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WEIGHTED GENERALIZED HÖLDER SPACES AS WELL-POSEDNESS CLASSES FOR SONINE INTEGRAL EQUATIONS

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ABSTRACT. For integral equations of the first kind

$$\mathbb{K}\varphi := \int_0^x k\,(x-t)\,\varphi\,(t)\,dt = f\,(x)\,, \quad x \in (0,b)$$

where $0 < b < \infty$, in the case of a certain class of almost decreasing Sonine kernels $k(t)$ we prove weighted estimates of continuity moduli $\omega(\mathbb{K}\varphi, h)$ and $\omega(\mathbb{K}^{-1}f, h)$. This allows us to show that the weighted generalized Hölder spaces $H^{\omega}(\rho)$ and $H^{\omega_1}(\rho)$ are suitable well-posedness classes for these integral equations of the first kind under the choice $\omega_1(h) = hk(h)\omega(h)$.

1. Introduction.

We consider integral equations of the first kind

$$\mathbb{K}\varphi := \int_0^x k\,(x-t)\,\varphi\,(t)\,dt = f\,(x)\,, \quad x \in (0,b), \tag{1.1}$$

where $0 < b < \infty$, and $k(x) \in L_1(0,b)$.

As is well known, one of the main problems for integral equations of the first kind is to find "nice" well-posedness classes. Spaces of integrable functions do not suit well for this purpose in the following

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sense: when looking for solutions φ, for instance, in the space L_{p_1}, the range $\mathbb{K}(L_{p_1})$ does not coincide with any space L_{p_2}. Therefore, the scale of the spaces L_p, as it is, cannot provide well-posedness classes: the range $\mathbb{K}(L_{p_1})$, if imbedded into a certain L_{p_2}, is usually a subset of L_{p_2}. The same is true for weighted L_p-spaces.

We show – for a rather wide class of kernels – that there exists a scale of spaces, within which it is possible to have both the space for solutions φ and the space of right-hand sides f. For this goal we consider spaces of functions continuous for $x > 0$ with possible singularity at $x = 0$ - the situation rather typical for applications. The continuity properties of functions will be characterized in terms of their continuity modulus, while behavior at the origin will be described in terms of weight functions "fixed" to the origin. That is, we consider the generalized Hölder spaces $H_0^\omega(\rho)$ (see definitions in Subsection 2.3).

The main result of the paper is the following: given ω and ρ from certain classes, there exists an exact isomorphism:

$$\mathbb{K}(H_0^\omega(\rho)) = H_0^{\omega_1}(\rho), \quad \text{where} \quad \omega_1(x) = xk(x)\omega(x). \tag{1.2}$$

This isomorphism is proved for a certain class of positive almost decreasing Sonine kernels. We recall that a kernel $k(x) \in L_1(0, b)$ is called a *Sonine kernel*, if it is a divisor of the unit with respect to the operation of convolution, that is, there exists a kernel $\ell(x) \in L_1(0, b)$ such that the relation

$$\int_0^x \ell\,(x-t)\,k\,(t)\,dt = 1, \tag{1.3}$$

is valid for almost all $x \in (0, b)$.

We refer to the original papers [**20, 21**] by N.Sonine, the paper [**11**] on imbedding theorems for ranges of operators of form (**1.1**), including the case of Sonine kernels, and recent papers [**15, 16**] on inversion of equations with Sonine kernels within the framework of L_p-spaces.

The class of Sonine kernels is sufficiently wide. We refer to [**15, 16**] for classical examples which typically involve weighted special functions with singularity at the origin; as shown in [**11**], any kernel for instance of the form $a(x)x^{\alpha-1}\ln^m \frac{2b}{x}$, $0 < \alpha < 1$, where $a(x)$ is an absolutely continuous function with $a(0) \neq 0$, is a Sonine kernel.

Isomorphism **(1.2)** was known earlier for the simplest example of Sonine kernels, $k(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}, 0 < \alpha < 1$, which corresponds to fractional integration operator $\mathbb{K} = I^{\alpha}_{0+}$. In the case of power characteristics $\omega(x) = x^{\lambda}$ the embedding $I^{\alpha}_{0+}(H^{\lambda}_0) \to H^{\lambda+\alpha}_0, \lambda + \alpha < 1$, in the non-weighted case goes back to G.Hardy and J.Littlewood [**3**] (see [**17**], Theorem 3.1). The isomorphism $I^{\alpha}_{0+}(H^{\lambda}_0(\rho)) = H^{\lambda+\alpha}_0(\rho)$ with power weight ρ was proved in [**10, 12**], see [**17**], Theorem 13.13. A simpler proof was given in [**5**]. An extension to general characteristics $\omega(x)$ for the same example $k(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}$ was given in [**7, 8, 9, 18**] (see [**17**], Theorems 13.15-13.18), and in [**6**]; such an isomorphism for general weights was proved in [**19**], Theorem 6.

The case of general kernels was considered in [**19**], where only the imbedding $\mathbb{K}(H^{\omega}_0(\rho)) \to H^{\omega_1}_0(\rho)$ was studied in terms somewhat different from those in this paper. One of the results obtained in this paper, Theorem $\mathbf{C}_1$, is a certain refinement of imbedding theorem proved in [**19**].

The main goal of this paper is to establish the exact isomorphism (1.2) based on the study of properties of operators inverse to Sonine integral operators in [**15, 16**] and technique of weighted estimations of continuity moduli developed in [**19**].

N o t a t i o n :

$C_0([0,b]) = \{f \in C([0,b]) : \ f(0) = 0\}$;

$H^{\omega}_0(\rho)$, see (2.12);

V_{λ} is defined by Definition 2.4;

W is defined in (2.1);

W_{μ} is defined by Definition 2.2;

Z^0, Z_1, see (2.7);

Φ is the Zygmund-Bari-Stechkin class, see Definition 2.7;

2. Preliminaries. Throughout this paper b will denote a fixed positive number.

2.1. *Classes W_{μ} and V_{λ}.* The following definition goes back to S.Bernstein [**2**].

Definition 2.1. A non-negative function $f(x)$ defined on an interval $[0,b]$ is called almost increasing (**a.i.**) on this interval, or almost decreasing (**a.d.**), respectively, if there exists a constant $C \geq 1$ such that

$$f(x) \leq Cf(y) \qquad \text{for all} \qquad 0 \leq x \leq y \leq b,$$

$$f(y) \leq Cf(x) \qquad \text{for all} \qquad 0 \leq x \leq y \leq b,$$

respectively.

We denote for brevity

$$\begin{aligned} W &= W([0,b]) \\ &= \{f \in C_0([0,b]) : \ f(x) > 0, x > 0, \ f(x) \text{ is a.i. on } [0,b].\} \end{aligned} \tag{2.1}$$

As in [**19**], we introduce the following class of weight functions.

Definition 2.2. By $W_\mu = W_\mu([0,b]), \mu > 0$, we denote the class of functions $\rho \in W([0,b])$, which have the properties:

(1) $\frac{\rho(x)}{x^\mu}$ is a.d.;

(2) there exists a constant $C > 0$ such that

$$\left|\frac{\rho(x)-\rho(y)}{x-y}\right| \leq C\frac{\rho(x^*)}{x^*}, \quad x^* = \max(x,y), \quad x,y \in [0,b]. \tag{2.2}$$

Property (1) of functions $\rho \in W_\mu$, that is,

$$\rho(x) \leq C\left(\frac{x}{y}\right)^\mu \rho(y), \quad 0 < y \leq x \leq b, \tag{2.3}$$

will be often used in the sequel, as well as property (2). The latter, in the case $0 < \mu \leq 1$ is equivalent to

$$\left|\frac{\rho(x)-\rho(y)}{x-y}\right| \leq C\min\left\{\frac{\rho(x)}{x}, \frac{\rho(y)}{y}\right\}. \tag{2.4}$$

Note that inequality (2.4) in the case $0 < \mu \leq 1$ is satisfied automatically with $C = \mu$ if $\rho(x)$ is increasing (not just almost increasing) and $\frac{\rho(x)}{x^\mu}$ is decreasing (not just almost decreasing).

Lemma 2.3. *Let* $\rho \in W_\mu([0,b]), \mu > 0$*. Then*

$$\left|\frac{\rho(x)-\rho(y)}{\rho(y)}\right| \leq C\left(\frac{x}{y}\right)^{\gamma-1}\frac{x-y}{y}, \quad 0 < y \leq x \leq b, \tag{2.5}$$

where $\gamma = \max(1,\mu)$.

Proof. Let $0 < \mu \leq 1$. By (2.4) we have $|\rho(x)-\rho(y)| \leq C(x-y)\frac{\rho(y)}{y}$ which yields (2.5). Let $\mu \geq 1$. Then by (2.2) and (2.3) we get

$$\left|\frac{\rho(x)-\rho(y)}{\rho(y)}\right| \leq C\frac{x-y}{x}\left(\frac{x}{y}\right)^{\mu} = C\frac{x-y}{y}\left(\frac{x}{y}\right)^{\mu-1},$$

which is (2.5) for $\mu \geq 1$. ◻

We also need the following class of positive a.d. kernels bounded beyond the origin introduced in [19] (note that the condition (2.2) in [19] must be read as condition (2.6) below).

Definition 2.4. A non-negative kernel $k(x)$ is said to belong to the class $V_\lambda = V_\lambda([0,b]),\ \lambda > 0,$ if

(1) $k(x) > 0$ for $0 < x \leq b$;

(2) $x^\lambda k(x)$ is a.i. on $[0,b]$;

(3) $x^{\lambda-\varepsilon}k(x)$ is a.d. on $[0,b]$ for every $\varepsilon > 0$;

(4) condition of the type (2.2) is satisfied:

$$\left|\frac{k(x+h)-k(x)}{h}\right| \leq C\frac{k(x)}{x+h}, h > 0 \quad \text{for all} \quad x, x+h \in [0,b]. \tag{2.6}$$

From conditions (2-3) of the above definition it follows that kernels $k(x) \in V_\lambda$ have the properties

$$k(y) \leq Ck(x)\left(\frac{x}{y}\right)^{\lambda}, \qquad 0 < y \leq x \leq b, \tag{2.7}$$

$$k(y) \le Ck(x)\left(\frac{x}{y}\right)^{\lambda-\varepsilon}, \quad 0 < x \le y \le b \tag{2.8}$$

for every $\varepsilon > 0$.

Observe that condition (2.2) fits to a.i. functions, while condition (2.6) fits to a.d. functions, which may be easily seen by power examples: $\rho(x) = x^{\mu}$, $\mu > 0$ and $k(x) = x^{-\alpha}$, $\alpha > 0$. Note also that the power kernel $k(x) = \frac{1}{x^{\alpha}}$, $\alpha > 0$, belongs to any V_{λ} with $\lambda > \alpha$. The same is true for power-logarithmic kernels

$$k(x) = \frac{\left(\ln \frac{2b}{x}\right)^{\theta}}{x^{\alpha}}, \qquad b < \infty$$

with any exponent θ. Condition (2.6) is satisfied for a wide class of a.d. functions, see Section 9.

Remark 2.5. If a non-negative function $k(x)$ satisfies condition (2.6) and there exists $k'(x)$, then

$$|k'(x)| \le C\frac{k(x)}{x}, \quad 0 < x \le b. \tag{2.9}$$

Lemma 2.6. *The inequality*

$$f(x) \le C\int_x^b \frac{f(t)}{t}\,dt, \quad 0 < x \le \frac{b}{2} \tag{2.10}$$

with the constant $C > 0$ not depending on f, holds for all non negative functions $f(x)$ on $[0,b]$ such that there exists a $\lambda \in \mathbb{R}^1$ such that $x^{\lambda} f(x)$ is a.i. on $([0,b])$.

Proof. The proof is direct:

$$\int_x^b \frac{f(t)}{t}\,dt \;\ge\; Cx^{\lambda} f(x)\int_x^b \frac{dt}{t^{1+\lambda}} \;\ge\; Cx^{\lambda} f(x)\int_x^{2x} \frac{dt}{t^{1+\lambda}} \;=\; Cf(x).$$

□

2.2. *Zygmund-Bari-Stechkin class* Φ. The class Φ defined below was introduced in [1] and is known as Bari-Stechkin or Zygmund-Bari-Stechkin class, see also a study of properties of functions $\omega \in \Phi$ in **[4, 13, 14]**.

Definition 2.7. The class Φ is defined as $\Phi = \Phi([0,b]) := \mathcal{Z}^0 \cap \mathcal{Z}_1$, where $\mathcal{Z}^0 = \mathcal{Z}^0([0,b])$ is the class of functions $\omega \in W([0,b])$ satisfying the condition

$$(\mathbb{Z}^\beta) \qquad \int_0^h \frac{\omega(x)}{x} dx \leq c\omega(h), \quad 0 < h \leq b$$

and $\mathcal{Z}_1 = \mathcal{Z}_1([0,b])$ is the class of functions $\omega \in W([0,b])$ satisfying the condition

$$(\mathbb{Z}_\gamma) \qquad \int_h^b \frac{\omega(x)}{x^2} dx \leq c\frac{\omega(h)}{h}, \quad 0 < h \leq b$$

where $c = c(\omega) > 0$ does not depend on $h \in (0,b]$.

It is known ([1]) that there exist exponents $0 < \lambda_1 < \lambda_2 < 1$ and constants $c_1 > 0, c_2 > 0$ such that

$$c_1 x^{\lambda_2} \leq \omega(x) \leq c_2 x^{\lambda_1}, \quad 0 \leq x \leq b. \tag{2.11}$$

As shown in [13] (see also [14]), in (2.11) one may take any $\lambda_1 < m_\omega$ and any $\lambda_2 > M_\omega$, where the numbers

$$m_\omega = \sup_{x>1} \frac{\ln \left[\underline{\lim}_{h\to 0} \frac{\omega(xh)}{\omega(h)}\right]}{\ln\ x}, \quad M_\omega = \inf_{x>1} \frac{\ln \left[\overline{\lim}_{h\to 0} \frac{\omega(xh)}{\omega(h)}\right]}{\ln\ x}$$

$m_\omega \leq M_\omega$, are known as the lower and upper indices of a function $\omega \in W$. Besides this, the membership of a function $\omega \in W$ in the Bari-Stechkin class Φ is characterized by the condition $m_\omega > 0, M_\omega < 1$.

2.3. *Weighted generalized Hölder spaces.* Let

$$\omega(f,h) = \max_{\substack{x,y\in[0,b] \\ |x-y|\leq h}} |f(x) - f(y)|$$

be the continuity modulus of a function f. By $H^\omega = H^\omega([0,b])$ we denote the generalized Hölder space

$$H^\omega = \{f(x): \ \omega(f,h) \le c\omega(h), \quad 0<h<b\}.$$

The function $\omega(h)$, referred to in the sequel as *the characteristic function of the space, or characteristic*, will be supposed to belong to the Zygmund-Bari-Stechkin class Φ.

We define the weighted space $H_0^\omega(\rho)$ as

$$H_0^\omega(\rho) \ = \ \Big\{f(x) : \rho(x)f(x) \in H^\omega \ , \quad \lim_{x\to 0} \rho(x)f(x) = 0\Big\}. \tag{2.12}$$

When equipped with the norm

$$\|f\|_{H_0^\omega(\rho)} = \|\rho f\|_{H_0^\omega} = \|\rho f\|_{C([0,b])} + \sup_{h>0} \frac{\omega(\rho f, h)}{\omega(h)},$$

this is a Banach space.

2.4. *The operator inverse to a Sonine operator.* In [16] (see also [15] for the case $b=\infty$) there was constructed the operator inverse to a Sonine operator under the following assumptions on the initial Sonine kernel $k(t)$ and its associate Sonine kernel $\ell(t)$:

monotonicity near the origin: *there exists a neighborhood* $0 < x < \varepsilon_0$ *where*

$$k(x) \ge 0, \quad \ell(x) \ge 0 \quad \text{and} \quad k(x)\downarrow, \quad \ell(x)\downarrow, \quad 0<x\le\varepsilon_0. \tag{2.13}$$

absolute integrability of $k'(x)$ and $\ell'(x)$ beyond the origin: it is assumed that derivatives exist in the generalized sense and

$$\int_\delta^b |k'(x)|\,dx < \infty, \quad \int_\delta^b |\ell'(x)|\,dx < \infty. \tag{2.14}$$

for any $0<\delta<b$.

The expression for the inverse operator generalizes the known Marchaud form ([17], p.224) of fractional differentiation; it has the form

$$\mathbb{K}^{-1}f := \lim_{\varepsilon\to 0}\mathbb{K}_\varepsilon^{-1}f = \ell(x)f(x) + \lim_{\varepsilon\to 0}\int\limits_\varepsilon^x \ell'(t)[f(x-t)-f(x)]\,dt. \tag{2.15}$$

The following results were obtained in [16].

Theorem 2.8. *Let $k(x)$ be a Sonine kernel satisfying assumptions* (2.13) *and* (2.14) *on $[0,b], 0 < b < \infty$. Then for any $f = \mathbb{K}\varphi$ with $\varphi \in L_p(0,b), 1 < p < \infty$ the inversion is given by $\varphi(x) = \mathbb{K}^{-1}f$, where the convergence of the integral in $\mathbb{K}^{-1}f = \lim\limits_{\varepsilon\to 0}\mathbb{K}_\varepsilon^{-1}f$ is treated in the L_p-sense:*

$$\lim_{\varepsilon\to 0}\left\|\mathbb{K}_\varepsilon^{-1}f - \varphi\right\|_{L_p(0,b)}. \tag{2.16}$$

Theorem 2.9. *Let a Sonine kernel $k(x)$ satisfy assumptions* (2.13) *and* (2.14) *on $[0,b], 0 < b < \infty$. A function $f \in L_1(0,b)$ belongs to the range $\mathbb{K}(L_p)$, $1 < p < \infty$, if and only if*

$$\ell(x)f(x) \in L_p(0,b) \tag{2.17}$$

and one of the following conditions is fulfilled:

$$\lim_{\substack{\varepsilon\to 0\\(L_p)}}\Psi_\varepsilon f \in L_p(0,b) \quad \text{or} \quad \sup_{0<\varepsilon<b}\|\Psi_\varepsilon f\|_{L_p(0,b)} < \infty. \tag{2.18}$$

where

$$\Psi_\varepsilon f(x) = \int\limits_\varepsilon^x \ell'(t)[f(x-t)-f(x)]\,dt \quad \text{for} \quad x > \varepsilon$$

and $\quad \Psi_\varepsilon f(x) = 0 \quad$ otherwise.

3. Formulation of the main results. The main contributions of this paper are Theorems **A-D**.

Theorem **A** gives an estimate of Zygmund type which characterizes the "improvement" of the behavior of the continuity modulus of a function f after the application of the operator $\mathbb{K}$ or its weighted version $\rho\mathbb{K}\frac{1}{\rho}$ to f. Theorem **B** shows worsening of the continuity modulus of a function f after the application of the inverse operator $\mathbb{K}^{-1}$ or $\rho\mathbb{K}^{-1}\frac{1}{\rho}$ to f.

These estimates allow us to obtain in Theorems $\mathbf{C}_1$ and $\mathbf{C}_2$ results on mapping properties of the operators $\mathbb{K}$ and $\mathbb{K}^{-1}$ within the frameworks of weighted generalized Hölder spaces. The statements of Theorems $\mathbf{C}_1$ and $\mathbf{C}_2$ are exact in the sense that they allow us to obtain in Theorem D a statement on the isomorphism between the spaces $H_0^{\omega}(\rho)$ and $H_0^{\omega_1}(\rho)$ realized by the Sonine operator $\mathbb{K}$.

Remark 3.1. Zygmund type estimates (3.2, 3.3) and (3.6, 3.7) in Theorems A and B are understood in the usual sense: they are valid under the assumption that the right-hand-sides of the estimates exist; in (3.6, 3.7), for instance, this implies that

$$\int_0^{\delta} \frac{\ell(t)\omega(f,t)}{t^{\max(1,\mu)}}dt < \infty \quad \text{for some} \quad \delta > 0. \tag{3.1}$$

Theorem A. *Let $k(x) \in V_\lambda, 0 < \lambda < 1$, and $\varphi(x) \in C_0([0,b])$. Then*

$$\omega\left(\mathbb{K}\varphi, h\right) \leq Chk(h)\omega(\varphi,h) + ch\int_h^b \frac{k(t)\ \omega(\varphi,t)}{t}dt, \qquad 0 < h \leq b. \tag{3.2}$$

For the weight $\rho \in W_\mu, 0 < \mu < 1+\lambda$, the following weighted estimate also holds

$$\omega\left(\rho\mathbb{K}\frac{\varphi}{\rho}, h\right) \leq Ch^{\gamma}k(h)\int_0^h \frac{\omega(\varphi,t)}{t^{\gamma}}\,dt + Ch\int_h^b \frac{k(t)\ \omega(\varphi,t)}{t}\,dt, \tag{3.3}$$
$$0 < h \leq b$$

where $\gamma = \max(1,\mu)$.

To formulate Theorem B below, we introduce an additional assumption on smoothness of the kernel $\ell(x)$ beyond the singular point $x = 0$. We suppose that

$$\ell(x) \in C^2([\delta, b]) \quad \text{for every} \quad \delta \in (0, b), \tag{3.4}$$

and there exists $\delta_0 > 0$ such that $\ell'(x)$ satisfies condition of type (2.6):

$$\left| \frac{\ell'(x) - \ell'(t)}{x - t} \right| \le c \frac{|\ell'(t)|}{x}, \qquad 0 < t < x \le \delta_0. \tag{3.5}$$

We show in Section 9 that condition (3.5) is satisfied for a large class of kernels, in particular for those which occur in applications.

In the following theorem ε_0 is a number from (2.13).

Theorem B. *Let the kernel $\ell(t)$ satisfy the conditions*

(1) *$\ell'(x)$ fulfills integrability condition* (2.14) *on $[\delta, b]$ for every $\delta > 0$,*

(2) *$\ell(x)$ is positive, decreasing and satisfying the condition in* (2.6) *on $(0, \delta_0]$ for some δ_0.*

Then for any $f \in C_0([0, b])$ the estimate

$$\omega\left(\mathbb{K}^{-1} f, h\right) \le C \int_0^h \frac{\ell(t)\omega(f, t)}{t} dt, \quad 0 < h \le \varepsilon_0 \tag{3.6}$$

is valid. If $\ell(t)$ satisfies additional assumptions (3.4 - 3.5)*, then the following weighted estimate holds*

$$\omega\left(\rho \mathbb{K}^{-1} \frac{f}{\rho}, h\right) \le C h^{\gamma - 1} \int_0^h \frac{\ell(t)\ \omega(f, t)}{t^\gamma} dt + c h \int_h^b \frac{|\ell(t)| \omega(f, t)}{t^2} dt, \quad 0 < h \le \varepsilon_0 \tag{3.7}$$

where $\rho \in W_\mu([0, b])$, $0 < \mu < 2$, and $\gamma = \max(1, \mu)$.

Theorem C_1**.** *Let $k(t) \in V_\lambda$ and the characteristic $\omega(t)$ satisfy the conditions*

$$\omega(t) \in \mathcal{Z}^0 \quad and \quad \omega_1(t) := tk(t)\omega(t) \in \mathcal{Z}_1. \tag{3.8}$$

Then the operator $\mathbb{K}$ is bounded from the space H_0^ω to the space $H_0^{\omega_1}$. The operator $\mathbb{K}$ is also bounded from $H_0^\omega(\rho)$ to $H_0^{\omega_1}(\rho)$ with $\rho \in W_\mu, 0 < \mu < 1 + \lambda$, if in the case $\mu > 1$ the following additional condition is satisfied:

$$t^{1-\mu}\omega(t) \in \mathcal{Z}^0. \tag{3.9}$$

Theorem C_2**.** *Let*

(1) *the kernel $\ell(x)$ satisfy assumptions of Theorem* **B***;*

(2) *the weight function $\rho(x)$ belong to $\in W_\mu([0, b]),\ 0 < \mu < 2$;*

(3) *the characteristic $\omega(x)$ meet the conditions*

$$x^{-\max(0,\mu-1)}\omega(x) \in \mathcal{Z}^0 \quad \text{and} \quad \omega(x) \in \mathcal{Z}_1. \tag{3.10}$$

Then the operator $\mathbb{K}^{-1}$ maps continuously the space $H_0^{\omega_2}(\rho)$ with $\omega_2(x) = \frac{\omega(x)}{\ell(x)}$ into $H_0^\omega(\rho)$.

Theorem D. *Let $k(x)$ and $\ell(x)$ be a pair of associated Sonine kernels and let the following conditions be satisfied*

(1) *$k(x) \in V_\lambda, 0 < \lambda < 1$;*

(2) *$\ell(x)$ satisfies assumptions of Theorem* **B***;*

(3) *$\rho(x) \in W_\mu, 0 < \mu < 1 + \lambda$;*

(4) *$x^{1-\gamma}\omega(x) \in \mathcal{Z}^0,\ \ \omega_1(x) := xk(x)\omega(x) \in \mathcal{Z}_1$, where $\gamma = \max(1, \mu)$.*

Then the operator $\mathbb{K}$ maps isomorphically the space $H_0^\omega(\rho)$ onto the space $H_0^{\omega_1}(\rho)$. The non-weighted case is contained in the above statement with $\rho \equiv 1$ under conditions (1), (2) *and* (4) *with $\gamma = 1$.*

Remark 3.2. Note that in all the statements on action of the inverse operator, which "worsens" the behavior of the continuity modulus, we do not impose on the kernel $\ell(t)$ the condition that it belongs to V_λ.

We needed the V_λ-condition only for the kernel $k(t)$ of the operator $\mathbb{K}$ which "improves" the behavior of continuity modulus.

Remark 3.3. Comparing characteristics $\omega_1(x)$ and $\omega_2(x)$ of Theorems **C1** and **C2**, observe that

$$H^{\omega_1} \subseteq H^{\omega_2}, \quad \|f\|_{H^{\omega_2}} \leq C\|f\|_{H^{\omega_1}}$$

because

$$\omega_1(x) \leq \omega_2(x) \quad \Longleftrightarrow \quad xk(x)\ell(x) \leq 1 \tag{3.11}$$

for small x. Inequality (3.11) holds for arbitrary associated Sonine kernels positive and a.d. near the origin. Indeed, from Sonine condition (1.3), we obtain:

$$1 = \int_0^x k(t)\ell(x-t)dt \geq c_1 k(x) \int_0^x \ell(x-t)dt \geq c_1 c_2 k(x)\ell(x)x. \tag{3.12}$$

Note that the characteristics $\omega_1(x)$ and $\omega_2(x)$ are even equivalent, that is, the inequality

$$xk(x)\ell(x) \geq c_0 > 0 \tag{3.13}$$

also holds, if we additionally assume that $k(x) \in V_\alpha$ and $\ell(x) \in V_\beta$ for some $\alpha, \beta \in (0,1)$, which is seen from the following estimation

$$1 = \int_0^x t^\alpha k(t)(x-t)^\beta \ell(x-t)\frac{dt}{t^\alpha (x-t)^\beta} \tag{3.14}$$

$$\leq c_3 x^{\alpha+\beta} k(x)\ell(x) \int_0^x \frac{dt}{t^\alpha (x-t)^\beta} = c_3 B(1-\alpha, 1-\beta) xk(x)\ell(x).$$

Remark 3.4. Assumptions on the almost monotonicity of the kernels $k(x), \ell(x)$ in Theorems A-D are satisfied in various applications, in particular, in the examples mentioned in Section 9. However we should mention that the condition of positivity of the kernels is not always satisfied globally on a given interval $[0,b]$, but is always fulfilled in a neighborhood of the origin. Therefore, in the case where the kernels may have negative values, the statements of Theorems **A**- **D** are proved on any interval $[0,a], a<b$, up to the first zero of the kernels $k(t), \ell(t)$. The estimation of the continuity moduli of convolutions with non-positive kernels requires a more elaborate technique (in the region where the kernel changes the sign, the ideas of almost monotonicity are not applicable). The authors hope to develop this approach in another paper.

4. Principal lemmas. The proof of Zygmund type estimates in Theorems **A** and **B** will be essentially based on the crucial technical lemmas below on estimation of the following integrals

$$I(k,\varphi;x,h)=(x+h)^{\gamma-1}\int\limits_x^{x+h}\frac{k(x+h-t)|\varphi(t)|}{t^\gamma}\,dt, \tag{4.1}$$

$$J_1(k,\varphi;x,h)=h(x+h)^{\gamma-1}\int\limits_0^x\frac{k(x-t)|\varphi(t)|}{t^\gamma(x+h-t)}\,dt. \tag{4.2}$$

and

$$J_2(k,\varphi;x,h)=h(x+h)^{\gamma-1}\int\limits_0^x\frac{k(x+h-t)|\varphi(t)|}{t^\gamma}\,dt, \tag{4.3}$$

where $1\le\gamma<2$.

Lemma 4.1. *Let $\varphi(x)\in C_0([0,b])$, $\gamma\in[1,2)$ and $k(x)$ be non-negative on $[0,b]$. Then*

$$\sup_{x\in[0,b]} I(k,\varphi;x,h)\le Ch^{\gamma-1}\int\limits_0^h\frac{k(t)\omega(\varphi,t)}{t^\gamma}\,dt, \qquad 0<h<b \tag{4.4}$$

if $k(t)$ is a.d. and

$$\sup_{x\in[0,b]} I(k_1,\varphi;x,h) \le Ch^{\gamma-1}k(h)\int_0^h \frac{\omega(\varphi,t)}{t^\gamma}\,dt, \qquad 0<h<b \tag{4.5}$$

if $k(t)$ is a.i.

Proof. To prove (4.4), we observe that $\varphi(0)=0$, so that

$$I(k,\varphi;x,h) \le (x+h)^{\gamma-1}\int_0^h \frac{\omega(\varphi,x+t)}{(x+t)^\gamma}k(h-t)\,dt \tag{4.6}$$

$$= (x+h)^{\gamma-1}\int_0^{\frac{h}{2}} \frac{\omega(\varphi,x+t)}{(x+t)^\gamma}k(h-t)dt \tag{4.7}$$

$$+ (x+h)^{\gamma-1}\int_0^{\frac{h}{2}} \frac{\omega(\varphi,x+h-t)}{(x+h-t)^\gamma}k(t)dt.$$

Let $x\le h$ first. Then $(x+h)^{\gamma-1}\le Ch^{\gamma-1}$. We observe that $h-t>t$ in the first integral in (4.7) and $x+h-t>t$ in the second one. Since the functions $\frac{\omega(\varphi,x)}{x^\gamma}$ and $k(x)$ are a.d., the estimates in (4.6 - 4.7) imply (4.4).

When $x\ge h$, we use $\left(\frac{x+h}{x+t}\right)^{\gamma-1}\le 2^{\gamma-1}$ in (4.6) and get

$$I(k,\varphi;x,h) \le C\int_0^h \frac{\omega(\varphi,x+t)}{x+t}k(h-t)dt. \tag{4.8}$$

Hence estimate (4.4) follows as above from estimates (4.6 - 4.7) with $\gamma=1$.

The proof of (4.5) is easier. From (4.6) we have

$$I(k,\varphi;x,h) \le C(x+h)^{\gamma-1}k(h)\int_0^h \frac{\omega(\varphi,x+t)}{(x+t)^\gamma}\,dt.$$

If $x \le h$, the estimation is obvious. If $x \ge h$, then $\frac{x+h}{x+t} \le 2$ and we obtain

$$I(k,\varphi;x,h) \le Ck(h)\int\limits_0^h \frac{\omega(\varphi,x+t)}{x+t}\,dt \le Ck(h)\int\limits_0^h \frac{\omega(\varphi,t)}{t}\,dt$$

$$\le Ch^{\gamma-1}k(h)\int\limits_0^h \frac{\omega(\varphi,t)}{t^\gamma}\,dt.$$

□

Lemma 4.2. *Let $k(x)$ be non-negative and a.d. on $[0,b]$ and let $1 \le \gamma < 2$. Then for $\varphi(t) \in C_0([0,b])$ and $0 < h < b$ the following estimate is valid*

$$\sup_{x\in[0,b]} J_1(k,\varphi;x,h) \le Ch^{\gamma-1}\int\limits_0^h \frac{k(t)\omega(\varphi,t)}{t^\gamma}\,dt + Ch\int\limits_h^b \frac{k(t)\omega(\varphi,t)}{t^2}\,dt. \tag{4.9}$$

If we additionally assume that $x^{\gamma-1}k(x)$ is a.d. on $[0,b]$, then also

$$\sup_{x\in[0,b]} J_2(k,\varphi;x,h) \le Ch^{\gamma}k(h)\int\limits_0^h \frac{\omega(\varphi,t)}{t^\gamma}\,dt + Ch\int\limits_h^b \frac{k(t)\omega(\varphi,t)}{t}\,dt. \tag{4.10}$$

Proof.

(i) Proof of inequality (4.9). We consider first the case where $x \le 4h$. Splitting the integration in $J_1(k,\varphi;x,h)$, we obtain

$$J_1(k,\varphi;x,h \le Ch^{\gamma-1}\int\limits_0^{\frac{x}{2}} \frac{k(t)\omega(\varphi,t)}{t^\gamma}\,dt + Ch^\gamma\int\limits_{\frac{x}{2}}^x \frac{k(x-t)\omega(\varphi,t)}{t^\gamma(x-t+h)}\,dt. \tag{4.11}$$

In the second term we use the fact that $\frac{\omega(\varphi,t)}{t} \le C\frac{\omega(\varphi,x-t)}{x-t}$ and get

$$J_1(k,\varphi;x,h) \le Ch^{\gamma-1}\int\limits_0^{2h} \frac{k(t)\omega(\varphi,t)}{t^\gamma}dt + Ch^{\gamma-1}\int\limits_0^{\frac{x}{2}} \frac{k(t)\omega(\varphi,t)}{t^\gamma}dt \tag{4.12}$$

which yields

$$J_1(k,\varphi;x,h) \le Ch^{\gamma-1} \int\limits_0^h \frac{k(t)\omega(\varphi,t)}{t^\gamma} dt. \tag{4.13}$$

We pass to a more difficult case where $x \ge 4h$. We split the integration as follows

$$J_1(k,\varphi;x,h) \le h(x+h)^{\gamma-1} \int\limits_0^h \frac{k(x-t)\omega(\varphi,t)}{t^\gamma(x+h-t)}\, dt$$

$$+h(x+h)^{\gamma-1} \int\limits_h^x \frac{k(x-t)\omega(\varphi,t)}{t^\gamma(x+h-t)}\, dt =: \; D_1(x,h) + D_2(x,h).$$

In the term $D_1(x,h)$ we use the fact that $k(t)$ is a.d. and the inequality $x+h-t > \frac{x+h}{2}$ and obtain

$$D_1(x,h) \le Ch(x+h)^{\gamma-2} \int\limits_0^h \frac{k(t)\omega(\varphi,t)}{t^\gamma} dt \le Ch^{\gamma-1} \int\limits_0^h \frac{k(t)\omega(\varphi,t)}{t^\gamma} dt. \tag{4.14}$$

For the term $D_2(x,h)$ we have

$$D_2(x,h) \le h(x+h)^{\gamma-1} \int\limits_h^{\frac{x}{2}} \frac{k(x-t)\omega(\varphi,t)}{t^\gamma(x+h-t)}\, dt$$

$$+h(x+h)^{\gamma-1} \int\limits_{\frac{x}{2}}^x \frac{k(x-t)\omega(\varphi,t)}{t^\gamma(x+h-t)}\, dt \;=: \; D_{21}(x,h) + D_{22}(x,h).$$

Observe that $x+h-t > \frac{x+h}{2}$ in the term $D_{21}(x,h)$. Consequently

$$D_{21}(x,h) \le Ch(x+h)^{\gamma-2} \int\limits_h^{\frac{x}{2}} \frac{|k(x-t)|\omega(\varphi,t)}{t^\gamma} dt.$$

It is easily seen that $\frac{(x+h)^{\gamma-2}}{t^\gamma} \leq \frac{1}{t^2}$. Taking also into account that $k(x-t) \leq Ck(t)$, we get

$$D_{21}(x,h) \leq Ch \int_h^b \frac{k(t)\omega(\varphi,t)}{t^2} dt. \tag{4.15}$$

We split the term $D_{22}(x,h)$ in the following way

$$D_{22}(x,h) = h(x+h)^{\gamma-1} \left(\int_{\frac{x}{2}}^{\frac{x+h}{2}} + \int_{\frac{x+h}{2}}^{x-h} + \int_{x-h}^{x} \right)$$
$$=: \ E_1(x,h) + E_2(x,h) + E_3(x,h).$$

In the term $E_1(x,h)$ we make use of the fact that $x+h-t > \frac{x+h}{2}$ and obtain

$$E_1(x,h) \leq h(x+h)^{\gamma-2} \int_{\frac{x}{2}}^{\frac{x+h}{2}} \frac{k(x-t)\omega(\varphi,t)}{t^\gamma} dt$$
$$\leq h^{\gamma-1} \int_0^{\frac{h}{2}} \frac{k(\frac{x}{2}-t)\omega(\varphi,t+\frac{x}{2})}{\left(t+\frac{x}{2}\right)^\gamma} dt.$$

Since the functions $\frac{\omega(\varphi,t)}{t^\gamma}$ and $k(t)$ are a.d. and $t+\frac{x}{2} > t$ and $\frac{x}{2}-t > t$, we obtain

$$E_1(x,h) \leq Ch^{\gamma-1} \int_0^h \frac{k(t)\omega(\varphi,t)}{t^\gamma} dt. \tag{4.16}$$

In the term $E_2(x,h)$ we observe that $t > \frac{x+h}{2}$ so that

$$\frac{\omega(\varphi,t)}{t^\gamma} = \frac{1}{t^{\gamma-1}} \frac{\omega(\varphi,t)}{t} \leq \frac{C}{(x+h)^{\gamma-1}} \frac{\omega(\varphi,x-t)}{x-t}$$

where there was taken into account that $t > x - t$ and $\frac{\omega(\varphi,t)}{t}$ is a.d. Therefore,

$$E_2(x,h) \leq Ch \int\limits_{\frac{x+h}{2}}^{x-h} \frac{k(x-t)\omega(\varphi,x-t)}{(x-t)(x+h-t)}\, dt = Ch \int\limits_{h}^{\frac{x-h}{2}} \frac{k(t)\omega(\varphi,t)}{t(t+h)} dt.$$

Hence

$$E_2(x,h) \leq Ch \int\limits_{h}^{b} \frac{k(t)\omega(\varphi,t)}{t^2} dt. \tag{4.17}$$

Finally, in the term $E_3(x,h)$ we use the fact that $t > x - h > \frac{x+h}{2}$ which yields $\frac{1}{t^\gamma} = \frac{1}{t^{\gamma-1}}\frac{1}{t} \leq \frac{C}{(x+h)^{\gamma-1}}\frac{1}{t}$. Then

$$\begin{aligned} E_3(x,h) &\leq Ch \int\limits_{x-h}^{x} \frac{k(x-t)\omega(\varphi,t)}{t(x-t+h)} dt \\ &= Ch \int\limits_{0}^{h} \frac{k(t)\omega(\varphi,x-t)}{(x-t)(t+h)} dt \leq C \int\limits_{0}^{h} \frac{k(t)\omega(\varphi,x-t)}{x-t} dt. \end{aligned}$$

Since $x - t > t$, hence it follows that

$$E_3(x,h) \leq C \int\limits_{0}^{h} \frac{k(t)\omega(\varphi,t)}{t} dt. \tag{4.18}$$

It remains to gather estimates (4.13 - 4.18) to arrive at estimate (4.9).

(ii) Proof of inequality (4.10). The proof of (4.10) follows more or less the same arguments but needs other estimations because the kernel $k_1(t) = tk(t)$ may be non-a.d. in contrast to $k(t)$ which is a.d. We omit some details, but give the main lines of the proof.

Let first $x \leq 4h$. Since $k(x)$ is a.d., we obtain

$$J_2(k,\varphi;x,h) \leq Ch^\gamma k(h) \int_0^h \frac{\omega(\varphi,t)}{t^\gamma}\, dt. \tag{4.19}$$

Let $x \geq 4h$. We have

$$\begin{aligned} J_2(k,\varphi;x,h) &\leq h(x+h)^{\gamma-1} \left(\int_0^h + \int_h^x \right) \frac{k(x+h-t)\omega(\varphi,t)}{t^\gamma}\, dt \\ &=: \ D_1(x,h) + D_2(x,h). \end{aligned}$$

In the term $D_1(x,h)$ we have $x+h-t > \frac{x+h}{2}$. Therefore, $k(x+h-t) \leq Ck\left(\frac{x+h}{2}\right)$. Then

$$D_1(x,h) \leq Ch(x+h)^{\gamma-1} k\left(\frac{x+h}{2}\right) \int_0^h \frac{\omega(\varphi,t)}{t^\gamma} dt.$$

Since $\gamma - 1 < \lambda$, the function $x^{\gamma-1}k(x)$ is a.d. Therefore,

$$D_1(x,h) \leq Ch^\gamma k(h) \int_0^h \frac{\omega(\varphi,t)}{t^\gamma}\, dt. \tag{4.20}$$

For the term $D_2(x,h)$ we split the integration as follows

$$\begin{aligned} D_2(x,h) &\leq Ch(x+h)^{\gamma-1} \left(\int_h^{\frac{x}{2}} + \int_{\frac{x}{2}}^x \right) \frac{k(x+h-t)\omega(\varphi,t)}{t^\gamma}\, dt \\ &=: \ D_{21}(x,h) + D_{22}(x,h). \end{aligned}$$

In the term $D_{21}(x,h)$ we have $x+h-t > \frac{1}{2}(x+h)$. Consequently

$$D_{21}(x,h) \leq Ch(x+h)^{\gamma-1} k\left(\frac{x+h}{2}\right) \int_h^{\frac{x}{2}} \frac{\omega(\varphi,t)}{t^\gamma} dt.$$

Since the function $x^{\gamma-1}k(x)$ is a.d., and $\frac{x+h}{2} \geq t$, we get

$$D_{21}(x,h) \leq Ch \int_h^b \frac{k(t)\omega(\varphi,t)}{t}dt. \tag{4.21}$$

We split the term $D_{22}(x,h)$ in the following way

$$\begin{aligned} D_{22}(x,h) &= Ch(x+h)^{\gamma-1}\left(\int_{\frac{x}{2}}^{\frac{x+h}{2}} + \int_{\frac{x+h}{2}}^{x-h} + \int_{x-h}^{x}\right) \\ &=: \ E_1(x,h) + E_2(x,h) + E_3(x,h). \end{aligned}$$

In the term $E_1(x,h)$ we have

$$E_1(x,h) = Ch(x+h)^{\gamma-1} \int_0^{\frac{h}{2}} \frac{k\left(\frac{x}{2}+h-t\right)\omega\left(\varphi, t+\frac{x}{2}\right)}{\left(t+\frac{x}{2}\right)^{\gamma}}dt.$$

Here the function $\frac{\omega(\varphi,t)}{t^{\gamma}}$ is a.d., so that

$$\begin{aligned} E_1(x,h) &\leq Ch(x+h)^{\gamma-1} \int_0^h \frac{k\left(\frac{x}{2}+h-t\right)\omega(\varphi,t)}{t^{\gamma}}dt \\ &\leq Ch(x+h)^{\gamma-1}k\left(\frac{x+h}{3}\right)\int_0^h \frac{\omega(\varphi,t)}{t^{\gamma}}dt \end{aligned} \tag{4.22}$$

where we have used the fact that $k(t)$ is a.d. and $\frac{x}{2}+h-t > \frac{x+h}{3}$, Making use of the fact that $x^{\gamma-1}k(x)$ is a.d., we get

$$E_1(x,h) \leq Ch^{\gamma}k\,(h)\int_0^h \frac{\omega(\varphi,t)}{t^{\gamma}}dt. \tag{4.23}$$

For the term

$$E_2(x,h) \leq Ch(x+h)^{\gamma-1} \int_{\frac{x+h}{2}}^{x-h} \frac{k(x+h-t)\omega(\varphi,t)}{t^{\gamma}}\,dt$$

we observe that $x+h \leq 2t$ so that

$$E_2(x,h) \leq Ch \int\limits_{\frac{x+h}{2}}^{x-h} \frac{k(x+h-t)\omega(\varphi,t)}{t}\, dt \tag{4.24}$$

$$= Ch \int\limits_{h}^{\frac{x-h}{2}} \frac{k(t+h)\omega(\varphi,x-t)}{x-t} dt \leq Ch \int\limits_{h}^{b} \frac{k(t)\omega(\varphi,t)}{t}\, dt.$$

Finally, in the term

$$E_3(x,h) = Ch(x+h)^{\gamma-1} \int\limits_{x-h}^{x} \frac{k(x+h-t)\omega(\varphi,t)}{t^\gamma}\, dt$$

we use the fact that $t > \frac{2}{5}(x+h)$ so that $\frac{1}{t^\gamma} \leq \frac{C}{t(x+h)^{\gamma-1}}$. Then

$$E_3(x,h) \leq Ch \int\limits_{x-h}^{x} \frac{k(x+h-t)\omega(\varphi,t)}{t} dt$$

$$= Ch \int\limits_{0}^{h} \frac{k_1(t)\omega(\varphi,x-t)}{x-t} dt \leq Chk(h) \int\limits_{0}^{h} \frac{\omega(\varphi,t)}{t} dt.$$

Hence

$$E_3(x,h) \leq Ch^\gamma k(h) \int\limits_{0}^{h} \frac{\omega(\varphi,t)}{t^\gamma} dt. \tag{4.25}$$

It remains to gather estimates (4.19 - 4.25) and we arrive at (4.10) in the case $1 < \gamma < 2$ □

5. Proof of Theorem A.

I. *Non-weighted part.* For the function $f(x) = \mathbb{K}\varphi(x)$ we represent the difference $\Delta_h f(x) = f(x+h) - f(x)$ with $x, x+h \in [0,b]$ as

$$\begin{aligned}\Delta_h f(x) = &\int_{-h}^{0} [\varphi(x-t)-\varphi(x)]k(t+h)\,dt \\ &- \int_{0}^{x} [\varphi(x-t)-\varphi(x)][k(t)-k(t+h)]\,dt \\ &+ \varphi(x)\left[\int_{-h}^{x} k(t+h)\,dt - \int_{0}^{x} k(t)dt\right].\end{aligned}$$

Hence

$$\begin{aligned}|\Delta_h f(x)| \le &\left|\int_{-h}^{0} [\varphi(x-t)-\varphi(x)]k(t+h)\,dt\right| \\ &+ \left|\int_{0}^{x} [\varphi(x-t)-\varphi(x)][k(t)-k(t+h)]\,dt\right| \\ &+ \left|\varphi(x)\int_{x}^{x+h} k(t)\,dt\right| \\ =:& A_1(x,h)+A_2(x,h)+A_3(x,h).\end{aligned} \tag{5.1}$$

Taking into account that $\omega(\varphi,t)$ is a.i. and making use of (2.7) we get

$$A_1(x,h) \le C\int_{0}^{h} \omega(\varphi,t)k(h-t)\,dt \le C\omega(\varphi,h)k(h)\int_{0}^{h}\left(\frac{h}{h-t}\right)^{\lambda} dt,$$

whence

$$A_1(x,h) \le Chk(h)\omega(\varphi,h). \tag{5.2}$$

For $A_2(x,h)$ by (2.6) we have

$$A_2(x.h) \le Ch\int_0^x \frac{\omega(\varphi,t)k(t)}{t+h}\,dt. \tag{5.3}$$

Let first $x \le h$. By property (2.7) we obtain

$$A_2(x,h) \le Ch^{1+\lambda}k(h)\int_0^x \frac{\omega(\varphi,t)\,dt}{t^\lambda(t+h)} \le Ch^{1+\lambda}k(h)\omega(\varphi,h)\int_0^h \frac{dt}{t^\lambda(t+h)}.$$

Hence

$$A_2(x,h) \le Chk(h)\omega(\varphi,h), \qquad x \le h \tag{5.4}$$

In the case where $x \ge h$ from (5.3) we have

$$A_2(x,h) \le Ch\int_0^h \frac{\omega(\varphi,t)k(t)}{t+h}\,dt + Ch\int_h^x \frac{\omega(\varphi,t)k(t)}{t+h}\,dt$$

where the first term is obviously estimated like in (5.4) so that

$$A_2(x,h) \le Chk(h)\omega(\varphi,h) + Ch\int_h^b \frac{\omega(\varphi,t)k(t)}{t}\,dt, \qquad x \ge h. \tag{5.5}$$

For the term $A_3(x,h)$ in the case $x \le h$ we have

$$A_3(x,h) \le C\omega(\varphi,h)k(x+h)(x+h)^\lambda \int\limits_x^{x+h} \frac{dt}{t^\lambda} \le C\omega(\varphi,h)k(h)h^\lambda \int\limits_0^{2h} \frac{dt}{t^\lambda}$$

so that

$$A_3(x,h) \le Chk(h)\omega(\varphi,h), \qquad x \le h. \tag{5.6}$$

Let us show that in the case $x > h$ one has

$$A_3(x,h) \le Ch\int\limits_h^b \frac{k(t)\omega(\varphi,t)}{t}\,dt, \qquad x > h. \tag{5.7}$$

We have

$$A_3(x,h) \le Ch\omega(\varphi,x)k(x) \tag{5.8}$$

and then estimate (5.7) is derived from (5.8) by means of Lemma 2.6:

$$A_3(x,h) \le Ch \int_x^{\delta_0} \frac{\omega(\varphi,t)k(t)}{t}\,dt \le Ch \int_h^b \frac{\omega(\varphi,t)\ k(t)}{t}\,dt, \tag{5.9}$$

$$h \le x \le \frac{\delta_0}{2}.$$

Collecting the estimates in (5.2) and (5.4 - 5.7), from (5.1) we obtain (3.2).

II. *Weighted part.* We have

$$\left(\rho\mathbb{K}\frac{\varphi}{\rho}\right)(x) = \mathbb{K}\varphi(x) + \mathbb{A}\varphi(x) \tag{5.10}$$

where

$$\mathbb{A}\varphi(x) = \int_0^x \frac{\rho(x)-\rho(t)}{\rho(t)}k(x-t)\varphi(t)dt =: \int_0^x A(x,t)\varphi(t)dt$$

and

$$A(x,t) = \frac{\rho(x)-\rho(t)}{\rho(t)}k(x-t). \tag{5.11}$$

The estimation of the continuity modulus of $\mathbb{K}\varphi(x)$ has already been done in the part 1 of the proof. It remains to estimate $\omega(\mathbb{A}\varphi,h)$. In the estimation of $\omega(\mathbb{A}\varphi,h)$ we follow some ideas of such estimations suggested in [5] for the case $k(x) = x^{\alpha-1},\ 0<\alpha<1, \rho(x) = x^{\mu},\ 0< \mu < 2-\alpha$. We have

$$\omega(\mathbb{A}\varphi,h) \le \sup_{x\in[0,b]} A_h\varphi(x) + \sup_{x\in[0,b]} A_h^1\varphi(x)) \tag{5.12}$$

where

$$A_h\varphi(x) := \int_x^{x+h} A(x+h,t)\varphi(t)dt, \qquad A_h^1\varphi(x) := \int_0^x A_1(x,h,t)\varphi(t)dt$$

and

$$A_1(x,h,t) = A(x+h,t) - A(x,t).$$

(i). Estimation of $A(x,t)$**.** The estimate

$$|A(x,t)| \le C\left(\frac{x}{t}\right)^{\gamma-1}\frac{(x-t)k(x-t)}{t}, \qquad \gamma = \max(1,\mu). \tag{5.13}$$

for $A(x,t)$ follows from inequality (2.5) of Lemma 2.3.

(ii). Estimation of $A_h\varphi(x)$**.** For the term $A_h\varphi(x)$ we have

$$|A_h\varphi(x)| \le Ch^{\gamma}k(h)\int\limits_0^h \frac{\omega(\varphi,t)}{t^{\gamma}}dt. \tag{5.14}$$

which follows from estimate (5.13) and inequality (4.5) applied to the kernel $tk(t)$.

(iii). Estimation of $A_1(x,h,t)$**.** For $A_1(x,h,t)$ the following estimate is valid

$$|A_1(x,h,t)| \le Ch\left(\frac{x+h}{t}\right)^{\gamma-1}\frac{|k(x+h-t)|}{t}. \tag{5.15}$$

$$\gamma = \max(1,\mu).$$

To prove (5.15), we split $A_1(x,h,t)$ as follows:

$$\begin{aligned} A_1(x,h,t) &= \frac{\rho(x+h)-\rho(x)}{\rho(t)}k(x+h-t) \\ &\quad + \frac{\rho(x)-\rho(t)}{\rho(t)}[k(x+h-t)-k(x-t)] \\ &=: \ A_{11}(x,h,t) + A_{12}(x,h,t). \end{aligned}$$

For $A_{11}(x,h,t)$ by (2.2) and (2.3) we have

$$\begin{aligned} |A_{11}(x,h,t)| &\le Ch\left(\frac{x+h}{t}\right)^{\mu-1}\frac{|k(x+h-t)|}{t} \\ &\le Ch\left(\frac{x+h}{t}\right)^{\gamma-1}\frac{|k(x+h-t)|}{t}. \end{aligned} \tag{5.16}$$

For $A_{12}(x,h,t)$, by (2.5) and (2.6) we obtain

$$|A_{12}(x,h,t)| \leq Ch\left(\frac{x}{t}\right)^{\gamma-1}\frac{(x-t)k(x-t)}{t(x+h-t)}. \tag{5.17}$$

We use the fact that the function $xk(x)$ is a.i., so that for $0 < x \leq \frac{\delta_0}{2}$ we have

$$|A_{12}(x,h,t)| \leq Ch\left(\frac{x+h}{t}\right)^{\gamma-1}\frac{k(x+h-t)}{t}, \tag{5.18}$$

and then from (5.16) and (5.18) we obtain (5.15).

(iv). Estimation of $A_h^1\varphi(x)$. By estimate (5.15) and equation (4.10) of Lemma 4.2 we have

$$|A_h^1\varphi(x)| \leq Ch^\gamma k(h)\int\limits_0^h \frac{\omega(\varphi,t)}{t^\gamma}\,dt + Ch\int\limits_h^b \frac{\omega(\varphi,t)|k(t)|}{t}\,dt. \tag{5.19}$$

Gathering estimates (5.14) and (5.19), from (5.12) we obtain estimate of type (3.3) for $\omega(\mathbb{A}\varphi,h)$ and therefore (3.3) holds for $\omega\left(\rho\mathbb{K}\frac{\varphi}{\rho},h\right)$ in view of (5.10) and already proved non-weighted estimate (3.2).

6. Proof of Theorem B.

6.1. *Auxiliary lemmas.*

Lemma 6.1. *Let a function $\ell(x)$ be bounded on $[\delta,b]$ for every $\delta>0$ and non-negative, almost decreasing and satisfying condition* (2.6) *on $(0,\delta_0]$ for some $\delta_0>0$. Then for any $f\in C_0([0,b])$ the estimate*

$$\omega(\ell f,h) \leq C\int\limits_0^h \frac{\ell(t)\omega(f,t)}{t}\,dt, \qquad 0<h\leq\frac{\delta_0}{2}. \tag{6.1}$$

is valid, where $C>0$ does not depend on $h\in\left(0,\frac{\delta_0}{2}\right)$.

Proof. We denote $F(x) = \ell(x)f(x)$. Let first $0 < x \leq \frac{\delta_0}{2}$ (and then $x + h \leq \delta_0$). We have so that

$$F(x+h) - F(x) = \ell(x+h)[f(x+h) - f(x)] + f(x)[\ell(x+h) - \ell(x)] \\ =: \Delta_1(x) + \Delta_2(x).$$

The estimate for $\Delta_1(x)$ is direct:

$$|\Delta_1(x)| \leq C\ell(x+h)\omega(f,h) \leq C\ell(h)\omega(f,h) \tag{6.2}$$

where the fact that $\ell(x)$ is a.d., was used. For Δ_2, taking into account (2.6) for ℓ and the fact that $|f(x)| = |f(x) - f(0)| \leq \omega(f,x)$, we obtain

$$|\Delta_2(x)| \leq Ch\frac{\ell(x)\omega(f,x)}{x+h}. \tag{6.3}$$

Observe that

$$\ell(x)\omega(f,x) \leq C\int_0^x \frac{\ell(t)\omega(f,t)}{t}\,dt \qquad \text{forall} \qquad x \in (0,b] \tag{6.4}$$

(which obviously follows from the fact that $\frac{\ell(x)\omega(x)}{x}$ is a.d.). Therefore, in the case $x \leq h$ we have

$$|\Delta_2(x)| \leq C\int_0^h \frac{\ell(t)\omega(f,t)}{t}\,dt. \tag{6.5}$$

In the remaining case $x \geq h$, from (6.3) we obtain

$$|\Delta_2(x)| \leq Ch\frac{\omega(f,x+h)}{x+h}\ell(x) \leq C\omega(f,h)\ell(x) \\ \leq C\omega(f,h)\ell(h) \leq C\int_0^h \frac{\ell(t)\omega(f,t)}{t}\,dt.$$

Therefore, (6.5) holds for all $x \in (0, \frac{\delta_0}{2}]$. To state that (6.5) and (6.2) prove estimate (6.1), it suffices to consider the case where $x \geq$

$\frac{\delta_0}{2}$ which is trivial. Indeed, from the above estimations of $\Delta_1(x)$ and $\Delta_2(x)$and boundedness of $\ell(x)$ beyond the origin, we see that $\sup\limits_{x\geq\delta_0} |F(x+h)-F(x)| \leq C\omega(f,h)$ which is obviously dominated by the right-hand side of (6.1). □

Lemma 6.2. *Let the kernel $\ell(t)$ satisfy the assumptions of Lemma* 6.1 *and let there exist almost everywhere the derivative $\ell'(x)$ satisfying the condition $|\ell'(x)| \leq C\frac{|\ell(x)|}{x}$. Let $\rho \in W_\mu$, $0<\mu<2$. Then for any $f \in C_0([0,b])$ satisfying condition* (3.1)*, the function*

$$g(x) \ := \int\limits_0^x \frac{\rho(x)-\rho(t)}{\rho(t)} \ell'(x-t) f(t)\, dt$$

is bounded on $[0,b]$ and

$$\lim_{x\to 0} g(x) = 0. \tag{6.6}$$

Proof. The function $g(x)$ admits the following estimate

$$|g(x)| \leq Cx^{\gamma-1}\int\limits_0^x \frac{|\ell(t)|\omega(f,x-t)}{(x-t)^\gamma}dt \tag{6.7}$$

$$= Cx^{\gamma-1}\int\limits_0^x \frac{|\ell(x-t)|\omega(f,t)}{t^\gamma}dt,$$

where $\gamma = \max(1,\mu)$. Indeed, let $0<\mu\leq 1$. Then by (2.4)

$$|g(x)| \leq C\int\limits_0^x |\ell(t)|\frac{\omega(f,x-t)}{x-t}dt$$

Let $1<\mu<2$. Then by (2.2) and (2.3) we get

$$|g(x)| \leq Cx^{\mu-1}\int\limits_0^x \frac{|\ell(t)|\omega(f,x-t)}{(x-t)^\mu}dt$$

which proves (6.7).

From (6.7) and boundedness of $\ell(t)$ on $[\delta_0, b]$, it is easily obtained that $g(x)$ is bounded on $[\delta_0, b]$. So we consider only $0 < x \le \delta_0$ below. From (6.7) we have

$$|g(x)| \le Cx^{\gamma-1} \int_0^{\frac{x}{2}} |\ell(t)| \frac{\omega(f, x-t)}{(x-t)^\gamma} dt + Cx^{\gamma-1} \int_{\frac{x}{2}}^{x} |\ell(t)| \frac{\omega(f, x-t)}{(x-t)^\gamma} dt =: \ I_1 + I_2. \tag{6.8}$$

Since $x - t > t$ in the term I_1 and $\frac{\omega(f,x)}{x^\gamma}$ is a.d., we obtain $I_1 \le Cx^{\gamma-1} \int_0^x \frac{|\ell(t)|\omega(f,t)}{t^\gamma} dt$. Similarly, since $t > x - t$ in I_2 and $\ell(t)$ is a.d. for small t, we have $I_2 \le Cx^{\gamma-1} \int_0^{\frac{x}{2}} \frac{|\ell(s)| \ \omega(f,s)}{s^\gamma} ds$. Therefore,

$$|g(x)| \le Cx^{\gamma-1} \int_0^x \frac{|\ell(t)| \ \omega(f,t)}{t^\gamma} dt. \tag{6.9}$$

By assumption (3.1), from (6.9) the statements of the lemma follow. □

6.2. Complete Proof of Theorem B.

I. *Non-weighted part.* The estimation of the continuity modulus of the first term $\ell(x)f(x)$ in (2.15) was already given in Lemma 6.1.

For the second term

$$\Psi(x) :=\int_0^x \ell'(t)[f(x-t) - f(x)] \ dt$$

in (2.15) we have

$$\Psi(x+h)-\Psi(x)=\int_0^x \ell'(t)[f(x+h-t)-f(x+h)+f(x)-f(x-t)]dt + \int_x^{x+h} \ell'(t)[f(x+h-t)-f(x+h)]dt =: B_1+B_2. \tag{6.10}$$

(a) *Estimation of* B_1.

In the case $x \le h$ we immediately get

$$|B_1| \le 2\int_0^h |\ell'(t)|\omega(f,t)\,dt$$

and then by Remark 2.5

$$|B_1| \le C\int_0^h \frac{\ell(t)\omega(f,t)}{t}\,dt. \tag{6.11}$$

Let $x \ge h$. We decompose the integral $\int\limits_0^x = \int\limits_0^h + \int\limits_h^x$ and use the estimate (6.11) in the first term:

$$|B_1| \le C\int_0^h \frac{\ell(t)}{t}\omega(f,t)\,dt + 2\omega(f,h)I_h(x), \tag{6.12}$$

where $I_h(x) = \int\limits_h^x |\ell'(t)|dt$.

To estimate $I_h(x)$ we observe that $|\ell'(t)| = -\ell'(t)$ for small $t \in (0,\varepsilon_0)$ according to (2.13). Therefore, for $h \le t \le x \le \varepsilon_0$ we have

$$I_h(x) = -\int_h^x \ell'(t)\,dt = \ell(h) - \ell(x) \le \ell(h).$$

When $h \geq \varepsilon_0$, we obviously have

$$I_h(x) \leq \int_{\varepsilon_0}^{b} |\ell'(t)|dt = C < \infty$$

according to (2.14). When $h \leq \varepsilon_0 \leq x$, we have

$$I_h(x) \leq \int_{h}^{\varepsilon_0} |\ell'(t)|dt + \int_{\varepsilon_0}^{b} |\ell'(t)|dt = \ell(h) - \ell(\varepsilon_0) + C \leq C\ell(h)$$

for small h. So $I_h(x) \leq C\ell(h)$ in all the cases and from (6.12) we arrive at the same estimate (6.11) in the case $x \geq h$ as well.

(b) *Estimation of* B_2. We have

$$|B_2(x)| \leq \int_{x}^{x+h} |\ell'(t)|\omega(f,t)dt \leq C \int_{0}^{h} \ell(x+t)\frac{\omega(f,x+t)}{x+t}dt.$$

Since both the functions $\ell(x)$ and $\frac{\omega(f,x)}{x}$ are a.d., we get

$$|B_2(x)| \leq C \int_{0}^{h} \frac{\ell(t)\omega(f,t)}{t}dt. \tag{6.13}$$

Gathering estimates (6.1), (6.4), (6.11) and (6.13), we get at (3.6).

II. *Weighted part.* We have

$$\left(\rho\mathbb{K}^{-1}\frac{f}{\rho}\right)(x) = \ell(x)f(x) + \int_{0}^{x} \left[\frac{\rho(x)}{\rho(x-t)}f(x-t) - f(x)\right]\ell'(t)dt \tag{6.14}$$
$$= \mathbb{K}^{-1}f(x) + \mathbb{B}f(x)$$

where

$$\mathbb{B}f(x) = \int_0^x \frac{\rho(x) - \rho(x-t)}{\rho(x-t)} \ell'(t) f(x-t) dt =: \int_0^x B(x,t) f(t) dt$$

and

$$B(x,t) = \frac{\rho(x) - \rho(t)}{\rho(t)} \ell'(x-t). \tag{6.15}$$

Estimate (3.6) for the continuity modulus of $\mathbb{K}^{-1} f(x)$ was already obtained in the first part of the theorem. It remains to estimate $\omega(\mathbb{B}f, h)$. It suffices to consider small values of $h : 0 < h \le \frac{\delta}{2}$, where δ is from assumptions (3.4 - 3.5). In the case $h \ge \frac{\delta}{2}$ the estimation of $\omega(\mathbb{B}f, h)$ is trivial, since the function $\mathbb{B}f(x)$ is bounded as proved in Lemma 6.2. Therefore, we assume that $h < \frac{\delta}{2}$ in the sequel.

We denote

$$B_1(x,h,t) = B(x+h,t) - B(x,t)$$

and have

$$\begin{aligned} \mathbb{B}f(x+h) - \mathbb{B}f(x) &= \int_x^{x+h} B(x+h,t) f(t) dt + \int_0^x B_1(x,h,t) f(t) dt \\ &=: B_h f(x) + B_h^1 f(x). \end{aligned} \tag{6.16}$$

(i). Estimation of $B(x,t)$. The following estimate is valid

$$|B(x,t)| \le C \left(\frac{x}{t}\right)^{\gamma-1} \frac{\ell(x-t)}{t}, \qquad \gamma = \max(1,\mu). \tag{6.17}$$

Indeed, when $0 < \mu \le 1$, by property (2.6) for $\ell(x)$ we obtain

$$|B(x,t)| \le C \frac{(x-t)|\ell'(x-t)|}{t} \le C \frac{\ell(x-t)}{t}, \qquad t < x.$$

When $1 < \mu < 2$, we use properties (2.2) and (2.3) and get (6.17) with $\gamma = \mu - 1$.

(ii). Estimation of $B_h f(x)$. For the term $B_h f(x)$ the estimate is valid

$$|B_h f(x)| \le C h^{\gamma-1} \int\limits_0^h \frac{\omega(f,t)}{t^\gamma}\ell(t)dt. \tag{6.18}$$

which follows from (6.17) and inequality (4.4) of Lemma 4.1.

(iii). Estimation of $B_1(x,h,t)$. For $B_1(x,h,t)$ the following estimate is valid

$$|B_1(x,h,t)| \le Ch\frac{\ell(x-t)}{t(x+h-t)}\left(\frac{x+h}{t}\right)^{\gamma-1}, \qquad \gamma = \max(1,\mu). \tag{6.19}$$

To prove (6.19), we split $B_1(x,h,t)$ as follows:

$$\begin{aligned} B_1(x,h,t) = &\frac{\rho(x+h)-\rho(x)}{\rho(t)}\ell'(x+h-t) \\ &+ \frac{\rho(x)-\rho(t)}{\rho(t)}[\ell'(x+h-t)-\ell'(x-t)] =: \ B_{11}+B_{12}. \end{aligned}$$

Making use also of properties (2.2 - 2.3), which yield

$$\frac{\rho(x+h)-\rho(x)}{\rho(t)} \le C\frac{h(x+h)^{\mu-1}}{t^\mu}, \tag{6.20}$$

and taking into account that $|\ell'(x+h-t)| \le C\frac{\ell(x+h-t)}{x+h-t}$, we obtain

$$|B_{11}| \le C\frac{h(x+h)^{\mu-1}}{t^\mu}\frac{\ell(x+h-t)}{x+h-t}. \tag{6.21}$$

To estimate B_{12}, we use (3.4 - 3.5). If $x < \frac{\delta}{2}$, then $x-t+h < \delta$ since $h < \frac{\delta}{2}$. So we may make use of (3.5) and get

$$|\ell'(x+h-t)-\ell'(x-t)| \le C\frac{h\ell'(x-t)}{x+h-t} \le C\frac{h\ell(x-t)}{(x-t)(x+h-t)}.$$

This inequality and estimate (2.5) yield

$$|B_{12}| \leq Ch\left(\frac{x}{t}\right)^{\gamma}\frac{\ell(x-t)}{t(x+h-t)}. \tag{6.22}$$

In the remaining case $x \geq \frac{\delta}{2}$ in the estimation of B_{12}, the arguments are similar if we consider separately the cases $0 < t < x - \frac{\delta}{2}$ and $x - \frac{\delta}{2} < t < x$. In the former case we use exactly the same arguments as above within the frameworks of assumptions (3.4 - 3.5), while in the latter case we may use the fact that $\ell(x) \in C^2\left(\left[\frac{\delta}{2}, b\right]\right)$ and arrive at the same estimate (6.22).

Gathering estimates (6.21) and (6.22), we obtain (6.19).

(iv). Estimation of $B_h^1 f(x)$. The estimate

$$|B_h^1 f(x| \leq Ch^{\gamma-1}\int\limits_0^h \frac{\ell(t)\omega(f,t)}{t^{\gamma}} + Ch\int\limits_h^b \frac{\ell(t)\omega(f,t)}{t^2}\,dt \tag{6.23}$$

immediately follows from (6.19) and inequality (4.9) of Lemma 4.2.

It remains to collect estimates (6.18) and (6.23) in order to obtain the final estimate (3.7) from relations (6.14, 6.16).

7. Proof of Theorems $\mathbf{C}_1$ and $\mathbf{C}_2$.

Proof of Theorem C_1: boundedness of the operator $\mathbb{K}$. We treat simultaneously the weighted and non-weighted ($\rho \equiv 1$) cases. By Zygmund type estimates (3.2), (3.3) we have

$$\omega\left(\rho\mathbb{K}\varphi, h\right) \leq C\|\rho\varphi\|_{H_0^{\omega}}\left[h^{\gamma}\int\limits_0^h \frac{k(t)\omega(t)}{t^{\gamma}}\,dt + Ch\int\limits_h^b \frac{\omega(t)k(t)}{t}\,dt\right] \tag{7.1}$$

whence

$$\omega\left(\rho\mathbb{K}\varphi, h\right) \leq Chk(h)\omega(h)\|\rho\varphi\|_{H_0^{\omega}} \tag{7.2}$$

by conditions (3.8) and (3.9).

It remains to check that $\rho\varphi|_{x=0} = 0$ for all $\varphi \in H_0^\omega(\rho)$. For $\varphi(x) = \frac{\varphi_0(x)}{\rho(x)}$ with $\varphi_0(x) \in C_0([0,b])$ we have

$$|\rho(x)(\mathbb{K}\varphi)(x)| \leq \rho(x) \int_0^x \frac{k(x-t)|\varphi_0(t)|}{\rho(t)}\, dt.$$

By properties (2.3) and (2.7) we obtain

$$\begin{aligned} |\rho(x)(\mathbb{K}\varphi)(x)| &\leq Ck(x) \int_0^x \left(\frac{x}{t}\right)^\mu \left(\frac{x}{x-t}\right)^\lambda \omega(\varphi_0,t)\, dt \\ &= Cxk(x) \int_0^1 \frac{\omega(\varphi_0,xt)\, dt}{t^\mu (1-t)^\lambda}. \end{aligned}$$

Hence

$$|\rho(x)(\mathbb{K}\varphi)(x)| \leq Cxk(x) \qquad \text{with} \qquad C = \int_0^1 \frac{\omega(\varphi_0,bt)\, dt}{t^\mu(1-t)^\lambda} < \infty.$$

Therefore, $\lim\limits_{x\to 0} \rho(x)(\mathbb{K}\varphi)(x) = 0$ since $xk(x) \leq Cx^{1-\lambda}$.

Proof of Theorem C_2: boundedness of the operator $\mathbb{K}^{-1}$. We have to prove that

$$\sup_{h>0} \frac{\omega\left(\rho\mathbb{K}^{-1}\frac{f}{\rho}, h\right)}{\omega(h)} \leq C\|f\|_{H_0^{\omega_2}}, \qquad f \in H_0^{\omega_2}. \tag{7.3}$$

Making use of estimates (3.6), (3.7) of Theorem B, we obtain

$$\begin{aligned} \omega\left(\rho\mathbb{K}^{-1}\frac{f}{\rho}, h\right) &\leq C\left[h^{\gamma-1}\int_0^h \frac{\ell(t)\omega_2(t)}{t^\gamma}dt + h\int_h^b \frac{\ell(t)\omega_2(t)}{t^2}dt\right]\|f\|_{H_0^{\omega_2}} \\ &= C\left[h^{\gamma-1}\int_0^h \frac{\omega(t)}{t^\gamma}dt + h\int_h^b \frac{\omega(t)}{t^2}dt\right]\|f\|_{H_0^{\omega_2}} \end{aligned}$$

By conditions (3.10) of the theorem, we get

$$\omega\left(\rho\mathbb{K}^{-1}\frac{f}{\rho}, h\right) \leq C\omega(h)\|f\|_{H_0^{\omega_2}}$$

which proves (7.3).

It remains to prove that $\rho\mathbb{K}^{-1}\frac{f}{\rho}\Big|_{x=0} = 0$ for $f \in H_0^{\omega_2}$. Making use of relation (6.14) and the expression for the inverse operator $\mathbb{K}^{-1}$, we have

$$\left|\left(\rho\mathbb{K}^{-1}\frac{f}{\rho}\right)(x)\right| \leq |\ell(x)f(x)| + \left|\int_0^x [f(x-t) - f(x)]\ell'(t)dt\right| + \left|\int_0^x \frac{\rho(x)-\rho(x-t)}{\rho(x-t)} f(x-t)\ell'(t)dt\right| = D_1 + D_2 + D_3.$$

Since $|\ell(x)f(x)| \leq |\ell(x)\omega(f,x)| \leq C|\ell(x)\omega_1(x)| = C|\omega(x)|$ and $\omega(x) \in \mathcal{Z}^0$, it follows that $D_1 \to 0$ as $x \to 0$. Also,

$$D_2 \leq C\int_0^x \omega(f,t)|\ell'(t)|dt \leq C\int_0^x \frac{\ell(t)\omega_2(t)}{t}dt = C\int_0^x \frac{\omega(t)}{t}dt \to 0$$

as $x \to 0$. As regards the term D_3, it was estimated in Lemma 6.2, so it also tends to zero as $x \to 0$.

8. Proof of Theorem D.

8.1. *Auxiliary lemma.*

Lemma 8.1. *Let $0 < \mu < 2$. If $x^{1-\gamma}\omega(x) \in \mathcal{Z}^0$, where $\gamma = \max(1, \mu)$, then there exists a $p_0 > 1$ such that*

$$\frac{\omega(t)}{x^\mu} \in L_p(0,b) \qquad \text{forall} \qquad p \in [1, p_0).$$

Proof. In the case $\mu \leq 1$ we use the fact that $\omega(x) \in \mathcal{Z}^0$ from which it follows that there exists a $\delta_1 \in (0,1)$ such that $\omega(x) \leq Cx^{\delta_1}$. Then $\frac{\omega(x)}{x^\mu} \leq \frac{C}{x^{\mu-\delta_1}} \in L_p(0,b)$, where $1 \leq p < \frac{1}{\mu-\delta_1}$ if $\delta_1 < \mu$ and $1 \leq p < \infty$ if $\delta_1 \geq \mu$.

In the case $\mu > 1$ we use the fact that $\frac{\omega(x)}{x^{\mu-1}} \in \mathcal{Z}^0$ so that $\omega(x) \leq x^{\mu-1+\delta_2}$ with $0 < \delta_2 < 1$. Then $\frac{\omega(x)}{x^\mu} \leq \frac{C}{x^{1-\delta_2}} \in L_p(0,b)$ with $1 \leq p < \frac{1}{1-\delta_2}$.

It remains to note that in the case $0 < \mu < 1$ we have $p_0 = \frac{1}{\mu-\delta_1}$ if $\delta_1 < \mu$ and $p_0 = \infty$ if $\delta_1 \geq \mu$, while in the case $1 < \mu < 2$ we have $p_0 = \frac{1}{1-\delta_2}$. □

8.2. Complete Proof of Theorem D. By Theorems $\mathbf{C}_1$ and $\mathbf{C}_2$, we have

$$\mathbb{K} \ : \ H_0^\omega(\rho) \to H_0^{\omega_1}(\rho), \quad \omega_1(x) = xk(x)\omega(x) \tag{8.1}$$

and

$$\mathbb{K}^{-1} \ : \ H_0^{\omega_2}(\rho) \to H_0^\omega(\rho), \quad \omega_2(x) = \frac{\omega(x)}{\ell(x)}. \tag{8.2}$$

Then by Remark 3.3, from (8.2) we also have

$$\mathbb{K}^{-1} \ : \ H_0^{\omega_1}(\rho) \to H_0^\omega(\rho), \quad \omega_1(x) = xk(x)\omega(x). \tag{8.3}$$

To state that the results in (8.1) and (8.3) already guarantee the existence of an isomorphism between the spaces $H_0^\omega(\rho)$ and $H_0^{\omega_1}(\rho)$, it remains to prove that the range of the operator $\mathbb{K}$ coincides with the space $H_0^{\omega_1}(\rho)$:

$$\mathbb{K}(H_0^\omega(\rho)) = H_0^{\omega_1}(\rho). \tag{8.4}$$

We do not have an independent characterization of the range $\mathbb{K}(H_0^\omega(\rho))$, but in the case of the Lebesgue spaces L_p, a characterization of the range $\mathbb{K}(L_p)$ is provided by Theorem 2.9. Therefore, to state that a function $f \in H_0^{\omega_1}(\rho)$ belongs to the range $\mathbb{K}(H_0^\omega(\rho))$, it suffices to prove that there exists $p > 1$ such that conditions (2.17) and (2.18) of Theorem 2.9 are satisfied for $f \in H_0^{\omega_1}(\rho)$. This will yield

$$H_0^{\omega_1}(\rho) \subset \mathbb{K}(L_p)$$

and then Theorem 2.9 and mapping (8.3) will guarantee that coincidence (8.4) holds.

Verification of condition (2.17). For $f \in H_0^{\omega_1}(\rho)$ we have $f = \frac{g}{\rho}$ with $g \in H_0^{\omega_1}$. Therefore,

$$|\ell(x) f(x)| \leq C \frac{\ell(x)\omega_1(x)}{\rho(x)} \leq C \frac{\omega(x)}{\rho(x)}$$

by (3.11). Since $\frac{\rho(x)}{x^\mu}$ is a.d., we have

$$\frac{1}{\rho(x)} \leq \frac{C}{x^\mu}. \tag{8.5}$$

Then

$$|\ell(x) f(x)| \leq C \frac{\omega(x)}{x^\mu} \in L_p$$

for any $p \in [1, p_0)$ by Lemma 8.1.

Verification of condition (2.18). For

$$\Psi_\varepsilon f(x) = \int_0^{x-\varepsilon} \ell'(x-t) \left[\frac{g(t)}{\rho(t)} - \frac{g(x)}{\rho(x)} \right] dt, \quad x > \varepsilon$$

we have

$$|\Psi_\varepsilon f(x)| \leq \frac{1}{\rho(x)} \int_0^x |\ell'(x-t)|\ |g(x) - g(t)| dt$$

$$+ \int_0^x |\ell'(x-t)| |g(t)| \left| \frac{1}{\rho(t)} - \frac{1}{\rho(x)} \right| dt \ =: \ F_1(x) + F_2(x).$$

By (8.5)

$$F_1(x) \leq \frac{C}{x^\mu} \int_0^x |\ell'(x-t)| \; \omega(g, x-t) dt \leq \frac{C}{x^\mu} \int_0^x \frac{\ell(t)\omega(g,t)}{t} dt.$$

We take into account that $\omega(g,t) \leq Ctk(t)\omega(t)$ and by (3.11) we have $tk(t)\ell(t) \leq 1$ for small t and consequently we arrive at

$$F_1(x) \leq \frac{C}{x^\mu} \int_0^x \frac{\omega(t)}{t} dt \leq C \frac{\omega(x)}{x^\mu} \in L_p$$

by Lemma 8.1.

It remains to estimate the term $F_2(x)$. By (2.2 - 2.4) and (8.5) we have

$$\left| \frac{1}{\rho(t)} - \frac{1}{\rho(x)} \right| \leq C \frac{x-t}{xt^\mu}$$

which yields

$$F_2(x) \leq \frac{C}{x} \int_0^x \frac{\ell(x-t)\omega(g,t)}{t^\mu} dt \leq \frac{C}{x} \int_0^{\frac{x}{2}} \frac{\ell(x-t)\omega_1(t)}{t^\mu} dt$$

$$+ \; \frac{C}{x} \int_{\frac{x}{2}}^x \frac{\ell(x-t)\omega_1(t)}{t^\mu} dt \; =: F_{21}(x) + F_{22}(x).$$

For the term $F_{21}(x)$ we observe that

$$F_{21}(x) \leq \frac{C}{x} \int_0^{\frac{x}{2}} \frac{\ell(t)\omega_1(t)}{t^\mu} \, dt \leq \frac{C}{x} \int_0^{\frac{x}{2}} \frac{\omega(t)}{t^\mu} \, dt$$

and then from the condition $t^{1-\mu}\omega(t) \in \mathcal{Z}^0$ it follows that

$$F_{21}(x) \leq C \frac{\omega(x)}{x^\mu} \in L_p$$

by Lemma 8.1.

For the term

$$F_{22}(x) = \frac{C}{x} \int_0^{\frac{x}{2}} \frac{\ell(t)\omega_1(x-t)}{(x-t)^\mu}\, dt$$

we distinguish the cases $0 < \mu \le 1$ and $1 < \mu \le 2$. In the case $0 < \mu \le 1$ we write

$$F_{22}(x) \le \frac{C}{x} \int_0^{\frac{x}{2}} \ell(t)k(x-t)(x-t)^{1-\mu}\omega(x-t)\, dt.$$

Since the function $x^{1-\mu}\omega(x)$ is increasing in the case $0 < \mu \le 1$, we obtain

$$F_{22}(x) \le C\frac{\omega(x)}{x^\mu} \int_0^{\frac{x}{2}} \ell(t)k(x-t)\, dt \le C\frac{\omega(x)}{x^\mu}.$$

In the case $1 < \mu < 2$, we observe that the function $\frac{\omega_1(x)}{x^\mu}$ is a.d. and $x - t > t$ so that

$$F_{22}(x) \le \frac{C}{x} \int_0^{\frac{x}{2}} \ell(t)\frac{\omega_1(t)}{t^\mu}\, dt = \frac{C}{x} \int_0^{\frac{x}{2}} \frac{tk(t)\ell(t)\omega(t)}{t^\mu}\, dt \le \frac{C}{x} \int_0^{\frac{x}{2}} \frac{\omega(t)}{t^\mu}\, dt.$$

Therefore, in all the cases

$$F_2(x) \le F_{21}(x) + F_{22}(x) \le C\frac{\omega(x)}{x^\mu}$$

and then

$$|\Psi_\varepsilon f(x)| \le F_1(x) + F_2(x) \le C\frac{\omega(x)}{x^\mu},$$

where $C > 0$ does not depend on ε. Hence we conclude that $\sup\limits_{\varepsilon>0} \|\Psi_\varepsilon\|_{L_p} < \infty$ for $1 < p < p_0$, where p_0 is from Lemma 8.1.

The conditions of Theorem 2.9 having been verified, Theorem **D** is proved. □

9. Appendix: Concerning condition (3.5) First we observe that condition (3.5) holds for instance, for functions

$$\ell(x) = x^{-\alpha}\left(ln\frac{A}{x}\right)^p,$$

on $[0,b], b<\infty$, where $\alpha\in(0,1), p\in\mathbb{R}^1, A>b$, which may be verified by direct differentiation of this function and checking condition (3.5). Note that in the case $p=1$ the associated Sonine kernel $k(x)$ is the special Volterra function studied in [22, 23] in connection with the solution of the integral equation of the first kind with a power-logarithmic kernel.

The same is also valid for similar functions which are obtained after a finite number of operations of addition, multiplication and substitution of the power function and the logarithmic function. In the following lemma we give a simple general condition sufficient for a function $\ell(t)$ to satisfy condition (3.5). This lemma covers many examples known as Sonine kernels, in particular the kernel

$$\ell(x) = \frac{I_\alpha(\sqrt{x})}{x^{\alpha/2}}$$

which occurs in applications, $I_\alpha(x) = \sum\limits_{k=0}^{\infty}\frac{\left(\frac{x}{2}\right)^{2k+\alpha}}{k!\Gamma(k+\alpha+1)}$ is the Bessel function of the second kind, as well as many others.

Lemma 9.1. *Let* $\ell(x)=\frac{a(x)}{x^\alpha}$ *where* $\alpha>0$ *and* $a(x)\in C^2([0,b]), 0<b<\infty$, *and* $a(0)\neq 0$. *Then condition* (3.5) *is satisfied for some* $\delta_0>0$.

Proof. Rewrite condition (3.5) in the form

$$|\ell'(x)-\ell'(\lambda x)|\le C(1-\lambda)|\ell'(\lambda x)|,$$
$$0<\lambda<1, \quad 0<x\le\delta.$$

For the function $\ell(x)=x^{-\alpha}a(x)$ this condition after some calculation takes the form

$$|\alpha a(\lambda x)-\alpha\lambda^{1+\alpha}a(x)+x\lambda^{\alpha+1}a'(x)-\lambda x a'(\lambda x)|\le c(1-\lambda)|\alpha a(x\lambda x)-\lambda x a'(\lambda x)|.$$

Under the notation $g(x)=\alpha a(x)-xa'(\lambda x)$, the last inequality turns into

(9.1) $$|g(\lambda x) - \lambda^{1+\alpha} g(x)| \le C(1-\lambda)|g(\lambda x)| \quad or$$
$$\left|\lambda^{1+\alpha}\frac{g(x)}{g(\lambda x)} - 1\right| \le C(1-\lambda).$$

Obviously

$$\left|\lambda^{1+\alpha}\frac{g(x)}{g(\lambda x)} - 1\right| \le |1-\lambda^{1+\alpha}| + \lambda^{1+\alpha}\left|\frac{g(x)-g(\lambda x)}{g(\lambda x)}\right|.$$

Therefore, to obtain (9.1), it suffices to show that

$$|g(x) - g(\lambda x)| \le C(1-\lambda)|g(\lambda x)|$$

which is valid for small $x \in [0, \delta_0])$ with some $\delta_0 > 0$, because $g(x) \in C^1([0,b])$ and $g(0) = \alpha a(0) \ne 0$. □

REFERENCES

1. N.K. Bari and S.B. Stechkin, *Best approximations and differential properties of two conjugate functions (in Russian)*, Proceedings of Moscow Math. Soc., **5**:483–522, (1956).

2. S.N. Bernstein, *Complete Works, Vol II. Constructive function theory* [1931-1953] *(in Russian)*, Izdat. Akad. Nauk SSSR, Moscow, (1954).

3. H.G. Hardy and J.E. Littlewood, *Some properties of fractional integrals*, I. Math. Z., **27(4)**:565–606, (1928).

4. N.K. Karapetiants and N.G. Samko, *Weighted theorems on fractional integrals in the generalized Hölder spaces $H_0^\omega(\rho)$ via the indices m_ω and M_ω*, Fract. Calc. Appl. Anal., **7(4)**, (2004).

5. N.K. Karapetiants and L.D. Shankishvili, *A short proof of Hardy-Littlewood-type theorem for fractional integrals in weighted Hölder spaces*, Fract. Calc. Appl. Anal., **2(2)**:177–192, (1999).

6. N.K. Karapetiants and L.D. Shankishvili, *Fractional integro-differentiation of the complex order in generalized Hölder spaces $H_0^\omega([0;1];\rho)$*, Integral Transforms Spec. Funct., **13(3)**:199–209, (2002).

7. Kh. M. Murdaev, *An estimate for continuity modulus of integrals and derivatives of fractional order (Russian)*, Deponierted in VINITI, Moscow, (1985), Depon. VINITI no 4209, 16 p.

8. Kh. M. Murdaev and S.G. Samko, *Mapping properties of fractional integro-differentiation in weighted generalized Hölder spaces $H_0^\omega(\rho)$ with the weight $\rho(x) = (x-a)^\mu(b-x)^\nu$) and given continuity modulus (Russian)*, Deponierted in VINITI, Moscow, (1986), Depon. VINITI no 3350-B, 25 p.

9. Kh. M. Murdaev and S.G. Samko, *Weighted estimates of continuity modulus of fractional integrals of function having a prescribed continuity modulus with weight (Russian)*, Deponierted in VINITI, Moscow, (1986), Depon. VINITI no 3351-B, 42 p.

10. B.S. Rubin, *Fractional integrals in Hölder spaces with weight, and operators of potential type*, Izv. Akad. Nauk Armjan. SSR Ser. Mat., **9(4)**:308324, (1974).

11. B.S. Rubin, *An imbedding theorem for images of convolution operators on a finite segment, and operators of potential type (in Russian)*, I. Izv. Vysch, Uchebn. Zaved., Matematika., **(1)**:5363, (1982).

12. B.S. Rubin, *Fractional integrals and Riesz potentials with radial density in spaces with power weight*, Izv. Akad. Nauk Armyan. SSR Ser. Mat., **21(5)**:488–503, 511, (1986).

13. N.G. Samko, *Singular integral operators in weighted spaces with generalized Hölder condition*, Proc. A. Razmadze Math. Inst, **120**:107134, (1999).

14. N.G. Samko, *On non-equilibrated almost monotonic functions of the Zygmund-Bary- Stechkin class*, Real Anal. Exch., **30(2)**, (2005).

15. S.G. Samko and R.P. Cardoso, *Integral equations of the first kind of Sonine type*, Int. J. Math. Math. Sci., **(57)**:36093632, (2003).

16. S.G. Samko and R.P. Cardoso, *Sonine integral equations of the first kind in* $L_p(0, b)$, Fract. Calc. Appl. Anal., **6(3)**:235258, (2003).

17. S.G. Samko, A.A. Kilbas, and O.I. Marichev, *Fractional Integrals and Derivatives. Theory and Applications*, London-New-York: Gordon & Breach. Sci. Publ., (Russian edition - Fractional Integrals and Derivatives and some of their Applications, Minsk: Nauka i Tekhnika, 1987.), (1993). 1012 pages.

18. S.G. Samko and Kh.M. Murdaev, *Weighted Zygmund estimates for fractional differentiation and integration, and their applications.* , Trudy Matem. Ibst. Steklov, **180**:197–198, 1987. translated in in Proc. Steklov Inst. Math., AMS, (1989), issue **3**, 233–235.

19. S.G. Samko and Z. Mussalaeva, *Fractional type operators in weighted generalized Hölder spaces.*, Proc. Georgian Acad.Sci., Mathem., **1(5)**:601–626, (1993).

20. N. Sonine, *Sur la generalization dune formula d'Abel*, Acta Mathem., **4**:170176, (1884).

21. N. Sonine, *Investigatons of cylinder functions and special polynomials. (Russian)*, Moscow: Gos. Izdat. Tekhn. Liter., (1954). 244 pages.

22. V. Volterra, *Teoria della potense, dei logarithmi e delle funzioni di compozione*, Atti Accad. dei Lincei, ser. **5, 11**:167–249, (1916).

23. V. Volterra and J. Peres, Paris, *Lecons sur la composition et les fonctions permutables*, Gauthier-Villars, (1924). 186 pages.

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