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THE NUMERICAL SOLUTION OF THE AXIALLY SYMMETRIC LINEAR SLOSHING PROBLEM BY THE BOUNDARY INTEGRAL EQUATION METHOD

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ABSTRACT. This paper is devoted to the numerical solution of the axially symmetric linear sloshing problem. A mathematical model of linear sloshing in a tank filled by an inviscid incompressible liquid is considered. The problem is rewritten as a linear evolution problem on a free surface with an operator coefficient. First, by Laguerre transformation with respect to time, we reduce the non-stationary problem to a sequence of operator equations. Then, using potential theory for the Laplace equation in a bounded domain with corners, a system of boundary integral equations of the second kind is obtained. Taking into account the axial symmetry, we obtain a system of one-dimensional integral equations of the second kind, the kernels of which are represented through the use of complete elliptic integrals of the first and second kinds. A non-linear mesh grading transformation is used to weaken the density singularities. The logarithmic singularity is avoided as well. The full discretization is realized by a Nyström method and results of numerical experiments are presented.

1. Introduction and Problem Statement. Sloshing is a free surface flow problem in a tank, which is subjected to forced oscillations. In this paper we deal with an unsteady potential flow of an inviscid incompressible liquid having a free surface, in a uniform gravitational field. This problem is rewritten and considered by us as an evolution problem. The additional complexity of the problem is the time dependence of the process. For the temporal discretization we use the Laguerre transformation **[1, 2]**. The obvious advantage of the Laguerre

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transformation is the explicit representation of the solution, with the aid of which we can determine the main characteristics of the motion – velocity, pressure and free boundary.

For the solution of this problem we propose to use the boundary integral equation method, which for most cases consists in the reduction of a boundary value problem in some domain to the corresponding integral equation on the boundary of this domain. The main importance of the boundary integral equation method lies in the reduction of the dimensionality of the problem. Note that in our case we already have some differential-operator equation on the boundary and the above method is used only for the explicit representation of the given operator equation.

The linear sloshing problem in an infinite chute with a rib is considered in [**3**]. The problem was reduced to an abstract second order differential equation in time with an operator coefficient on a free surface. For the discretization a collocation quadrature method and Cayley transform method are used. The same methods are considered in [**4**] for the numerical solution of the evolution problem on the boundary. In [**5**] the combination of Laguerre transformation and the integral equation method is used for the abstract evolution problem of second order in time on the smooth closed curve. Some results of the numerical analysis of gravity waves in a channel with a free boundary are presented in [**1**]. In [**6**] the authors announce a modified Nyström-Kress boundary element scheme which has some advantages in capturing the local singularities for admissible domains associated with two-dimensional sloshing in a rectangular tank. In particular, they consider solving the mixed Dirichlet-Neumann problem arising in sloshing simulations for two-dimensional sloshing in a rectangular rigid tank for the case of inviscid irrotational flows. By use of potential theory the problem is reduced to an equation on the boundary. Then after employing a special substitution the appropriate quadrature rules are used to obtain a linear algebraic system.

The surveys by Moiseev&Rumyantsev [**7**], Lukovsky [**8**], Morand& Ohayon [**9**], Ibrahim [**10**], Gavrilyuk&Timokha [**11**] describe the major of theoretical and experimental results for the nonlinear fluid sloshing problems.

The paper is organized in the following way. In Section 2 we apply the Laguerre transformation in time for the semi-discretization of non-stationary problem. As a result the infinite sequence of operator equations on the free boundary is obtained. Then by use of the potential theory the system of boundary integral equations of the second kind is derived. Taking into account the axial symmetry, we obtain the system of one-dimensional integral equations of the second kind. Section 3 is devoted to finding representations of the fundamental solution and its normal derivative in the axially symmetric case. Note that, in general, the integral equations have singularities in the densities at corners of the domain and logarithmic singularities in the kernels. In Section 4 we perform the non-linear mesh grading transformation to weaken the singularities in the densities. The logarithmic singularity is avoided by use of the special trigonometrical quadratures, which are described in Section 5. The full discretization is realized by a Nyström method. The results of numerical experiments are presented in Section 6.

Now we describe briefly the mathematical model for gravity waves in a tank (for details we refer to [**12, 13**]). Let D be a bounded domain in $\mathbb{R}^2$. We consider the motion of water, supposed here to be a perfect, incompressible and homogeneous fluid, in a tank with a fixed boundary $S \cup \tilde{S}$, where S is the wetted part of the tank surface and $\tilde{S}$ is the dry part, and a free boundary S_t (see Fig. 1). Here $S_t := \{(x_1, x_2, \eta(x_1, x_2, t)) \in \mathbb{R}^3 : (x_1, x_2) \in D, \ t > 0\}$, where $\eta : D \times [0, \infty) \to \mathbb{R}$. Note that surface S is located underneath S_t and condition $\partial S = \partial S_t$ holds on the plane $x_3 = 0$, where ∂S and ∂S_t denote boundaries of S and S_t, respectively. Thus, Ω_t with $\partial\Omega_t = S \cup S_t$ is the domain occupied by the liquid.

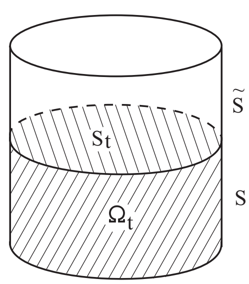

FIGURE 1. View of the domain Ω_t occupied by the liquid.

By scaling of equations we can replace the acceleration of gravity by 1 and assuming that the motion is irrotational we obtain the following equations for the potential function φ and the pressure p in Ω_t:

$$v = \text{grad}\,\varphi, \tag{1.1}$$

$$\triangle\varphi = 0, \tag{1.2}$$

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\text{grad}\,\varphi)^2 = -x_3 - \frac{p - p_0}{\rho_0}. \tag{1.3}$$

Here v is the velocity vector, the constant ρ_0 is the density of the fluid, and p_0 is the constant value of the pressure in the air. The potential function φ satisfies the following boundary conditions. On the free surface S_t the conditions of kinematic and dynamic type must be satisfied.

• *Kinematic boundary condition.* The fluid does not cross the free boundary, therefore the normal component of the velocity division coincides with the normal component of the surface division, from which we deduce

$$\frac{\partial\eta}{\partial t} + \frac{\partial\varphi}{\partial x_1}\frac{\partial\eta}{\partial x_1} + \frac{\partial\varphi}{\partial x_2}\frac{\partial\eta}{\partial x_2} = \frac{\partial\varphi}{\partial x_3} \quad \text{on } S_t. \tag{1.4}$$

• *Dynamic boundary condition.* Since the forces applied to both sides of the free surface are equal the pressure in the fluid and in the atmosphere must be equal, which leads to

$$\frac{\partial\varphi}{\partial t} + \frac{1}{2}(\text{grad}\varphi)^2 + \eta = 0 \quad \text{on } S_t. \tag{1.5}$$

On the base of the tank S we have only that the velocity of the fluid is tangential to the base, that is,

$$\frac{\partial\varphi}{\partial\nu} = 0, \tag{1.6}$$

where ν denotes the outward unit normal vector on S.

For the complete formulation of the problem we also need initial conditions. It should be noted here that the mathematical validation of the initial value problem is still an open question (even in the two-dimensional formulation) [14]. There is only a limited set of mathematical papers that report local existence theorems for the initial-boundary value problems. Almost all of these results are documented by Shinbrot [15], Ovsyannikov [16], Pawell&Günther [17] and Lukovsky&Timokha [18].

Since there is not a proven theory on how to formulate initial conditions for the free boundary value problem (1.1 - 1.6), we assume that φ and $\frac{\partial\varphi}{\partial t}$ are known at the initial time, that is,

$$\varphi|_{t=0} = \omega_0, \quad \frac{\partial\varphi}{\partial t}|_{t=0} = \omega_1 \quad \text{on } S_t. \tag{1.7}$$

Summarizing, we have to determine a function φ, which is harmonic with respect to the space variables and fulfills boundary conditions (1.4 - 1.6) and the initial conditions (1.7). Then the velocity, the pressure and the free surface can be obtained from (1.1), (1.3) and (1.5).

The problem described above is non-linear, but the hypothesis of small perturbations will allow us to linearize it. We assume that for small perturbations of the fluid, η and φ derivatives are small as well and that the free surface occupies the "equilibrium" position $x_3 = 0$. Thus the products of η and φ derivatives may be neglected, which leads to the linear problem for the velocity potential φ:

$$\Delta\varphi = 0 \quad \text{in } \Omega \times (0, \infty), \tag{1.8}$$

$$\frac{\partial^2\varphi}{\partial t^2} + \frac{\partial\varphi}{\partial x_3} = f \quad \text{on } \Gamma_1 \times (0, \infty), \tag{1.9}$$

$$\frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \Gamma_2 \times (0, \infty), \tag{1.10}$$

$$\varphi|_{t=0} = \omega_0, \quad \frac{\partial\varphi}{\partial t}|_{t=0} = \omega_1 \quad \text{on } \Gamma_1. \tag{1.11}$$

Here f is a function that describes the force field which acts on the moving fluid, $\Gamma_1 = \{(x_1, x_2, 0) \in \mathbb{R}^3 : (x_1, x_2) \in D\}$, $\Gamma_2 = S$ and Ω is the domain with boundary $\partial\Omega = \Gamma_1 \bigcup \Gamma_2$. Note that the domain Ω has a rib $\Gamma = \Gamma_1 \bigcap \Gamma_2$.

After the velocity potential is found, the free boundary can be calculated by the formula

$$x_3 = \eta(x_1, x_2, t) = -\frac{\partial\varphi}{\partial t}(x_1, x_2, 0, t). \tag{1.12}$$

Now we rewrite the problem (1.8 - 1.11) as the linear evolution problem on the boundary Γ_1 with operator coefficient. We seek the function $u : \Gamma_1 \times [0, \infty) \to \mathbb{R}$, which satisfies the evolution equation of second order

$$\frac{\partial^2 u}{\partial t^2} + Au = f \quad \text{in } \Gamma_1 \times [0, \infty) \tag{1.13}$$

and the initial conditions

$$u|_{t=0} = \omega_0, \quad \frac{\partial u}{\partial t}|_{t=0} = \omega_1 \quad \text{on } \Gamma_1. \tag{1.14}$$

Here ω_0, ω_1 and f are given functions on Γ_1. The operator A is defined as

$$Au = \frac{\partial \Psi}{\partial \nu} \quad \text{on } \Gamma_1, \tag{1.15}$$

where Ψ is the solution of interior mixed Dirichlet-Neumann boundary value problem

$$\Delta \Psi = 0 \quad \text{in } \Omega, \tag{1.16}$$

$$\Psi = u \ \text{ on } \Gamma_1, \quad \frac{\partial \Psi}{\partial \nu} = 0 \quad \text{on } \Gamma_2. \tag{1.17}$$

The operator A is called Dirichlet-to-Neumann map or Poincaré-Steklov operator and is a pseudo-differential operator. The existence and uniqueness questions for the solutions of the evolution problem (1.13) and (1.14) are discussed in [19, 20]. The authors show that for enough smoothness of input data this initial value problem has a unique solution in the corresponding space.

2. Semi-discretization in time and boundary integral equation method. It is known [21] that every bounded absolutely continuous function $g : (0, \infty) \to \mathbb{R}$ can be expanded in a uniformly convergent Fourier-Laguerre series

$$g(t) = \kappa \sum_{n=0}^{\infty} g_n L_n(\kappa t), \tag{2.1}$$

where L_n are the Laguerre polynomials, $\kappa > 0$ is some fixed parameter and g_n are the Fourier-Laguerre coefficients

$$g_n = \int_0^\infty g(t)e^{-\kappa t}L_n(\kappa t)\mathrm{d}t, \quad n = 0, 1, \ldots \tag{2.2}$$

We interpret the formula (2.2) as the direct Laguerre transformation for a given original function g and the formula (2.1) as the inverse transformation for given images g_n, $n = 0, 1, \ldots$ Note that the Laguerre polynomials form a complete orthogonal system with respect to the corresponding scalar product and the expansion (2.1) converges in the weighted L^2 norm [21].

The relation

$$L_n'' = \sum_{m=0}^{n-2}(n - m)L_m, \quad n = 2, 3, \ldots$$

follows from the classic properties of the Laguerre polynomials [23]. Then by the definition of the Fourier-Laguerre coefficients (2.2) and by partial integration we find the representation

$$\tilde{g}_n = -g'(0) - \kappa(n + 1)g(0) + \sum_{m=0}^{n}(n - m + 1)g_m, \quad n = 0, 1, \ldots \tag{2.3}$$

for the Fourier-Laguerre coefficients $\tilde{g}_n$ of the derivative g''. Now we use the Laguerre transformation with respect to the time variable for the problem (1.13) and (1.14), in other words, we seek the solution in the form

$$u(x, t) = \kappa \sum_{n=0}^{\infty} u_n(x)L_n(\kappa t). \tag{2.4}$$

Then, according to (2.3), the coefficients u_n must satisfy the sequence of operator equations

$$(\kappa^2 I + A)u_n = F_n - \sum_{m=0}^{n-1}\beta_{n-m}u_m \quad \text{on } \Gamma_1 \tag{2.5}$$

for $n = 0, 1, \ldots$ with $F_n = f_n + \omega_1 + \kappa(n+1)\omega_0$ and $\beta_n = \kappa^2(n+1)$. Here f_n are the Fourier-Laguerre coefficients for the function f.

Taking into account that the domain Ω is bounded, the solution of the boundary value problem (1.16) and (1.17) can be represented in the form of a single-layer potential

$$\Psi(x) = \int_{\partial\Omega} \mu(y)\Phi(x,y)\mathrm{d}s(y), \quad x \in \Omega, \tag{2.6}$$

where $\Phi(x,y) = (4\pi)^{-1}|x-y|^{-1}$ is the fundamental solution of (1.16) and μ is an unknown density. Then from properties of this potential [30] we have the following integral representation for the operator A:

$$(Av)(x) = \frac{1}{2}\mu(x) + \int_{\partial\Omega} \mu(y) \frac{\partial}{\partial\nu(x)}\Phi(x,y)\mathrm{d}s(y), \quad x \in \partial\Gamma_1\backslash\Gamma. \tag{2.7}$$

Thus the operator equations (2.5) can be reduced to the sequence of systems of integral equations of the second kind

$$\begin{cases} \frac{1}{2}\mu_n(x) + \int_{\partial\Omega} \mu_n(y) \left[\kappa^2\Phi(x,y) + \frac{\partial}{\partial\nu(x)}\Phi(x,y)\right] \mathrm{d}s(y) \\ = F_n(x) - \sum\limits_{m=0}^{n-1} \beta_{n-m} \int_{\partial\Omega} \mu_m(y)\Phi(x,y)\mathrm{d}s(y), \quad x \in \Gamma_1\backslash\Gamma, \\ \frac{1}{2}\mu_n(x) + \int_{\partial\Omega} \mu_n(y) \frac{\partial}{\partial\nu(x)}\Phi(x,y)\mathrm{d}s(y) = 0, \quad x \in \Gamma_2\backslash\Gamma, \end{cases} \tag{2.8}$$

where $n = 0, 1, \ldots$

In the next step we assume that the surface $\partial\Omega$ is created by the rotation of some curve $L := L_1 \bigcup L_2$ about the $0x_3$ axis, where L is given through a parametric representation (see Fig. 2).

In this case it is convenient to introduce the cylindrical coordinate system (r, z, φ).

Let $L_i := \{x_i(\xi) = (r_i(\xi), z_i(\xi)), (i-1)\pi \le \xi \le i\pi\}$ with $r_i \ge 0$ and $|x_i'(\xi)| > 0$ for all $\xi \in [(i-1)\pi, i\pi]$, $i = 1, 2$. Then the boundary $\partial\Omega$ can be represented in the form

$$\partial\Omega := \{x(\xi,\varphi) = (r(\xi)\cos(\varphi), r(\xi)\sin(\varphi), z(\xi)),\ 0 \le \xi, \varphi \le 2\pi\}, \tag{2.9}$$

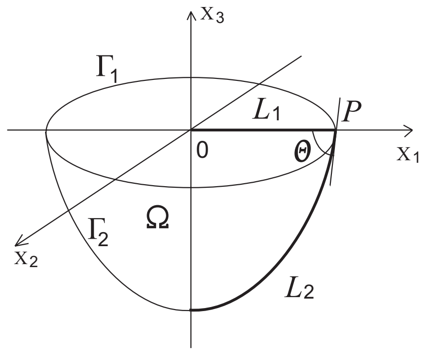

FIGURE 2. View of the domain Ω.

where $r(\xi) = r_i(\xi),\ z(\xi) = z_i(\xi)$ for $\xi \in [(i-1)\pi, i\pi]\,,\ i = 1, 2$.

We also assume that the functions F_n do not depend on φ. Then, due to above assumptions, we can write the integral equations (2.8) in the parametric form

$$(2.10)\quad \begin{cases} \dfrac{1}{2}\bar{\mu}_n(\xi) + \dfrac{1}{4\pi}\displaystyle\int_0^{2\pi} \bar{\mu}_n(\tau)Q(\tau)[\kappa^2\hat{\Phi}(\xi,\tau) + \hat{\Phi}_1(\xi,\tau)]\mathrm{d}\tau \\ = \bar{F}_n(\xi) - \dfrac{1}{4\pi}\displaystyle\sum_{m=0}^{n-1}\beta_{n-m}\int_0^{2\pi}\bar{\mu}_m(\tau)Q(\tau)\hat{\Phi}(\xi,\tau)\mathrm{d}\tau, \quad \xi \in [0,\pi), \\ \dfrac{1}{2}\bar{\mu}_n(\xi) + \dfrac{1}{4\pi}\displaystyle\int_0^{2\pi} \bar{\mu}_n(\tau)Q(\tau)\hat{\Phi}_1(\xi,\tau)\mathrm{d}\tau = 0, \quad \xi \in (\pi, 2\pi], \end{cases}$$

where $\bar{F}_n(\xi) = F_n(x_1(\xi)),\ \bar{\mu}_n(\xi) = \mu_n(x(\xi))$,
$Q(\xi) = r(\xi)\sqrt{[r'(\xi)]^2 + [z'(\xi)]^2}$. Here $\hat{\Phi}$ and $\hat{\Phi}_1$ denote the fundamental solution and its normal derivative, respectively, in the axially symmetric case.

3. Fundamental solution and its normal derivative in axially symmetric case. In this section we find a representation of the fundamental solution and its normal derivative in the axially symmetric case. After introducing the notation $R(\xi,\tau,\varphi) = ([r(\xi)]^2 + [r(\tau)]^2 -$

$2r(\xi)r(\tau)\cos(\varphi) + [z(\xi) - z(\tau)]^2)^{\frac{1}{2}}$ analogously to [22], we can write $\hat{\Phi}$ and $\hat{\Phi}_1$ as

$$\hat{\Phi}(\xi, \tau) = \int_0^{2\pi} \frac{\mathrm{d}\varphi}{R(\xi, \tau, \varphi)} \tag{3.1}$$

and

$$\hat{\Phi}_1(\xi, \tau) = \int_0^{2\pi} \frac{\partial}{\partial \nu(\xi)} \frac{\mathrm{d}\varphi}{R(\xi, \tau, \varphi)}, \tag{3.2}$$

where $\nu(\xi)$ denotes the unit normal vector at the point $x(\xi) = (r(\xi), z(\xi))$ directed into the exterior of the domain.

Next we substitute $\varphi = \pi + 2\alpha$ into the integral (3.1), after which it can be written in the following form

$$\hat{\Phi}(\xi, \tau) = \frac{4}{[p(\xi, \tau)]^{\frac{1}{2}}} K(k^2(\xi, \tau)), \tag{3.3}$$

where $p(\xi, \tau) = (r(\xi) + r(\tau))^2 + (z(\xi) - z(\tau))^2$, $k^2(\xi, \tau) = \dfrac{2r(\xi)r(\tau)}{p(\xi, \tau)}$ and $K(k) = \displaystyle\int_0^{\pi/2} [1 - k^2 \sin(\alpha)]^{-\frac{1}{2}} \mathrm{d}\alpha$ is the complete elliptic integral of the first kind [23].

For the case of $\hat{\Phi}_1$ from (3.2) we have

$$\hat{\Phi}_1(\xi, \tau) = \frac{\partial \hat{\Phi}(\xi, \tau)}{\partial r(\xi)} \frac{z'(\xi)}{|x'(\xi)|} - \frac{\partial \hat{\Phi}(\xi, \tau)}{\partial z(\xi)} \frac{r'(\xi)}{|x'(\xi)|}, \tag{3.4}$$

where

$$\frac{\partial \hat{\Phi}(\xi, \tau)}{\partial r(\xi)} = 4p(\xi, \tau)^{-\frac{3}{2}} \left(2r(\tau) \frac{E\left(k(\xi, \tau)\right) - K\left(k(\xi, \tau)\right)}{k^2(\xi, \tau)} \right. \\ \left. + \left(r(\tau) - r(\xi)\right) \frac{E\left(k(\xi, \tau)\right)}{1 - k^2\left(\xi, \tau\right)} \right)$$

and

$$\frac{\partial \hat{\Phi}(\xi,\tau)}{\partial z(\xi)} = 4p(\xi,\tau)^{-\frac{3}{2}}\left(z(\tau)-z(\xi)\right)\frac{E\left(k^2(\xi,\tau)\right)}{1-k^2(\xi,\tau)}.$$

Here $E(k) = \int\limits_0^{\pi/2} [1-k^2\sin(\alpha)]^{\frac{1}{2}} \mathrm{d}\alpha$ is the complete elliptic integral of the second kind [23].

Now, due to the expressions $\hat{\Phi}$ and $\hat{\Phi}_1$ through the complete elliptic integrals, the system (2.10) can be rewritten in the form

$$(3.5)\quad \begin{cases} \frac{1}{2}\bar{\mu}_{1,n}(\xi) + \frac{1}{\pi}\sum\limits_{j=1}^{2}\int\limits_{(j-1)\pi}^{j\pi} \bar{\mu}_{j,n}(\tau)\left\{K_{1j}(\xi,\tau)H^{11}_{1j}(\xi,\tau)\right. \\ \qquad\qquad \left. +E_{1j}(\xi,\tau)H^{21}_{1j}(\xi,\tau)\right\}\mathrm{d}\tau = \bar{F}_n(\xi) \\ -\frac{1}{\pi}\sum\limits_{m=0}^{n-1}\beta_{n-m}\sum\limits_{j=1}^{2}\int\limits_{(j-1)\pi}^{j\pi} \bar{\mu}_{j,m}(\tau)B_j(\xi,\tau)K_{1j}(\xi,\tau)\mathrm{d}\tau, \\ \qquad\qquad\qquad \xi\in[0,\pi), \\ \frac{1}{2}\bar{\mu}_{2,n}(\xi) + \frac{1}{\pi}\sum\limits_{j=1}^{2}\int\limits_{(j-1)\pi}^{j\pi} \bar{\mu}_{j,n}(\tau)\{K_{2j}(\xi,\tau)H^{12}_{2j}(\xi,\tau) \\ \qquad +E_{2j}(\xi,\tau)H^{22}_{2j}(\xi,\tau)\}\mathrm{d}\tau = 0, \quad \xi\in(\pi,2\pi]. \end{cases}$$

Here

$$\begin{aligned} \bar{\mu}_{i,n}(\xi) &= \bar{\mu}_n(\xi), \xi\in[(i-1)\pi, i\pi], \quad i=1,2, \\ K_{ij}(\xi,\tau) &= K(k_{ij}(\xi,\tau)), E_{ij}(\xi,\tau) = E(k_{ij}(\xi,\tau)), \quad i,j=1,2, \\ k^2_{ij}(\xi,\tau) &= k^2(\xi,\tau), \xi\in[(i-1)\pi, i\pi], \tau\in[(j-1)\pi, j\pi], \quad i,j=1,2, \\ H^{11}_{ij}(\xi,\tau) &= \frac{Q_j(\tau)}{\left(p_{ij}(\xi,\tau)\right)^{1/2}}\left(\kappa^2 - \frac{2r_j(\tau)}{k^2_{ij}(\xi,\tau)p_{ij}(\xi,\tau)}\frac{z'_i(\xi)}{|x'_i(\xi)|}\right), \\ H^{21}_{ij}(\xi,\tau) &= \frac{Q_j(\tau)}{\left(p_{ij}(\xi,\tau)\right)^{3/2}}\left(\frac{2r_j(\tau)}{k^2_{ij}(\xi,\tau)}\frac{z'_i(\xi)}{|x'_i(\xi)|} + \frac{\langle\nu(\xi), x_j(\xi)-x_i(\xi)\rangle}{1-k^2_{ij}(t,\tau)}\right) \\ &\qquad\qquad \text{for } i\neq j \text{ and } \quad \xi\neq\tau, \end{aligned}$$

$$H_{ii}^{21}(\xi,\xi) = Q_i(\xi)\left(\frac{2r_i(\xi)}{k_{ii}^2(\xi,\xi)\left(p_{ii}(\xi,\xi)\right)^{3/2}}\frac{z_i'(\xi)}{|x_i'(\xi)|} + \frac{r_i''(\xi)z_i'(t) - r_i'(\xi)z_i''(\xi)}{4r_i(\xi)|x_i(\xi)|^3}\right),$$

$$H_{ij}^{12}(\xi,\tau) = -\frac{Q_j(\tau)}{\left(p_{ij}(\xi,\tau)\right)^{3/2}}\frac{2r_j(\tau)}{k_{ij}^2(\xi,\tau)}\frac{z_i'(\xi)}{|x_i'(\xi)|}, \quad H_{ij}^{22}(\xi,\tau) = H_{ij}^{21}(\xi,\tau),$$

$$B_j(\xi,\tau) = \frac{Q_j(\tau)}{\left(p_{1j}(\xi,\tau)^{1/2}\right)}, \quad Q_i(\xi) = Q(\xi), \xi \in [(i-1)\pi, i\pi], \quad i = 1,2,$$

$$p_{ij}(\xi,\tau) = p(\xi,\tau), \xi \in [(i-1)\pi, i\pi], \tau \in [(j-1)\pi, j\pi], \quad i,j = 1,2.$$

Taking into account the representations of the complete elliptic integrals [23], we can write the following identities

$$K_{ij}(\xi,\tau) = K^1\left(\eta_{ij}(\xi,\tau)\right)\ln\frac{1}{\eta_{ij}^2(\xi,\tau)} + K^2\left(\eta_{ij}(\xi,\tau)\right),$$
$$E_{ij}(\xi,\tau) = E^1\left(\eta_{ij}(\xi,\tau)\right)\ln\frac{1}{\eta_{ij}^2(\xi,\tau)} + E^2\left(\eta_{ij}(\xi,\tau)\right), \tag{3.6}$$

where $\eta_{ij}(\xi,\tau) = 1 - k_{ij}^2(\xi,\tau),\ i,j = 1,2$ and $K^\ell,\ E^\ell,\ \ell = 1,2$ – functions, which are given in the view of power series. Since these series are slowly convergent for some values of argument, it is convenient to use a hyper accurate Chebyshev approximations (see [24]) for these functions

$$K^\ell\left(\eta_{ij}(\xi,\tau)\right) \approx \sum_{m=0}^{NK} a_{m\ell}\eta_{ij}^m(\xi,\tau), \quad E^\ell\left(\eta_{ij}(\xi,\tau)\right) \approx \sum_{m=0}^{NE} b_{m\ell}\eta_{ij}^m(\xi,\tau), \tag{3.7}$$

where $a_{m\ell}, b_{m\ell}$ are given coefficients. Note that, in particular, for $NK = NE = 10$ the maximal absolute error of calculations by formulas (3.6), using (3.7), has order 10^{-18}.

Thus we can rewrite the system (3.5) as

$$
(3.8)\quad \begin{cases}
\frac{1}{2}\bar{\mu}_{1,n}(\xi) + \frac{1}{\pi}\sum_{j=1}^{2}\int_{(j-1)\pi}^{j\pi} \bar{\mu}_{j,n}(\tau)\{N_{1j}^{11}(\xi,\tau)\ln\frac{1}{1-k_{1j}^2(\xi,\tau)} \\
\qquad\qquad + N_{1j}^{21}(\xi,\tau)\}\mathrm{d}\tau = \bar{F}_n(\xi) \\
-\frac{1}{\pi}\sum_{m=0}^{n-1}\beta_{n-m}\sum_{j=1}^{2}\int_{(j-1)\pi}^{j\pi} \bar{\mu}_{j,m}(\tau)\{M_j^1(\xi,\tau)\ln\frac{1}{1-k_{1j}^2(\xi,\tau)} \\
\qquad\qquad + M_j^2(\xi,\tau)\}\mathrm{d}\tau, \quad \xi \in [0,\pi), \\
\frac{1}{2}\bar{\mu}_{2,n}(\xi) + \frac{1}{\pi}\sum_{j=1}^{2}\int_{(j-1)\pi}^{j\pi} \bar{\mu}_{j,n}(\tau)\{N_{2j}^{12}(\xi,\tau)\ln\frac{1}{1-k_{2j}^2(\xi,\tau)} \\
\qquad\qquad + N_{2j}^{22}(\xi,\tau)\}\mathrm{d}\tau = 0, \quad \xi \in (\pi, 2\pi],
\end{cases}
$$

where $N_{ij}^{\ell k}(\xi,\tau) = K^{\ell}(\eta_{ij}(\xi,\tau))\, H_{ij}^{1k}(\xi,\tau) + E^{\ell}(\eta_{ij}(\xi,\tau))\, H_{ij}^{2k}(\xi,\tau)$, $M_j^{\ell}(\xi,\tau) = K^{\ell}(\eta_{1j}(\xi,\tau))\, B_j(\xi,\tau)$, $i,j,k,\ell = 1,2$.

4. Weakening of Singularities. It is known [25] that the density μ in (2.6) must have the form

$$
(4.1)\quad \mu(x) = O(|x-P|^{\lambda}), \quad x \to P, \quad \lambda = \min\left\{\frac{\pi}{2\Theta}, \frac{\pi}{2(2\pi-\Theta)}\right\} - 1
$$

near the point P, where Θ is the interior angle of the curve L (see Fig. 2), i.e., the density has a singularity at the corner.

For the purpose of weakening of the above singularity we follow [26, 28, 29] by making a special non-linear mesh grading transformation, the main idea of which is to introduce a parameterization γ, which varies more slowly than arc-length parameterization in the vicinity of the corner P. By forcing γ to vary slowly enough near the corner, the solutions $\bar{\mu}_{i,n}$ of (3.8) can be made as smooth as desired on $[0, 2\pi]$.

First we introduce a cubic polynomial

$$
(4.2)\qquad v(s) = \left(\frac{1}{q} - \frac{\pi}{2}\right)\left(\frac{\pi - 2s}{\pi}\right)^3 - \frac{1}{q}\left(\frac{\pi - 2s}{\pi}\right) + \frac{\pi}{2},
$$

where $0 \le s \le \pi$ and $q \ge 2$. Then, setting

$$w(s) = \pi \frac{[v(s)]^q}{[v(s)]^q + [v(\pi - s)]^q}, \quad 0 \le s \le \pi, \tag{4.3}$$

we define the mesh grading transformation

$$\gamma(s) = \begin{cases} \gamma_1(s) = w(s), & 0 \le s \le \pi, \\ \gamma_2(s) = \pi + w(s - \pi), & \pi \le s \le 2\pi. \end{cases} \tag{4.4}$$

Then clearly

$$\gamma \in C^{q-1}[0, 2\pi], \quad \gamma^{(\ell)}(0) = \gamma^{(\ell)}(\pi) = \gamma^{(\ell)}(2\pi) = 0, \;\; \ell = 1, \dots, q-1.$$

Now, using the transformation (4.4), we perform the substitution

$$\xi = \begin{cases} \gamma_1(s), & 0 \le s \le \pi, \\ \gamma_2(s), & \pi \le s \le 2\pi, \end{cases} \qquad \tau = \begin{cases} \gamma_1(\sigma), & 0 \le \sigma \le \pi, \\ \gamma_2(\sigma), & \pi \le \sigma \le 2\pi, \end{cases} \tag{4.5}$$

and rewrite the integral equations (3.8) in the form

$$\begin{cases} \frac{1}{2}\varphi_{1,n}(s) + \frac{\gamma_1'(s)}{\pi} \sum_{j=1}^{2} \int\limits_{(j-1)\pi}^{j\pi} \varphi_{j,n}(\sigma) \{ \bar{N}_{1j}^{11}(s,\sigma) \ln \frac{1}{1 - \bar{k}_{1j}^2(s,\sigma)} \\ \qquad + \bar{N}_{1j}^{21}(s,\sigma) \} \mathrm{d}\sigma = g_n(s) \\ - \frac{\gamma_1'(s)}{\pi} \sum_{m=0}^{n-1} \beta_{n-m} \sum_{j=1}^{2} \int\limits_{(j-1)\pi}^{j\pi} \varphi_{j,m}(\sigma) \{ \bar{M}_j^1(s,\sigma) \ln \frac{1}{1 - \bar{k}_{1j}^2(s,\sigma)} \\ \qquad + \bar{M}_j^2(s,\sigma) \} \mathrm{d}\sigma, \quad s \in [0,\pi], \\ \frac{1}{2}\varphi_{2,n}(s) + \frac{\gamma_2'(s)}{\pi} \sum_{j=1}^{2} \int\limits_{(j-1)\pi}^{j\pi} \varphi_{j,n}(\sigma) \{ \bar{N}_{2j}^{12}(s,\sigma) \ln \frac{1}{1 - \bar{k}_{2j}^2(s,\sigma)} \\ \qquad + \bar{N}_{2j}^{22}(s,\sigma) \} \mathrm{d}\sigma = 0, \quad s \in [\pi, 2\pi]. \end{cases} \tag{4.6}$$

Here we have introduced the functions

$$g_n(s) = F_n(\gamma_1(s))\gamma_1'(s), \;\; \varphi_{i,n}(s) = \bar{\mu}_{i,n}(\gamma_i(s))\gamma_i'(s), \;\; i = 1, 2,$$

and the notations

$$\bar{N}^{\ell k}_{ij}(s,\sigma) = N^{\ell k}_{ij}(\gamma_i(s),\gamma_j(\sigma)), \quad \bar{M}^k_j(s,\sigma) = M^k_j(\gamma_1(s),\gamma_j(\sigma)),$$
$$\bar{k}^2_{ij}(s,\sigma) = k^2_{ij}(\gamma_i(s),\gamma_j(\sigma)), \quad i,j,\ell,k = 1,2.$$

In the next step we want to extend each of the functions γ_1 and γ_2 to be 2π-periodic. For this in addition we define

$$\gamma_1(s) = \begin{cases} \gamma_1(s), & 0 \le s \le \pi, \\ \gamma_1(2\pi - s), & \pi \le s \le 2\pi \end{cases} \tag{4.7}$$

$$\gamma_2(s) = \begin{cases} \gamma_2(2\pi - s), & 0 \le s \le \pi, \\ \gamma_2(s), & \pi \le s \le 2\pi, \end{cases} \tag{4.8}$$

together with

$$\gamma_i(s+2\pi) = \gamma_i(s), \quad i = 1,2. \tag{4.9}$$

Thus γ_i are even and 2π-periodic and clearly these properties extend to all functions in the system (4.6). Now we can write this system as

$$\begin{cases} \dfrac{1}{2}\varphi_{1,n}(s) + \dfrac{\gamma_1'(s)}{2\pi}\displaystyle\sum_{j=1}^{2}\int_0^{2\pi}\varphi_{j,n}(\sigma)\{\bar{N}^{11}_{1j}(s,\sigma)\ln\frac{1}{1-\bar{k}^2_{1j}(s,\sigma)} \\ \qquad\qquad + \bar{N}^{21}_{1j}(s,\sigma)\}\mathrm{d}\sigma = g_n(s) \\ -\dfrac{\gamma_1'(s)}{2\pi}\displaystyle\sum_{m=0}^{n-1}\beta_{n-m}\sum_{j=1}^{2}\int_0^{2\pi}\varphi_{j,m}(\sigma)\{\bar{M}^1_j(s,\sigma)\ln\frac{1}{1-\bar{k}^2_{1j}(s,\sigma)} \\ \qquad\qquad + \bar{M}^2_j(s,\sigma)\}\mathrm{d}\sigma, \quad s \in [0,2\pi], \\ \dfrac{1}{2}\varphi_{2,n}(s) + \dfrac{\gamma_2'(s)}{2\pi}\displaystyle\sum_{j=1}^{2}\int_0^{2\pi}\varphi_{j,n}(\sigma)\{\bar{N}^{12}_{2j}(s,\sigma)\ln\frac{1}{1-\bar{k}^2_{2j}(s,\sigma)} \\ \qquad\qquad + \bar{N}^{22}_{2j}(s,\sigma)\}\mathrm{d}\sigma = 0, \quad s \in [0,2\pi]. \end{cases} \tag{4.10}$$

The main importance of these transformations lies in the fact that now the desired functions $\varphi_{i,n}$ are in $C^{q-1}[0,2\pi]$.

Note that the kernels of the system (4.10) contain logarithmic singularities, which occur for $i = j$ and $s = \sigma$, because $1 - \bar{k}_{ij}^2 = \dfrac{|\bar{x}_i(s) - \bar{x}_j(\sigma)|^2}{\bar{p}_{ij}(s,\sigma)}$. For these kernels we perform the transformation

$$\ln \frac{\bar{p}_{ii}(s,\sigma)}{|\bar{x}_i(s) - \bar{x}_i(\sigma)|^2} = -\ln 4(\cos(s) - \cos(\sigma))^2 + b_i(s,\sigma), \tag{4.11}$$

where

$$b_i(s,\sigma) = 2\ln \frac{2|\cos(s) - \cos(\sigma)|\,(\bar{p}_{ii}(s,\sigma))^{\frac{1}{2}}}{|\bar{x}_i(s) - \bar{x}_j(\sigma)|}$$

with the diagonal terms

$$b_i(s,s) = 2\ln \frac{4|\sin(s)|\bar{r}_i(s)}{|\bar{x}_i'(s)|}.$$

Obviously, the function b is not defined at the four corners and the center of the square $[0,2\pi] \times [0,2\pi]$ and we will take this fact into account later. Next we note that all functions in (4.10) are even and therefore the following identity

$$\int_0^{2\pi} \varphi(\sigma) \ln\left[4(\cos(s) - \cos(\sigma))^2\right] \mathrm{d}\sigma = 2\int_0^{2\pi} \varphi(\sigma) \ln\left(4\sin^2\frac{s-\sigma}{2}\right) \mathrm{d}\sigma \tag{4.12}$$

for $s \in [0,2\pi]$ holds. Now, with the aid of (4.12), the system (4.10) can be rewritten in the form

$$
(4.13)\quad \begin{cases}
\frac{1}{2}\varphi_{1,n}(s) + \frac{\gamma_1'(s)}{2\pi}\int_0^{2\pi}\varphi_{1,n}(\sigma)\left\{L_{11}^{11}(s,\sigma)\ln\left(4\sin^2\frac{s-\sigma}{2}\right)\right. \\
\qquad\left. +L_{11}^{21}(s,\sigma)\right\}\mathrm{d}\sigma + \frac{\gamma_1'(s)}{2\pi}\int_0^{2\pi}\varphi_{2,n}(\sigma)L_{12}^{31}(s,\sigma)\mathrm{d}\sigma \\
= g_n(s) - \frac{\gamma_1'(s)}{2\pi}\sum_{m=0}^{n-1}\beta_{n-m}\left[\int_0^{2\pi}\varphi_{2,m}(\sigma)A_{12}^{31}(s,\sigma)\mathrm{d}\sigma\right. \\
\left. +\int_0^{2\pi}\varphi_{1,m}(\sigma)\left\{A_{11}^{11}(s,\sigma)\ln\left(4\sin^2\frac{s-\sigma}{2}\right)+A_{11}^{21}(s,\sigma)\right\}d\sigma\right], \\
\qquad\qquad s\in[0,2\pi], \\
\frac{1}{2}\varphi_{2,n}(s) + \frac{\gamma_2'(s)}{2\pi}\int_0^{2\pi}\varphi_{2,n}(\sigma)\left\{L_{22}^{12}(s,\sigma)\ln\left(4\sin^2\frac{s-\sigma}{2}\right)\right. \\
\left. +L_{22}^{22}(s,\sigma)\right\}\mathrm{d}\sigma + \frac{\gamma_2'(s)}{2\pi}\int_0^{2\pi}\varphi_{1,n}(\sigma)L_{21}^{32}(s,\sigma)\mathrm{d}\sigma = 0, \\
\qquad\qquad s\in[0,2\pi],
\end{cases}
$$

where

$$
\begin{aligned}
L_{11}^{11}(s,\sigma) &= -2\bar{N}_{11}^{11}(s,\sigma), \\
L_{11}^{21}(s,\sigma) &= \bar{N}_{11}^{11}(s,\sigma)b_1(s,\sigma) + \bar{N}_{11}^{21}(s,\sigma), \\
L_{12}^{31}(s,\sigma) &= \bar{N}_{12}^{11}(s,\sigma)\ln\frac{1}{1-\bar{k}_{12}^2(s,\sigma)} + \bar{N}_{12}^{21}(s,\sigma), \\
A_{11}^{11}(s,\sigma) &= -2\bar{M}_1^1(s,\sigma), \\
A_{11}^{21}(s,\sigma) &= \bar{M}_1^1(s,\sigma)b_1(s,\sigma) + \bar{M}_1^2(s,\sigma), \\
A_{12}^{31}(s,\sigma) &= \bar{M}_2^1(s,\sigma)\ln\frac{1}{1-\bar{k}_{12}^2(s,\sigma)} + \bar{M}_2^2(s,\sigma), \\
L_{22}^{12}(s,\sigma) &= -2\bar{N}_{22}^{12}(s,\sigma), \\
L_{22}^{22}(s,\sigma) &= \bar{N}_{22}^{12}(s,\sigma)b_2(s,\sigma) + \bar{N}_{22}^{22}(s,\sigma), \\
L_{21}^{32}(s,\sigma) &= \bar{N}_{21}^{21}(s,\sigma)\ln\frac{1}{1-\bar{k}_{21}^2(s,\sigma)} + \bar{N}_{21}^{22}(s,\sigma).
\end{aligned}
$$

Let H_e^p, $p \geq 0$ denote the Sobolev spaces of even 2π-periodic functions. The following theorem establishes the existence of solutions of the system (4.13).

Theorem 4.1. *Assume that* $q \geq 2$. *Then for every* $g_n \in H_e^0[0,2\pi]$ *there exist unique solutions* $\varphi_{1,n}, \varphi_{2,n} \in H_e^0[0,2\pi]$ *of the system* (4.13).

Proof. The system (4.13) has only trivial solutions for $g_n = 0$. To prove this we will use the induction method. Let us consider the first system of the sequence (4.13):

$$\begin{cases} \frac{1}{2}\varphi_{1,0}(s) + \frac{\gamma_1'(s)}{2\pi}\int\limits_0^{2\pi}\varphi_{1,0}(\sigma)\left\{L_{11}^{11}(s,\sigma)\ln\left(4\sin^2\frac{s-\sigma}{2}\right) + L_{11}^{21}(s,\sigma)\right\}\mathrm{d}\sigma \\ \qquad + \frac{\gamma_1'(s)}{2\pi}\int\limits_0^{2\pi}\varphi_{2,0}(\sigma)L_{12}^{31}(s,\sigma)\mathrm{d}\sigma = 0, \quad s \in [0,2\pi], \\ \frac{1}{2}\varphi_{2,0}(s) + \frac{\gamma_2'(s)}{2\pi}\int\limits_0^{2\pi}\varphi_{2,0}(\sigma)\left\{L_{22}^{12}(s,\sigma)\ln\left(4\sin^2\frac{s-\sigma}{2}\right) + L_{22}^{22}(s,\sigma)\right\}\mathrm{d}\sigma \\ \qquad + \frac{\gamma_2'(s)}{2\pi}\int\limits_0^{2\pi}\varphi_{1,0}(\sigma)L_{21}^{32}(s,\sigma)\mathrm{d}\sigma = 0, \quad s \in [0,2\pi]. \end{cases}$$

Obviously, this system corresponds to the following mixed boundary value problem for the Laplace equation

$$\Delta U = 0 \quad \text{in } \Omega, \tag{4.14}$$

$$U + \frac{\partial U}{\partial \nu} = 0 \quad \text{on } \Gamma_1, \tag{4.15}$$

$$\frac{\partial U}{\partial \nu} = 0 \quad \text{on } \Gamma_2. \tag{4.16}$$

With use of Green's formulas, considering that $\partial\Omega$ is continuous (see [27]), it is easy to show that the boundary value problem (4.14 - 41.6) has only the trivial solution which implies $\varphi_{1,0} = 0$, $\varphi_{2,0} = 0$. Then due to induction we obtain that the systems (4.13) have only trivial solutions for $g_n = 0$ and $n > 0$.

Let us define the following operators

$$
\begin{aligned}
(S_i\psi)(s) &:= -\frac{1}{\pi}\int_0^{2\pi} \bar{N}_{ii}^{1i}(s,\sigma)\ln\left(4\sin^2\frac{s-\sigma}{2}\right)\psi(\sigma)\mathrm{d}\sigma,\\
(B_i\psi)(s) &:= \frac{1}{2\pi}\int_0^{2\pi} \bar{N}_{ii}^{1i}(s,\sigma)b_i(s,\sigma)\psi(\sigma)\mathrm{d}\sigma,\\
(C_{ij}\psi)(s) &:= \frac{1}{2\pi}\int_0^{2\pi} L_{ij}^{3i}(s,\sigma)\psi(\sigma)\mathrm{d}\sigma,\\
(D_i\psi)(s) &:= \frac{1}{2\pi}\int_0^{2\pi} \bar{N}_{ii}^{2i}(s,\sigma)\psi(\sigma)\mathrm{d}\sigma,\\
(\bar{S}\psi)(s) &:= -\frac{1}{\pi}\int_0^{2\pi} \bar{M}_1^{1}(s,\sigma)\ln\left(4\sin^2\frac{s-\sigma}{2}\right)\psi(\sigma)\mathrm{d}\sigma,\\
(\bar{B}\psi)(s) &:= \frac{1}{2\pi}\int_0^{2\pi} \bar{M}_1^{1}(s,\sigma)b_1(s,\sigma)\psi(\sigma)\mathrm{d}\sigma,\\
(\bar{C}\psi)(s) &:= \frac{1}{2\pi}\int_0^{2\pi} A_{11}^{31}(s,\sigma)\psi(\sigma)\mathrm{d}\sigma,\\
(\bar{D}\psi)(s) &:= \frac{1}{2\pi}\int_0^{2\pi} \bar{M}_1^{2}(s,\sigma)\psi(\sigma)\mathrm{d}\sigma
\end{aligned}
$$

for $s \in [0, 2\pi]$. Then we can write (4.13) in the operator form

$$
\left(\frac{1}{2}\mathbf{I}+\mathbf{L}\right)\vec{\varphi}_n = \vec{g}_n - \sum_{m=0}^{n-1}\beta_{n-m}\mathbf{K}\vec{\varphi}_m, \tag{4.17}
$$

where $\mathbf{I}$ is 2×2 identity matrix, $\vec{\varphi}_n := (\varphi_{1,n}, \varphi_{2,n})^T$, $\vec{g}_n := (\alpha^{-1}g_n, 0)^T$,

$$
\begin{aligned}
\mathbf{L} &:= \begin{pmatrix} \gamma_1'(S_1+B_1+D_1) & \gamma_1' C_{12} \\ \gamma_2' C_{21} & \gamma_2'(S_2+B_2+D_2) \end{pmatrix},\\
\mathbf{K} &:= \begin{pmatrix} \gamma_1'(\bar{S}+\bar{B}+\bar{D}) & \gamma_1'\bar{C} \\ 0 & 0 \end{pmatrix}.
\end{aligned}
$$

The operators $\bar{D},\ \bar{C},\ D_i,\ C_{ij},\ i,j=1,2$ have continuous kernels and therefore are compact in $H_e^0[0,2\pi]$. For the analysis of $S_i,\ B_i,\ \bar{S},\ \bar{B}$ we refer to the results obtained in [28, 29] by Mellin transform techniques.

It was shown that operators $S_i + B_i,\ i = 1,2$ and $\bar{S} + \bar{B}$ are bounded from $H_e^0[0,2\pi]$ to $H_e^1[0,2\pi]$ and therefore compact in $H_e^0[0,2\pi]$.

Thus the operator $\mathbf{L}$ is compact in $H_e^0[0,2\pi] \times H_e^0[0,2\pi]$. Then from Riesz-Shauder theory and by induction the proof of the theorem is complete. $\square$

5. Full discretization by a Nyström method. For the numerical solution of the integral equations (4.13) we use a Nyström method [30]. We consider the following two quadrature rules on equidistant grids $s_k^\ell = kh^\ell$, $h^\ell = \dfrac{\pi}{M_\ell}$, $k = 0, \dots, 2M_\ell - 1$, $M_\ell \in \mathbb{N}$, $\ell = 1,2$

$$\frac{1}{2\pi}\int_0^{2\pi} f(s)\mathrm{d}s \approx \frac{1}{2M_\ell}\sum_{j=0}^{2M_\ell-1} f(s_j^\ell), \tag{5.1}$$

$$\frac{1}{2\pi}\int\limits_0^{2\pi} f(\sigma)\ln\left(4\sin^2\frac{s-\sigma}{2}\right)\mathrm{d}\sigma \approx \sum_{j=0}^{2M_\ell-1} R_j^\ell(s) f(s_j^\ell), \tag{5.2}$$

where

$$R_j^\ell(s) = -\frac{1}{M_\ell}\left[\sum_{m=1}^{M_l-1}\frac{1}{m}\cos(s-s_j^\ell) + \frac{1}{M_\ell}\cos M_\ell(s-s_j^\ell)\right].$$

These quadratures are obtained by replacing the function f by its trigonometric interpolation polynomial with respect to the points s_k^ℓ, $k = 0, \dots, 2M_l - 1$ and exact integration [30]. The use of the above quadratures for the integrals in (4.13) and the collocation in the quadrature points leads to a sequence of linear systems. As we remarked earlier, all functions in (4.13) are even with respect to π and $\varphi_{\ell,n}(0) = 0$ for $\ell = 1,2$. Therefore we can write these linear systems in the form

$$
(5.3)\quad \begin{cases}
\dfrac{1}{2}\varphi_{1k}^{n} + \gamma_1'(s_k^1)\displaystyle\sum_{p=1}^{M_1-1}\varphi_{1p}^{n}\left\{L_{11}^{11}(s_k^1, s_p^1)\left(R_p^1(s_k^1) + R_{2M_1-p}^1(s_k^1)\right)\right.\\
\quad \left. +\dfrac{1}{M_1}L_{11}^{21}(s_k^1, s_p^1)\right\} + \dfrac{\gamma_1'(s_k^1)}{M_2}\displaystyle\sum_{p=1}^{M_2-1}\varphi_{2p}^{n}L_{12}^{31}(s_k^1, s_p^2)\\
\qquad\qquad\qquad\qquad = G_n(s_k^1),\ k = 1, \dots, M_1 - 1,\\
\dfrac{1}{2}\varphi_{2k}^{n} + \gamma_2'(s_k^2)\displaystyle\sum_{p=1}^{M_2-1}\varphi_{2p}^{n}\left\{L_{22}^{12}(s_k^2, s_p^2)\left(R_p^2(s_k^2) + R_{2M_2-p}^2(s_k^2)\right)\right.\\
\quad \left. +\dfrac{1}{M_2}L_{22}^{22}(s_k^2, s_p^2)\right\} + \dfrac{\gamma_2'(s_k^2)}{M_1}\displaystyle\sum_{p=1}^{M_1-1}\varphi_{1p}^{n}L_{21}^{32}(s_k^2, s_p^1) = 0,\\
\qquad\qquad\qquad\qquad k = 1, \dots, M_2 - 1,
\end{cases}
$$

where

$$
G_n(s_k^1) = g_n(s_k^1) - \gamma_1'(s_k^1)\sum_{m=0}^{n-1}\beta_{n-m}\left[\frac{1}{M_2}\sum_{p=1}^{M_2-1}\varphi_{2p}^{m}A_{12}^{31}(s_k^1, s_p^2)\right.
$$
$$
\left. + \sum_{p=1}^{M_1-1}\varphi_{1p}^{m}\left\{A_{11}^{11}(s_k^1, s_p^1)\left(R_p^1(s_k^1) + R_{2M_1-p}^1(s_k^1)\right) + \frac{1}{M_1}A_{11}^{21}(s_k^1, s_p^1)\right\}\right],
$$

$\varphi_{\ell k}^{n} \approx \varphi_{\ell,n}(s_k^{\ell})$, $k = 1, \dots, M_\ell - 1$, $\ell = 1, 2$ and $n = 0, 1, \dots$ Here we note that we do not need to calculate the function b at the singular points. The approximate value of the function u_n can be calculated by the formula

$$
(5.4)\quad \tilde{u}_n(x_1(s)) = \sum_{p=1}^{M_1-1}\varphi_{1p}^{n}\left[L_{11}^{11}(s, s_p^1)\left\{R_p^1(s) + R_p^1(\pi - s)\right\}\right.
$$
$$
\left. +\frac{1}{M_1}L_{11}^{21}(s, s_p^1)\right] + \frac{1}{M_2}\sum_{p=1}^{M_2-1}\varphi_{2p}^{n}L_{12}^{31}(s, s_p^2).
$$

The convergence analysis and the error estimate for above method can be carried out in the same way as in [26, 28]. Specifically, from the corresponding results in [26, 28] the next theorem about error estimates follows in our case.

Theorem 5.1. *Assume that the corner of the curve L has the interior angle $(1-\rho)\pi$ with $0 < |\rho| < 1$ and assume that $f_n, \omega_\ell \in H^{p+5/2}(L)$ for $p \in \mathbb{N}$ and $q \geq 2$. Then for $q > (p+1/2)(1+|\rho|)$ there holds the following error estimate*

$$\|\varphi_\ell^n - \tilde{\varphi}_\ell^n\|_{H_e^0[0,2\pi]} \leq C_n M^{-p}$$

for the exact solution φ_ℓ^n and the numerical solution $\tilde{\varphi}_\ell^n$ obtained by Nyström method. Here $M = \min\{M_1, M_2\}$, $C_n > 0$ and $\ell = 1, 2$.

Finally, for the numerical solution of the evolution problem (1.13 - 1.14) according to (2.4) we have the following approximation

$$u_N^M(x_1(s), t) = \kappa \sum_{n=0}^{N} \tilde{u}_n(x_1(s)) L_n(\kappa t). \tag{5.5}$$

Now, in order to find the velocity potential $\varphi(x,t)$ corresponding to the problem (1.8 - 1.11), we have to solve the following non-stationary mixed Dirichlet-Neumann boundary value problem

$$\Delta\varphi = 0 \quad \text{in } \Omega \times (0,\infty), \tag{5.6}$$

$$\varphi = u \quad \text{on } \Gamma_1 \times (0,\infty), \tag{5.7}$$

$$\frac{\partial\varphi}{\partial\nu} = 0 \quad \text{on } \Gamma_2 \times (0,\infty), \tag{5.8}$$

where u is the solution of the evolution problem (1.13 - 1.14).

For the solving of (5.6 - 5.8) we also use the combination of the Laguerre transform and boundary integral equation method. As result, we have the following representation

$$\varphi(x,t) = \kappa \sum_{n=0}^{\infty} \varphi_n(x) L_n(\kappa t), \tag{5.9}$$

where the Fourier-Laguerre coefficients φ_n satisfy the sequence of mixed boundary value problems

$$\begin{aligned} \Delta\varphi_n &= 0 \quad && \text{in } \Omega, \\ \varphi_n &= u_n && \text{on } \Gamma_1, \\ \frac{\partial\varphi_n}{\partial\nu} &= 0 && \text{on } \Gamma_2 \end{aligned}$$

for $n = 0, 1, \ldots$ The numerical solution of these problems can be found by potential theory using a single-layer potential. Then we obtain the system of integral equations (2.8) with $\alpha = 0$, $\beta_n = 0$ and $f_n = u_n$ for $n = 0, 1, \ldots$ This system can be solved by the method used in the previous sections. Note that according to the representation (1.12) we have the numerical representation

$$x_N^M(x_1(s), t) = \kappa \sum_{n=1}^{N} \tilde{u}_n(x_1(s)) \sum_{m=0}^{n-1} L_m(\kappa t).$$

for the free boundary.

6. Numerical experiments.

1. *Robin-Neumann mixed boundary value problem for the Laplace equation.* The first system of the integral equations in the sequence (2.8) corresponds to the following mixed boundary value problem for the Laplace equation

(6.1) $$\Delta U = 0 \quad \text{in } \Omega,$$

(6.2) $$U + \frac{\partial U}{\partial \nu} = f_0 \quad \text{on } \Gamma_1,$$

(6.3) $$\frac{\partial U}{\partial \nu} = 0 \quad \text{on } \Gamma_2.$$

Assume that the boundaries Γ_1 and Γ_2 are created by the rotation of the curves L_1 and L_2, respectively, which are given as (see Fig. 3 on page 432)

$$L_1 := \left\{ x_1(\xi) = \left(\xi/\sqrt{2}\pi, 0 \right), \ 0 \le \xi \le \pi \right\},$$
$$L_2 := \left\{ x_2(\xi) = \left(-\cos(3\xi/4), \sin(3\xi/4) - 1/\sqrt{2} \right), \pi \le \xi \le 2\pi \right\}$$

and the boundary function $f_0 = 1$. Obviously, the exact solution is $U_{ex} = 1$. Table 1 shows the H^0-error between the exact solution U_{ex} and the numerical solution U_M

$$\varepsilon_M = \left(\frac{\pi}{M} \sum_{k=1}^{M-1} \left[U_{ex}\left(x_1(s_k^1)\right) - U_M\left(x_1(s_k^1)\right) \right]^2 \left| x_1'(s_k^1) \right| \right)^{1/2}$$

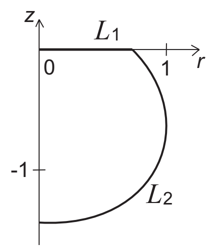

FIGURE 3. View of the domain Ω

and the convergence order

$$ord_M = \frac{\ln \varepsilon_M - \ln \varepsilon_{M/2}}{\ln 2}, \tag{6.5}$$

which are obtained by the above method. Here $M_1 = M_2 = M$.

M	$q = 3$	$q = 3$	$q = 4$	$q = 4$	$q = 5$	$q = 5$
	ϵ_M	ord_M	ϵ_M	ord_M	ϵ_M	ord_M
8	6.10×10^{-3}		9.50×10^{-3}		6.70×10^{-3}	
		2.0		6.4		5.3
16	1.50×10^{-3}		1.11×10^{-4}		1.68×10^{-4}	
		2.3		4.0		3.5
32	3.09×10^{-4}		6.93×10^{-6}		1.45×10^{-5}	
		2.3		3.5		4.2
64	6.70×10^{-5}		6.09×10^{-7}		8.20×10^{-7}	
		2.4		3.2		4.2
128	1.17×10^{-5}		6.43×10^{-8}		4.48×10^{-8}	
		2.4		3.2		4.2
256	2.24×10^{-6}		6.99×10^{-9}		5.31×10^{-9}	

Table 1. ε_M-errors and convergence orders for the Example **1.**

The numerical results illustrate the expected improvement of the convergence order for increasing values of the grading exponent q. Note that expected rates according to the Theorem 5.1 are equal to 1.9, 2.7, 3.5 for $q = 3, 4, 5$, respectively.

2. *Evolution problem on the free boundary.* Now we consider the non-stationary problem (1.13) and (1.14).

2.1. Assume that the boundaries Γ_1 and Γ_2 are created by the rotation of the curves L_1 and L_2, respectively, given as (see Fig. 4)

$$L_1 := \{x_1(\xi) = (\xi/\pi, 0)\,,\ 0 \le \xi \le \pi\}\,,$$
$$L_2 := \{x_2(\xi) = (\sin(\xi/2)\,, \cos(\xi/2)/3)\,,\ \pi \le \xi \le 2\pi\}\,.$$

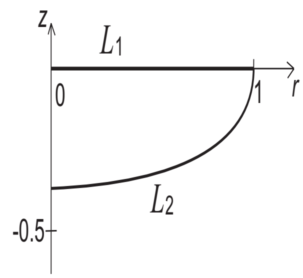

FIGURE 4. View of the domain Ω

Assume that $f(x,t) = \mathrm{e}^{-t}$ and initial functions $w_0 = 1$ and $w_1 = -1$. In this case the problem (1.13) and (1.14) has an exact solution given by $u_{ex}(x,t) = \mathrm{e}^{-t}$. For our algorithm we used the Fourier-Laguerre expansion

$$\mathrm{e}^{-t} = \kappa \sum_{n=0}^{\infty} \frac{1}{(\kappa+1)^{n+1}} L_n(\kappa t).$$

Table 2 demonstrates the H^0-error between the exact solution and the numerical solutions that is calculated numerically as

$$\text{(6.6)}\ \varepsilon_{NM} = \left(h_1 h \sum_{i=0}^{K} a_i \sum_{j=1}^{M-1} \left[u_N^M\left(x_1(s_j^1), t_i\right) - u_{ex}\left(t_i\right)\right]^2 \left|x_1'(s_j^1)\right| \right)^{1/2}$$

with $t_i = ih_1$, $h_1 = T/K$, $K \in \mathrm{N}$, $a_0 = a_K = 0.5$ and $a_i = 1$ for $i = 1, \dots, K-1$. Here $M_1 = M_2 = M$, $T = 2$, $K = 20$ and $q = 5$ in all cases.

M	$N = 10$	$N = 20$	$N = 30$
16	2.31×10^{-3}	2.32×10^{-3}	2.30×10^{-3}
32	2.30×10^{-4}	2.33×10^{-4}	2.32×10^{-4}
64	2.19×10^{-5}	9.51×10^{-5}	2.19×10^{-5}
128	2.98×10^{-6}	2.21×10^{-6}	2.20×10^{-6}

Table 2. ε_{NM}-errors for the Example **2.1.**

2.2. Let $f(x_1(s), t) = \epsilon s \cos(\delta t)$, $\epsilon, \delta > 0$ and the functions in the initial conditions (1.14) have the form $\omega_0(s) = 0$ and $\omega_1(s) = s$ for $s \in [0, \pi]$. Note that we consider the same boundaries as in Example **2.1**.

Table 3 shows the numerical solution of evolution problem (1.13) and (1.14) for various discretization parameters. Here we used $\alpha = 1$, $\kappa = 2$, $\epsilon = 0.1$, $\delta = 1$ and $M_1 = M_2 = M$, $q = 5$.

The expected convergence of the Nyström method according to the Theorem 5.1 and fast convergence of the Fourier-Laguerre series are clearly exhibited.

t	M	$N = 10$	$N = 20$	$N = 30$
0.0	32	-0.0007519	0.0002682	0.0002662
	64	-0.0007433	0.0002695	0.0002693
	128	-0.0007426	0.0002697	0.0002694
0.5	32	0.7783628	0.7788306	0.7787446
	64	0.7783628	0.7788301	0.7787434
	128	0.7783682	0.7788301	0.7787434
1.0	32	1.4462481	1.4452752	1.4449508
	64	1.4462499	1.4452847	1.4449591
	128	1.4462500	1.4452855	1.4449597
1.5	32	1.8954851	1.8992701	1.8997868
	64	1.8955412	1.8993248	1.8998538
	128	1.8955457	1.8993293	1.8998586
2.0	32	2.1852703	2.1807374	2.1806811
	64	2.1855055	2.1809513	2.1806893
	128	2.1855247	2.1809687	2.1809104

Table 3. Numerical results for the Example **2.2.**

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