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THE BEDROSIAN IDENTITY AND HOMOGENEOUS SEMI-CONVOLUTION EQUATIONS

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ABSTRACT. We characterize a class of functions satisfying the classical Bedrosian identity or the circular Bedrosian identity by certain homogeneous semi-convolution equations. The structure of solutions of these equations is then studied using translation invariant subspaces of Hardy spaces and additive positive definite kernels. The results obtained provide some insight into the Bedrosian identity and a construction of intrinsic mode functions for the time-frequency analysis of nonlinear and nonstationary signals.

1. Introduction. The Hilbert transform is defined for each function $f \in L^p(\mathbf{R})$, $1 \le p \le \infty$, at $x \in \mathbf{R}$ as

$$(1.1)\quad (Hf)(x) := \text{p.v.}\,\frac{1}{\pi}\int_{\mathbf{R}} \frac{f(y)}{x-y}\,dy := \lim_{\substack{\varepsilon\to 0^+\\ N\to\infty}} \frac{1}{\pi}\int_{\varepsilon\le |y-x|\le N} \frac{f(y)}{x-y}\,dy,$$

whenever the Cauchy principal value of the above singular integral exists. In engineering analysis, people often face the need for calculating the Hilbert transform of a product of functions. A simple method for computing such products under certain conditions was found by Bedrosian [**2**]: If two functions $f, g \in L^2(\mathbf{R})$ satisfy either $\operatorname{supp}\hat{f} \subseteq$

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$[0,\infty)$, $\operatorname{supp}\hat{g} \subseteq [0,\infty)$ or $\operatorname{supp}\hat{f} \subseteq [-a,a]$, $\operatorname{supp}\hat{g} \subseteq (-\infty,-a] \cup [a,\infty)$ for some positive number a, then there holds

$$[H(fg)](x) = f(x)(Hg)(x), \quad \text{almost everywhere } \; x \in \mathbf{R}. \tag{1.2}$$

Here $\operatorname{supp}\hat{f}$ denotes the support of the Fourier transform $\hat{f}$ of f. Recall that the Fourier transform $\mathcal{F}$ is defined for $h \in L^1(\mathbf{R})$ as

$$\hat{h}(\xi) := (\mathcal{F}h)(\xi) := \int_{\mathbf{R}} h(x)e^{-i\xi x}\,dx, \quad \xi \in \mathbf{R},$$

and for $h \in L^2(\mathbf{R})$ by the standard approximation process, [**16**]. Formula (1.2) is known as the *Bedrosian identity* in the literature and the result above is called the *Bedrosian theorem*.

A reason for the wide application of the Hilbert transform in signal theory is its importance in the time-frequency analysis, [**8, 10, 11, 17, 23**]. For a given real signal $f \in L^2(\mathbf{R})$, we form its *analytic signal* $Af := f + iHf$. Denote by sgn the signum function taking values 1, -1 and 0 for $\xi > 0$, $\xi < 0$ and $\xi = 0$, respectively. By the well-known fact, see, e.g., [**9**, page 324], that for each $f \in L^2(\mathbf{R})$

$$(Hf)\hat{}\,(\xi) = -i\,\operatorname{sgn}(\xi)\hat{f}(\xi), \quad \text{almost everywhere } \; \xi \in \mathbf{R},$$

$\operatorname{supp}(Af)\hat{} \subseteq [0,\infty)$. Hence, Af can be extended to a holomorphic function on the upper half-plane, [**15**]. The mathematical term *analytic* is applied to Af for this reason. We further decompose Af into

$$(Af)(t) = \rho(t)e^{i\theta(t)}, \quad t \in \mathbf{R}. \tag{1.3}$$

The $\rho(t)$ and $\theta(t)$ above are called the instantaneous amplitude and phase of the signal f at time t, respectively. However, the instantaneous phase θ is physically meaningful only if its derivative is nonnegative. This suggested introducing the intrinsic mode function (IMF) in [**17**]. IMFs are expected to have the property that the instantaneous amplitude and phase derived from (1.3) have sound physical meaning. They are fundamental for the Hilbert-Huang transform (HHT) for the time-frequency analysis of nonlinear and nonstationary signals, [**17**]. We shall present in Section 4 a method of constructing IMFs by understanding the equation

$$H(\rho(\cdot)e^{i\theta(\cdot)})(t) = \rho(t)(He^{i\theta(\cdot)})(t), \quad t \in \mathbf{R}, \tag{1.4}$$

which together with (1.2) has attracted much interest from engineering [6, 7, 11, 17, 22, 23].

Another variation of the Bedrosian identity comes from the practical application of analytic signals. In engineering applications, a real-valued signal f is typically represented by its finite samples

$$S_f := \{f(nt_0) : n = 0, 1, \dots, N\}, \tag{1.5}$$

where t_0 is a selected sampling interval and Nt_0 is the sampling time. A popular method, see e.g., [**20**], of calculating the Hilbert transform of f using the samples (1.5) works by multiplying the discrete time Fourier transform (DTFT) of S_f with sgn and then applying the inverse DTFT. The mathematical principle of this algorithm is easy to detect. Suppose we have an observation of a signal $f \in L^2(\mathbf{R})$ in a finite time duration, which we assume without loss of generality to be $[0, 2\pi]$. This observation is first extended to a periodic function $\tilde{f}(t) := f(t - 2\pi[t/2\pi])$, $t \in \mathbf{R}$, where $[x]$ denotes the biggest integer that is less than or equal to $x \in \mathbf{R}$. An approximation of the analytic signal of f is finally constructed as

$$\tilde{A}\tilde{f} := \tilde{f} + i\widetilde{H}\tilde{f}, \tag{1.6}$$

where $\widetilde{H}$ is the circular Hilbert transform defined for each $f \in L^1_{2\pi}$ at $t \in [0, 2\pi]$ as

$$\begin{aligned} (\widetilde{H}f)(t) &:= \text{p.v.}\, \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t-s) \cot \frac{s}{2}\, ds \\ &:= \lim_{\varepsilon \to 0^+} \frac{1}{2\pi} \int_{\varepsilon \le |s| \le \pi} f(t-s) \cot \frac{s}{2}\, ds \end{aligned} \tag{1.7}$$

if the Cauchy principal value of the above singular integral exists. Here with χ_A being the characteristic function of a subset $A \subseteq \mathbf{R}$, we denote by $L^p_{2\pi}$, $1 \le p \le \infty$, the set of all the 2π-periodic functions f on $\mathbf{R}$ such that $f\chi_{[0,2\pi]} \in L^p[0, 2\pi]$. The above algorithm based on DTFT suggests the necessity to study the condition for the following *circular Bedrosian identity* in $L^2_{2\pi}$

$$[\widetilde{H}(fg)](t) = f(t)(\widetilde{H}g)(t), \quad \text{almost everywhere } t \in [0, 2\pi]. \tag{1.8}$$

In particular, our interest in the Bedrosian identities is stimulated by the recent mathematical progress on the Bedrosian identity [**25, 31**]. Before introducing these results, we collect here some notations that are frequently referred to throughout the paper. Let $\mathbf{N}$ be the set of all the positive integers, $\mathbf{Z}$ the set of all the integers and $\mathbf{Z}_+ := \mathbf{N}\cup\{0\}$. We shall enumerate finite sets with $\mathbf{Z}_p := \{0, \ldots, p-1\}$ and $\mathbf{N}_p := \{1, \ldots, p\}$ for $p \in \mathbf{N}$. Following the custom in the literature, we also denote by $\mathbf{R}$ and $\mathbf{C}$ the set of real numbers and the set of complex numbers, respectively. Finally, we set $\mathbf{R}_+ := \{x \in \mathbf{R} : x \geq 0\}$, $\mathbf{R}_- := \{x \in \mathbf{R} : x \leq 0\}$, $\mathbf{C}_R := \{z \in \mathbf{C} : \mathrm{Re}\,(z) > 0\}$ and $\mathbf{C}_+ := \{z \in \mathbf{C} : \mathrm{Im}\,(z) > 0\}$.

The following characterization of the classical Bedrosian identity was given in [**31**].

Theorem A. *If $f, f', g \in L^2(\mathbf{R})$, then the Hilbert transform of the function fg satisfies the Bedrosian identity* (1.2) *if and only if*

$$\int_{-1}^{0}\int_{\mathbf{R}} \frac{\xi}{t^2} e^{ix\xi(t+1)/t} \hat{f}\left(\frac{\xi}{t}\right)\hat{g}(\xi)\, d\xi\, dt = 0. \tag{1.9}$$

A sufficient condition was derived from (1.9) in the same paper, which states that if $f, g \in L^2(\mathbf{R})$ are such that

$$\mu\left(\{t\xi : t \in [-1,0],\ \xi \in \mathrm{supp}\,\hat{f}\} \cap \mathrm{supp}\,\hat{g}\right) = 0, \tag{1.10}$$

then the Bedrosian identity (1.2) holds, where μ denotes the Lebesgue measure on $\mathbf{R}$. Note that condition (1.10) covers the sufficient conditions in the Bedrosian theorem.

The circular Bedrosian theorem for the circular Bedrosian identity (1.8) was established in [**25**]. Note first that if $f \in L^2_{2\pi}$ then $\widetilde{H}f$ has the following form in terms of the Fourier coefficient of f

$$(\widetilde{H}f)(t) = \sum_{k\in\mathbf{Z}} -i\ \mathrm{sgn}(k) c_k(f) e^{ikt}, \text{ almost everywhere } t \in [0, 2\pi], \tag{1.11}$$

where $c_k(f)$ is the kth Fourier coefficient of f defined as

$$c_k(f) := \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-ikt}\, dt. \tag{1.12}$$

Theorem B. *Let $f, g \in L^2_{2\pi}$ and $K \in \mathbf{Z}_+$. If $c_n(f) = 0$ for $|n| > K$ and $c_n(g) = 0$ for $|n| \leq K$ or if $c_n(f) = 0$ for $n < -K$ and $c_n(g) = 0$ for $n \leq K$, then the circular Bedrosian identity* (1.8) *holds.*

We shall show in Section 2 that the classical Bedrosian identity (1.2) is equivalent to two homogeneous semi-convolution equations. The characterizations of right translation invariant subspaces of $L^2(\mathbf{R}_+)$ and additive positive definite kernels on $\mathbf{R}_+$ are then used to investigate these equations. The results will provide some insight into the Bedrosian identity (1.2). For example, we are able to show that condition (1.10) is unnecessary. The circular Bedrosian identity (1.8) is studied in Section 3. Especially, we shall prove that the sufficient condition in Theorem B is unnecessary for (1.8). In Section 4, we apply the method and theory developed in Sections 2 and 3 to the construction of intrinsic mode functions.

2. The Bedrosian identity in $L^2(\mathbf{R})$. In this section, we focus on the classical Bedrosian identity (1.2) where $f, g \in L^2(\mathbf{R})$ and "almost everywhere $x \in \mathbf{R}$" implies that the equation $H(fg) = fHg$ holds everywhere on $\mathbf{R}$ except for a subset of $\mathbf{R}$ that has zero Lebesgue measure. Note that we have used this convention several times in the introduction. Our study will be based on a necessary and sufficient condition for identity (1.2). A similar condition was first obtained in [**7**] under the assumption that $f, g \in L^2(\mathbf{R}) \cap L^\infty(\mathbf{R})$.

2.1 *A necessary and sufficient condition.* We begin with two lemmas from [**9**, Chapter 8].

Lemma 2.1. *There exists a positive constant c such that for each $f \in L^1(\mathbf{R})$ and $y > 0$*

$$\mu\{x : |(Hf)(x)| > y\} \leq \frac{c}{y}\|f\|_{L^1(\mathbf{R})}.$$

Lemma 2.2. *Let $f \in L^p(\mathbf{R})$, $1 < p \leq 2$. Then $Hf \in L^p(\mathbf{R})$, $H^2 f = -f$ almost everywhere and the Fourier transform of Hf is given by*

$$(Hf)\hat{}(\xi) = -i\ \mathrm{sgn}(\xi)\hat{f}(\xi), \quad \textit{almost everywhere } \xi \in \mathbf{R}. \tag{2.1}$$

For $p = 1$*, formula* (2.1) *remains valid for all* $\xi \in \mathbf{R}$ *provided that* $Hf \in L^1(\mathbf{R})$.

Lemma 2.2 exhibits that if $Hf \in L^1(\mathbf{R})$, then $-i\ \mathrm{sgn}(\cdot)\hat{f}$ is the Fourier transform of some function in $L^1(\mathbf{R})$. Our next result shows that this is also sufficient for Hf to be integrable on $\mathbf{R}$.

Lemma 2.3. *Let* $f \in L^1(\mathbf{R})$*. If there exists* $g \in L^1(\mathbf{R})$ *such that*

$$-i\ \mathrm{sgn}(\xi)\hat{f}(\xi) = \hat{g}(\xi), \quad \xi \in \mathbf{R}, \tag{2.2}$$

then $Hf = g$ *almost everywhere.*

Proof. Suppose $f \in L^1(\mathbf{R})$ and there exists $g \in L^1(\mathbf{R})$ such that (2.2) holds. We can choose a sequence of functions $\{\phi_n : n \in \mathbf{N}\}$ that are infinitely differentiable and compactly supported such that for each $h \in L^1(\mathbf{R})$, see e.g., [**16**, pages 188–189],

$$\lim_{n\to\infty} \|\phi_n * h - h\|_{L^1(\mathbf{R})} = 0,$$

where the convolution $\phi_n * h$ of ϕ_n and h is defined by

$$\phi_n * h := \int_{\mathbf{R}} \phi_n(\cdot - y)h(y)\, dy.$$

It is also clear that for each $h \in L^1(\mathbf{R})$, $\phi_n * h \in L^2(\mathbf{R})$, and hence

$$(\phi_n * h)\hat{}(\xi) = \hat{\phi}_n(\xi)\hat{h}(\xi), \quad \text{almost everywhere } \xi \in \mathbf{R}.$$

The above equation implies that

$$-i\ \mathrm{sgn}(\xi)(\phi_n * f)\hat{}(\xi) = (\phi_n * g)\hat{}(\xi), \quad \text{almost everywhere } \xi \in \mathbf{R}.$$

By Lemma 2.2, we have

$$H(\phi_n * f) = \phi_n * g \quad \text{almost everywhere.} \tag{2.3}$$

Since $\phi_n * f$ converges to f in $L^1(\mathbf{R})$, by Lemma 2.1, $H(\phi_n * f)$ converges to Hf in Lebesgue measure. There hence exists a subsequence $\{\phi_{n_i} :$

$i \in \mathbf{N}\}$ for which $H(\phi_{n_i} * f)$ converges to Hf almost everywhere on $\mathbf{R}$ with respect to the Lebesgue measure. This combined with (2.3) yields that $Hf = g$ almost everywhere and completes the proof. □

Theorem 2.4. *Functions $f, g \in L^2(\mathbf{R})$ satisfy the Bedrosian identity (1.2) if and only if*

$$\int_{\mathbf{R}_+} \hat{f}(\xi + \eta)\hat{g}(-\eta)\, d\eta = 0, \quad \xi \in \mathbf{R}_+ \tag{2.4}$$

and

$$\int_{\mathbf{R}_-} \hat{f}(\xi + \eta)\hat{g}(-\eta)\, d\eta = 0, \quad \xi \in \mathbf{R}_-. \tag{2.5}$$

Proof. Since $fg, fHg \in L^1(\mathbf{R})$, by Lemmas 2.2 and 2.3, $H(fg) = fHg$ almost everywhere if and only if

$$(fHg)\hat{\ }(\xi) = -i\ \mathrm{sgn}(\xi)(fg)\hat{\ }(\xi), \quad \xi \in \mathbf{R}. \tag{2.6}$$

By the fact that for all $f_1, f_2 \in L^2(\mathbf{R})$

$$(f_1 f_2)\hat{\ } = \frac{1}{2\pi}\hat{f}_1 * \hat{f}_2 \quad \text{and} \quad (Hf_1)\hat{\ } = -i\ \mathrm{sgn}(\cdot)\hat{f}_1 \ \text{ almost everywhere,}$$

equation (2.6) has the form

$$\int_{\mathbf{R}} \hat{f}(\xi - \eta)\hat{g}(\eta)(\mathrm{sgn}(\xi) - \mathrm{sgn}(\eta))\, d\eta = 0, \quad \xi \in \mathbf{R}.$$

Clearly, the above equation can be divided into the following three equations

$$\int_{\mathbf{R}_+} \hat{f}(\xi + \eta)\hat{g}(-\eta)\, d\eta = 0, \quad \xi \in \mathbf{R}_+ \setminus \{0\}, \tag{2.7}$$

$$\int_{\mathbf{R}_-} \hat{f}(\xi + \eta)\hat{g}(-\eta)\, d\eta = 0, \quad \xi \in \mathbf{R}_- \setminus \{0\}, \tag{2.8}$$

$$\int_{\mathbf{R}_+} \hat{f}(\eta)\hat{g}(-\eta)\, d\eta = \int_{\mathbf{R}_-} \hat{f}(\eta)\hat{g}(-\eta)\, d\eta. \tag{2.9}$$

By the continuity of $\int_{\mathbf{R}_+} \hat{f}(\cdot+\eta)\hat{g}(-\eta)\,d\eta$ and $\int_{\mathbf{R}_-} \hat{f}(\cdot+\eta)\hat{g}(-\eta)\,d\eta$, f, g satisfy equations (2.7), (2.8) and (2.9) if and only if they satisfy (2.4) and (2.5). $\square$

2.2. *Particular solutions.* By Theorem 2.4, we shall consider the two Hilbert spaces $L^2(\mathbf{R}_+)$ and $L^2(\mathbf{R}_-)$. For simplicity, we denote by $f^+ := f\chi_{\mathbf{R}_+}$ and $f^- := f\chi_{\mathbf{R}_-}$ for each $f \in L^2(\mathbf{R})$. Equations (2.4) and (2.5) can be interpreted as

$$\overline{(\hat{g}(-\cdot)^+)} \perp \operatorname{span}\{\tau_y^*(\hat{f}^+) : y \in \mathbf{R}_+\}, \text{ in } L^2(\mathbf{R}_+) \tag{2.10}$$

and

$$\overline{(\hat{g}(-\cdot)^-)} \perp \operatorname{span}\{\tau_y^*(\hat{f}^-) : y \in \mathbf{R}_-\}, \text{ in } L^2(\mathbf{R}_-), \tag{2.11}$$

where $\bar{f}$ denotes the conjugate of f, τ^* is the dual of the translation operator τ which is defined for each function $f \in L^2(\mathbf{R})$ and $y \in \mathbf{R}$ as $\tau_y f := f(\cdot - y)$. We remark that the sufficient condition (1.10) can also be derived from (2.10) and (2.11) by noting that

$$\bigcup_{y\in\mathbf{R}_+} \operatorname{supp}(\tau_y^*(\hat{f}^+)) = \left\{t\xi : t \in [0,1], \xi \in \mathbf{R}_+ \cap \operatorname{supp}(\hat{f})\right\},$$
$$\operatorname{supp}(\hat{g}(-\cdot)^+) = \{\xi : \xi \in \mathbf{R}_+, -\xi \in \operatorname{supp}\hat{g}\}.$$

Equations (2.4) and (2.5) are independent of each other in the sense that only the values of $\hat{f}$ and $\hat{g}(-\cdot)$ on $\mathbf{R}_+$ are used in equation (2.4) while equation (2.5) only involves the values of $\hat{f}$ and $\hat{g}(-\cdot)$ on $\mathbf{R}_-$. For this reason, we shall work in the space $L^2(\mathbf{R}_+)$ only and consider the general homogeneous semi-convolution equation

$$\int_{\mathbf{R}_+} \psi(x+y)\phi(x)\,dx = 0, \quad y \in \mathbf{R}_+, \tag{2.12}$$

where $\psi, \phi \in L^2(\mathbf{R}_+)$, or its equivalent form

$$\bar{\phi} \perp \operatorname{span}\{\tau_y^*\psi : y \in \mathbf{R}_+\}, \text{ in } L^2(\mathbf{R}_+). \tag{2.13}$$

The results to be obtained for (2.13) can be applied directly to equations (2.10) and (2.11).

A subspace $\mathcal{M}$ of $L^2(\mathbf{R}_+)$ is said to be *left translation invariant* if for each $y \in \mathbf{R}_+$, $\tau_y^*(\mathcal{M}) \subseteq \mathcal{M}$. The homogeneous semi-convolution equation (2.12) is closely related to the closed left translation invariant subspace of $L^2(\mathbf{R}_+)$. To see this, we set for each $\psi \in L^2(\mathbf{R}_+)$, $\tau^*(\psi) := \overline{\operatorname{span}}\{\tau_y^*\psi : y \in \mathbf{R}_+\}$, the closure of span $\{\tau_y^*\psi : y \in \mathbf{R}_+\}$ in $L^2(\mathbf{R}_+)$. Clearly, $\tau^*(\psi)$ is the smallest closed left translation invariant subspace of $L^2(\mathbf{R}_+)$ that contains ψ. For a specified $\psi \in L^2(\mathbf{R}_+)$, there exists nontrivial $\phi \in L^2(\mathbf{R}_+)$ satisfying (2.13) if and only if $\tau^*(\psi)$ is a proper subspace of $L^2(\mathbf{R}_+)$. For instance, if the following sufficient condition in the Bedrosian theorem

$$\operatorname{supp} \hat{f} \subseteq [-a, a], \quad \operatorname{supp} \hat{g} \subseteq (-\infty, -a] \cup [a, \infty)$$

is valid for some $a \in \mathbf{R}_+$, then

$$\tau^*(\hat{f}^+) \subseteq \mathcal{M}_a := \{h \in L^2(\mathbf{R}_+) : \operatorname{supp} h \subseteq [0, a]\}$$

and $\hat{g}(-\cdot)^+$ is in the orthogonal complement of $\mathcal{M}_a$. The Bedrosian theorem is essentially a consequence of the trivial fact that $\mathcal{M}_a$ is a closed left translation invariant subspace of $L^2(\mathbf{R}_+)$. This suggests that if we impose other particular requirements on $\tau^*(\psi)$ we may obtain more sufficient conditions for (2.13). The following theorem provides a characterization of functions ψ for which $\tau^*(\psi)$ is finite dimensional.

Theorem 2.5. *Set $\psi \in L^2(\mathbf{R}_+)$. Then the space $\tau^*(\psi)$ is finite-dimensional if and only if there exists $p \in \mathbf{N}$, $\{\lambda_j : j \in \mathbf{N}_p\} \subseteq \mathbf{C}_R$, $\{n_j : j \in \mathbf{N}_p\} \subseteq \mathbf{Z}_+$ and $\{c_{jk} : j \in \mathbf{N}_p, k \in \mathbf{Z}_{n_j}\} \subseteq \mathbf{C}$ such that*

$$\psi(t) = \sum_{j \in \mathbf{N}_p} \sum_{k \in \mathbf{Z}_{n_j}} c_{jk} t^k e^{-\lambda_j t}, \quad t \in \mathbf{R}_+. \tag{2.14}$$

It was established in [**1**] that $\tau^*(\psi)$ is finite dimensional if and only if ψ is of form (2.14). With the additional requirement that $\psi \in L^2(\mathbf{R}_+)$, the λ_j in (2.14) can be chosen from $\mathbf{C}_R$.

Suppose $\psi \in L^2(\mathbf{R}_+)$ has the form (2.14). The second step in solving equation (2.12) is to find $\phi \in L^2(\mathbf{R}_+)$ such that $\bar{\phi} \perp \tau^*(\psi)$. To this end, we introduce the *Laplace transform* $\mathcal{L}$ defined for each $h \in L^2(\mathbf{R}_+)$ at $s \in \mathbf{C}_R$ as

$$(\mathcal{L}h)(s) := \int_{\mathbf{R}_+} e^{-st} h(t)\, dt.$$

Proposition 2.6. *Let ψ be given by (2.14) with $c_{jn_j-1} \neq 0$, $\lambda_j \neq \lambda_k$, $j,k \in \mathbf{N}_p$. Then $\phi \in L^2(\mathbf{R}_+)$ satisfies $\bar{\phi} \perp \tau^*(\psi)$ if and only if for all $j \in \mathbf{N}_p$ and $k \in \mathbf{Z}_{n_j}$*

$$\frac{d^k(\mathcal{L}\phi)}{ds^k}(\lambda_j) = 0. \tag{2.15}$$

Proof. Set $\mathcal{I} := \{(j,k) : j \in \mathbf{N}_p, k \in \mathbf{Z}_{n_j}\}$ and $q := \#\mathcal{I}$, the cardinality of $\mathcal{I}$. We first prove that

$$\tau^*(\psi) = M := \operatorname{span}\{t^k e^{-\lambda_j t} : (j,k) \in \mathcal{I}\}.$$

It is clear that $\tau^*(\psi) \subseteq M$. To show the inverse inclusion, we observe for all $y \in \mathbf{R}_+$ that

$$\psi(t+y) = \sum_{j\in\mathbf{N}_p} \sum_{k\in\mathbf{Z}_{n_j}} t^k e^{-\lambda_j t} e^{-\lambda_j y} P_{jk}(y), \tag{2.16}$$

where

$$P_{jk}(y) := \sum_{l=k}^{n_j-1} c_{jl} \binom{l}{k} y^{l-k}.$$

We define a matrix associated with each $\{y_{jk} : (j,k) \in \mathcal{I}\} \in \mathbf{C}^q$ by setting

$$B(y_{jk} : (j,k) \in \mathcal{I}) := \{e^{-\lambda_{j'} y_{jk}} P_{j'k'}(y_{jk}) : (j,k), (j',k') \in \mathcal{I}\}.$$

Since $c_{jn_j-1} \neq 0$ for each $j \in \mathbf{N}_p$, P_{jk} is a nontrivial polynomial of degree $n_j - 1 - k$, $k \in \mathbf{Z}_{n_j}$. One can see by this fact that the function $(y_{jk} : (j,k) \in \mathcal{I}) \to \det(B(y_{jk} : (j,k) \in \mathcal{I}))$ is a nontrivial entire function on $\mathbf{C}^q$. By the well-known fact that the real zeros of an entire function on $\mathbf{C}^q$ form a set of zero Lebesgue measure on $\mathbf{R}^q$, we can choose $\{y_{jk} : (j,k) \in \mathcal{I}\} \subseteq \mathbf{R}_+$ such that $B(y_{jk} : (j,k) \in \mathcal{I})$ is nonsingular. Substituting these y_{jk} into equation (2.16) yields that

$$\{t^k e^{-\lambda_j t} : (j,k) \in \mathcal{I}\} \subseteq \tau^*(\psi).$$

This relation implies that $M \subseteq \tau^*(\psi)$.

We conclude that $\bar{\phi} \perp \tau^*(\psi)$ if and only if for all $(j,k) \in \mathcal{I}$

$$\int_{\mathbf{R}_+} t^k e^{-\lambda_j t} \phi(t)\, dt = 0. \tag{2.17}$$

The proposition is proved by the observation that (2.15) is equivalent to (2.17). $\square$

The application of Theorem 2.5 and Proposition 2.6 to equations (2.10) and (2.11) yields a class of functions f, g satisfying the Bedrosian identity (1.2).

Proposition 2.7. *Let $f, g \in L^2(\mathbf{R})$. If there exists $\{\lambda_j : j \in \mathbf{N}_p\} \subseteq \mathbf{C}_R$ and $\{\gamma_j : j \in \mathbf{N}_q\} \subseteq \mathbf{C}_R$ such that λ_j is the zero of $\mathcal{L}(\hat{g}(-\cdot)^+)$ of order n_j, $j \in \mathbf{N}_p$, γ_j is the zero of $\mathcal{L}(\hat{g}^+)$ of order m_j, $j \in \mathbf{N}_q$ and $\hat{f}$ has the form*

$$\begin{aligned} \hat{f}(\xi) = u(\xi) \sum_{j \in \mathbf{N}_p} \sum_{k \in \mathbf{Z}_{n_j}} c_{jk} \xi^k e^{-\lambda_j \xi} \\ + u(-\xi) \sum_{j \in \mathbf{N}_q} \sum_{k \in \mathbf{Z}_{m_j}} c'_{jk} \xi^k e^{\gamma_j \xi}, \quad \xi \in \mathbf{R}, \end{aligned} \tag{2.18}$$

where u is the Heaviside function defined by $u(x) = 1$ for $x \geq 0$ and $u(x) = 0$ for $x < 0$, $\{c_{jk} : j \in \mathbf{N}_p, k \in \mathbf{Z}_{n_j}\} \subseteq \mathbf{C}$ and $\{c'_{jk} : j \in \mathbf{N}_q, k \in \mathbf{Z}_{m_j}\} \subseteq \mathbf{C}$ are arbitrary constants, then the Bedrosian identity (1.2) *holds.*

Let $f \in L^2(\mathbf{R})$ be given by (2.18). If there exist nonzero constants c_{jk} and c'_{jk} in (2.18), then $\operatorname{supp} \hat{f} = \mathbf{R}$. Since $\tau^*(\hat{f}^+)$ and $\tau^*(\hat{f}(-\cdot)^+)$ are of finite dimensions, there must exist $g \in L^2(\mathbf{R})$ satisfying equations (2.10) and (2.11). This implies that condition (1.10) is unnecessary for the Bedrosian identity (1.2). We give an explicit example below to make this clearer. Set

$$\hat{f}(\xi) := e^{-|\xi|}, \quad \hat{g}(\xi) := e^{-|\xi|} - \frac{3}{2}e^{-2|\xi|}, \quad \xi \in \mathbf{R}. \tag{2.19}$$

It can be verified by Proposition 2.7 that f, g satisfy the Bedrosian identity. Therefore, the functions $f, g \in L^2(\mathbf{R})$ given above satisfy the Bedrosian identity (1.2) while we have the property that $\operatorname{supp} \hat{f} = \operatorname{supp} \hat{g} = \mathbf{R}$.

The example above tells us that $f, g \in L^2(\mathbf{R})$ are of low Fourier frequency and high Fourier frequency respectively is unnecessary for them to satisfy the Bedrosian identity (1.2). On the other hand, we shall prove in a special case that if $f \in L^2(\mathbf{R})$ is of low Fourier frequency then it is necessary for $g \in L^2(\mathbf{R})$ to have high Fourier frequency to satisfy the Bedrosian identity. We begin with a technical lemma.

Lemma 2.8. *Let $\psi \in L^2(\mathbf{R}_+)$ be such that* $\operatorname{supp} \psi \subseteq [0, 1]$. *If there exists $k \in \mathbf{N}$ and $\varepsilon \in (0, 1)$ such that $\psi\chi_{[1-\varepsilon,1]} \in C^{(k)}[1 - \varepsilon, 1]$ and $\psi^{(k-1)}(1) \neq 0$ then $\phi \in L^2(\mathbf{R}_+)$ satisfies the homogeneous semi-convolution equation* (2.12) *if and only if* $\operatorname{supp} \phi \subseteq [1, \infty)$.

Proof. Let $\psi \in L^2(\mathbf{R}_+)$ have all the properties described in the assumption. It is clear that if $\phi \in L^2(\mathbf{R}_+)$ satisfies $\operatorname{supp} \phi \subseteq [1, \infty)$ then (2.12) holds. On the other hand, suppose $\phi \in L^2(\mathbf{R}_+)$ satisfies (2.12), or equivalently,

$$\int_0^{1-y} \psi(x + y)\phi(x)\, dx = 0, \quad y \in [0, 1]. \tag{2.20}$$

Let k be the smallest positive integer such that there exists $\varepsilon \in (0, 1)$ satisfying $\psi\chi_{[1-\varepsilon,1]} \in C^{(k)}[1 - \varepsilon, 1]$ and $\psi^{(k-1)}(1) \neq 0$. We can choose $\varepsilon \in (0, 1)$ small enough so that

$$\varepsilon\|\psi^{(k)}\|_{L^\infty[1-\varepsilon,1]} < |\psi^{(k-1)}(1)|. \tag{2.21}$$

Differentiate the lefthand side of equation (2.20) with respect to y on $(1 - \varepsilon, 1)$ k times to get at each $y \in (1 - \varepsilon, 1)$ for which $1 - y$ is a Lebesgue point of ϕ that

$$\psi^{(k-1)}(1)\phi(1 - y) - \int_0^{1-y} \psi^{(k)}(x + y)\phi(x)\, dx = 0.$$

This equation implies that we can define a function $\varphi \in C[0,\varepsilon]$ such that

$$\varphi(y) = \phi(y), \text{ almost everywhere } y \in [0,\varepsilon]$$

and

$$\varphi(y) = \frac{1}{\psi^{(k-1)}(1)} \int_0^y \psi^{(k)}(x+1-y)\varphi(x)\,dx, \quad y \in [0,\varepsilon].$$

We observe that φ is a fixed point of the mapping $K : C[0,\varepsilon] \to C[0,\varepsilon]$ that is defined for each $h \in C[0,\varepsilon]$ as

$$(Kh)(y) := \frac{1}{\psi^{(k-1)}(1)} \int_0^y \psi^{(k)}(x+1-y)h(x)\,dx, \quad y \in [0,\varepsilon].$$

By (2.21), K is a contraction mapping, which implies that $\varphi(y) = 0$ for each $y \in [0,\varepsilon]$. It follows that $\phi(y) = 0$ almost everywhere $y \in [0,\varepsilon]$.

Equation (2.20) can then be rewritten as

$$\int_0^{1-y} \psi(x+y)\phi(x+\varepsilon)\,dx = 0, \quad y \in [\varepsilon, 1].$$

The arguments above can be repeated to show that $\phi(y) = 0$ almost everywhere $y \in [0,1]$. $\square$

Recall that a function h on $\mathbf{R}$ is *real-analytic* if it possesses derivatives of all orders and agrees with its Taylor series in a neighborhood of every point. A nontrivial real-analytic function h has the property that at each $x \in \mathbf{R}$ there exists $n \in \mathbf{Z}_+$ such that $h^{(n)}(x) \neq 0$. Proposition 2.9 is a consequence of this fact, Theorem 2.4 and Lemma 2.8.

Proposition 2.9. *Let $f, g \in L^2(\mathbf{R})$. Suppose there exists $a, b \in \mathbf{R}_+$ such that* $\operatorname{supp} \hat{f} \subseteq [-a,b]$ *and* $\hat{f}\chi_{[-a,b]}$ *is the restriction on* $[-a,b]$ *of a nontrivial real-analytic function. Then the Bedrosian identity* (1.2) *holds if and only if* $\operatorname{supp} \hat{g} \subseteq (-\infty,-b] \cup [a,\infty)$.

2.3 *Relations with right translation invariant subspaces of $L^2(\mathbf{R}_+)$.* As shown by the Bedrosian theorem, the roles of f, g in the Bedrosian identity (1.2) are different. The purpose of this subsection is to determine for what $g \in L^2(\mathbf{R})$ would there exist nontrivial $f \in L^2(\mathbf{R})$ satisfying the Bedrosian identity (1.2). The study will be based on a characterization of closed right translation invariant subspaces of $L^2(\mathbf{R}_+)$ due to Lax, [**18**]. We need to introduce the Hardy spaces on the upper half-plane, also the Hardy spaces on the unit disc for later use in Section 3, from [**12, 15, 26**].

Set $U := \{z \in \mathbf{C} : |z| < 1\}$ and $\mathbf{T} := \{z \in \mathbf{C} : |z| = 1\}$. Let $\mathbf{H}(U)$ and $\mathbf{H}(\mathbf{C}_+)$ be the set of all the holomorphic functions on U and $\mathbf{C}_+$, respectively. We introduce the Hardy spaces by setting for $0 < p < \infty$

$$\mathbf{H}^p(U) := \left\{h \in \mathbf{H}(U) : \sup\left\{\int_0^{2\pi} |h(\mathrm{r}e^{it})|^p \, dt : r \in (0,1)\right\} < \infty\right\},$$
$$\mathbf{H}^p(\mathbf{C}_+) := \left\{h \in \mathbf{H}(\mathbf{C}_+) : \sup\left\{\int_{\mathbf{R}} |h(x+iy)|^p \, dx : y > 0\right\} < \infty\right\}$$

and for $p = \infty$

$$\mathbf{H}^\infty(U) := \{h \in \mathbf{H}(U) : \sup\{|h(z)| : z \in U\} < \infty\},$$
$$\mathbf{H}^\infty(\mathbf{C}_+) := \{h \in \mathbf{H}(\mathbf{C}_+) : \sup\{|h(z)| : z \in \mathbf{C}_+\} < \infty\}.$$

If $h \in \mathbf{H}^p(U)$ or $\mathbf{H}^p(\mathbf{C}_+)$, $0 < p \le \infty$, then h has a nontangential boundary limit, which we still denote by h. For instance, an inner function on U is a function $h \in \mathbf{H}^\infty(U)$ for which $|h| = 1$ almost everywhere on $\mathbf{T}$. An interesting class of inner functions on U is the Blaschke products. Such functions are given by

$$B(z) := z^k \prod_{n \in \mathbf{N}} \frac{z_n - z}{1 - \bar{z}_n z} \frac{|z_n|}{z_n}, \quad z \in U$$

where $k \in \mathbf{Z}_+$, $\{z_n : n \in \mathbf{N}\}$ is a sequence in $U \backslash \{0\}$ such that $\sum_{n \in \mathbf{N}} (1 - |z_n|) < \infty$. A function $h \in \mathbf{H}(U)$ is called an outer function if it is of the form

$$h(z) = c \exp\left\{\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} \log \varphi(e^{it}) \, dt\right\}, \quad z \in U,$$

where $c \in \mathbf{T}$, φ is a positive measurable function on $\mathbf{T}$ such that $\log \varphi \in L^1(\mathbf{T})$. The Blaschke products, inner functions and outer functions on $\mathbf{C}_+$ are obtained from their counterparts on U through the Cayley transform

$$\mathcal{K}(w) := \frac{i-w}{i+w}, \quad w \in \mathbf{C}_+.$$

There is a canonical factorization in Hardy spaces. For each nontrivial $h \in \mathbf{H}^p(U)$, $0 < p \leq \infty$,

$$Q_h(z) := \exp\left\{\frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\log|h(e^{it})|\,dt\right\}, \ \ z \in U$$

is an outer function in $\mathbf{H}^p(U)$, and there exists a unique inner function M_h on U such that $h = M_h Q_h$. Similarly, for each nontrivial $h \in \mathbf{H}^p(\mathbf{C}_+)$, $0 < p \leq \infty$,

$$Q_h(z) := \exp\left\{\frac{1}{i\pi}\int_{\mathbf{R}} \frac{(1+tz)\log|h(t)|}{(t-z)(1+t^2)}\,dt\right\}, \ \ z \in \mathbf{C}_+$$

is an outer function in $\mathbf{H}^p(\mathbf{C}_+)$, and there exists a unique inner function M_h on $\mathbf{C}_+$ such that $h = M_h Q_h$. In both cases, we shall call Q_h, M_h in the factorization the outer factor and inner factor of h, respectively.

Let us return to the Bedrosian identity (1.2). We introduce a map $\mathcal{H}_{\mathbf{C}_+}$ from $L^2(\mathbf{R}_+)$ to $\mathbf{H}^2(\mathbf{C}_+)$ by setting for each $\phi \in L^2(\mathbf{R}_+)$

$$(\mathcal{H}_{\mathbf{C}_+}\phi)(z) := \frac{1}{\sqrt{2\pi}}\int_{\mathbf{R}_+} e^{izx}\phi(x)\,dx, \ \ z \in \mathbf{C}_+.$$

If we further impose an inner product $(\cdot,\cdot)$ on $\mathbf{H}^2(\mathbf{C}_+)$ by setting for each $h_1, h_2 \in \mathbf{H}^2(\mathbf{C}_+)$

$$(h_1, h_2) := \int_{\mathbf{R}} h_1(t)\overline{h_2(t)}\,dt,$$

then $\mathcal{H}_{\mathbf{C}_+}$ is an isomorphism from $L^2(\mathbf{R}_+)$ to $\mathbf{H}^2(\mathbf{C}_+)$. The translation operator τ is turned into a multiplication operator, which we denote by Λ, on $\mathbf{H}^2(\mathbf{C}_+)$

$$(\Lambda_y h)(z) := e^{iyz}h(z), \quad y \in \mathbf{R}_+, \ z \in \mathbf{C}_+, \quad h \in \mathbf{H}^2(\mathbf{C}_+).$$

In terms of the isomorphism $\mathcal{H}_{\mathbf{C}_+}$, the relation between the translation operator on $L^2(\mathbf{R}_+)$ and the multiplication operator on $\mathbf{H}^2(\mathbf{C}_+)$ is shown below.

$$\mathcal{H}_{\mathbf{C}_+}\tau_y = \Lambda_y \mathcal{H}_{\mathbf{C}_+}, \ \ y \in \mathbf{R}_+.$$

A subspace $\mathcal{M}$ of $\mathbf{H}^2(\mathbf{C}_+)$ is called *right translation invariant* if $\Lambda_y(\mathcal{M}) \subseteq \mathcal{M}$ for each $y \in \mathbf{R}_+$. We set for each $h \in \mathbf{H}^2(\mathbf{C}_+)$, $\Lambda(h) := \overline{\operatorname{span}}\{\Lambda_y h : y \in \mathbf{R}_+\}$ which is the smallest closed right translation invariant subspace of $\mathbf{H}^2(\mathbf{C}_+)$ that contains h. It was established in [**18**] that $\mathcal{M}$ is a closed right translation invariant subspace of $\mathbf{H}^2(\mathbf{C}_+)$ if and only if there exists an inner function $h_{\mathcal{M}} \in \mathbf{H}(\mathbf{C}_+)$ such that

$$\mathcal{M} = \{h_{\mathcal{M}}\varphi : \varphi \in \mathbf{H}^2(\mathbf{C}_+)\}.$$

Moreover, the $h_{\mathcal{M}}$ above is uniquely determined by $\mathcal{M}$, save for multiplication by a complex constant in $\mathbf{T}$. One can obtain the following result using this fact and the same arguments as those for Theorem 17.23 in [**26**, page 350].

Lemma 2.10. *Suppose M_h is the inner factor of $h \in \mathbf{H}^2(\mathbf{C}_+)$. Then*

$$\Lambda(h) = \{M_h\varphi : \varphi \in \mathbf{H}^2(\mathbf{C}_+)\}.$$

Moreover, $\Lambda(h) = \mathbf{H}^2(\mathbf{C}_+)$ if and only if h is an outer function in $\mathbf{H}^2(\mathbf{C}_+)$.

Theorem 2.11. *Let $g \in L^2(\mathbf{R})$. Then there does not exist nontrivial $f \in L^2(\mathbf{R})$ satisfying the Bedrosian identity* (1.2) *if and only if both $\mathcal{H}_{\mathbf{C}_+}(\hat{g}^+)$ and $\mathcal{H}_{\mathbf{C}_+}(\hat{g}(-\cdot)^+)$ are outer functions in $\mathbf{H}^2(\mathbf{C}_+)$.*

Proof. It suffices to show that for a given $\phi \in L^2(\mathbf{R}_+)$, there does not exist a nontrivial $\psi \in L^2(\mathbf{R}_+)$ satisfying (2.12) if and only if $\mathcal{H}_{\mathbf{C}_+}\phi$ is an outer function in $\mathbf{H}^2(\mathbf{C}_+)$. It is clear that (2.12) can be rewritten as

$$\bar{\psi} \perp \{\tau_y\phi : y \in \mathbf{R}_+\}, \ \text{ in } \ L^2(\mathbf{R}_+).$$

Since $\mathcal{H}_{\mathbf{C}_+}$ is an isomorphism, the above equation is equivalent to

$$\mathcal{H}_{\mathbf{C}_+}(\bar{\psi}) \perp \Lambda(\mathcal{H}_{\mathbf{C}_+}\phi), \ \text{ in } \ \mathbf{H}^2(\mathbf{C}_+). \tag{2.22}$$

There does not exist a nontrivial ψ satisfying (2.22) if and only if $\Lambda(\mathcal{H}_{\mathbf{C}_+}\phi) = \mathbf{H}^2(\mathbf{C}_+)$. The result now follows directly from Lemma 2.10. $\square$

If either $\mathcal{H}_{\mathbf{C}_+}(\hat{g}^+)$ or $\mathcal{H}_{\mathbf{C}_+}(\hat{g}(-\cdot)^+)$ is not an outer function then a method of constructing nontrivial $f \in L^2(\mathbf{R})$ satisfying the Bedrosian identity will be provided in subsection 2.5. To this end, we need to characterize additive positive definite kernels on $\mathbf{R}_+$.

2.4. *Additive positive definite kernels.* This subsection will make preparations for the next one where we shall show that there exists $f \in L^2(\mathbf{R})$ for which there is no nontrivial $g \in L^2(\mathbf{R})$ satisfying the Bedrosian identity (1.2) and give a method of constructing $f \in L^2(\mathbf{R})$ from a given $g \in L^2(\mathbf{R})$ so that f, g would satisfy identity (1.2). These will be done through the additive positive definite kernels on $\mathbf{R}_+$.

We first explain our motivation. For each $\psi \in L^1(\mathbf{R}_+)$ we define an operator L_ψ on $L^2(\mathbf{R}_+)$ by setting for each $\phi \in L^2(\mathbf{R}_+)$

$$L_\psi(\phi) := \int_{\mathbf{R}_+} \psi(\cdot + x)\phi(x)\,dx.$$

Then ϕ satisfies the homogeneous semi-convolution equation (2.12) if and only if $L_\psi(\phi) = 0$. If L_ψ is a strictly positive self-adjoint operator on $L^2(\mathbf{R}_+)$ then there does not exist a nontrivial $\phi \in L^2(\mathbf{R}_+)$ such that $L_\psi(\phi) = 0$. Suppose we have ψ_1, ψ_2 such that L_{ψ_1}, L_{ψ_2} are positive self-adjoint, then $L_{\psi_1} + L_{\psi_2} = L_{\psi_1+\psi_2}$ has more chance to become strictly positive. This suggests that we should look for $\psi \in L^1(\mathbf{R}_+)$ such that L_ψ is positive self-adjoint and use them to construct strictly positive self-adjoint operators on $L^2(\mathbf{R}_+)$. We shall study the construction in a general setting.

With $\mathcal{X}$ being $\mathbf{R}_+$ or $\mathbf{Z}_+$ in mind, we let $\mathcal{X}$ be a locally compact metric space with a Borel measure dx on $\mathcal{X}$ such that every nonempty open subset of $\mathcal{X}$ has nonzero measure and every compact subset of $\mathcal{X}$ has finite measure. Suppose there exists a sequence of compact subsets $\{\mathcal{X}_n \subseteq \mathcal{X} : n \in \mathbf{N}\}$ such that $\mathcal{X} = \cup_{n\in\mathbf{N}}\mathcal{X}_n$ and $\mathcal{X}_n \subseteq \mathcal{X}_{n+1}, n \in \mathbf{N}$. We also suppose that there exists a continuous commutative operator $+ : \mathcal{X} \times \mathcal{X} \to \mathcal{X}$. The image of this operator at each $(x, y) \in \mathcal{X} \times \mathcal{X}$ will be denoted by $x + y$. We require that there exist an element of $\mathcal{X}$, denoted by 0, such that $0 + x = x$ for each $x \in \mathcal{X}$.

Recall that $C_0(\mathcal{X})$ is the space of all the continuous functions f on $\mathcal{X}$ such that for each $\varepsilon > 0$, the set $\{x \in \mathcal{X} : |f(x)| \geq \varepsilon\}$ is compact. We point out that a continuous function $f \in C_0(\mathcal{X})$ if and only if for each sequence $\{x_n : n \in \mathbf{N}\} \subseteq \mathcal{X}$ that has no clustering points

$$\lim_{n\to\infty} f(x_n) = 0.$$

Denote by $\mathcal{B}(\mathcal{X})$ the set of all the regular complex Borel measures on $\mathcal{X}$ and $\mathcal{B}_+(\mathcal{X})$ the set of all the finite regular positive Borel measures on $\mathcal{X}$. It is well known that the dual of $C_0(\mathcal{X})$ is $\mathcal{B}(\mathcal{X})$, namely, $C_0(\mathcal{X})^* = \mathcal{B}(\mathcal{X})$. We shall make use of the fact that a continuous functional T on $C_0(\mathcal{X})$ satisfies $T(f) \geq 0$ for each $f \in C_0(\mathcal{X})$ with $f \geq 0$ if and only if there exists $\nu \in \mathcal{B}_+(\mathcal{X})$ such that for each $f \in C_0(\mathcal{X})$, see e.g., [**26**, pages 40–41]

$$T(f) = \int_{\mathcal{X}} f(x)\, d\nu(x).$$

Finally, we assume that there is a function $E : \mathcal{X} \times \mathcal{X} \to \mathbf{R}$ such that for all $x, y, y_1, y_2 \in \mathcal{X}$

$$E(x,y) = E(y,x), \quad E(x, y_1 + y_2) = E(x, y_1)E(x, y_2),$$

and that the class of functions $E_x := E(x,\cdot)$, $x \in \mathcal{X}$, is uniformly bounded and has the properties that $E_0 = 1$, span $\{E_x : x \in \mathcal{X}, x \neq 0\}$ is dense in $C_0(\mathcal{X})$.

A continuous function $f : \mathcal{X} \times \mathcal{X} \to \mathbf{C}$ is called a *positive definite kernel* ([**19**]) on $\mathcal{X}$ if for each sequence $\{x_j : j \in \mathbf{N}_n\} \subseteq \mathcal{X}$ the matrix $\{f(x_j, x_k) : j, k \in \mathbf{N}_n\}$ is positive semi-definite. We call $f \in C(\mathcal{X})$ an *additive positive definite kernel* if $f(\cdot + \cdot)$ is a positive definite kernel on $\mathcal{X}$. Let $F(\mathcal{X}) := \{c + g : c \in \mathbf{R}_+, g \in C_0(\mathcal{X})\}$ and $P(\mathcal{X})$ be the set of all the functions in $F(\mathcal{X})$ that are additive positive definite kernels on $\mathcal{X}$.

We start the characterization of additive positive definite kernels on $\mathcal{X}$ with an extension of a result of Mercer [**19**].

Lemma 2.12. *A function $f \in F(\mathcal{X})$ is an additive positive definite kernel if and only if for each $g \in L^1(\mathcal{X}, dx)$*

$$\int_{\mathcal{X}}\int_{\mathcal{X}} f(x+y) g(x)\overline{g(y)}\, dx\, dy \geq 0. \tag{2.23}$$

Proof. By Lemma 1 in [**5**], $f \in C(\mathcal{X})$ is an additive positive definite kernel if and only if for each compact set $\mathcal{Z} \subseteq \mathcal{X}$ and $g \in L^1(\mathcal{Z}, dx)$

$$\int_{\mathcal{Z}} \int_{\mathcal{Z}} f(x+y) g(x) \overline{g(y)}\, dx\, dy \geq 0.$$

Therefore, if $f \in F(\mathcal{X})$ satisfies (2.23) for each $g \in L^1(\mathcal{X}, dx)$ then $f \in P(\mathcal{X})$. On the other hand, suppose $f \in P(\mathcal{X})$. Noting that for each $g \in L^1(\mathcal{X}, dx)$

$$\int_{\mathcal{X}} \int_{\mathcal{X}} f(x+y) g(x) \overline{g(y)}\, dx\, dy = \lim_{n \to \infty} \int_{\mathcal{X}_n} \int_{\mathcal{X}_n} f(x+y) g(x) \overline{g(y)}\, dx\, dy$$

proves the lemma. $\square$

Lemma 2.13. *Let $\nu \in \mathcal{B}(\mathcal{X})$. Then the function*

$$f_\nu := \int_{\mathcal{X}} E(\cdot, y)\, d\nu(y)$$

belongs to $P(\mathcal{X})$ if and only if $\nu \in \mathcal{B}_+(\mathcal{X})$.

Proof. Suppose $\nu \in \mathcal{B}_+(\mathcal{X})$ then for each sequence $\{x_j : j \in \mathbf{N}_n\} \subseteq \mathcal{X}$ and $\{c_j : j \in \mathbf{N}_n\} \subseteq \mathbf{C}$

$$\begin{aligned}\sum_{j,k \in \mathbf{N}_n} c_j \bar{c}_k f_\nu(x_j + x_k) &= \int_{\mathcal{X}} \sum_{j,k \in \mathbf{N}_n} c_j \bar{c}_k E(x_j, y) E(x_k, y)\, d\nu(y) \\ &= \int_{\mathcal{X}} |\sum_{j \in \mathbf{N}_n} c_j E(x_j, y)|^2\, d\nu(y) \geq 0.\end{aligned}$$

Since

$$f_\nu = \nu(\{0\}) + \int_{\mathcal{X} \setminus \{0\}} E(\cdot, y)\, d\nu(y),$$

to show that $f_\nu \in P(\mathcal{X})$ it suffices to show that

$$\int_{\mathcal{X} \setminus \{0\}} E(\cdot, y)\, d\nu(y) \in C_0(\mathcal{X}).$$

Let $\{x_n : n \in \mathbf{N}\} \subseteq \mathcal{X}$ be a sequence without clustering points. Since E_{x_n}, $n \in \mathbf{N}$, are uniformly bounded and for each $y \in \mathcal{X} \setminus \{0\}$

$$\lim_{n\to\infty} E(x_n, y) = 0,$$

by the Lebesgue dominated convergence theorem [**26**, page 26],

$$\lim_{n\to\infty} \int_{\mathcal{X}\setminus\{0\}} E(x_n, y)\, d\nu(y) = 0.$$

We conclude that $f_\nu \in P(\mathcal{X})$ if $\nu \in \mathcal{B}_+(\mathcal{X})$.

On the other hand, suppose that $f_\nu \in P(\mathcal{X})$. For each $g \in C_0(\mathcal{X})$ with $g \geq 0$, by the density of span $\{E_x : x \in \mathcal{X}\setminus\{0\}\}$ in $C_0(\mathcal{X})$, for each $\varepsilon > 0$ we can choose $\{x_j : j \in \mathbf{N}_n\} \subseteq \mathcal{X}\setminus\{0\}$ and $\{c_j : j \in \mathbf{N}_n\} \subseteq \mathbf{R}$ such that

$$\|\sqrt{g} - \sum_{j\in\mathbf{N}_n} c_j E(x_j, \cdot)\|_{L^\infty(\mathcal{X},dx)} < \varepsilon.$$

Therefore,

$$\int_{\mathcal{X}} g(x) - \Bigg(\sum_{j\in\mathbf{N}_n} c_j E(x_j, x)\Bigg)^2 d\nu(x) \geq -\varepsilon\|\nu\|(2\|\sqrt{g}\|_{L^\infty(\mathcal{X},dx)} + \varepsilon),$$

where $\|\nu\|$ is the total variance of ν on $\mathcal{X}$. By the calculation that

$$\begin{aligned}\int_{\mathcal{X}} \Bigg(\sum_{j\in\mathbf{N}_n} c_j E(x_j, x)\Bigg)^2 d\nu(x) &= \int_{\mathcal{X}} \sum_{j,k\in\mathbf{N}_n} c_j c_k E(x_j + x_k, x)\, d\nu(x)\\ &= \sum_{j,k\in\mathbf{N}_n} c_j c_k f_\nu(x_j + x_k) \geq 0,\end{aligned}$$

we have

$$\int_{\mathcal{X}} g(x)\, d\nu(x) \geq 0. \tag{2.24}$$

The measure ν hence has the property that for each $g \in C_0(\mathcal{X})$ with $g \geq 0$, (2.24) holds. We conclude that $\nu \in \mathcal{B}_+(\mathcal{X})$ and prove the lemma. $\square$

Lemma 2.14. *The set $\{f_\nu : \nu \in \mathcal{B}_+(\mathcal{X})\}$ is closed under uniformly convergence. Namely, if f_{ν_n} where $\nu_n \in \mathcal{B}_+(\mathcal{X})$, $n \in \mathbf{N}$, converges uniformly to $f \in C(\mathcal{X})$ then there exists some $\nu \in \mathcal{B}_+(\mathcal{X})$ such that $f = \int_{\mathcal{X}} E(\cdot, y)\, d\nu(y)$.*

Proof. Suppose that $\{f_{\nu_n} : \nu_n \in \mathcal{B}_+(\mathcal{X}), n \in \mathbf{N}\}$ converges to $f \in C(\mathcal{X})$ uniformly. Firstly,

$$\lim_{n\to\infty} f_{\nu_n}(0) = \lim_{n\to\infty} \|\nu_n\| = f(0).$$

This equation means that $\|\nu_n\|$, $n \in \mathbf{N}$, are bounded. The Alaoglu theorem states that the closed unit ball of $\mathcal{B}(\mathcal{X})$ is compact in the weak-star topology. There hence exists a subsequence $\{n_j : j \in \mathbf{N}\} \subseteq \mathbf{N}$ and a $\nu \in \mathcal{B}(\mathcal{X})$ such that for each $g \in C_0(\mathcal{X})$

$$\lim_{j\to\infty} \int_{\mathcal{X}} g(x)\, d\nu_{n_j}(x) = \int_{\mathcal{X}} g(x)\, d\nu(x).$$

We hence have for each $x \in \mathcal{X}\backslash\{0\}$ that

$$f(x) = \int_{\mathcal{X}} E(x, y)\, d\nu(y). \tag{2.25}$$

By the continuity of both sides of the equation above, (2.25) holds for all $x \in \mathcal{X}$. Since for each sequence $\{x_j : j \in \mathbf{N}_m\} \subseteq \mathcal{X}$ and $\{c_j : j \in \mathbf{N}_m\} \subseteq \mathbf{C}$

$$\sum_{j,k\in\mathbf{N}_m} c_j \bar{c}_k f(x_j + x_k) = \lim_{n\to\infty} \sum_{j,k\in\mathbf{N}_m} c_j \bar{c}_k f_{\nu_n}(x_j + x_k) \geq 0,$$

f is an additive positive definite kernel on $\mathcal{X}$. By the second part of the proof of Lemma 2.13, we have $\nu \in \mathcal{B}_+(\mathcal{X})$, which completes the proof. $\square$

Theorem 2.15. *A function $f \in F(\mathcal{X})$ is an additive positive kernel on $\mathcal{X}$ if and only if there exists $\nu \in \mathcal{B}_+(\mathcal{X})$ such that*

$$f = \int_{\mathcal{X}} E(\cdot, y)\, d\nu(y). \tag{2.26}$$

Proof. By Lemma 2.13, if there exists $\nu \in \mathcal{B}_+(\mathcal{X})$ such that f is given by (2.26) then $f \in P(\mathcal{X})$.

Now suppose $f \in F(\mathcal{X})$ is an additive positive definite kernel on $\mathcal{X}$. Since $\{f + (1/m) : m \in \mathbf{N}\}$ converges uniformly to f, by Lemma 2.14, it suffices to show that for each $m \in \mathbf{N}$ there exists $\nu_m \in \mathcal{B}_+(\mathcal{X})$ such that

$$f + \frac{1}{m} = \int_{\mathcal{X}} E(\cdot, y)\, d\nu_m(y).$$

Let $g := f + (1/m)$. Since span $\{E_x : x \in \mathcal{X} \backslash \{0\}\}$ is dense in $C_0(\mathcal{X})$, the set $\{f_\nu : \nu \in \mathcal{B}(\mathcal{X})\}$ is dense in $F(\mathcal{X})$ with respect to the norm of $L^\infty(\mathcal{X}, dx)$. As a consequence, we have a sequence $\{f_{\nu_n} : \nu_n \in \mathcal{B}(\mathcal{X})\}$ that is convergent to g in $L^\infty(\mathcal{X}, dx)$. For each $h \in L^1(\mathcal{X}, dx)$ we have

$$\int_{\mathcal{X}}\int_{\mathcal{X}} f_{\nu_n}(x+y)h(x)\overline{h(y)}\, dx\, dy - \int_{\mathcal{X}}\int_{\mathcal{X}} g(x+y)h(x)\overline{h(y)}\, dx\, dy \\ \geq -\|f_{\nu_n} - g\|_{L^\infty(\mathcal{X}, dx)} \|h\|^2_{L^1(\mathcal{X}, dx)}.$$

Note that

$$\int_{\mathcal{X}}\int_{\mathcal{X}} g(x+y)h(x)\overline{h(y)}\, dx\, dy \geq \frac{1}{m}\|h\|^2_{L^1(\mathcal{X}, dx)}.$$

By Lemma 2.12, there exists $N \in \mathbf{N}$ such that for each $n > N$, f_{ν_n} is an additive positive definite kernel on $\mathcal{X}$. This implies, by Lemma 2.13, $\nu_n \in \mathcal{B}_+(\mathcal{X})$ for $n > N$. By Lemma 2.14, there exists $\nu_m \in \mathcal{B}_+(\mathcal{X})$ such that

$$g = \int_{\mathcal{X}} E(\cdot, y)\, d\nu_m(y).$$

This equation completes the proof. $\square$

We shall use Theorem 2.15 to characterize additive positive definite kernels in $F(\mathbf{R}_+)$ and $F(\mathbf{Z}_+)$. To this end, we remark that by Theorem 6.2 in [**30**, Chapter 2], span $\{e^{-xt} : x > 0\}$ and span $\{E_n : n \in \mathbf{N}\}$ is dense in $C_0(\mathbf{R}_+)$ and $C_0(\mathbf{Z}_+)$ respectively, where for each $n \in \mathbf{N}$, $E_n := \{e^{-nm} : m \in \mathbf{Z}_+\}$.

Theorem 2.16. *A function $f \in P(\mathbf{Z}_+)$ if and only if there exists $\lambda := \{\lambda_m : m \in \mathbf{Z}_+\} \in \ell^1(\mathbf{Z}_+)$ with $\lambda_m \geq 0, m \in \mathbf{Z}_+$ such that*

$$f(n) = \sum_{m \in \mathbf{Z}_+} \lambda_m e^{-nm}, \quad n \in \mathbf{Z}_+. \tag{2.27}$$

Theorem 2.17. *A continuous function f is a bounded additive positive definite kernel on $\mathbf{R}_+$ if and only if there exists $\nu \in \mathcal{B}_+(\mathbf{R}_+)$ such that*

$$f(x) = \int_{\mathbf{R}_+} e^{-xt}\, d\nu(t), \quad x \in \mathbf{R}_+. \tag{2.28}$$

Proof. By Theorem 2.15, it suffices to show that if f is a bounded additive positive definite kernel on $\mathbf{R}_+$ then $f \in P(\mathbf{R}_+)$. We assume that f is not the zero function on $\mathbf{R}_+$. Since f is an additive positive definite kernel on $\mathbf{R}_+$, we have for each $x, y \in \mathbf{R}_+$ that $f(x) \geq 0$ and

$$f(2x)f(2y) \geq f^2(x+y). \tag{2.29}$$

We claim that f has no zero on $\mathbf{R}_+$. Otherwise, suppose there exists $x_0 \in \mathbf{R}_+$ such that $f(x_0) = 0$. Then we set $y = 0$ and $x = x_0/2^n$ for $n \in \mathbf{N}$ successively in (2.29) to find that $f(x_0/2^n) = 0$ for each $n \in \mathbf{N}$, which implies by the continuity of f that $f(0) = 0$. By (2.29), $f(y) = 0$ for each $y \in \mathbf{R}_+$, contradicting with the assumption that f is nontrivial. Therefore, (2.29) can be rewritten as

$$\log f(2x) + \log f(2y) \geq 2 \log f(x+y). \tag{2.30}$$

For all $x_0 \in \mathbf{R}_+$ and $\delta > 0$, setting $x = (x_0 + 2\delta)/2$ and $y = x_0/2$ in (2.30) yields that

$$\log f(x_0 + 2\delta) - \log f(x_0 + \delta) \geq \log f(x_0 + \delta) - \log f(x_0).$$

We deduce from the above inequality for each $n \geq 2$ that

$$\log f(x_0 + n\delta) \geq (n-1)[\log f(x_0 + \delta) - \log f(x_0)] + \log f(x_0 + \delta).$$

Since f is bounded, $\log f(x_0 + \delta) - \log f(x_0) \leq 0$. This means that f is nonincreasing on $\mathbf{R}_+$. There hence exists $c \in \mathbf{R}_+$ such that

$$\lim_{x \to \infty} f(x) = c.$$

To complete the proof, we decompose f into $f = c + (f - c)$ with $f - c \in C_0(\mathbf{R}_+)$. $\square$

Theorem 2.17 shows that bounded additive positive definite kernels on $\mathbf{R}_+$ turn out to be the only functions $f \in C(\mathbf{R}_+)$ such that $f(\|\cdot - \cdot\|^2)$ is a positive definite kernel on $\mathbf{R}^d$ for all $d \in \mathbf{N}$, [**27**]. Here $\| \cdot - \cdot \|$ is the distance on $\mathbf{R}^d$.

We end up this subsection with a discussion on existing results on additive positive definite kernels on $\mathbf{Z}_+$ and $\mathbf{R}$, [**30**]. The problem of Hanburger is to determine for what $f \in C(\mathbf{Z}_+)$ will there exist a nondecreasing function ψ such that

$$f(n) = \int_{\mathbf{R}} t^n d\psi(t). \tag{2.31}$$

It was proved in [**30**] that there exists a nondecreasing function ψ such that (2.31) holds if and only if f is an additive positive definite kernel on $\mathbf{Z}_+$. Another result ([**4, 29, 30**]) states that for all $-\infty \le a < b \le \infty$, f is continuous on (a, b) and satisfies for all $a < a' < b' < b$ and real functions $g \in C[(a'/2), (b'/2)]$ that

$$\int_{a'/2}^{b'/2} \int_{a'/2}^{b'/2} f(x+y) g(x) g(y)\, dx\, dy \ge 0$$

if and only if f can be represented in the form

$$f(x) = \int_{\mathbf{R}} e^{-xt}\, d\psi(t), \quad x \in \mathbf{R},$$

where ψ is nondecreasing and the above integral converges for $a < x < b$. It is difficult to prove Theorems 2.16 and 2.17 from these two results or to conjecture form (2.27) from (2.31). Finally, we remark that our methods of proof are different from those in [**4, 29, 30**].

2.5 *Studying the Bedrosian identity with kernels in* $P(\mathbf{R}_+)$. Let us return to the homogeneous semi-convolution equation

$$\int_{\mathbf{R}_+} \psi(x+y)\phi(x)\, dx = 0, \quad y \in \mathbf{R}_+. \tag{2.32}$$

We shall set $\psi \in P(\mathbf{R}_+) \cap L^1(\mathbf{R}_+)$. This suggests that $\psi \in C_0(\mathbf{R}_+)$ and hence $\psi \in L^2(\mathbf{R}_+)$. By Theorem 2.17, ψ must have the form

$$\psi(x) = \int_{\mathbf{R}_+} e^{-xt}\, d\nu(t), \quad x \in \mathbf{R}_+, \tag{2.33}$$

where $\nu \in \mathcal{B}_+(\mathbf{R}_+)$. By Tonelli's theorem [**14**, page 67], we impose the requirement that

$$\int_{\mathbf{R}_+} \frac{d\nu(t)}{t} < \infty \tag{2.34}$$

to make $\psi \in L^1(\mathbf{R}_+)$. This requirement implies that $\nu(\{0\}) = 0$.

Our proof for the next result is adapted from [**21**]. Recall that the support of a Borel measure ν on a topological space Z, denoted by $\operatorname{supp}\nu$, is defined as

$$\operatorname{supp}(\nu) := \bigcap \left\{S \subseteq Z : S \text{ is closed}, \ \nu(S^C) = 0\right\}.$$

Lemma 2.18. *Let ψ be defined by* (2.33) *with ν satisfying condition* (2.34). *Then*

$$\tau^*(\psi) = \overline{\operatorname{span}}\,\{e^{-xt} : t \in \operatorname{supp}(\nu)\backslash\{0\}\}.$$

Proof. It suffices to show that $g \in L^2(\mathbf{R}_+)$ satisfies $g \perp \tau^*(\psi)$ if and only if $g \perp e^{-xt}$ for each $t \in \operatorname{supp}(\nu)\backslash\{0\}$. Suppose for each $t \in \operatorname{supp}(\nu)\backslash\{0\}$ that

$$\int_{\mathbf{R}_+} g(x)e^{-xt}\,dx = 0. \tag{2.35}$$

Then we observe by the Fubini theorem that for each $y \in \mathbf{R}_+$

$$\int_{\mathbf{R}_+} g(x)\psi(x+y)\,dx = \int_{\mathbf{R}_+} e^{-yt}\,d\nu(t) \int_{\mathbf{R}_+} g(x)e^{-xt}\,dx = 0. \tag{2.36}$$

Conversely, suppose for each $y \in \mathbf{R}_+$ there holds (2.36). Integrating the second term in (2.36) with respect to $\overline{g(y)}dy$ on $\mathbf{R}_+$ and using the Fubini theorem yields that

$$\int_{\mathbf{R}_+} \left| \int_{\mathbf{R}_+} g(x)e^{-xt}\,dx \right|^2 d\nu(t) = 0. \tag{2.37}$$

Since $\int_{\mathbf{R}_+} g(x)e^{-xt}\,dx$ is continuous on $(0,\infty)$, by the definition of $\operatorname{supp}\nu$, equation (2.35) holds true for each $t \in \operatorname{supp}(\nu)\backslash\{0\}$. The proof is complete. $\square$

The following results follow immediately from Lemma 2.18.

Lemma 2.19. *Let ψ be defined by* (2.33) *with ν satisfying* (2.34). *Then $\phi \in L^2(\mathbf{R}_+)$ satisfies the homogeneous semi-convolution equation* (2.32) *if and only if $\mathcal{L}\phi$ vanishes on* $\operatorname{supp}(\nu)\backslash\{0\}$.

Lemma 2.19 suggests a method to construct $f \in L^2(\mathbf{R})$ satisfying the Bedrosian identity (1.2) from a given $g \in L^2(\mathbf{R})$ for which either $\mathcal{H}_{\mathbf{C}_+}(\hat{g}^+)$ or $\mathcal{H}_{\mathbf{C}_+}(\hat{g}(-\cdot)^+)$ is not an outer function in $\mathbf{H}^2(\mathbf{C}_+)$. Let g be a specified function in $L^2(\mathbf{R})$. First determine the two sets $A_+ := \{t \in \mathbf{R}_+\backslash\{0\} : \mathcal{L}(\hat{g}^+)(t) = 0\}$ and $A_- := \{t \in \mathbf{R}_+\backslash\{0\} : \mathcal{L}(\hat{g}(-\cdot)^+)(t) = 0\}$, then find two measures $\nu_+, \nu_- \in \mathcal{B}_+(\mathbf{R}_+)$ satisfying (2.34) that are supported on A_- and A_+, respectively. The function f satisfying (1.2) is finally given by

$$\hat{f}(\xi) := c_1 u(\xi)\int_{\mathbf{R}_+} e^{-\xi t}\,d\nu_+(t) + c_2 u(-\xi)\int_{\mathbf{R}_+} e^{\xi t} d\,\nu_-(t), \quad \xi \in \mathbf{R},$$

or equivalently,

$$f(x) = \frac{c_1}{2\pi}\int_{\mathbf{R}_+} \frac{1}{t - ix}\,d\nu_+(t) + \frac{c_2}{2\pi}\int_{\mathbf{R}_+} \frac{1}{t + ix}\,d\nu_-(t), \quad x \in \mathbf{R},$$

where c_1, c_2 are arbitrary complex constants. Example (2.19) falls into this kind of construction.

We next give examples of f for each of which there does not exist nontrivial $g \in L^2(\mathbf{R})$ satisfying the Bedrosian identity (1.2). For this purpose, we introduce the concept of uniqueness subsets of $\mathbf{C}_R$, which are subsets $A \subseteq \mathbf{C}_R$ such that there does not exist a nontrivial holomorphic function on $\mathbf{C}_R$ that vanishes on A. For example, if A has a clustering point in $\mathbf{C}_R$ then A is a uniqueness subset of $\mathbf{C}_R$.

Theorem 2.20. *Let f be given by*

$$f(x) := \int_{\mathbf{R}_+} \frac{1}{t - ix}\,d\nu_+(t) + \int_{\mathbf{R}_+} \frac{1}{t + ix}\,d\nu_-(t), \quad x \in \mathbf{R},$$

where $\nu_+, \nu_- \in \mathcal{B}_+(\mathbf{R}_+)$ *satisfy condition* (2.34). *If both* $\operatorname{supp}(\nu_+)$ *and* $\operatorname{supp}(\nu_-)$ *are uniqueness subsets of* $\mathbf{C}_R$ *then there does not exist nontrivial* $g \in L^2(\mathbf{R})$ *satisfying the Bedrosian identity* (1.2).

Proof. Since ν_+, ν_- satisfy (2.34), we have

$$\hat{f}(\xi) = 2\pi u(\xi) \int_{\mathbf{R}_+} e^{-\xi t}\, d\nu_+(t) + 2\pi u(-\xi) \int_{\mathbf{R}_+} e^{\xi t}\, d\nu_-(t), \quad \xi \in \mathbf{R}.$$

By Lemma 2.19, $g \in L^2(\mathbf{R})$ satisfies (1.2) if and only if $\mathcal{L}(\hat{g}(-\cdot)^+)$ and $\mathcal{L}(\hat{g}^+)$ vanish on $\operatorname{supp}(\nu_+)\backslash\{0\}$ and $\operatorname{supp}(\nu_-)\setminus\{0\}$, respectively. Since $\mathcal{L}(\hat{g}(-\cdot)^+)$ and $\mathcal{L}(\hat{g}^+)$ are holomorphic on $\mathbf{C}_R$, this is possible if and only if $g = 0$. ◻

3. The circular Bedrosian identity in $L^2_{2\pi}$. We study in this section the circular Bedrosian identity (1.8) where $f, g \in L^2_{2\pi}$.

3.1. *A necessary and sufficient condition.* Similar results as Lemmas 2.1 and 2.2 hold for the circular Hilbert transform [**9**, Chapter 9].

Lemma 3.1. *There exists a constant* c *such that for each* $f \in L^1_{2\pi}$ *and* $y > 0$

$$\mu\{x \in [0, 2\pi] : |(\tilde{H}f)(x)| > y\} \le \frac{c}{y}\|f\|_{L^1_{[0,2\pi]}}.$$

Lemma 3.2. *If* $f \in L^p_{2\pi}$, $1 < p < \infty$, *then* $\tilde{H}f \in L^p_{2\pi}$, $\tilde{H}^2 f = -f + c_0(f)$ *almost everywhere and the Fourier coefficients of* $\tilde{H}f$ *are given by*

$$c_k(\tilde{H}f) = -i\,\operatorname{sgn}(k) c_k(f), \quad k \in \mathbf{Z}. \tag{3.1}$$

For $p = 1$, *formula* (3.1) *remains true under the additional assumption that* $\tilde{H}f \in L^1_{2\pi}$.

The same arguments as those for Lemma 2.3 are able to yield the following result based on Lemmas 3.1 and 3.2.

Lemma 3.3. *Let $f \in L^1_{2\pi}$. Then $\widetilde{H}f \in L^1_{2\pi}$ if and only if there exists $g \in L^1_{2\pi}$ such that*

$$-i\ \mathrm{sgn}(k)c_k(f) = c_k(g), \quad k \in \mathbf{Z}. \tag{3.2}$$

Moreover, if (3.2) *holds then $\widetilde{H}f = g$ almost everywhere.*

We shall give a characterization of the circular Bedrosian identity (1.8). Let $\mathbf{I}$ be a countable index set. An element $\psi \in \ell^2(\mathbf{I})$ will be viewed as a continuous function on $\mathbf{I}$. The kth component of ψ is denoted by $\psi(k)$ for each $k \in \mathbf{I}$. For convenience, we also introduce a map $\widetilde{F}$ from $L^2_{2\pi}$ to $\ell^2(\mathbf{Z})$ by setting for each $f \in L^2_{2\pi}$

$$(\widetilde{F}f)(k) := c_k(f), \ \ k \in \mathbf{Z}$$

and two adjoint maps $\widetilde{F}_+, \widetilde{F}_-$ from $L^2_{2\pi}$ to $\ell^2(\mathbf{Z}_+)$ by setting

$$(\widetilde{F}_+f)(0) := \frac{c_0(f)}{2}, \quad (\tilde{F}_+f)(k) := c_k(f), \ \ k \in \mathbf{N}$$

and

$$(\widetilde{F}_-f)(0) := \frac{c_0(f)}{2}, \ \ (\widetilde{F}_-f)(k) := c_{-k}(f), \ \ k \in \mathbf{N}.$$

The shift operator S on $\ell^2(\mathbf{Z}_+)$ is defined for each $\psi \in \ell^2(\mathbf{Z}_+)$ as

$$(S\psi)(0) := 0, \quad (S\psi)(k) := \psi(k-1), \ \ k \in \mathbf{N}.$$

The adjoint operator S^* of S is hence given for each $\psi \in \ell^2(\mathbf{Z}_+)$ by

$$(S^*\psi)(k) := \psi(k+1), \ \ k \in \mathbf{Z}_+.$$

Using Lemmas 3.2, 3.3 and similar arguments as those in the proof of Theorem 2.4, we get a characterization of the circular Bedrosian identity.

Theorem 3.4. *Let $f, g \in L^2_{2\pi}$. Then the circular Bedrosian identity* (1.8) *holds if and only if f, g satisfy the following three equations*:

$$\sum_{j\in\mathbf{Z}_+} (\widetilde{\mathcal{F}}_+f)(j)(\widetilde{\mathcal{F}}_-g)(j) = \sum_{j\in\mathbf{Z}_+} (\widetilde{\mathcal{F}}_-f)(j)(\widetilde{\mathcal{F}}_+g)(j), \tag{3.3}$$

$$\sum_{j\in\mathbf{Z}_+} (S^{*k}\widetilde{\mathcal{F}}_+f)(j)(\widetilde{\mathcal{F}}_-g)(j) = 0, \ \ k \in \mathbf{N}, \tag{3.4}$$

$$\sum_{j\in\mathbf{Z}_+} (S^{*k}\widetilde{\mathcal{F}}_-f)(j)(\widetilde{\mathcal{F}}_+g)(j) = 0, \ \ k \in \mathbf{N}. \tag{3.5}$$

3.2. *Particular solutions.* We start the discussion of this subsection by making a simple observation based on Theorem 3.4. Theorem B in the introduction says that if $f \in L^2_{2\pi}$ is of low Fourier frequency and $g \in L^2_{2\pi}$ is of high Fourier frequency, then they satisfy the circular Bedrosian identity (1.8). Our next result shows that if f is of low Fourier frequency, then it is necessary for $g \in L^2_{2\pi}$ to have high Fourier frequency in order to satisfy the circular Bedrosian identity.

Proposition 3.5. *Let $f, g \in L^2_{2\pi}$. If there exists $k_1, k_2 \in \mathbf{N}$ such that $\operatorname{supp} \widetilde{\mathcal{F}} f \subseteq [-k_1, k_2]$ and $(\widetilde{\mathcal{F}} f)(-k_1)(\widetilde{\mathcal{F}} f)(k_2) \neq 0$, then the circular Bedrosian identity* (1.8) *holds if and only if*

$$\operatorname{supp} \widetilde{\mathcal{F}} g \subseteq (-\infty, -k_2] \cup [k_1, \infty) \tag{3.6}$$

and

$$(\widetilde{\mathcal{F}} f)(k_2)(\widetilde{\mathcal{F}} g)(-k_2) - (\widetilde{\mathcal{F}} f)(-k_1)(\widetilde{\mathcal{F}} g)(k_1) = 0. \tag{3.7}$$

Proof. It can be verified directly that if (3.6) and (3.7) hold, then equations (3.3), (3.4) and (3.5) are satisfied. Conversely, suppose $\widetilde{H}(fg) = f \widetilde{H} g$ almost everywhere, or equivalently, equations (3.3), (3.4) and (3.5) hold. We substitute $k = k_2 - j$ for $j = 0, \dots, k_2 - 1$ successively into equation (3.4) to find that $(\widetilde{\mathcal{F}} g)(-j) = 0$ for $j = 0, \dots, k_2 - 1$. Similarly, $(\widetilde{\mathcal{F}} g)(j) = 0$ for $j = 0, \dots, k_1 - 1$. Equation (3.7) is now a direct consequence of equation (3.3). ◻

We impose in this subsection the requirement that

$$(\widetilde{F}_+ f, \overline{\widetilde{F}_- g}) = (\widetilde{F}_- f, \overline{\widetilde{F}_+ g}) = 0,$$

in order to satisfy (3.3), where $(\cdot, \cdot)$ is the inner product on $\ell^2(\mathbf{Z}_+)$, $\bar{\phi}$ denotes the conjugate of $\phi \in \ell^2(\mathbf{Z}_+)$. As a consequence, it suffices to solve the homogeneous discrete semi-convolution equations below to find $f, g \in L^2_{2\pi}$ satisfying the circular Bedrosian identity (1.8)

$$(S^{*k} \widetilde{F}_+ f, \overline{\widetilde{F}_- g}) = 0, \quad (S^{*k} \widetilde{F}_- f, \overline{\widetilde{F}_+ g}) = 0, \ \ k \in \mathbf{Z}_+.$$

Therefore, we shall consider the following general homogeneous discrete semi-convolution equation

$$(S^{*k}\psi, \bar{\phi}) = 0, \quad k \in \mathbf{Z}_+, \tag{3.8}$$

where $\psi, \phi \in \ell^2(\mathbf{Z}_+)$. Let ψ be a specified element in $\ell^2(\mathbf{Z}_+)$. We set

$$S^*(\psi) := \overline{\text{span}}\,\{S^{*k}\psi : k \in \mathbf{Z}_+\}.$$

There exists nontrivial $\phi \in l^2(\mathbf{Z}_+)$ satisfying (3.8) if and only if $S^*(\psi)$ is not dense in $\ell^2(\mathbf{Z}_+)$. Theorem 3.6 gives a characterization for $S^*(\psi)$ to be finite dimensional.

Theorem 3.6. *Let $\psi \in \ell^2(\mathbf{Z}_+)$. Then $S^*(\psi)$ is finite dimensional if and only if ψ is given by*

$$\psi(n) = \varphi(n) + \sum_{j \in \mathbf{N}_p} \sum_{k \in \mathbf{Z}_{n_j}} c_{jk} n^k \lambda_j^n, \ ; \ n \in \mathbf{Z}_+, \tag{3.9}$$

where $\varphi \in \ell^2(\mathbf{Z}_+)$ is of finite support, $p \in \mathbf{N}$, $\{\lambda_j : j \in \mathbf{N}_p\} \subseteq U \setminus \{0\}$, $\{n_j : j \in \mathbf{N}_p\} \subseteq \mathbf{Z}_+$, $\{c_{jk} : j \in \mathbf{N}_p, k \in \mathbf{Z}_{n_j}\} \subseteq \mathbf{C}$.

Proof. Suppose ψ is defined by (3.9) with $\{\lambda_j : j \in \mathbf{N}_p\} \subseteq U \setminus \{0\}$. Then $S^*(\psi)$ is of finite dimensions because

$$S^*(\psi) \subseteq \text{span}\,\{\{\varphi\} \cup \{E_{jk} : j \in \mathbf{N}_p, k \in \mathbf{Z}_{n_j}\}\},$$

where $E_{jk} := \{n^k \lambda_j^n : n \in \mathbf{Z}_+\} \in \ell^2(\mathbf{Z}_+)$. Conversely, suppose that $S^*(\psi)$ is of finite dimensions. Then there exists $q \in \mathbf{N}$ such that

$$S^*(\psi) \subseteq \text{span}\,\{S^{*k}\psi : k \in \mathbf{Z}_q\}.$$

This inclusion relation implies that there exists $\{c_k : k \in \mathbf{Z}_q\} \subseteq \mathbf{C}$ such that

$$S^{*q}\psi = \sum_{k \in \mathbf{Z}_q} c_k S^{*k}\psi,$$

or equivalently,

$$\psi(n+q) = \sum_{k \in \mathbf{Z}_q} c_k \psi(n+k), \quad n \in \mathbf{Z}_+.$$

By the theory of linear difference equations, see e.g., [**13**], ψ must be of form (3.9). The requirement that $\psi \in \ell^2(\mathbf{Z}_+)$ enables us to choose $\{\lambda_j : j \in \mathbf{N}_p\} \subseteq U \setminus \{0\}$ in (3.9). $\square$

Let ψ be of form (3.9). We would like to associate each element in $\ell^2(\mathbf{Z}_+)$ with a function in $\mathbf{H}^2(U)$ before we solve $\phi \in \ell^2(\mathbf{Z}_+)$ from (3.8). It is known that if $h \in \mathbf{H}(U)$ is expanded as

$$h(z) = \sum_{n \in \mathbf{Z}_+} a_n z^n, \quad z \in U,$$

then $h \in \mathbf{H}^2(U)$ if and only if $\{a_n : n \in \mathbf{Z}_+\} \in \ell^2(\mathbf{Z}_+)$, [**26**]. By this fact, we define a map $\mathcal{H}_U : \ell^2(\mathbf{Z}_+) \to \mathbf{H}^2(U)$ by setting for each $\varphi \in \ell^2(\mathbf{Z}_+)$

$$(\mathcal{H}_U \varphi)(z) := \sum_{n \in \mathbf{Z}_+} \varphi(n) z^n, \quad z \in U. \tag{3.10}$$

Similar arguments as those in the proof of Proposition 2.6 yield our next result.

Proposition 3.7. *If ψ is given by* (3.9) *with $\varphi \equiv 0$, $c_{jn_j-1} \neq 0$, $j \in \mathbf{N}_p$ and $|\lambda_k| > |\lambda_{k+1}|$, $k \in \mathbf{N}_{p-1}$, then $\phi \in \ell^2(\mathbf{Z}_+)$ satisfies* (3.8) *if and only if for each $j \in \mathbf{N}_p$, λ_j is the zero of $\mathcal{H}_U(\phi)$ of order n_j.*

We apply Theorem 3.6 and Proposition 3.7 to give a particular class of solutions for the circular Bedrosian identity (1.8).

Proposition 3.8. *Let $p, q \in \mathbf{N}$, $\{\lambda_j : j \in \mathbf{N}_p\} \subseteq U \setminus \{0\}$, $\{\gamma_j : j \in \mathbf{N}_q\} \subseteq U \setminus \{0\}$, $\{n_j : j \in \mathbf{N}_p\} \subseteq \mathbf{Z}_+$, $\{m_j : j \in \mathbf{N}_q\} \subseteq \mathbf{Z}_+$, $\{c_{jk} : j \in \mathbf{N}_p, k \in \mathbf{Z}_{n_j}\} \subseteq \mathbf{C}$, $\{c'_{jk} : j \in \mathbf{N}_q, k \in \mathbf{Z}_{m_j}\} \subseteq \mathbf{C}$ be such that*

$$\sum_{j \in \mathbf{N}_p} c_{j0} = \sum_{j \in \mathbf{N}_q} c'_{j0}.$$

Then the circular Bedrosian identity (1.8) *is satisfied if $f, g \in L^2_{2\pi}$ are given by*

$$c_n(f) := u(n) \sum_{j \in \mathbf{N}_p} \sum_{k \in \mathbf{Z}_{n_j}} c_{jk} n^k \lambda_j^n + u(-n) \sum_{j \in \mathbf{N}_q} \sum_{k \in \mathbf{Z}_{m_j}} c'_{jk} (-n)^k \gamma_j^{-n}, \quad n \in \mathbf{Z}$$

and

$$g(t) := g_1(e^{-it}) \prod_{j\in\mathbf{N}_p} (e^{-it}-\lambda_j)^{n_j} + g_2(e^{it}) \prod_{j\in\mathbf{N}_q} (e^{it}-\gamma_j)^{m_j}, \quad t \in [0,2\pi],$$

where $g_1, g_2 \in \mathbf{H}^2(U)$ *satisfy*

$$g_1(0) \prod_{j\in\mathbf{N}_p} (-\lambda_j)^{n_j} = g_2(0) \prod_{j\in\mathbf{N}_q} (-\gamma_j)^{m_j}.$$

We shall use Proposition 3.8 to give an explicit example of functions satisfying the circular Bedrosian identity (1.8). This example is constructed as

$$\begin{aligned} f(t) &:= \frac{1}{1-\frac{1}{2}e^{it}} + \frac{1}{1-\frac{1}{3}e^{-it}}, \\ g(t) &:= 2\frac{e^{-it}-1/2}{1-(1/3)e^{-it}} + 3\frac{e^{it}-1/3}{1-(1/2)e^{it}}, \quad t\in[0,2\pi]. \end{aligned} \tag{3.11}$$

That f, g given by (3.11) satisfy identity (1.8) follows directly from Proposition 3.8. Since $\operatorname{supp}(\tilde{F}f) = \operatorname{supp}(\tilde{F}g) = \mathbf{Z}$, all the Bedrosian type sufficient conditions for (1.8) are unnecessary.

3.3. *Relations with shift invariant subspaces of* $\ell^2(\mathbf{Z}_+)$. The purpose of this subsection is to determine for what $g \in L^2_{2\pi}$ would there exist nonconstant $f \in L^2_{2\pi}$ satisfying (1.8). Our main tool is the Beurling theorem characterizing the closed shift invariant subspace of $\mathbf{H}^2(U)$, [**3, 26**].

With the inner product $(\cdot,\cdot)$ on $\mathbf{H}^2(U)$ defined for all $f, g \in \mathbf{H}^2(U)$ as

$$(f,g) := \frac{1}{2\pi}\int_0^{2\pi} f(e^{it})\overline{g(e^{it})}\,dt,$$

the operator $\mathcal{H}_U$ defined by (3.10) becomes an isomorphism from $\ell^2(\mathbf{Z}_+)$ to $\mathbf{H}^2(U)$. The shift operator S is turned into a multiplication operator, which is denoted by $\mathbf{S}$, on $\mathbf{H}^2(U)$

$$(\mathbf{S}h)(z) := zh(z), \quad h \in \mathbf{H}^2(U), \quad z \in U.$$

The shift operator on $\ell^2(\mathbf{Z}_+)$ and the multiplication operator on $\mathbf{H}^2(U)$ are related through the isomorphism $\mathcal{H}_U$ as follows

$$\mathcal{H}_U S = \mathbf{S}\mathcal{H}_U.$$

We call a subspace $\mathcal{M}$ of $\mathbf{H}^2(U)$ a *shift invariant subspace* if $\mathbf{S}(\mathcal{M}) \subseteq \mathcal{M}$. For each $h \in \mathbf{H}^2(U)$, we denote by $\mathbf{S}(h)$ the smallest closed shift invariant subspace that contains h. It is clear that

$$\mathbf{S}(h) = \overline{\operatorname{span}}\,\{\mathbf{S}^k h : k \in \mathbf{Z}_+\}.$$

Equations (3.3), (3.4) and (3.5) are transformed into

$$(\mathcal{H}_U(S^*\overline{\widetilde{F}_+ f}), \mathcal{H}_U(S^*\widetilde{F}_- g)) = (\mathcal{H}_U(S^*\overline{\widetilde{F}_- f}), \mathcal{H}_U(S^*\widetilde{F}_+ g)), \tag{3.12}$$

$$\mathcal{H}_U(S^*\overline{\widetilde{F}_+ f}) \perp \mathbf{S}(\mathcal{H}_U(\widetilde{F}_- g)) \tag{3.13}$$

and

$$\mathcal{H}_U(S^*\overline{\widetilde{F}_- f}) \perp \mathbf{S}(\mathcal{H}_U(\widetilde{F}_+ g)). \tag{3.14}$$

There is a celebrated characterization of the closed shift invariant subspace of $\mathbf{H}^2(U)$ due to Beurling [**3**]. We shall make use of some consequences of this characterization [**26**, pages 348–350].

Lemma 3.9. *Let M_h be the inner factor of a function $h \in \mathbf{H}^2(U)$. Then $\mathbf{S}(h) = \{M_h\varphi : \varphi \in \mathbf{H}^2(U)\}$, $\mathbf{S}(h) = \mathbf{H}^2(U)$ if and only if h is an outer function on U. Moreover, for two inner functions h_1, h_2 on U, $\mathbf{S}(h_1) = \mathbf{S}(h_2)$ if and only if $h_1 = ch_2$ for some $c \in \mathbf{T}$.*

Another observation is needed to prove the main result of this subsection. If φ is a function on $\mathbf{T}$ then $\varphi(e^{i\cdot})$ is defined on $[0, 2\pi]$. For simplicity, we shall not distinguish between these two functions.

Lemma 3.9. *Let $h \in \mathbf{H}^2(U)$. Then $\mathbf{S}(h)^\perp$ has exactly one dimension if and only if the inner factor M_h of h is a Möbius transformation, namely, there exists $\lambda \in U$ and $c \in \mathbf{T}$ such that*

$$M_h(z) = c\frac{z - \lambda}{1 - \bar{\lambda}z}, \quad z \in U. \tag{3.15}$$

Proof. If M_h has form (3.15), by Lemma 3.9, $\varphi \in \mathbf{S}(h)^{\perp}$ if and only if for each $n \in \mathbf{Z}_+$

$$\int_0^{2\pi} \overline{\varphi(e^{it})} \frac{e^{it} - \lambda}{1 - \bar{\lambda} e^{it}} e^{int} \, dt = 0.$$

The equation above has the following form in terms of the Fourier coefficient of φ

$$-\lambda \overline{(\tilde{\mathcal{F}}\varphi)(n)} + \sum_{k \in \mathbf{N}} (1 - |\lambda|^2) \bar{\lambda}^{k-1} \overline{(\tilde{\mathcal{F}}\varphi)(n+k)} = 0. \tag{3.16}$$

Comparing two consecutive equations of (3.16) yields that

$$\overline{(\tilde{\mathcal{F}}\varphi)(n+1)} = \lambda \overline{(\tilde{\mathcal{F}}\varphi)(n)}, \quad n \in \mathbf{Z}_+.$$

Consequently, we have

$$\mathbf{S}(h)^{\perp} = \operatorname{span} \left\{ \frac{1}{1 - \bar{\lambda} z} \right\},$$

which is of one dimension.

On the other hand, suppose that $\mathbf{S}(h)^{\perp}$ is one dimensional. We first prove that M_h has a zero on U. Let $\varphi \in \mathbf{H}^2(U)$ be such that $\mathbf{S}(h)^{\perp} = \operatorname{span}\{\varphi\}$. Since $\mathbf{S}(h)$ is shift invariant, $S^* \widetilde{\mathcal{F}} \varphi \subseteq \operatorname{span}\{\widetilde{\mathcal{F}}\varphi\}$. By Theorem 3.6, there exists $c \in \mathbf{C} \setminus \{0\}$ and $\lambda \in U$ such that

$$(\widetilde{\mathcal{F}}\varphi)(n) = c\lambda^n, \quad n \in \mathbf{Z}_+,$$

where we set $0^0 := 1$ if it appears. Since $M_h \perp \varphi$, a simple computation shows that $M_h(\bar{\lambda}) = 0$. Therefore, there exists an inner function φ on U such that $M_h = \varphi m_\lambda$ where

$$m_\lambda(z) := \frac{z - \bar{\lambda}}{1 - \lambda z}, \quad z \in U.$$

It follows from this equation that $\mathbf{S}(h) \subseteq \mathbf{S}(m_\lambda)$. Since both $\mathbf{S}(h)^{\perp}$ and $\mathbf{S}(m_\lambda)^{\perp}$ are one dimensional, we have $\mathbf{S}(h) = \mathbf{S}(M_h) = \mathbf{S}(m_\lambda)$. By Lemma 3.9, M_h is a Möbius transformation. $\square$

Theorem 3.11. *Let $g \in L^2_{2\pi}$. Then there does not exist a nonconstant $f \in L^2_{2\pi}$ satisfying the circular Bedrosian identity* (1.8) *if and only if one of the following conditions holds*:

(1) *both $\mathcal{H}_U(\widetilde{F}_+g)$ and $\mathcal{H}_U(\widetilde{F}_-g)$ are outer functions on U*,

(2) *one of $\mathcal{H}_U(\widetilde{F}_+g)$ and $\mathcal{H}_U(\widetilde{F}_-g)$ is an outer function and the inner factor of the other is a Möbius transformation.*

Proof. If condition (1) above is satisfied, then by Lemma 3.9,

$$\mathbf{S}(\mathcal{H}_U(\widetilde{F}_-g)) = \mathbf{S}(\mathcal{H}_U(\widetilde{F}_+g)) = \mathbf{H}^2(U).$$

If $f \in L^2_{2\pi}$ satisfies the circular Bedrosian identity then by equations (3.13) and (3.14),

$$S^*\overline{\widetilde{F}_+f} = S^*\overline{\widetilde{F}_-f} = 0.$$

The function f is hence a constant function. Suppose $\mathcal{H}_U(\widetilde{F}_+g)$ is an outer function, the inner factor of $h := \mathcal{H}_U(\widetilde{F}_-g)$ is a Möbius transformation and $f \in L^2_{2\pi}$ satisfies identity (1.8). By equation (3.14), $S^*\overline{\widetilde{F}_-f} = 0$. Since $\mathbf{S}(h)^\perp$ is one dimensional, f is nonconstant if and only if

$$\mathcal{H}_U(S^*\widetilde{F}_-g) \in \mathbf{S}(h).$$

The above formula holds if and only if there exists $\varphi \in \mathbf{H}^2(U)$ such that

$$Q_hM_h - h(0) = zM_h\varphi,$$

where Q_h and M_h are the outer factor and inner factor of h, respectively. Since M_h is a Möbius transformation, it has a zero on U. As a consequence, $h(0) = 0$, which implies that $Q_h = z\varphi$. This is impossible since Q_h has no zeros on U. It is concluded that only constant $f \in L^2_{2\pi}$ would satisfy (1.8) if condition (1) or (2) holds.

There are two possibilities if neither (1) nor (2) is valid. First, if neither $\mathcal{H}_U(\widetilde{F}_+g)$ nor $\mathcal{H}_U(\widetilde{F}_-g)$ is an outer function, then there exist nontrivial elements $\psi_1, \psi_2 \in \ell^2(\mathbf{Z}_+)$ such that

$$\mathcal{H}_U(\psi_1) \perp \mathbf{S}(\mathcal{H}_U(\widetilde{F}_-g)) \quad \text{and} \quad \mathcal{H}_U(\psi_2) \perp \mathbf{S}(\mathcal{H}_U(\widetilde{F}_+g)).$$

Let $f \in L^2_{2\pi}$ be defined by

$$S^*\overline{\widetilde{F}_+f} = c_1\psi_1, \quad S^*\overline{\widetilde{F}_-f} = c_2\psi_2,$$

where c_1, c_2 are constants chosen to satisfy (3.12), that is,

$$c_1(\mathcal{H}_U(\psi_1), \mathcal{H}_U(S^*\widetilde{F}_-g)) - c_2(\mathcal{H}_U(\psi_2), \mathcal{H}_U(S^*\widetilde{F}_+g)) = 0. \tag{3.17}$$

Second, suppose that $\mathcal{H}_U(\widetilde{F}_+g)$ is an outer function and the inner factor of $\mathcal{H}_U(\widetilde{F}_-g)$ is not a Möbius transformation. By Lemma 3.10, there exist linear independent elements $\psi_1, \psi_2 \in \ell^2(\mathbf{Z}_+)$ for which $\mathcal{H}_U(\psi_1), \mathcal{H}_U(\psi_2)$ are orthogonal to $\mathbf{S}(\mathcal{H}_U(\widetilde{F}_-g))$. Let $f \in L^2_{2\pi}$ be defined by

$$S^*\overline{\widetilde{F}_+f} = c_3\psi_1 + c_4\psi_2, \quad S^*\overline{\widetilde{F}_-f} = 0,$$

where c_3, c_4 are constants such that

$$c_3(\mathcal{H}_U(\psi_1), \mathcal{H}_U(S^*\widetilde{F}_-g)) + c_4(\mathcal{H}_U(\psi_2), \mathcal{H}_U(S^*\widetilde{F}_-g)) = 0. \tag{3.18}$$

Both (3.17) and (3.18) have a nontrivial solution (c_1, c_2) or (c_3, c_4). We are hence able to find nonconstant $f \in L^2_{2\pi}$ satisfying the circular Bedrosian identity. $\square$

If $g \in L^2_{2\pi}$ does not satisfy condition (1) or (2) in Theorem 3.11, then the additive positive definite kernels on $\mathbf{Z}_+$ can be used to construct nontrivial $f \in L^2_{2\pi}$ satisfying the circular Bedrosian identity. Also, a class of functions $f \in L^2_{2\pi}$ for each of which there does not exist a nontrivial $g \in L^2_{2\pi}$ satisfying the circular Bedrosian identity can be obtained from additive positive definite kernels on $\mathbf{Z}_+$. However, since these would be trivial extensions of the real line case, we would not present them.

4. Applications to intrinsic mode functions.

Originally, an intrinsic mode function (IMF) is defined to be a real function such that the numbers of its zeros and local extrema differ at most by one and that its local mean is zero, [**17**]. For a study on IMFs along this direction, see [**28**]. As a different approach, it was suggested in [**32**] that real functions $f \in L^2(\mathbf{R})$ such that

$$(f + iHf)(t) = \rho(t)e^{i\theta(t)}, \quad t \in \mathbf{R} \tag{4.1}$$

where for all $t \in \mathbf{R}$

$$\rho(t) \geq 0, \quad \frac{d\theta(t)}{dt} \geq 0 \tag{4.2}$$

be taken as a basic atom for the Hilbert-Huang transform [**17**]. Following [**32**], we still refer to functions f satisfying (4.1) and (4.2) as IMFs. A method of constructing such IMFs is to solve functions $\rho \in L^2(\mathbf{R})$ and $\theta \in C^1(\mathbf{R})$ from the nonlinear singular integral equation

$$H(\rho(\cdot)\cos\theta(\cdot))(t) = \rho(t)\sin\theta(t), \quad \text{almost everywhere } \ t \in \mathbf{R}, \tag{4.3}$$

with the constraint that ρ, θ satisfy (4.2) for all $t \in \mathbf{R}$. In the circular case, the method is to find $\rho \in L^2_{2\pi}$ and 2π-periodic function $\theta \in C^1(\mathbf{R})$ such that

$$\widetilde{H}(\rho(\cdot)\cos\theta(\cdot))(t) = \rho(t)\sin\theta(t), \quad \text{almost everywhere } \ t \in [0, 2\pi] \tag{4.4}$$

with ρ, θ satisfying (4.2) for all $t \in [0, 2\pi]$.

Functions ρ, θ satisfying (4.3) or (4.4) have been characterized using the boundary value of functions in Hardy spaces, [**24**]. Solutions of (4.3) and (4.4) with explicit expression are desirable in engineering applications. In the unimodular case $\rho \equiv 1$, an interesting class of functions θ satisfying (4.3) or (4.4) is provided in [**24**] using finite Blaschke products. Specifically, they are given in the real line case by

$$e^{i\theta(x)} = \frac{e^{i2\arctan(x)} - \lambda}{1 - \lambda e^{i2\arctan(x)}}, \quad x \in \mathbf{R} \tag{4.5}$$

and in the circular case by

$$e^{i\theta(t)} = e^{it}\frac{e^{it} - \lambda}{1 - \lambda e^{it}}, \quad t \in [0, 2\pi], \tag{4.6}$$

where $\lambda \in (0, 1)$. The purpose of this section is to extend the construction of IMFs by characterizing ρ satisfying (4.3) or (4.4).

Proposition 4.1. *Let θ be given in* (4.5). *Then a nonnegative function $\rho \in L^2(\mathbf{R})$ satisfies*

$$H(\rho\cos\theta) = \rho\sin\theta \ \ \textit{almost everywhere} \tag{4.7}$$

if and only if it has the form

$$\rho(t) = \frac{c}{1+b^2t^2}, \quad t \in \mathbf{R},$$

where $b := (1+\lambda)/(1-\lambda)$, c *is an arbitrary nonnegative constant.*

Proof. Suppose $\rho \in L^2(\mathbf{R})$ satisfies (4.7). Set $\tilde{c} := \cos\theta + 1$ and rewrite (4.7) into

$$H(\rho\tilde{c}) - H\rho = \rho\sin\theta \text{ almost everywhere.} \tag{4.8}$$

Note the relations that $H\tilde{c} = \sin\theta$ [**25**] and

$$(\mathcal{F}\tilde{c})(\xi) = \frac{2\pi}{b}\exp(-|\frac{\xi}{b}|), \quad \xi \in \mathbf{R}.$$

By Lemma 2.2, equation (4.8) holds if and only if

$$\begin{aligned}\int_{\mathbf{R}} \exp\bigg(-\bigg|\frac{\eta}{b}\bigg|\bigg)\hat{\rho}(\xi-\eta)(\operatorname{sgn}(\xi)-\operatorname{sgn}(\eta))\,d\eta \\ = b\operatorname{sgn}(\xi)\hat{\rho}(\xi), \quad \xi \in \mathbf{R}.\end{aligned} \tag{4.9}$$

We observe from equation (4.9) that (4.8) holds if and only if ρ satisfies the following two inhomogeneous semi-convolution equations

$$2\int_{\mathbf{R}_+} \exp\bigg(-\frac{\eta}{b}\bigg)\hat{\rho}(\xi+\eta)\,d\eta = b\hat{\rho}(\xi), \quad \xi \in \mathbf{R}_+ \setminus \{0\}, \tag{4.10}$$

$$2\int_{\mathbf{R}_-} \exp\bigg(-\bigg|\frac{\eta}{b}\bigg|\bigg)\hat{\rho}(\xi+\eta)\,d\eta = b\hat{\rho}(\xi), \quad \xi \in \mathbf{R}_- \setminus \{0\}. \tag{4.11}$$

A change of variables transforms (4.10) into

$$2\exp\bigg(\frac{\xi}{b}\bigg)\int_{\xi}^{\infty} \exp\bigg(-\frac{s}{b}\bigg)\hat{\rho}(s)\,ds = b\hat{\rho}(\xi), \quad \xi \in \mathbf{R}_+ \setminus \{0\}.$$

This equation shows that $\hat{\rho} \in C^1(0,\infty)$ and satisfies for each $\xi \in \mathbf{R}_+ \setminus \{0\}$ that

$$b\hat{\rho}'(\xi) + \operatorname{sgn}(\xi)\hat{\rho}(\xi) = 0.$$

Likewise, we can obtain the above equation for each $\xi \in \mathbf{R}_- \setminus \{0\}$. Therefore, if $\rho \in L^2(\mathbf{R})$ satisfies (4.7), then there exist constants $c_1, c_2 \in \mathbf{C}$ such that

$$\hat{\rho}(\xi) = c_1 u(\xi)\exp\bigg(-\frac{\xi}{b}\bigg) + c_2 u(-\xi)\exp\bigg(\frac{\xi}{b}\bigg), \quad \xi \in \mathbf{R} \setminus \{0\}. \tag{4.12}$$

Conversely, it can be verified directly that if ρ is given above, then it satisfies equations (4.10) and (4.11). Applying the inverse Fourier transform yields that $\hat{\rho}$ has the form (4.12) if and only if there exists $c_3, c_4 \in \mathbf{C}$ such that

$$\rho(t) = \frac{c_3}{1 - ibt} + \frac{c_4}{1 + ibt}, \quad t \in \mathbf{R}.$$

Simple computations show that ρ given above is nonnegative if and only if $c_3 = \overline{c_4}$, $\mathrm{Re}\,(c_3) \geq 0$ and $\mathrm{Im}\,(c_3) = 0$. This completes the proof. $\square$

We next characterize nonnegative $\rho \in L^2_{2\pi}$ satisfying (4.4). Considering the local symmetry of IMFs, we would like to impose the additional requirement that

$$\int_0^{2\pi} \rho(t) \cos\theta(t)\, dt = 0. \tag{4.13}$$

Proposition 4.2. *Let θ be given by* (4.6) *with $\lambda \in (0, 1)$. Then a nonnegative function $\rho \in L^2_{2\pi}$ satisfies* (4.4) *and* (4.13) *if and only if it is given by*

$$\rho(t) = \mathrm{Re}\left(\frac{c_1 e^{-it}}{1 - \lambda e^{-it}}\right) + c_2, \quad t \in [0, 2\pi],$$

where $c_1 \in \mathbf{C}$, $c_2 \in \mathbf{R}$ are constants such that

$$\lambda \mathrm{Re}\,(c_1) - |c_1| + c_2(1 - \lambda^2) \geq 0. \tag{4.14}$$

Proof. It was proved in [**24**] that

$$\widetilde{H} e^{i\theta(\cdot)} = -i e^{i\theta(\cdot)}.$$

We point out by this fact that a real function $\rho \in L^2_{2\pi}$ satisfies (4.4) and (4.13) if and only if

$$\widetilde{H}(\rho e^{i\theta(\cdot)}) = \rho \widetilde{H} e^{i\theta(\cdot)} \text{ almost everywhere.} \tag{4.15}$$

Note that the following expansion holds

$$e^{i\theta(t)} = -\lambda e^{it} + \sum_{n\in\mathbf{N}} (\lambda^{n-1} - \lambda^{n+1}) e^{i(n+1)t}, \;\; t \in [0, 2\pi].$$

By Theorem 3.4, (4.15) holds if and only if for each $k \in \mathbf{N}$

$$-\lambda c_{-k}(\rho) + \sum_{j\in\mathbf{N}} c_{-k-j}(\rho)(\lambda^{j-1} - \lambda^{j+1}) = 0. \tag{4.16}$$

It is then observed that (4.16) holds if and only if there exists the constant $c_1' \in \mathbf{C}$ such that

$$c_{-k}(\rho) = c_1' \lambda^{k-1}, \;\; k \in \mathbf{N}.$$

As a consequence, ρ satisfies (4.15) if and only if there exists $\varphi \in \mathbf{H}^2(U)$ such that

$$\rho(t) = \frac{c_1' e^{-it}}{1 - \lambda e^{-it}} + \varphi(e^{it}), \;\; t \in [0, 2\pi]. \tag{4.17}$$

We next observe that ρ given by (4.17) is a real function if and only if there exists $c_2 \in \mathbf{R}$ such that

$$\varphi(e^{it}) = \frac{\bar{c_1'} e^{it}}{1 - \bar{\lambda} e^{it}} + c_2, \;\; t \in [0, 2\pi].$$

Finally, we verify by elementary analysis that

$$\rho(t) = \mathrm{Re}\left(\frac{c_1 e^{-it}}{1 - \lambda e^{-it}}\right) + c_2, \;\; t \in [0, 2\pi],$$

where $c_1 := 2c_1'$, is nonnegative on $[0, 2\pi]$ if and only if condition (4.14) is satisfied. $\square$

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