde{\rho}_{1,\delta}, \dots, \tilde{\rho}_{n_h,\delta}], \\ \mathbb{N}_{h,\delta}(f_\rho) &= [\tilde{f}_{1,\delta}, \dots, \tilde{f}_{n_h,\delta}], \end{aligned}$$

and

$$\overline{\mathbb{N}}_{\bar{h},\delta}(k_\rho) = [\overline{\mathbb{N}}^{(1)}_{\bar{h},\delta}(k_\rho), \dots, \overline{\mathbb{N}}^{(n_{\bar{h}})}_{\bar{h},\delta}(k_\rho)].$$

with

$$\overline{\mathbb{N}}^{(i)}_{\bar{h},\delta}(k_\rho) = [\tilde{k}_{i,1\delta}, \dots \tilde{k}_{i,n_{\bar{h}}\delta}] \qquad (1 \le i \le n_{\bar{h}}).$$

If $u_{h,\bar{h},\delta}$ is well-defined, we can write

$$u_{h,\bar{h},\delta} = \phi_{h,\bar{h},\delta}\big(\mathbb{N}_{h,\bar{h},\delta}([\rho, k, f])\big),$$

where

$$\operatorname{card} \mathbb{N}_{h,\bar{h},\delta} = n_{\bar{h}}^2 + 2h_n \asymp \left(\frac{m+1}{\bar{h}}\right)^{2d} + \left(\frac{m+1}{h}\right)^2.$$

**5. Error analysis of the noisy modified Galerkin method.** In this section, we establish an error bound for the noisy modified Galerkin method. As mentioned above, since the problem is unsolvable when $d < l$ and $r_1 = 1$, we only need to consider the case of $d = l$ or $r_1 \ge 2$. To derive our error bound, we first establish the uniform weak coercivity of the bilinear forms $B(\cdot, \cdot; \rho, k_\rho)$ for $[\rho, k] \in \mathscr{R} \times \mathscr{K}$. Once we know that the bilinear forms are uniformly weakly coercive, we can obtain an abstract error estimate, as a variant of the First Strang Lemma (see, e.g., [**4**, pg. 186]). The remaining task is then to estimate the various terms appearing in this abstract error estimate.

So, the first task is to establish uniform weak coercivity. Before doing so, we lay some groundwork.

The first thing we need to do is to recall approximation properties of the quasi-interpolation operators introduced in the previous section:

**Lemma 5.1.** *Let $\mathscr{S}_h$ and $\mathscr{S}_{\bar h} \otimes \mathscr{S}_{\bar h}$ be the spline spaces of degree $m$ described in the previous section. For any $a \in [1, \infty]$ and $q \in \mathbb{Z}^{++}$, there exists $M_1 > 0$ (independent of $h$ and $\bar h$) such that for any $r \in \{0, \ldots, \min\{m, q, 2\}\}$, the following hold:*

1. *Let $w \in W^{q,a}(I^d)$. Then*
$$\|w - Q_h w\|_{W^{r,a}(I^d)} \le M_1 h^{\min\{m+1,q\}-r} \|w\|_{W^{q,a}(I^d)}.$$
2. *Let $w \in C^q(I^d)$. Then*
$$\|w - Q_h w\|_{C(I^d)} \le M_1 h^{\min\{m+1,q\}} \|w\|_{C^q(I^d)}.$$
3. *Let $w \in W^{q,a}(I^{2d})$. Then*
$$\|w - Q_{\bar h \otimes \bar h} w\|_{W^{r,a}(I^{2d})} \le M_1 \bar h^{\,\min\{m+1,q\}-r} \|w\|_{W^{q,a}(I^{2d})}.$$

*Proof.* See, e.g., [**17**]. ◻

Next, we need to establish an auxiliary lemma, which shows that the inverses of certain operators are uniformly bounded. By [**24**, Lemma 3.1], there exists $C_\circ > 0$ such that

$$\|k_\rho\|_{C^{\min\{r_1,r_2\}}(I^{2d})} \le C_o \qquad \forall\, [\rho, k] \in \mathscr{R} \times \mathscr{K}.$$

Let

$$h_0 = \left( \frac{1}{2M_1 C_o c_4} \right)^{1/\min\{m+1, r_1, r_2\}}.$$

Note that for any $\rho \in \mathscr{R}$ and any $g \in C(I^{2d})$, the adjoint $T^*_{\rho,g}$ of $T_{\rho,g}$ is given by

$$T^*_{\rho,g} w = J(\cdot, \rho) \int_{I^d} g(x, \cdot) w(x)\, dx \qquad \forall\, w \in L_1(I^d).$$

**Lemma 5.2.** *Let* $h \in (0, h_0]$ *and* $k \in \mathscr{K}$. *Then* $I - T^*_{\rho, Q_{h\otimes h} k_\rho}$ *is invertible on* $L_1(I^d)$, *with*

$$\|(I - T^*_{\rho, Q_{h\otimes h} k_\rho})^{-1}\|_{\mathrm{Lin}[L_1(I^d)]} \le 2\, c_4.$$

*Proof.* Let $\rho \in \mathscr{K}$. Then

$$\|T^*_{\rho,g}\|_{\mathrm{Lin}[L_1(I^d)]} \le \|J(\cdot,\rho)\|_{C(I^d)} \|g\|_{C(I^{2d})} \le \kappa_{d,l} \|g\|_{C(I^{2d})} \quad \forall\, g \in C(I^{2d}),$$

where $\kappa_{d,l}$ is defined in (2.5). For $[\rho, k] \in \mathscr{R} \times \mathscr{K}$ and $h \in (0, h_0]$, we may use Lemma 5.1 to find that

$$\begin{aligned}\|T^*_{\rho,(I - Q_{\bar h\otimes \bar h})k_\rho}\|_{\mathrm{Lin}[L_1(I^d)]} &\le \kappa_{d,l} \|(I - Q_{\bar h\otimes \bar h}) k_\rho\|_{L_\infty(I^{2d})} \\ &\le \kappa_{d,l} M_1 h^{\min\{m+1, r_1, r_2\}} \|k_\rho\|_{C^{\min\{r_1,r_2\}}(I^{2d})} \\ &\le \kappa_{d,l} M_1 h_0^{\min\{m+1, r_1, r_2\}} C_o \le \frac{1}{2c_4}\end{aligned}$$

We now have

$$\|T^*_{k_\rho - \Pi_{h\otimes h} k_\rho}\|_{\mathrm{Lin}[L_1(I^d)]} \|(I - T^*_{k_\rho})^{-1}\|_{\mathrm{Lin}[L_1(I^d)]} \le \frac{1}{2\, c_4} \cdot c_4 = \tfrac{1}{2}.$$

Using [**10**, Lemma 1.3.14], we see that $I - T^*_{\Pi_{h\otimes h} k_\rho}$ is invertible, with

$$\begin{aligned}&\|(I - T^*_{\Pi_{h\otimes h} k_\rho})^{-1}\|_{\mathrm{Lin}[L_1(I^d)]} \\ &\qquad \le \frac{\|(I - T^*_{k_\rho})^{-1}\|_{\mathrm{Lin}[L_1(I^d)]}}{1 - \|T^*_{k_\rho - \Pi_{h\otimes h} k_\rho}\|_{\mathrm{Lin}[L_1(I^d)]} \|(I - T^*_{k_\rho})^{-1}\|_{\mathrm{Lin}[L_1(I^d)]}} \le 2\, c_4,\end{aligned}$$

as required. □

We now establish uniform weak coercivity.

**Lemma 5.3.** *There exist* $h_1 > 0$ *and* $\gamma > 0$ *such that the following holds: for any* $[\rho, k] \in \mathscr{R} \times \mathscr{K}$, *any* $h \in (0, h_1]$, *and any* $v \in \mathscr{S}_h$, *there exists nonzero* $w \in \mathscr{S}_h$ *such that*

$$B(v, w; \rho, k_\rho) \ge \gamma \|v\|_{C(I^d)} \|w\|_{L_1(I^d)}.$$

*Proof.* Let $[\rho, k] \in \mathscr{R} \times \mathscr{K}$ and $h \in (0, h_0]$. Let $v \in \mathscr{S}_h$. If $v = 0$, then this inequality holds for any nonzero $w \in \mathscr{S}_h$. So, we may restrict our attention to the case $v \neq 0$.

By [**22**, Lemma 10], there exists nonzero $g \in L_1(I^d)$ such that

$$\langle v, g \rangle \geq \tfrac{1}{2} \|v\|_{C(I^d)} \|g\|_{L_1(I^d)}.$$

Choose

$$w = (I - T^*_{Q_{\bar{h} \otimes \bar{h}} k_\rho})^{-1} \mathscr{P}_h g.$$

Since $T^*_{Q_{\bar{h} \otimes \bar{h}} k_\rho} : \mathscr{S}_h \to \mathscr{S}_h$, we may use Lemma 4.1 and Lemma 5.2 to see that $w$ is a well-defined element of $\mathscr{S}_h$, and that

$$\|w\|_{L_1(I^d)} \leq 2\pi_1 c_4 \|g\|_{L_1(I^d)}.$$

Hence

$$\langle (I - T_{\rho, Q_{\bar{h} \otimes \bar{h}} k_\rho}) v, w \rangle \geq \tfrac{1}{2} \|v\|_{C(I^d)} \|g\|_{L_1(I^d)} \geq \frac{1}{4\pi_1 c_4} \|v\|_{C(I^d)} \|w\|_{L_1(I^d)},$$

from which we see that $w \neq 0$.

Using the Minkowski inequality, we find that

$$\begin{aligned} |\langle T_{\rho, (I - Q_{\bar{h} \otimes \bar{h}}) k_\rho} v, w \rangle| &\leq \|(I - Q_{\bar{h} \otimes \bar{h}}) k_\rho\|_{L_\infty(I^{2d})} \|v\|_{C(I^d)} \|w\|_{L_1(I^d)} \\ &\leq M_1 C_o h_0^{\min\{m+1, r_1, r_2\}} \|v\|_{C(I^d)} \|w\|_{L_1(I^d)}. \end{aligned}$$

Combining the last two inequalities and setting

$$h_1 = \min\left\{ \frac{1}{8\pi_1 c_4 M_1 C_o}, h_0 \right\} \qquad \text{and} \qquad \gamma = \frac{1}{8\pi_1 c_4},$$

the lemma follows. □

Since the bilinear forms $B(\cdot, \cdot; \rho, k)$ are uniformly weakly coercive for $k \in \mathscr{K}$, we have the following variant of the First Strang Lemma found in [**4**, pg. 186] and [**21**, pp. 310–312]:

**Lemma 5.4.** *Suppose there exist $\delta_0 \in (0, 1]$ and $h_2 \in (0, h_1]$ such that the following holds: for any $\delta \in [0, \delta_0]$, any $h, \bar{h} \in (0, h_2]$, any $[\rho, k] \in \mathscr{R} \times \mathscr{K}$, and any $v, w \in \mathscr{S}_h$, we have*

$$|B(v, w; \rho, k_\rho) - \tilde{B}_{h, \bar{h}, \delta}(v, w; \rho, k_\rho)| \leq \tfrac{1}{2} \gamma \, \|v\|_{C(I^d)} \|w\|_{L_1(I^d)}$$

*where $\gamma$ is as in Lemma 5.3. Then there exists $M_2 > 0$ such that the following hold for any $\delta \in [0, \delta_0]$ and any $h, \bar{h} \in (0, h_2]$:*

1. *The noisy modified Galerkin method is well-defined. That is, there exists a unique $u_{h,\bar{h},\delta} \in \mathscr{S}_h$ such that*

$$\tilde{B}_{h,\bar{h},\delta}(u_{h,\bar{h},\delta}, w; \rho, k_\rho) = f_{h,\delta}(w; \rho) \qquad \forall\, w \in \mathscr{S}_h.$$

2. *Let $u_\rho = S([f, k])$. Then*

$$\begin{aligned} &\|u_\rho - u_{h,\bar{h},\delta}\|_{C(I^d)} \\ &\qquad \le M_2 \inf_{v \in \mathscr{S}_h} \Bigg[ \|u_\rho - v\|_{C(I^d)} \\ &+ \sup_{w \in \mathscr{S}_h} \left( \frac{|B(v, w; \rho, k_\rho) - \tilde{B}_{h,\bar{h},\delta}(v, w; \rho, k_\rho)|}{\|w\|_{L_1(I^d)}} + \frac{|\langle f, w\rangle - f_{h,\delta}(w; \rho)|}{\|w\|_{L_1(I^d)}} \right) \Bigg]. \end{aligned}$$

We now estimate the quantities appearing in the second part of Lemma 5.4. First, we estimate the difference between the bilinear forms $B(\cdot, \cdot; \rho, k_\rho)$ and $\tilde{B}_{h,\bar{h},\delta}(\cdot, \cdot; \rho, k_\rho)$.

**Lemma 5.5.** *Suppose that $d = l$ or $r_1 \ge 2$. Let $m \ge \max\{r_1, r_2\} - 1$ and let $q \ge r_1 - 1$ in (4.5) and (4.6). There exists $M_3 > 0$ such that for any positive $h$, $\bar{h}$, and $\delta$, for any $[\rho, k] \in \mathscr{K}$, and for any $v, w \in \mathscr{S}_h$, we have*

$$\begin{aligned} &|B(v, w; \rho, k_\rho) - \tilde{B}_{h,\bar{h},\delta}(v, w; \rho, k_\rho)| \\ &\qquad\qquad \le M_3 (h^{r_1 - 1} + \bar{h}^{\,r_2} + \delta) \|v\|_{C(I^d)} \|w\|_{L_1(I^d)}. \end{aligned}$$

*Proof.* Given $h$, $\bar{h}$, $\delta$, $\rho$, $k$, $v$, and $w$ as in the statement of the lemma, define

$$\begin{aligned} A_1 &= \langle (T_{\rho,k_\rho} - T_{\rho,k_\rho;\bar{h}}) v, w \rangle, \\ A_2 &= \langle (T_{\rho,k_\rho;\bar{h}} - T_{\rho,k_\rho;h,\bar{h}}) v, w \rangle, \\ A_3 &= \langle (T_{\rho,k_\rho;h,\bar{h}} - T_{\rho,k_\rho;h,\bar{h},\delta}) v, w \rangle, \\ A_4 &= \langle (T_{\rho,k_\rho;h,\bar{h},\delta} - \tilde{T}_{\rho,k_\rho;h,\bar{h},\delta}) v, w \rangle, \end{aligned}$$

where

$$T_{\rho,k_\rho;\bar{h}}v = \int_{I^d} (Q_{\bar{h}\otimes\bar{h}}k_\rho)(\cdot,y)v(y)J(y;\rho)\,dy$$

and

$$T_{\rho,k_\rho;h,\bar{h}}v = \int_{I^d} (Q_{\bar{h}\otimes\bar{h}}k_\rho)(\cdot,y)v(y)J(y;Q_h\rho)\,dy.$$

Then

$$|B(v,w;\rho,k_\rho) - B_{h,\bar{h},\delta}(v,w;\rho,k_\rho)| \le |A_1| + |A_2| + |A_3| + |A_4|. \tag{5.1}$$

We first estimate $|A_1|$. From (2.5), we see that

$$\|J(\cdot,\rho)\|_{L_1(I^d)} \le \kappa_{d,l}.$$

Using Lemma 5.1, we obtain

$$\begin{aligned} |A_1| &\le \|(T_{\rho,k_\rho} - T_{\rho,k_\rho;\bar{h}})v\|_{C(I^d)}\|w\|_{L_1(I^d)} \\ &\le \kappa_{d,l}\|(I - Q_{\bar{h}\otimes\bar{h}})k_\rho\|_{L_\infty(I^{2d})}\|v\|_{C(I^d)}\|w\|_{L_1(I^d)} \\ &\preccurlyeq \bar{h}^{\,r_2}\|v\|_{C(I^d)}\|w\|_{L_1(I^d)}. \end{aligned}$$

Next, we estimate $|A_2|$. We have

$$\begin{aligned} A_2 &= \int_{I^d} \left[ \int_{I^d} (Q_{\bar{h}\otimes\bar{h}}k_\rho)(x,y)v(y)[J(y;\rho) - J(y;Q_h\rho)]\,dy \right] w(x)\,dx \\ &= \int_{I^d}\int_{I^d} \omega(x,y)[\det A(y;\rho) - \det A(y;Q_h\rho)]\,dy\,dx, \end{aligned}$$

where

$$\omega(x,y) = \frac{(Q_{\bar{h}\otimes\bar{h}}k_\rho)(x,y)}{J(y;\rho) + J(y;Q_h\rho)}.$$

Let $\Pi_d$ denote the set of all permutations of $\{1,\dots,d\}$. For $\pi \in \Pi_d$, define

$$\begin{aligned} &b_{\pi,i,j}(x,y) \\ &\quad = \omega(x,y)\bar{a}_{\pi_1,1}(y)\dots\bar{a}_{\pi_{i-1},i-1}(y)a_{\pi_{i+1},i+1}(y)\dots a_{\pi_d,d}(y)(\partial_i\rho_j)(y) \end{aligned}$$

and

$$\begin{aligned} &\bar{b}_{\pi,i,j}(x,y) \\ &= \omega(x,y)\bar{a}_{\pi_1,1}(y)\dots\bar{a}_{\pi_{i-1},i-1}(y)a_{\pi_{i+1},i+1}(y)\dots a_{\pi_d,d}(y)(\partial_{\pi_i}\bar{\rho}_j)(y), \end{aligned}$$

where $\partial_i = (\partial/\partial y_i)$ and $a_{i,j}$ and $\bar{a}_{i,j}$ respectively denote the $(i,j)$th components of $A(\cdot,\rho)$ and $A(\cdot,Q_h\rho)$.

As on [**23**, pp. 455–456], we find

$$A_2 = \sum_{\pi\in\Pi_{\mathbf{d}}} (-1)^{|\pi|} \sum_{i=1}^{d}\sum_{j=1}^{d} \theta_{\pi,\mathbf{i},\mathbf{j}},$$

with $|\pi|$ denoting the sign of $\pi \in \Pi_d$ and with

$$\begin{aligned} \theta_{\pi,\mathbf{i},\mathbf{j}} = &\int_{I^d} \left[ \int_{I^d} b_{\pi,\mathbf{i},\mathbf{j}}(x,y) v(y) \partial_{\pi_i}\big(\rho_j(y) - \bar{\rho}_j(y)\big)\, dy \right] w(x)\, dx \\ &+ \int_{I^d} \left[ \int_{I^d} \bar{b}_{\pi,\mathbf{i},\mathbf{j}}(x,y) v(y) \partial_{i}\big(\rho_j(y) - \bar{\rho}_j(y)\big)\, dy \right] w(x)\, dx \end{aligned}$$

for $\pi \in \Pi_d$ and $i,j \in \{1,\dots,d\}$. Since

$$\begin{aligned} |\theta_{\pi,\mathbf{i},\mathbf{j}}| &\le \|b_{\pi,\mathbf{i},\mathbf{j}}\|_{L_\infty(I^{2d})} \|\rho_j - \bar{\rho}_j\|_{W^{1,\infty}(I^d)} \|v\|_{C(I^d)} \|w\|_{L_1(I^d)} \\ &\preccurlyeq h^{r_1-1} \|v\|_{C(I^d)} \|w\|_{L_1(I^d)}, \end{aligned}$$

we have

$$|A_2| \preccurlyeq h^{r_1-1} \|v\|_{C(I^d)} \|w\|_{L_1(I^d)} \qquad \text{if } r_1 \ge 1 \text{ or } r_2 \ge 1. \tag{5.3}$$

We next note that

$$|A_3| \preccurlyeq \delta \|v\|_{C(I^d)} \|w\|_{L_1(I^d)}, \tag{5.4}$$

the details being substantially the same as in the proof of the analogous bound for $|A_2|$ in [**22**, Lemma 13].

We now estimate $|A_4|$. Of course, $A_4 = 0$ when $d = l$, so we only need to consider the case $d < l$. For a cube $K \in \mathcal{Q}_h$, let

$$\begin{aligned} \theta_K = \int_{I^d} \Bigg\{ \int_K & (Q_{h,\bar{h},\delta} k_\rho)(x,y) \\ & \big[J(y; Q_{h,\delta}\rho) - R_q\big(A(y; Q_{h,\delta}\rho), A(y^{(K)}; Q_{h,\delta}\rho)\big)\big] v(y)\, dy \Bigg\} w(x)\, dx. \end{aligned}$$

Recalling the definition (4.6), along with the error estimate (4.4), we find that

$$|\theta_K| \preccurlyeq h^{r_1-1} \int_{I^d} \left[ \int_K |v(y)|\, dy \right] |w(x)|\, dx.$$

Hence

$$\begin{aligned} |A_4| &\le \sum_{K \in \mathscr{Q}_h} |\theta_K| \preccurlyeq h^{r_1-1} \int_{I^d} \left[ \int_{I^d} |v(y)|\, dy \right] |w(x)|\, dx \\ &= h^{r_1-1} \|v\|_{L_1(I^d)} \|w\|_{L_1(I^d)} \\ &\le h^{r_1-1} \|v\|_{C(I^d)} \|w\|_{L_1(I^d)}. \end{aligned} \tag{5.5}$$

Finally, substituting (5.2)–(5.5) into (5.1), we get the estimate in the statement of our lemma. □

Next, we need to estimate the difference between the linear forms $\langle f_\rho, \cdot \rangle$ and $f_{h,\delta}(\cdot, \rho)$. Before doing this, we recall a result concerning the smoothness of composite functions:

**Lemma 5.6.** *Let $\rho \in \mathscr{R}$ and $v \in C^{r_3}(I^l)$. There exists $M_4 > 0$, independent of $\rho$, such that*

$$\|v_\rho\|_{C^{\min\{r_1, r_3\}}(I^d)} \le M_4 \|v\|_{C^{r_3}(I^l)} \qquad \forall\, v \in C^{r_3}(I^l).$$

*Proof.* This is [**24**, Lemma 3.1] □

Using this lemma, we are now able to estimate the difference between the linear forms $\langle f_\rho, \cdot \rangle$ and $f_{h,\delta}(\cdot, \rho)$.

**Lemma 5.7.** *Let $m \ge r_3 - 1$. There exists $M_5 > 0$ such that*

$$|\langle f_\rho, w \rangle - f_{h,\delta}(w; \rho)| \le M_5 (h^{\min\{r_1, r_3\}} + \delta) \|f\|_{C^{r_3}(I^l)} \|w\|_{L_1(I^d)}$$

*for all $\rho \in \mathscr{R}$, $f \in C^{r_3}(I^l)$, $h > 0$, $\delta \ge 0$, and $w \in \mathscr{S}_h$.*

*Proof.* Choose $\rho \in \mathscr{R}$, $f \in C^{r_3}(I^l)$, $h > 0$, $\delta \ge 0$, and $w \in \mathscr{S}_h$. Then

$$|\langle f_\rho, w \rangle - f_{h,\delta}(w; \rho)| \le |A_1| + |A_2|, \tag{5.6}$$

where

$$A_1 = \langle f_\rho - Q_h f_\rho, w \rangle$$

and

$$A_2 = \langle Q_h f_\rho, w \rangle - f_{h,\delta}(w; \rho).$$

Using Lemmas 5.1 and 5.6, we have

$$\begin{aligned} |A_1| &\le \|f_\rho - Q_h f_\rho\|_{C(I^d)} \|w\|_{L_1(I^d)} \\ &\le M_1 h^{\min\{r_1, r_3\}} \|f_\rho\|_{C^{\min\{r_1,r_3\}}(I^d)} \|w\|_{L_1(I^d)} \\ &\le M_1 M_4 h^{\min\{r_1,r_3\}} \|f\|_{C^{r_3}(I^l)} \|w\|_{L_1(I^d)}. \end{aligned}$$

Moreover, we also have

$$|A_2| \preccurlyeq \delta \|w\|_{L_1(I^d)},$$

the proof following that of the bound for $|A_4|$ in [**22**, Lemma 14]. Our lemma follows from these last two inequalities, along with (5.6). ◻

Our final preparatory step is to note that a "shift theorem" relates the smoothness of $(I - T_{\rho,k_\rho})^{-1} f$ to the smoothnesses of $\rho$, $k$, and $f$.

**Lemma 5.8.** *Let $[\rho, k] \in \mathscr{R} \times \mathscr{K}$ and $f \in C^{r_3}(I^l)$.. Then*

$$\|(I - T_{\rho,k_\rho})^{-1} f_\rho\|_{C^{\min\{r_1,r_2,r_3\}}(I^d)} \le M_6 \|f\|_{C^{r_3}(I^l)},$$

*where $M_6 > 0$ is independent of $\rho$, $k$, and $f$.*

*Proof.* Given such $\rho$, $k$, and $f$, use the Hölder version of [**22**, Lemma 16], along with Lemma 5.6, to see that

$$\begin{aligned} \|(I - T_{\rho,k_\rho})^{-1} f_\rho\|_{C^{\min\{r_1,r_2,r_3\}}(I^d)} &\preccurlyeq \|f_\rho\|_{C^{\min\{r_1,r_2,r_3\}}(I^d)} \\ &\le M_4 \|f\|_{C^{r_3}(I^l)}, \end{aligned}$$

as required. ◻

We are now ready to show that the noisy modified Galerkin method is well-defined, as well as to give an upper bound on its error.

**Theorem 5.1.** *Suppose that $d = l$ or $r_1 \geq 2$. Let $m \geq \max\{r_1, r_2\} - 1$ and let $q \geq r_1 - 1$ in (4.5) and (4.6). Choose $h_2 > 0$ and $\delta_0 > 0$ such that*

$$M_3(h_1^{r_1-1} + h_2^{r_2} + \delta_0) \leq \tfrac{1}{2}\gamma,$$

*where $h_1$ and $\gamma$ are as in Lemma 5.3. There exists $M_7 > 0$ such that for any $h \in (0, h_1]$, $\bar{h} \in (0, h_2]$, and $\delta \in [0, \delta_0]$:*

1. *The noisy modified Galerkin method is well-defined.*
2. *We have the error bound*

$$e(\phi_{h,\bar{h},\delta}, \mathbb{N}_{h,\bar{h},\delta}) \leq M_7(h^{\min\{r_1-1,r_2,r_3\}} + \bar{h}^{\,r_2} + \delta).$$

*Proof.* Let $h$, $\bar{h}$, and $\delta$ be as described. Choose $[\rho, k, f] \in \mathscr{F}$, and let $u_\rho = S([\rho, k, f])$. Using Lemmas 5.4 and 5.5, we immediately see that $u_{h,\bar{h},\delta} = \phi_{h,\bar{h},\delta}\big(\mathbb{N}_{h,\bar{h},\delta}([\rho, k, f])\big)$ is well-defined. It only remains to prove the error bound. Let $r = \min\{r_1 - 1, r_2, r_3\}$, and set $v = Q_h u_\rho$. Using Lemmas 5.1 and 5.8, along with the conditions defining the class $\mathscr{F}$, we have

$$\begin{aligned}\|u_\rho - v\|_{C(I^d)} &\leq M_1 h^r \|u_\rho\|_{C^r(I^d)} \leq M_1 h^r \|u_\rho\|_{C^{\min\{r_1,r_2,r_3\}}(I^d)} \\ &\leq M_1 M_6 h^r \|f\|_{C^{r_3}(I^l)} \leq M_1 M_6 h^r.\end{aligned}$$

The desired result follows once we substitute this inequality, along with the results of Lemmas 5.5 and 5.7, into the error bound of Lemma 5.4. □

**6. Minimizing the error of the noisy modified Galerkin method.** Let $n \in \mathbb{Z}^+$, and consider noisy modified Galerkin methods using at most $n$ noisy function evaluations. How can we choose the parameters $h$ and $\bar{h}$ that will minimize the error of the noisy modified Galerkin method?

Recall that

$$\operatorname{card} \mathbb{N}_{h,\bar{h},\delta} \asymp n_{\bar{h}}^2 + n_h,$$

where

$$n_h = \left(\frac{m+1}{h}\right) \qquad \text{and} \qquad n_{\bar{h}} = \left(\frac{m+1}{\bar{h}}\right)^d.$$

It will be useful to rewrite this bound in terms of a proportionality constant, so that we have

$$\operatorname{card} \mathbb{N}_{h,\bar{h},\delta} \le C_{\mathrm{card}} \left( n_{\bar{h}}^2 + n_h \right).$$

As in the proof of Theorem 5.1, let

$$r = \min\{r_1 - 1, r_2, r_3\}.$$

Let

$$\tau = \frac{\max\{r_1, r_2\}}{\min\{r_1, r_2\}}.$$

We define parameters $\kappa$ and $\bar{\kappa}$ as follows:

1. Suppose that $r_2 < 2r$, so that $r_2 < 2\min\{r_1, r_2\}$. Take

$$\kappa = \left( \frac{n}{\tau^{2d} C_{\mathrm{card}}} \right)^{r_2/(2\min\{r_1,r_2\})} \qquad \text{and} \qquad \bar{\kappa} = \sqrt{\frac{n}{\theta 2d C_{\mathrm{card}}} - \kappa}.$$

2. Suppose that $r_2 = 2r$. Take

$$\kappa = \frac{n}{2\theta^{2d} C_{\mathrm{card}}} \qquad \text{and} \qquad \bar{\kappa} = \sqrt{\frac{n}{2\theta^{2d} C_{\mathrm{card}}}}.$$

3. Suppose that $r_2 > 2r$. Take

$$\bar{\kappa} = \left( \frac{n}{\theta^{2d} C_{\mathrm{card}}} \right)^{r/r_2} \qquad \text{and} \qquad \kappa = \frac{n}{\theta^{2d} C_{\mathrm{card}}} - \bar{\kappa}^2.$$

With these definitions of $\kappa$ and $\bar{\kappa}$, define meshsizes

$$h = \frac{\min\{r_1, r_2\}}{\kappa^{1/d}} \qquad \text{and} \qquad \bar{h} = \frac{\min\{r_1, r_2\}}{\bar{\kappa}^{1/d}}. \tag{6.1}$$

Since the degree of the spline space satisfies

$$m = \max\{r_1, r_2\} - 1,$$

we find that

$$n_h = \tau^d \kappa \qquad \text{and} \qquad n_{\bar{h}} = \tau^d \bar{\kappa}.$$

In the sequel, we shall assume without loss of generality that $h$ and $\bar{h}$ have been chosen so that $n_h$ and $n_{\bar{h}}$ are positive integers. With these choices of $h$ and $\bar{h}$, let

$$\mathbb{N}_{n,\delta} = \mathbb{N}_{h,\bar{h},\delta} \qquad \text{and} \qquad \phi_{n,\delta} = \phi_{h,\bar{h},\delta}.$$

Then

$$\text{card}\mathbb{N}_{n,\delta} \leq C_{\text{card}} \left(n_{\bar{h}}^2 + n_h\right) \leq C_{\text{card}} \theta^{2d} (\bar{\kappa}^2 + \kappa) \leq n.$$

Using a case-by-case analysis (as in [**22**, Thm. 18]), we have

**Theorem 6.1.** *Suppose that $d = l$ or $r_1 \geq 2$. Let $m = \max\{r_1, r_2\} - 1$ and let $q \geq r_1 - 1$ in (4.5) and (4.6). Then there exists $n_0 \in \mathbb{Z}^{++}$ and $\delta_0 > 0$ such that $\phi_{n,\delta}$ is well-defined for $n \geq n_0$ and $\delta \in [0, \delta_0]$. Furthermore, there exists a positive constant $M_8$ such that*

$$e(\phi_{n,\delta}, \mathbb{N}_{n,\delta}) \leq M_8 (n^{-\mu_2} + \delta) \qquad \forall\, n \geq n_0, \delta \in [0, \delta_0],$$

*where*

$$\mu_2 = \min\left\{\frac{r_1 - 1}{d}, \frac{r_2}{2d}, \frac{r_3}{d}\right\}.$$

Comparing Theorems 3.1 and 6.1, we find the following bounds on the $n$th minimal error of noisy information:

**Corollary 6.1.** *Let*

$$\mu_1 = \min\left\{\frac{r_1}{d}, \frac{r_2}{2d}, \frac{r_3}{d}\right\} \qquad \textit{and} \qquad \mu_2 = \min\left\{\frac{r_1 - 1}{d}, \frac{r_2}{2d}, \frac{r_3}{d}\right\}.$$

1. *If $d < l$ and $r_1 = 1$, then*

$$r(n, \delta) \asymp 1.$$

2. *If $d = l$ or $r_1 \geq 2$, then*

$$\left(\frac{1}{n}\right)^{\mu_1} + \delta \preccurlyeq r(n, \delta) \preccurlyeq \left(\frac{1}{n}\right)^{\mu_2} + \delta.$$

Using Corollary 6.1, we see that for some values of the parameters $r_1$, $r_2$, $r_3$, $d$, $l$, and $p$, we can obtain the tight bounds on the minimal noisy error that are given in the Introduction. However, tight bounds for the remaining cases remain an open problem.

**7. Open questions.** The results of this paper lead to several open questions. The most obvious is that the lower and upper bounds are not always tight; we hope to remedy this problem in the near future.

The remaining questions deal with

1. solving problems over more general domains, and
2. solving problems when $k$ and $f$ belong to Sobolev spaces.

We treat these questions in turn.

7.1. *More general domains.* We often want to solve the Fredholm problem over a domain that is not the diffeomorphic image of a cube, such as

- the boundary of a given region, and
- a smooth domain (such as a ball or sphere).

How can we solve such problems?

One simple idea for solving problems over smooth domains would be to let $\Omega$ be the image of a ball, rather than a cube. This approach was studied (for the surface approximation and integration problems) in [**24**, §5]; it appears that the results of this paper also apply to the case where the domain is the image of a ball, the main difference being a slight extra complication appearing in the definitions of certain integrals that will appear in the noisy Galerkin method.

A more general technique would be to use oriented cellulated regions [**7**, pp. 369–370], which essentially means that the domains are finite unions of images of cubes. This would allow us to handle smooth regions, as well as domains that are boundaries of regions in $\mathbb{R}^d$. We hope to analyze the complexity of Fredholm problems over such regions in a future paper.

7.2. *Sobolev space data.* To a certain extent, the problem studied in this paper generalizes that of [**22**], since that paper dealt with problems defined over $I^d$, whereas the current paper deals with problems defined over more general domains. However, there is one important difference between these papers. In [**22**], we assumed that $k$ and $f$ belonged to Sobolev spaces, whereas the current paper assumes that they belong to Hölder spaces. The reason for this change is that we need to know the smoothness of composite functions $v \circ \rho$, where $v$ is a possible right-hand side for our Fredholm problem and $\rho \in \mathscr{R}$. We know the smoothness for the Hölder case, but not for the Sobolev case.

In the Hölder case studied in this paper, this information was provided by Lemma 5.6. This lemma was actually proved in [**24**]. The proof consists of using the Faa di Bruno formula [**5**, Thm. 2.1] for derivatives of a composite function, along with the fact that the restriction of a $C^r(\Omega)$ function to some $\Omega' \subset \Omega$ is a $C^r(\Omega')$ function, even if $\Omega'$ is of lower dimension than $\Omega$.

Now suppose that we consider the Sobolev case. Here, the set $\mathscr{R}$ will be as it was in the rest of this paper, but $\mathscr{K}$ and the set of right-hand-side functions $f$ will be Sobolev classes. That is, we let $\mathscr{K}$ consist of the functions $k \in C^{r_2}(I^{2l})$ satisfying

$$\|k\|_{C^{r_2}(I^{2l})} \le c_3,$$

along with the uniform invertibility condition

$$\|(I - T_{\rho,k_\rho})^{-1}\|_{\mathrm{Lin}[L_p(I^d)]} \le c_4 \qquad \forall\, \rho \in \mathscr{R}$$

that are suitable for the Sobolev case. Our right-hand-side functions $f$ will be the unit ball of $W^{r_3,p}(I^l)$. We measure error in the $L_p(I^d)$-sense.

If we were to try to establish a Sobolev version of Lemma 5.6, we would once again start with the Faa di Bruno formula. We would be done once we knew the Sobolev $\rho(I^d)$-smoothness of a Sobolev $I^l$ function for $\rho \in \mathscr{R}$.

First, suppose that $d = l$. We can then use the Sobolev space change-of-variables theorem [**1**, Thm. 3.35] to see that there exists $C > 0$ such that

$$\|v\|_{W^{\min\{r_1, r_3/p\}}(\rho(I^d))} \le C\|v\|_{W^{r_3,p}(I^d)} \qquad \forall\, v \in W^{r_3,p}(I^d)$$

for any $\rho \in \mathscr{R}$. This is sufficient to establish the Sobolev version of Lemma 5.6 that we need for the case $d = l$. It is not too difficult to see that the results in §5 now extend to this case. The main difference is that $C(I^d)$ and $L_\infty(I^d)$ should be replaced by $L_p(I^d)$, and that $L_1(I^d)$ should be replaced by $L_{p'}(I^d)$, where $p' = p/(p-1)$.

Now suppose that $d < l$. Let $v \in W^{r_3,p}(I^l)$. The Sobolev trace theorem tells us that $v \in W^{r_3-1/p,p}(\partial I^l)$. Now consider a face of $\partial I^l$, such as $I^{l-1} \times \{0\}$. Then the restriction of $v$ to $I^{l-1} \times \{0\}$ lies in the space $W^{r_3-1/p,p}(I^{l-1} \times \{0\})$. Treating such a face as $I^{l-1}$ and repeating this argument, we eventually find that $v \in W^{r_3-(l-d)/p,p}(\mathrm{id}(I^d))$.

However, we are interested in the Sobolev smoothness of $f$ over $\rho(I^d)$, rather than over $\mathrm{id}(I^d)$. For $\rho$ sufficiently smooth (i.e., $r_1 \geq r_3$), this result is covered by [**3**, § 5.5, Remark 21]. But we need a more general result, holding for all admissible values of $r_1$ and $r_3$.

Based on the discussion above, we state

**Conjecture 7.1.** *There exists $C > 0$ such that*

$$\|v\|_{W^{\min\{r_1, r_3-(l-d)/p\}}(\rho(I^d))} \leq C\|v\|_{W^{r_3,p}(I^l)} \qquad \forall\, v \in W^{r_3,p}(I^l)$$

*for any $\rho \in \mathscr{R}$.*

We emphasize that this conjecture holds if $d = l$ or $r_1 \leq r_3$; it only needs to be proved when $d < l$ and $r_1 < r_3$.

If this conjecture holds, it can be shown that the results of this paper may be extended to the Sobolev case, provided that we redefine

$$\mu_1 = \min\left\{\frac{r_1}{d}, \frac{r_2}{2d}, \frac{r_3 - (l-d)/p}{d}\right\} \qquad \text{and} \qquad \mu_2 = \min\left\{\frac{r_1 - 1}{d}, \frac{r_2}{2d}, \frac{r_3 - (l-d)/p}{d}\right\}.$$

**Acknowledgments.** I am happy to thank Robert Adams and Hans Triebel for their email conversations with me. I would also like to thank the referees for their helpful insights.

## REFERENCES

**1.** R. A. Adams. *Sobolev Spaces*, Academic Press, New York, (1975).

**2.** A. K. Aziz, M. R. Dorr, and R. B. Kellogg, *A new approximation method for the Helmholtz equation in an exterior domain*, SIAM J. Numer. Anal., **19(5)** (1982), 899-908.

**3.** V. I. Burenkov, *Sobolev Spaces on Domains*, B. G. Teuber, Stuttgart-Leipzig, (1998).

**4.** P. G. Ciarlet, *The Finite Element Method For Elliptic Problems*, North-Holland, Amsterdam, (1978).

**5.** G. M. Constantine and T. H. Savits, *A multivariate Faa di Bruno formula with applications*, Trans. Amer. Math. Soc., **348** (1996), 503-520.

**6.** R. Courant and D. Hilbert, *Methods of Mathematical Physics*, Wiley, Volume **2**, (1989).

**7.** C. H. Edwards, Jr., *Advanced Calculus of Several Variables*, Academic Press, New York, (1973).

**8.** K. V. Emelyanov and A. M. Ilin, *Number of arithmetic operations necessary for the approximate solution of Fredholm integral equations*, USSR Comp. Math. Math. Phys., **7(4)** (1967), 259-267.

**9.** K. Frank, S. Heinrich, and S. Pereverzev, *Information complexity of multivariate Fredholm integral equations in Sobolev classes*, J. Complexity, **12(1)** (1996), 17-34.

**10.** W. Hackbusch, *Integral Equations: Theory and Numerical Treatment*, volume **120** of International Series of Numerical Mathematics, Birkhauser, Basel, (1995).

**11.** T. Jiang, *The worst case complexity of the Fredholm equation of the second kind with non-periodic free term and noise information*, Numer. Funct. Anal. Optim., **19(34)** (1998), 329-343.

**12.** E. Novak, *Deterministic and Stochastic Error Bounds in Numerical Analysis*, Volume **1349** of Lecture Notes in Mathematics. Springer-Verlag, New York, (1988).

**13.** J. T. Oden and J. N. Reddy, *An introduction to the mathematical theory of finite elements*, Wiley- Interscience, New York, (1976).

**14.** S. V. Pereverzev, *Complexity of the Fredholm problem of second kind*, In Optimal Recovery, pages 255-272. Nova Science, New York, (1992).

**15.** L. Plaskota, *Noisy Information and Computational Complexity*, Cambridge University Press, Cambridge, (1996).

**16.** ———, *Worst case complexity of problems with random information noise*, J. Complexity, **12** (1996), 416-439.

**17.** M. H. Schultz, *$L_1$-multivariate approximation theory*, SIAM J. Numer. Anal, **6** (1969), 161-183.

**18.** A. Yu. Shadrin, *The $L_1$-norm of the $L_2$-spline projector is bounded independently of the knot sequence: a proof of de Boors conjecture*, Acta Math., **187(1)** (2001), 59-137.

**19.** J. F. Traub, G. W. Wasilkowski, and H. Wozniakowski, *Information-Based Complexity*, Academic Press, New York, (1988).

**10.** A. G. Werschulz, *What is the complexity of the Fredholm problem of the second kind?*, Journal of Integral Equations, **9** (1985), 213-241.

**21.** ———, *The Computational Complexity of Differential and Integral Equations: An Information-Based Approach*, Oxford University Press, New York, (1991).

**22.** ———, *Where does smoothness count the most for Fredholm equations of the second kind with noisy information?*, J. Complexity, **19(6)** (2003), 758-798.

**23.** A. G. Werschulz and H. Wozniakowski, *What is the complexity of surface integration?*, Journal of Complexity, **17** (2001), 442-466.

**24.** ——— *Surface approximation is sometimes easier than surface integration*, Constructive Approximation, **20(2)** (2004), 267-302.

Department of Computer and Information Sciences, Fordham University, New York, NY 10023, Department of Computer Science, Columbia University, New York, NY 10027
**Email address: agw@cs.columbia.edu**