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A REGULARITY THEOREM FOR A VOLTERRA INTEGRAL EQUATION OF THE THIRD KIND

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ABSTRACT. An existence and smoothness theorem is given for a Volterra integral equation of the form

$$f(x)v(x) = \phi(x) - \int_0^x K(x,\xi)v(\xi)\,d\xi,$$

where $f(x)$ has a zero at $x = 0$, and the kernel $K(x,\xi)$ has a kind of square root behavior at the diagonal $x = \xi$.

1. Introduction. In this paper we will consider a special class of third kind linear Volterra integral equation, i.e.

$$f(x)v(x) = \phi(x) - \int_0^x K(x,\xi)v(\xi)\,d\xi. \tag{1}$$

It is easy to see that this type of equation can be written as an equation of the second kind ($f(x) \equiv 1$), which, in general, has a singular kernel, if the function $f(x)$ has zeroes. Seminal works dealing with this type of equations are [5] and [6]. The idea of [6] is to split up the kernel $K(x,\xi)$ into a constant part, w.l.o.g. one can take 1, and a function $\Gamma(x,\xi)$. Evans constructed then an approximating series for the solution. Each term of this series is constructed in two steps (for more details see the proof of Proposition 2.1 below) : First one has to solve a singular differential equation, stemming from the constant part of the kernel; in the second step one has to evaluate an integral in order to get the

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inhomogeneity of the next ODE. In Evans' paper the function $\Gamma(x, \xi)$ is basically assumed to be C^1, see [6 condition 1b) on p. 431. This is convenient in order to see the integral equation as a perturbation of the underlying singular ODE.

In our paper we are interested in the consequences of the appearance of a certain "non-smoothness" in the function $\Gamma(x, \xi)$, i.e. we shall assume that it has the structure

$$\Gamma(x, \xi) = \Gamma_1(x, \xi) + \sqrt{1 - \frac{\xi}{x}}\Gamma_2(x, \xi), \tag{2}$$

for smooth Γ_1, Γ_2. As in [6], we will assume that $f(x)$ has only one zero at $x = 0$, but in contrast to Evans - who derived existence and continuity results for the dependent variable $u(x) = f(x)v(x)$ - we are interested in the smoothness of $v(x)$, which has obviously worse smoothness properties than $u(x)$.

The aim of the paper is to investigate the impact of the square-root in the kernel on the regularity of the solution. In order to see the effect more clearly, we will assume smooth "coefficient functions" f, Γ_1 and Γ_2. We will state sufficient conditions for the regularity of the solution, and we will see that this regularity depends crucially on the numerical value of Γ_2 (and its derivatives) at the point $(x, \xi) = (0, 0)$, and on the order of decrease to zero of the function $\phi(x)$ for $x \to 0$.

Let us briefly comment on the motivation to consider such an equation. The origin is a so called singular optimal control problem (see e.g. [7]) over an infinite time interval, which appears in Mathematical Economics (see [8]). In this type of problems the value function is often the solution of a free boundary value problem. The free boundary, say $b(t)$, is fixed by the so called "smooth fitting conditions ", which guarantee that the value function is smooth enough, even over the boundary. It turns out that $b(t)$ is described as the solution of a nonlinear integral equation, involving a Greens function

$$G(r, x, x') = \frac{1}{\sqrt{4\pi r}}\theta(r)e^{-\frac{(x-x')^2}{4r}}$$

of the heat operator (θ denotes the Heaviside function).

One possibility to approach such an equation is, to linearize the problem, and to apply some kind of Newton scheme. One observes that the Fréchet derivative of the nonlinear integral operator is exactly of the form

$$f(x)v(x) + \int_0^x (1 + \Gamma(x,\xi))\, v(\xi)\, d\xi.$$

Here the zero of the function $f(x)$ at $x = 0$ stems from the transformation of the infinite time interval to a finite one, and the origin of the square root in Γ is the factor $\frac{1}{\sqrt{r}}$ in the Greens function G of the heat operator. However, a glance at our main theorem 2.1, inequality (10) reveals that we face a so called "loss of derivatives" in the linear problem. Therefore a straightforward application of Newtons scheme is not possible, and one is forced to use more sophisticated methods as the Nash-Moser technique (see e.g. [12] for an introduction to this subject).

Let us remark that we confine here to the case, where the function $f(x)$ has positive sign in a neighborhood of zero. Beside the non-uniqueness of the solution, which appears for the negative sign, at least in the case Evans is considering, one observes that the behavior with respect to smoothness is also different (see the remark after the proof of Corollary 2.1). So we postpone this case to future research.

Let us also mention that standard theory, as it is formulated in [9], chapter 9, for the existence of solutions in certain L^p-spaces is, in general, not applicable, since the kernel of our equation (viewing it as an equation for $u = fv$) is, in general, not a kernel of type L^P (see the definition 9.2.2 in [9]). In order to apply this theory one would have to assume a function $f(x)$, tending very slowly to zero for $x \to 0$. (One would have to check, e.g., the sufficient conditions of Proposition 2.7, [9] for the kernel to be of L^p-type and then apply their Theorem 3.6 for existence of a solution.)

Finally we want to cite some more papers and a monograph, dealing with singular Volterra equations, respectively with the smoothness of their solutions: [2, 3, 4, 10, & 11].

2. The Result. In order to formulate our main result in a concise way, we summarize the regularity assumption we will use in the rest of the paper (if not stated otherwise) for the coefficient functions and the inhomogeneity of our integral equation in the following

Standing assumption: *The functions* $f(x), \phi(x), \Gamma_1(x,\xi), \Gamma_2(x,\xi)$ *fulfill the following regularity assumptions:*

$$f(x) \in C^\infty([0,1]), \quad f(0) = 0, \tag{3}$$
$$f(x) > 0 \text{ for } x \in (0,\epsilon] \text{ and for some } \epsilon > 0.$$

Let $x_0 := \min\{\inf\{x > 0 | f(x) = 0\}, 1\}$ *and* $0 < b < x_0$. *Assume that*

$$\phi(x) \in C^{s+1}([0,b]), \quad \phi(0) = 0, \tag{4}$$
$$r^* := \min\{\inf\{k | \phi^{(k+1)}(0) \neq 0\}, s\},$$

with $s \in \mathbf{N}_0$.

For the kernel functions we impose

$$\Gamma_1(x,\xi), \Gamma_2(x,\xi) \in C^\infty(\mathcal{G}), \tag{5}$$

where $\mathcal{G} = \{(x,\xi) | 0 \leq x \leq b, 0 \leq \xi \leq x\}$, *and* C^∞ *means here existence and continuity of all partial derivatives.*

Finally we assume

$$\Gamma_1(0,0) = 0, \quad \Gamma_2(0,0) \neq 0. \tag{6}$$

Note that by convention $r^* := s$ in (4), if $\phi^{(k+1)}(0) = 0$ for $k = 0, 1, 2, ..., s$, and that the assumption (6) on $\Gamma_2(0,0)$ is made, because without this assumptions the kernel would have a different kind of non-smoothness than the one we want to consider here. Moreover, it was already observed in [6] that there is no continuous solution of the equation, if $\phi(0) \neq 0$. Now we formulate and prove our main result, namely

Theorem 2.1. *Under the standing assumptions above consider the following integral equation*

$$f(x)v(x) = \phi(x) - \int_0^x \left(1 + \Gamma_1(x,\xi) + \sqrt{1 - \frac{\xi}{x}}\,\Gamma_2(x,\xi)\right) v(\xi)\, d\xi. \tag{7}$$

There exist $\gamma_{r^*,n} \in \mathbf{R}^+$*,* $s \geq n \geq r^*$*, (for* γ_{r^*,r^*} *we have*

$$\gamma_{r^*,r^*} = \frac{2 \cdot 4 \cdots (2r^*+2)}{3 \cdot 5 \cdots (2r^*+3)} \tag{8}$$

), such that, under the assumption

$$\left| \frac{\partial^i}{\partial x^i} \frac{\partial^j}{\partial \xi^j} \Gamma_2(0,0) \right| < \frac{1}{\gamma_{r^*,n}}, \quad i+j \leq n-r^*, \tag{9}$$

we have a unique solution $v \in C^n([0,b])$*, and*

$$||v||_{C^n([0,b])} \leq C\, ||\phi||_{C^{n+1}([0,b])}, \tag{10}$$

for some positive constant C*, which depends on* n *(and of course on the coefficient functions* f *and* Γ_1, Γ_2*).*

Proof. As usual we take the norm

$$||v||_{C^s([0,b])} = \max_{0 \leq i \leq s} \{ \sup_{x \in [0,b]} |f^{(i)}(x)| \}.$$

The proof of the theorem uses mainly the following local result, whose proof we will give as soon as we have finished the proof of the theorem.

Proposition 2.1. *Under the assumption of Theorem* 2.1*, the equation*

$$f(x)v(x) = \phi(x) - \int_0^x \left(1 + \Gamma_1(x,\xi) + \sqrt{1 - \frac{\xi}{x}}\Gamma_2(x,\xi) \right) v(\xi)\, d\xi$$

has a unique solution $v \in C^n([0,\bar{x}])$*, for* $\overline{x}$ *sufficiently small, and we have*

$$\|v||_{C^n([0,\bar{x}])} \leq C\, |\phi||_{C^{n+1}([0,\overline{x}])}$$

for some positive constant C*.*

The aim is now to continue the local solution provided by the proposition to a solution on the interval $[0, b]$. In order to do this, we rewrite the integral equation

$$f(x)v(x) = \phi(x) - \int_0^{\overline{x}} (1 + \Gamma(x, \xi))\, v(\xi)\, d\xi - \int_{\overline{x}}^{x} (1 + \Gamma(x, \xi))\, v(\xi)\, d\xi.$$

Putting the known local solution into the inhomogeneity, we arrive at

$$f(x)v(x) = \rho(x) - \int_{\overline{x}}^{x} (1 + \Gamma(x, \xi))\, v(\xi)\, d\xi. \tag{11}$$

Clearly $\rho(x) \in C^{n+1}([\overline{x}, b])$, and we consider now the equation on the interval $[\overline{x}, b]$. We first show that (11) has a solution in $C([\overline{x}, b])$. Transformation to the dependent variable $u(x) = v(x)f(x)$ gives

$$\begin{aligned} u(x) = \rho(x) &- \int_{\overline{x}}^{x} (1 + \Gamma_1(x, \xi))\, \frac{u(\xi)}{f(\xi)}\, d\xi \\ &- \int_{\overline{x}}^{x} \Gamma_2(x, \xi) \sqrt{1 - \frac{\xi}{x}}\, \frac{u(\xi)}{f(\xi)}\, d\xi. \end{aligned} \tag{12}$$

Further transforming the independent variables via $t = x - \overline{x}, s = \xi - \overline{x}$, respectively using $\overline{u}(t) = u(\overline{x} + t), \overline{f}(t) = f(\overline{x} + t)$ etc., yields

$$\overline{u}(t) = \overline{\rho}(t) - \int_0^t \left(1 + \overline{\Gamma_1}(t, s)\right) \frac{\overline{u}(s)}{\overline{f}(s)}\, ds - \int_0^t \overline{\Gamma_2}(t, s) \frac{\sqrt{t - s}}{\sqrt{\overline{x} + t}} \frac{\overline{u}(s)}{\overline{f}(s)}\, ds.$$

We can now apply, e.g. Theorem 1.3.1 of [3], to get $u \in C([\overline{x}, b])$. Next we show that the solution u fulfills $u \in C^1(]\overline{x}, b])$. This will follow, if we show that the r.h.s. of (12) is in $C^1(]\overline{x}, b])$, if $u \in C([\overline{x}, b])$. Since for the first two terms this is clear, we remain with the last integral. We show in the subsequent lemma in general that this second integral is $C^{m+1}(]\overline{x}, b])$, if $u \in C^m([\overline{x}, b])$.

Lemma 2.1. *Let $u \in C^m([\overline{x}, b])$ and $\tilde{\Gamma}(x, \xi) \in C^\infty(\{(x, \xi) | \overline{x} \le x \le b, \overline{x} \le \xi \le x\})$, where C^∞ means here again continuity of all partial derivatives. Then*

$$\int_{\overline{x}}^{x} \tilde{\Gamma}(x, \xi) \sqrt{1 - \frac{\xi}{x}}\, u(\xi)\, d\xi \in C^{m+1}(]\overline{x}, b]), \quad \overline{x} > 0.$$

Proof. Since $x \neq 0$, it suffices to show the assertion for

$$\int_{\overline{x}}^{x} \sqrt{x-\xi}\,u(\xi)\,d\xi = \int_{0}^{x-\overline{x}} \sqrt{z}\,u(x-z)\,dz.$$

m-times differentiation yields

$$\sum_{k=0}^{m-1} \delta_k (x-\overline{x})^{\frac{3}{2}+k-m} u^{(k)}(\overline{x}) + \int_{0}^{x-\overline{x}} \sqrt{z}\,u^{(m)}(x-z)\,dz,$$

or

$$\sum_{k=0}^{m-1} \delta_k (x-\overline{x})^{\frac{3}{2}+k-m} u^{(k)}(\overline{x}) + \int_{\overline{x}}^{x} \sqrt{x-\xi}\,u^{(m)}(\xi)\,d\xi,$$

for some constants δ_k. Differentiation once again gives

$$\sum_{k=0}^{m-1} (\frac{3}{2}+k-m)\delta_k (x-\overline{x})^{\frac{1}{2}+k-m} u^{(k)}(\overline{x}) + \int_{\overline{x}}^{x} \frac{1}{2\sqrt{x-\xi}} u^{(m)}(\xi)\,d\xi,$$

which is clearly in $C(]\overline{x}, b])$. □

Applying the Lemma for $m = 0$, gives $u \in C^1(]\overline{x}, b])$. But since the point $\overline{x}$ can be chosen arbitrarily - as long as it is near enough to 0 - and the solutions v and therefore u are unique on the intervals $[0, \overline{x}]$ and $[\overline{x}, b]$, we get $u \in C^1([0, b])$.

By induction we finally arrive at $u \in C^n([0, b])$, and therefore, using Proposition 2.1, $v \in C^n([0, b])$. We note that actually $\phi \in C^{n+1}([0, b])$ is sufficient for all our arguments, and apply the closed graph theorem in order to get the asserted inequality of the theorem. Indeed, we have to show that, if $\phi_k(x) \to \phi(x)$ in $C^{n+1}([0, b])$ for $k \to \infty$, and the same holds true for $v(\phi_k)$ in $C^n([0, b])$, where $v(\phi_k)$ denotes the solution of the equation with inhomogeneity $\phi_k(x)$, then v fulfills the equation with $\phi(x)$ as inhomogeneity. But this is clearly true. □

In the following corollary we formulate a C^∞-result for an inhomogeneity tending to zero faster than any power function (for $x \to 0$).

Corollary 2.1. *Assume, in addition to the* Standing Assumptions *on f and Γ, $\phi \in C^\infty([0,b])$ and $\phi^{(n)}(0) = 0$ for all $n \in \mathbb{N}_0$.*

Then we have for the solution v of equation (7)

$$||v||_{C^n} \le C_n ||\phi||_{C^{n+1}}, \quad \textit{for all } n \in \mathbb{N},$$

and some positive constants C_n. Moreover $v \in C^\infty([0,b])$.

Proof. We simply choose $r^* = n = s$ sufficiently large, s.t. we have $\Gamma_2(0,0) < \frac{1}{\gamma_{n,n}}$ (this is always possible since $\lim_{n\to\infty} \gamma_{n,n} = 0$. To see this, observe that $\ln(\frac{1}{\gamma_{n,n}}) = \sum_{k=1}^{n+1} \ln(\frac{2k+1}{2k}) > \frac{1}{2}\sum_{k=1}^{n+1} \frac{1}{2k}$, which tends to infinity for $n \to \infty$.) and apply Theorem 2.1. □

Remark: Note that, if we replace the first 1 by -1 in our integral equation (7)(this is the Case (32") in Evans' paper [6]), one gets in general - besides the non-uniqueness of the solution - a completely different behavior with respect to smoothness of the solution. E.g., if we set in our equation $\Gamma_2 \equiv 0$ it is easy to see that our method gives a C^∞ solution for the + sign, if the inhomogeneity is C^∞. This is not the case for the other sign, since we have the following example.

Example: $(xv)' = x + 2v$, which has the solution $v(x) = Dx + x\ln(x)$ for some constant D. We postpone this case to future research.

We start now with the proof of the central tool in the proof of our main theorem, namely

Proof of Proposition 2.1. Similarly as in [6], we construct an approximating sequence for the solution. Let

$$v(x) = v_{(0)}(x) + \overline{v}_{(1)}(x),$$

with

$$f(x)v_{(0)}(x) = \phi(x) - \int_0^x v_{(0)}(\xi)\,d\xi.$$

Hence

$$f(x)\overline{v}_{(1)}(x) = \phi_{(1)}(x) - \int_0^x (1 + \Gamma(x,\xi))\,\overline{v}_{(1)}(\xi)\,d\xi,$$

with $\phi_{(1)}(x) = -\int_0^x \Gamma(x,\xi)v_{(0)}(\xi)\,d\xi$.

In the same way

$$\overline{v}_{(1)}(x) = v_{(1)}(x) + \overline{v}_{(2)}(x),$$

with

$$f(x)v_{(1)}(x) = \phi_{(1)}(x) - \int_0^x v_{(1)}(\xi)\,d\xi,$$

and

$$f(x)\overline{v}_{(2)}(x) = \phi_{(2)}(x) - \int_0^x (1+\Gamma(x,\xi))\,\overline{v}_{(2)}(\xi)\,d\xi,$$

where $\phi_{(2)}(x) = -\int_0^x \Gamma(x,\xi)v_{(1)}(\xi)\,d\xi$.

Continuing in this way one gets

$$\overline{v}_{(m)}(x) = v_{(m)}(x) + \overline{v}_{(m+1)}(x), \tag{13}$$

with

$$f(x)v_{(m)}(x) = \phi_{(m)}(x) - \int_0^x v_{(m)}(\xi)\,d\xi, \tag{14}$$

and

$$f(x)\overline{v}_{(m+1)}(x) = \phi_{(m+1)}(x) - \int_0^x (1+\Gamma(x,\xi))\,\overline{v}_{(m+1)}(\xi)\,d\xi, \tag{15}$$

where

$$\phi_{(m+1)}(x) = -\int_0^x \Gamma(x,\xi)v_{(m)}(\xi)\,d\xi. \tag{16}$$

The solution of (14) is unique and given by the formula (see [6], eq. (29))

$$v_{(m)}(x) = \frac{1}{f(x)} e^{\int_x^b \frac{d\xi}{f(\xi)}} \int_0^x \phi'_{(m)}(\xi) e^{-\int_\xi^b \frac{dz}{f(z)}}\,d\xi. \tag{17}$$

From (16) we infer

$$\phi'_{(m+1)}(x) = -\frac{d}{dx}\left(\int_0^x \Gamma(x,\xi)v_{(m)}(\xi)\,d\xi\right).$$

Using the definition $\Psi_{(m)}(x) = \phi'_{(m)}(x)$, we arrive at

$$v_{(m)}(x) = \frac{1}{f(x)} e^{\int_x^b \frac{d\xi}{f(\xi)}} \int_0^x \Psi_{(m)}(\xi) e^{-\int_\xi^b \frac{dz}{f(z)}}\, d\xi, \tag{18}$$

and

$$\Psi_{(m+1)}(x) = -\frac{d}{dx}\left(\int_0^x \Gamma(x,\xi) v_{(m)}(\xi)\, d\xi\right). \tag{19}$$

So the procedure works as follows

$$\Psi_{(0)} = \phi'_{(0)} = \phi' \to v_{(0)} \to \Psi_{(1)} \to v_{(1)} \ldots,$$

where the arrows mean mappings given by equation (18) and (19), respectively. We will need some asymptotic properties of the operators, defined by equation (18) and (19). These will be provided by the following two lemmata, the proof of which we postpone to the next section.

Lemma 2.2. *Consider*

$$f(x)v(x) = \phi(x) - \int_0^x v(\xi)\, d\xi.$$

Assume that

$$\phi(x) \in C^{s+1}([0,b]), \quad \phi(0) = 0, \quad r := \min\{\inf\{k|\phi^{(k+1)}(0) \neq 0\}, s\},$$

and f as in the Standing assumptions *at the beginning of section 2. We then have for $0 \le x \le b$*

$$\sup_{0\le s\le x} |v^{(n)}(s)| \le \begin{cases} F_{nn}(x) \sup_{0\le s\le x} |\Psi^{(n)}(s)| & \text{if } 0 \le n \le r \\ \sum_{i=r}^n F_{ni}(x) \sup_{0\le s\le x} |\Psi^{(i)}(s)| & \text{if } r < n \le s, \end{cases}$$

where the F_{ni} are nonnegative continuous functions (depending only on f) on $[0,b]$ and

$$F_{nn}(0) = \frac{1}{1+(n+1)f_1}$$

holds with $f_1 = f'(0)$. *We also have*

$$v(x) \in C^s([0,b]).$$

Lemma 2.3. *Consider*

$$\Psi(x) := -\frac{d}{dx}\int_0^x \Gamma(x,\xi)v(\xi)\,d\xi.$$

Let $\Gamma(x,\xi)$ *be as in the* Standing Assumptions, *and let*

$$v \in C^s([0,b]), \quad r := \min\{\inf\{k | v^{(k)}(0) \neq 0\}, s\}.$$

Then we have for $0 \leq x \leq b$*:*

$$\sup_{0\leq s\leq x} |\Psi^{(n)}(s)| \leq \begin{cases} G_{nn}(x)\sup_{0\leq s\leq x}|v^{(n)}(s)| & \text{if } 0 \leq n \leq r \\ \sum_{l=r}^{n} G_{nl}(x)\sup_{0\leq s\leq x}|v^{(l)}(s)| & \text{if } r < n \leq s, \end{cases}$$

for some nonnegative continuous functions G_{nl} *(depending only on* Γ*) on* $[0,b]$. *Moreover* $G_{nl}(x)$ *can be made arbitrarily small if*

$$\max_{i+j\leq n-l}\sup_{0\leq s\leq x, 0\leq\xi\leq s}\left|\frac{\partial^i}{\partial\xi^i}\frac{\partial^j}{\partial s^j}\Gamma_1(s,\xi)\right| + \max_{i+j\leq n-l}\sup_{0\leq s\leq x, 0\leq\xi\leq s}\left|\frac{\partial^i}{\partial\xi^i}\frac{\partial^j}{\partial s^j}\Gamma_2(s,\xi)\right|$$

is small enough, and if x *is small enough. In addition* $G_{nn}(0) = \gamma_{n,n}\Gamma_2(0,0)$ *holds, where* $\gamma_{n,n}$ *is defined in Theorem* 2.1.

Finally we also have

$$\Psi(x) \in C^s([0,b]).$$

We continue with the proof of Proposition 2.1 and apply now Lemma 2.2 resp. 2.3 on the mappings $\Psi_{(m)} \to v_{(m)}$ respectively $v_{(m)} \to \Psi_{(m+1)}$. Note first that "the r" of $\Psi_{(m)}$ and $v_{(m)}$ always fulfill $r \geq r^*$ during the iteration. To get an estimate for $||v_{(m)}||_{C^n}$ in terms of the norm of the inhomogeneity $||\Psi_{(m)}||_{C^n}$ we use Lemma 2.2. It allows to find for all $\epsilon > 0$ an $x_1(\epsilon)$, s.t.

$$(21)\qquad ||v_{(m)}||_{C^n([0,x_1(\epsilon)])} \leq \begin{cases} (1+\epsilon)||\Psi_{(m)}||_{C^n([0,x_1(\epsilon)])} & \text{if } n = r^* \\ K_n||\Psi_{(m)}||_{C^n([0,x_1(\epsilon)])} & \text{if } r^* < n \leq s, \quad K_n > 0. \end{cases}$$

In a similar way one gets, by considering now the operator mapping $v_{(m)}$ to $\Psi_{(m+1)}$ via equation (19), and using Lemma 2.3

$$(22)\qquad ||\Psi_{(m+1)}||_{C^n([0,x_2(\epsilon)])} \leq \begin{cases} (\epsilon + \Gamma_2(0,0)\gamma_{r^*,r^*})||v_{(m)}||_{C^n([0,x_2(\epsilon)])} & \text{if } n = r^* \\ \tilde{q}||v_{(m)}||_{C^n([0,x_2(\epsilon)])} & \text{if } r^* < n \leq s, \end{cases}$$

where $\tilde{q} < 1$ holds if $\gamma_{r^*,n}$ in assumption (9) is chosen large enough. Combining (21) and (22), we arrive at

$$||v_{(m)}||_{C^n([0,x_3(q)])} \leq q||v_{(m-1)}||_{C^n([0,x_3(q)])}$$
$$\begin{cases} \text{if } n = r^* & \text{and } \Gamma_2(0,0)\gamma_{r^*,r^*} < 1 \\ \text{if } r^* < n \leq s & \text{and } \sup_{i+j \leq n-r^*} \left| \frac{\partial^i}{\partial x^i}\frac{\partial^j}{\partial \xi^j}\Gamma_2(0,0)\right| < \frac{1}{\gamma_{r^*,n}} \end{cases}$$

for a $q < 1$. We define now

$$v = \sum_{k=0}^{\infty} v_{(k)}.$$

Obviously this gives

$$||v||_{C^n([0,x_3])} \leq C||\Psi||_{C^n([0,x_3])},$$

or

$$||v||_{C^n([0,x_3])} \leq C||\phi||_{C^{n+1}([0,x_3])},$$

for some positive constant C.

The fact that v fulfills the integral equation is shown analogously to the argument given by Evans. For convenience of the reader we repeat it. Let

$$v = v_{(0)} + v_{(1)} + v_{(2)} + \cdots + v_{(k)} + V_{(k+1)},$$

and plug in the expression

$$L(x) := f(x)v(x) - \phi(x) + \int_0^x (1 + \Gamma(x,\xi))\, v(\xi)\, d\xi.$$

The result is

$$\begin{aligned} L(x) = f(x)V_{(k+1)}(x) - \phi(x) + \int_0^x (1+\Gamma(x,\xi))\, V_{(k+1)}(\xi)\, d\xi \\ + f(x)v_{(0)}(x) + \int_0^x \Gamma(x,\xi) v_{(0)}(\xi)\, d\xi + \int_0^x v_{(0)}(\xi)\, d\xi \\ + \cdots \\ + f(x)v_{(k)}(x) + \int_0^x \Gamma(x,\xi) v_{(k)}(\xi)\, d\xi + \int_0^x v_{(k)}(\xi)\, d\xi, \end{aligned}$$

or

$$\begin{aligned} L(x) = f(x)V_{(k+1)}(x) - \phi_{(k+1)}(x) + \int_0^x (1+\Gamma(x,\xi))\, V_{(k+1)}(\xi)\, d\xi \\ + f(x)v_{(0)}(x) - \phi(x) + \int_0^x v_{(0)}(\xi)\, d\xi \\ + f(x)v_{(1)}(x) - \phi_{(1)}(x) + \int_0^x v_{(1)}(\xi)\, d\xi \\ + \cdots \\ + f(x)v_{(k)}(x) - \phi_{(k)}(x) + \int_0^x v_{(k)}(\xi)\, d\xi, \end{aligned}$$

or

$$L(x) = f(x)V_{(k+1)}(x) - \phi_{(k+1)}(x) + \int_0^x (1+\Gamma(x,\xi))\, V_{(k+1)}(\xi)\, d\xi.$$

Clearly this tends to zero w.r.t. $||\cdot||_{C([0,x_3])}$ for $k \to \infty$. $\square$

3. Asymptotics. In this section we prove the asymptotic properties of the two operators, used in the proof of Proposition 2.1. We start with

Proof of Lemma 2.2. From the discussion in [6], p. 432f, it follows that the continuous solution of our equation and the continuous solution of the ODE

$$(f(x)v(x))' = \Psi(x) - v(x) \tag{23}$$

coincide. $C^{s+1}(]0,b])$-property of this solution is clear from the solution formula (17). So it remains to determine its asymptotic behavior for $x \to 0$. Applying Lemma 4.1 gives immediately the assertion for $n = 0$. Differentiating (23) yields

$$(f(x)v(x))'' + v'(x) = \Psi'(x),$$

or with $m(x) := v'(x)$,

$$f(x)m'(x) + (1 + 2f'(x))m(x) = \Psi'(x) - f''(x)v(x).$$

Lemma 4.1 gives

$$\begin{aligned}
\sup_{0\le s\le x} |m(s)| &\le F_1(x)\left(\sup_{0\le s\le x} |\Psi'(s)| + \sup_{0\le s\le x} |f''(s)v(s)|\right)\\
&\le F_1(x)\left(\sup_{0\le s\le x} |\Psi'(s)| + \sup_{0\le s\le x} |f''(s)| \sup_{0\le s\le x} |\Psi(s)| F_0(x)\right)\\
&= F_1(x) \sup_{0\le s\le x} |\Psi'(s)|\\
&\quad + F_1(x)F_0(x) \sup_{0\le s\le x} |f''(s)| \sup_{0\le s\le x} |\Psi(s)|,
\end{aligned}$$

We now distinguish two cases:

Case a: $r = 0$ In this case we simply rewrite the last right hand side as

$$F_{11}(x) \sup_{0\le s\le x} |\Psi'(s)| + F_{10}(x) \sup_{0\le s\le x} |\Psi(s)|.$$

Case b: $r \ge 1$ In this case we get the following estimate

$$\begin{aligned}
\sup_{0\le s\le x} |m(s)| &\le F_1(x) \sup_{0\le s\le x} |\Psi'(s)| + F_{10}(x) \sup_{0\le s\le x} |\Psi(s)|\\
&\le F_1(x) \sup_{0\le s\le x} |\Psi'(s)| + F_{10}(x) \sup_{0\le s\le x} \int_0^s |\Psi'(t)|\, dt\\
&= F_1(x) \sup_{0\le s\le x} |\Psi'(s)| + F_{10}(x) \int_0^x |\Psi'(t)|\, dt\\
&\le F_1(x) \sup_{0\le s\le x} |\Psi'(s)| + xF_{10}(x) \sup_{0\le s\le x} |\Psi'(s)|
\end{aligned}$$

Defining now $F_{11}(x) := F_1(x) + xF_{10}(x)$ gives, since F_{10} is continuous,

$$\lim_{x\to 0} F_{11}(x) = \frac{1}{1+2f_1}$$

yielding the assertion of the lemma for $n = 1$. Further differentiation and induction completes the proof. $\square$

Proof of Lemma 2.3. In the following G_l will denote generic nonnegative continuous functions on $[0, b]$, which may vary from place to place. These functions depend on Γ_1 and Γ_2 and their derivatives. Moreover $G_l(x)$ can be made arbitrarily small if

$$\max_{i+j\le n-l} \sup_{0\le s\le x, 0\le\xi\le s} \left| \frac{\partial^i}{\partial\xi^i} \frac{\partial^j}{\partial s^j} \Gamma_1(s,\xi) \right| + \max_{i+j\le n-l} \sup_{0\le s\le x, 0\le\xi\le s} \left| \frac{\partial^i}{\partial\xi^i} \frac{\partial^j}{\partial s^j} \Gamma_2(s,\xi) \right|$$

is small enough, and if x is small enough.

We first consider $\frac{d}{dx}\int_0^x \Gamma_1(x,\xi)v(\xi)\,d\xi$. C^s-property is clear and so we estimate now $\frac{d^{n+1}}{dx^{n+1}}\int_0^x \Gamma_1(x,\xi)v(\xi)\,d\xi$.

This last expression can be written as

$$\Gamma_1(x,x)v^{(n)}(x) + \sum_{l=0}^{n-1} c_l g_l(\Gamma_1) v^{(l)}(x) + \int_0^x \left(\frac{\partial^{n+1}}{\partial x^{n+1}} \Gamma_1(x,\xi) \right) v(\xi)\,d\xi,$$

where the g_l depend linearly on the partial derivatives of Γ_1 w.r.t. x and ξ of total order $n - l$ evaluated at (x, x), and c_l are constants. Hence

$$\begin{aligned} \sup_{0\le s\le x} \left| \frac{d^{n+1}}{ds^{n+1}} \int_0^s \Gamma_1(s,\xi)v(\xi)\,d\xi \right| &\le \sup_{0\le s\le x} |\Gamma_1(s,s)| \sup_{0\le s\le x} \left| v^{(n)}(s) \right| \\ &\quad + \sum_{l=0}^{n-1} G_l(x) \sup_{0\le s\le x} \left| v^{(l)}(s) \right| \\ &= G_n(x) \sup_{0\le s\le x} \left| v^{(n)}(s) \right| + \sum_{l=0}^{n-1} G_l(x) \sup_{0\le s\le x} \left| v^{(l)}(s) \right| \end{aligned} \tag{24}$$

with $G_n(0) = 0$, since $\Gamma_1(0,0) = 0$.

The integral part with Γ_2 needs some rearrangement. Assume first $n \le r$. By n-times integration by parts we find

$$\int_0^x \sqrt{1-\frac{\xi}{x}}\Gamma_2(x,\xi)v(\xi)\,d\xi = \frac{1}{\sqrt{x}}\tilde{\gamma}_n \int_0^x (x-\xi)^{n+\frac{1}{2}}\tilde{R}(x,\xi)\,d\xi.$$

Here we used the notation

$$\tilde{R}(x,\xi) = \frac{\partial^n}{\partial\xi^n}R(x,\xi) = \frac{\partial^n}{\partial\xi^n}\left(\Gamma_2(x,\xi)v(\xi)\right),$$

and

$$\tilde{\gamma}_n = \frac{2^n}{3\cdot 5\cdots(2n+1)}.$$

For $r < n \le s$, we get the additional term

$$S(x) = \delta_{r+1}x^{r+1}\frac{\partial^r}{\partial\xi^r}R(x,0) + \ldots + \delta_n x^n \frac{\partial^{n-1}}{\partial\xi^{n-1}}R(x,0).$$

for some constants $\delta_i, r+1 \le i \le n$. We differentiate this additional term $(n+1)$-times, which gives after some calculations

$$\sup_{0\le s\le x}|S^{(n+1)}(s)| \le \sum_{l=r}^{n-1}|v_l|G_l(x). \tag{25}$$

with $v_l = v^{(l)}(0)$. It remains to differentiate the integral term: The expression

$$\frac{d^{n+1}}{dx^{n+1}}\left(\frac{1}{\sqrt{x}}\tilde{\gamma}_n\int_0^x(x-\xi)^{n+\frac{1}{2}}\tilde{R}(x,\xi)\,d\xi\right)$$

consists of terms of the form

$$\text{const.}x^{-\frac{1}{2}-l_1}\int_0^x(x-\xi)^{n+\frac{1}{2}-l_2}\frac{\partial^{l_3}}{\partial x^{l_3}}\tilde{R}(x,\xi)\,d\xi \quad l_1+l_2+l_3 = n+1.$$

If $l_3 \geq 1$, and therefore $l_1 + l_2 \leq n$, we can find the following upper estimate of the absolute value of this term

$$
\begin{aligned}
(26) \quad & \text{const.} x^{-\frac{1}{2}-l_1} \int_0^x (x-\xi)^{n+\frac{1}{2}-l_2}\, d\xi \sum_{l=0}^{n} \sup_{0 \leq s \leq x} |v^{(l)}(s)| H_l(x) \\
& \leq \text{const.} \sum_{l=0}^{n} \sup_{0 \leq s \leq x} |v^{(l)}(s)| H_l(x) x^{-\frac{1}{2}-l_1} \int_0^x (x-\xi)^{n+\frac{1}{2}-l_2}\, d\xi \\
& \leq \text{const.} \sum_{l=0}^{n} x H_l(x) \sup_{0 \leq s \leq x} |v^{(l)}(s)| = \text{const.} \sum_{l=0}^{n} G_l(x) \sup_{0 \leq s \leq x} |v^{(l)}(s)|,
\end{aligned}
$$

where the H_l are some generic nonnegative continuous functions, depending on Γ_2 and its derivatives, and where $G_l(0) = 0$ holds for $l = 0, 1, 2, ..., n$.

On the other hand, for $l_3 = 0$, we find

$$
\begin{aligned}
(27) \quad & \sum_{l_1=0}^{n+1} \binom{n+1}{l_1} \left(-\frac{1}{2}\right)_{l_1} x^{-\frac{1}{2}-l_1} \tilde{\gamma}_n \\
& \int_0^x \left(n+\frac{1}{2}\right)_{l_2} (x-\xi)^{n+\frac{1}{2}-l_2} \sum_{l=0}^{n} c_l \left(\frac{\partial^{n-l}}{\partial \xi^{n-l}} \Gamma_2(x,\xi)\right) v^{(l)}(\xi)\, d\xi \\
& \leq \sum_{l=0}^{n} G_l(x) \sup_{0 \leq s \leq x} |v^{(l)}(s)| \sum_{l_1=0}^{n+1} \binom{n+1}{l_1} \left(-\frac{1}{2}\right)_{l_1} x^{-\frac{1}{2}-l_1} \tilde{\gamma}_n \\
& \qquad \int_0^x \left(n+\frac{1}{2}\right)_{l_2} (x-\xi)^{n+\frac{1}{2}-l_2}\, d\xi \\
& = \sum_{l=0}^{n} G_l(x) \sup_{0 \leq s \leq x} |v^{(l)}(s)| \frac{d^{n+1}}{dx^{n+1}} \left(x^{-\frac{1}{2}} \tilde{\gamma}_n \int_0^x (x-\xi)^{n+\frac{1}{2}}\, d\xi\right) \\
& = \sum_{l=0}^{n} G_l(x) \sup_{0 \leq s \leq x} |v^{(l)}(s)| \tilde{\gamma}_n \frac{(n+1)!}{n+3/2} = \sum_{l=0}^{n} G_l(x) \sup_{0 \leq s \leq x} |v^{(l)}(s)|,
\end{aligned}
$$

where $(a)_l$ denotes Pochhammer symbols, c_l are natural numbers with $c_n = 1$ and where the version of $G_n(0)$ in the last line of (27) fulfills $G_n(0) = \gamma_{n,n}\Gamma_2(0,0)$.

Combining (24),(25), (26) and (27) we get

$$\sup_{0\le s\le x} |\Psi^{(n)}(s)| \le \sum_{l=0}^{n} G_l(x) \sup_{0\le s\le x} |v^{(l)}(s)|$$

with $G_n(0) = \gamma_{n,n}\Gamma_2(0,0)$.

Finally, since $v_0 = v_1 = ... = v_{r-1} = 0$, we arrive at

$$\sup_{0\le s\le x} |\Psi^{(n)}(s)| \le \begin{cases} G_n(x)\sup_{0\le s\le x}|v^{(n)}(s)| & \text{if } 0\le n\le r \\ \sum_{l=r}^{n} G_l(x)\sup_{0\le s\le x}|v^{(l)}(s)| & \text{if } r<n\le s, \end{cases}$$

with $G_n(0) = \gamma_{n,n}\Gamma_2(0,0)$. Since we considered in this proof only fixed n, we switch now in the notation from G_l to G_{nl}, which concludes the proof. □

4. Appendix .

Lemma 4.1. *All continuous solutions on* $[0, b]$ *of*

$$f(x)m'(x) + (1 + nf'(x))m(x) = \rho(x), \quad n \in \mathbf{N}$$

fulfill

$$\sup_{0\le s\le x} |m(s)| \le \sup_{0\le s\le x} |\rho(s)| F_{n-1}(x).$$

Here $\rho(x)$ *is assumed to be continuous on* $[0, b]$*, and* f *is as in the* Standing Assumptions *of section* 2*. Moreover* $F_{n-1}(x)$ *is a continuous nonnegative function on* $[0, b]$ *with*

$$F_{n-1}(0) = \frac{1}{1 + nf'(0)}$$

Proof. The unique continuous solution $m(x)$ is given by

$$m(s) = \frac{1}{f(s)^n} e^{\int_s^b \frac{d\xi}{f(\xi)}} \int_0^s \rho(\xi) f^{n-1}(\xi) e^{-\int_\xi^b \frac{dz}{f(z)}}\, d\xi.$$

Since f is nonnegative on $[0, b]$ this gives the estimate

$$|m(s)| \leq \sup_{0\leq\xi\leq s} |\rho(\xi)| \frac{1}{f^n(s)} e^{\int_s^b \frac{d\xi}{f(\xi)}} \int_0^s f^{n-1}(\xi) e^{-\int_\xi^b \frac{dz}{f(z)}}\, d\xi$$

or

$$\sup_{0\leq s\leq x} |m(s)| \leq \sup_{0\leq\xi\leq x} |\rho(\xi)| \sup_{0\leq s\leq x} \frac{1}{f^n(s)} e^{\int_s^b \frac{d\xi}{f(\xi)}} \int_0^s f^{n-1}(\xi) e^{-\int_\xi^b \frac{dz}{f(z)}}\, d\xi.$$

The last right hand side can be written as

$$\sup_{0\leq\xi\leq x} |\rho(\xi)| F_{n-1}(x),$$

if we define $F_{n-1}(x) := \sup_{0\leq s\leq x} \tilde{F}_{n-1}(s)$ and

$$\tilde{F}_{n-1}(s) := \frac{1}{f^n(s)} e^{\int_s^b \frac{d\xi}{f(\xi)}} \int_0^s f^{n-1}(\xi) e^{-\int_\xi^b \frac{dz}{f(z)}}\, d\xi$$

A simple application of de l'Hospitals rule shows

$$\lim_{x\to 0} \tilde{F}_{n-1}(x) = \frac{1}{1 + nf'(0)}.$$

Clearly the same holds true for $F_{n-1}(x)$, which proves the lemma. □

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