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# IDENTIFICATION OF A TEMPERATURE DEPENDENT HEAT CONDUCTIVITY VIA ADAPTIVE GRID REGULARIZATION

ANDREAS NEUBAUER

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*This paper is dedicated to the 70 th birthday of Zuhair Nashed*

ABSTRACT. In this paper we treat the identification of a temperature dependent heat conductivity in an elliptic partial differential equation from a single boundary measurement. We are especially interested in conductivities with discontinuities. Therefore, we apply the recently developed *adaptive grid regularization* method. After showing results about the convergence of the method we present numerical results that demonstrate in a convincing way that this method is a powerful tool to identify discontinuous heat conductivities.

**1. Introduction.** In this paper we want to deal with the identification of the parameter $a$ in

$$\begin{aligned} -\operatorname{div}(a(u)\nabla u) &= f \quad \text{in } \Omega \\ a(u)\frac{\partial u}{\partial n} &= h \quad \text{on } \Gamma = \partial\Omega \end{aligned} \tag{1.1}$$

from a single measurement of $u$ at the boundary $\Gamma$.

Usually, one assumes that $a$ varies spatially on $x \in \Omega$. However, there are interesting problems such as in the cooling process of a steel strand where the heat conductivity depends merely on the temperature. The stationary case leads to the nonlinear elliptic equation above.

Equation (1.1), however with mixed boundary conditions instead of pure Neumann ones, was considered in [6], i.e., a Neumann condition

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was given on $\Gamma_1$ and a Dirichlet condition $u = g$ on $\Gamma_2$ with $\Gamma = \Gamma_1 \cup \Gamma_2$. The identification of $a$ was done from a single measurement of $u$ on $\Gamma_1$. Kügler showed that an identification is possible on the range of $u$ at $\Gamma_1$ provided that $g$ is constant and that $a \in H^1(I)$, where $I \subset \mathbb{R}$ is a finite interval covering $\mathcal{R}(\mu|_{\Gamma_1})$. Moreover, he regularized the ill-posed problem of identifying $a$ via Tikhonov regularization and gave smoothness conditions that guarantee convergence rates for the regularized solutions.

We are interested in identifying parameters $a$ that are not even continuous but have jumps. It is well known that standard regularization methods (cf., e.g., [4]) do not yield good results for discontinuous solutions. Better suited regularization methods for such problems are bounded variation regularization techniques or methods that have similar behavior (see, e.g., [1,5,7]). In [8,9] the author introduced a new finite-dimensional regularization technique based on adaptive grids. Numerical results in these papers show that this method is an efficient and fast tool to identify discontinuous solutions of ill-posed problems. The convergence analysis of this method was done in [10].

We will apply this method to the identification problem mentioned above. It will turn out that this inverse problem can be reformulated via a linear equation which makes the solution via regularization techniques much easier.

First of all we will show in the next section that the direct problem has a solution even for discontinuous parameters $a$. In Section 3, we will describe the adaptive grid regularization method and give convergence results when applied to the above identification problem. In the last section we will present numerical results.

**2. The direct and the inverse problem** We consider equation (1.1), where $\Omega$ is an open bounded convex subset of $\mathbb{R}^d$ $(d = 1, 2, 3)$ with Lipschitz boundary $\Gamma$, $f \in L^2(\Omega)$, $h \in L^2(\Gamma)$, and the parameter $a$ satisfies the conditions

$$\begin{aligned} &0 < \underline{a} < a < \overline{a} < \infty \quad \text{and} \\ &a \text{ is continuous except at most countably many points} \end{aligned} \tag{2.1}$$

for some positive constants $\underline{a}, \overline{a}$. The direct problem consists in finding

a weak solution of (1.1), i.e., in looking for $u \in H^1(\Omega)$ satisfying

$$\langle a(u)\nabla u, \nabla v\rangle = \langle f, v\rangle + \int_\Gamma h\, v\, d\sigma$$

for all $v \in H^1(\Omega)$, where $\langle\cdot,\cdot\rangle$ denotes the inner product in $L^2(\Omega)$. Obviously, a solution can only exist if $f$ and $h$ satisfy the condition

$$\int_\Omega f\, dx + \int_\Gamma h\, d\sigma = 0, \tag{2.2}$$

which we assume to hold in the following. We will show in the next proposition that (1.1) has a weak solution and that all solutions may be expressed via the unique solution $w \in \mathcal{V}$ of

$$\langle \nabla w, \nabla v\rangle = \langle f, v\rangle + \int_\Gamma h\, v\, d\sigma \quad \text{for all } v \in \mathcal{V} \tag{2.3}$$

with $\mathcal{V} := \{v \in H^1(\Omega) : \int_\Omega v\, dx = 0\}$. Note that existence and uniqueness of $w$ follow immediately from the Lax-Milgram Lemma.

**Proposition 2.1.** *Let $a$ satisfy* (2.1) *and let $f \in L^2(\Omega)$ and $h \in L^2(\Gamma)$ satisfy* (2.2). *Then equation* (1.1) *has a weak solution and all weak solutions $u_c$ may be calculated via*

$$u_c := A^{-1}(w + c), \tag{2.4}$$

*where $w \in \mathcal{V}$ is the unique solution of* (2.3)*, $c \in \mathbb{R}$ is a constant and $A$ is defined via*

$$A(s) := \int_0^s a(\xi)\, d\xi. \tag{2.5}$$

*Proof.* First of all note that, due to (2.1), $A$ is continuous and strictly monotonically increasing. Thus, $A^{-1}$ exists and is also continuous and strictly monotonically increasing. Moreover, for any function $u \in H^1(\Omega)$ it holds that $A(u) \in H^1(\Omega)$ with $\nabla A(u) = a(u)\nabla u$.

Therefore, it follows with the definition of $w$ that any weak solution of (1.1) satisfies that $A(u) = w + c$ for some constant $c \in \mathbb{R}$ and hence $u = A^{-1}(w + c)$.

It remains to be shown that $u_c := A^{-1}(w+c)$ is an element of $H^1(\Omega)$ for all $c \in \mathbb{R}$: It is an immediate consequence of (2.1) that

$$\underline{a}\, s \le A(s) \le \overline{a}\, s \quad \text{for } s \ge 0 \quad \text{and} \quad \overline{a}\, s \le A(s) \le \underline{a}\, s \quad \text{for } s < 0$$

and hence that

$$\begin{aligned} &\overline{a}^{-1}\, t \le A^{-1}(t) \le \underline{a}^{-1}\, t \quad \text{for } t \ge 0 \quad \text{and} \\ &\underline{a}^{-1}\, t \le A^{-1}(t) \le \overline{a}^{-1}\, t \quad \text{for } t < 0. \end{aligned} \tag{2.6}$$

This implies that $u_c \in L^2(\Omega)$. We still have to show that $u_c$ is differentiable. This is trivial if $a$ is continuous. Since then it holds that $A^{-1}$ is continuously differentiable with $(A^{-1})'(t) = (a(A^{-1}(t))^{-1}$ which by the chain rule yields that $\nabla u_c = a(u_c)^{-1}\nabla w$. We will now show that this is still valid even if $a$ is discontinuous in at most countably many points.

Let $k_n$ be the piecewise linear hat function defined by $k_n(0) = n$ and $k_n(\pm\, n^{-1}) = 0$. Then we define $a_n := k_n * a$, i.e.,

$$a_n(t) = \int_{t-\frac{1}{n}}^{t} (n - n^2(t-\xi))a(\xi)\, d\xi + \int_{t}^{t+\frac{1}{n}} (n + n^2(t-\xi))a(\xi)\, d\xi\,.$$

Obviously, $a_n$ is continuous, $\underline{a} \le a_n \le \overline{a}$, and

$$|a_n(t_n) - a(t)| \le \sup_{\xi \in [t-\frac{1}{n}, t+\frac{1}{n}]} |a(\xi) - a(t)|.$$

Therefore,

$$a_n(t_n) \xrightarrow[n\to\infty]{} a(t) \quad \text{if } t_n \to t \text{ and } a \text{ is continuous in } t\,. \tag{2.7}$$

We now define $u_{c,n} := A_n^{-1}(w+c)$, where $A_n$ is defined as in (2.5) with $a$ replaced by $a_n$. Then, we know from the considerations above that $u_{c,n} \in H^1(\Omega)$ and

$$\int_\Omega u_{c,n}(x)\nabla\phi(x)\, dx = -\int_\Omega \frac{\nabla w(x)}{a_n(u_{c,n}(x))}\phi(x)\, dx \quad \text{for all} \quad \phi \in C_c^1(\Omega). \tag{2.8}$$

We have to show that this formula also holds in the limiting case. Since, due to (2.1) and (2.6), the estimates

$$|u_{c,n}| \le \frac{|w+c|}{\underline{a}} \qquad \text{and} \qquad \left|\frac{\nabla w}{a_n(u_{c,n})}\right| \le \frac{|\nabla w|}{\underline{a}}$$

are valid, by the Lebesgue Dominated Convergence Theorem it suffices to show that

$$u_{c,n} \xrightarrow{pw} u_c \quad \text{a.e.} \qquad \text{and} \qquad \frac{\nabla w}{a_n(u_{c,n})} \xrightarrow{pw} \frac{\nabla w}{a(u_c)} \quad \text{a.e.} \tag{2.9}$$

at least for a subsequence (here pw stands for pointwise).

Note that (2.1) and (2.7) imply that $A_n$ converges pointwise towards $A$. Since the estimates in (2.6) also hold for $A_n^{-1}$, for an abitrary subsequence of $A_n^{-1}(t)$ there is another subsequence (again denoted by $A_n^{-1}(t)$) and an element $s \in \mathbb{R}$ such that $A_n^{-1}(t) \to s$ as $n \to \infty$. The estimate

$$\begin{aligned} |A(s)-t| &\le |A(s)-A_n(s)| + |A_n(s)-t| \\ &\le |A(s)-A_n(s)| + \Big|\int_{A_n^{-1}(t)}^{s} a_n(\xi)\,d\xi\Big| \\ &\le |A(s)-A_n(s)| + \overline{a}\,|s - A_n^{-1}(t)| \to 0 \end{aligned}$$

now shows that $s = A^{-1}(t)$ and that, hence, $A_n^{-1}$ converges pointwise towards $A^{-1}$. This already implies that the left assertion in (2.9) is true for all $x \in \Omega$. Due to (2.7), we now also obtain that the right assertion holds for elements $x \in \Omega$ with $a$ being continuous in $u_c(x)$.

Now let $t$ be such that $a$ is not continuous at $t$ and let $D_t := \{x \in \Omega : u_c(x) = t\}$. From the definition of $u_c$ we then obtain that $w|_{D_t}$ is constant. Now it follows from [2, Remark 3.93] that $\nabla w = 0$ a.e. in $D_t$. Thus, the second assertion in (2.9) also holds a.e. in $D_t$. Since $a$ is discontinuous in at most countably many points, the assertion holds a.e. in $\Omega$.

Finally, this yields that $u_c \in H^1(\Omega)$ with $\nabla u_c = a(u_c)^{-1}\nabla w$. □

Obviously, the solution of a Neumann problem is not unique. However, it follows from the definition (2.4) similarly to (2.6) that

$$\frac{c_2-c_1}{\overline{a}} \le u_{c_2}(x) - u_{c_1}(x) \le \frac{c_2-c_1}{\underline{a}} \quad \text{for} \quad c_2 > c_1, \quad x \in \Omega\,.$$

This implies that $\int_\Omega u_c(x)\,dx$ depends Lipschitz continuously on $c$ and that

$$\lim_{c\to\pm\infty}\int_\Omega u_c(x)\,dx = \pm\infty\,.$$

Now, by the Intermediate Value Theorem, it follows as for the solution $w$ of (2.3) that there exists a unique solution $u_c$ such that $\int_\Omega u_c(x)\,dx = 0$, i.e., $u_c \in \mathcal{V}$.

Let us now turn to the inverse problem, i.e., from a given single measurement of $u$ at the boundary we want to identify $a$. The first question that arises is of course if this problem has a unique solution. The next proposition states similarly to [6, Theorem 3.1] that an identification is possible only on the range of $u$ at $\Gamma$, i.e., on $\mathcal{R}(u|_\Gamma) = \mathcal{R}(\gamma u)$. Here $\gamma$ denotes the trace operator.

**Proposition 2.2.** *Let $a_1$ and $a_2$ satisfy* (2.1) *and let $f \in L^2(\Omega)$ and $h \in L^2(\Gamma)$ satisfy* (2.2). *Moreover, let $u_1$ and $u_2$ be weak solutions in $H^1(\Omega)$ of equation* (1.1) *with $a$ replaced by $a_1$ and $a_2$, respectively.*

*If $\gamma u_1 = \gamma u_2$, then $a_1 = a_2$ a.e. on $\mathcal{R}(u_1|_\Gamma)$.*

*If, in addition, $\gamma u_1$ is continuous on $\Gamma$ and not constant, then $I := \mathcal{R}(u_1|_\Gamma)$ is an interval and $a_1 = a_2$ for all points in $I$ where $a_1$ and $a_2$ are continuous.*

*Proof.* It follows immediately from Proposition 2.1 that

$$\int_0^{u_1} a_1(\xi)\,d\xi - \int_0^{u_2} a_2(\xi)\,d\xi = c_1 - c_2$$

Since, due to (2.1), it holds that $\gamma A_i(u) = A_i(\gamma u)$, $i = 1, 2$, for all $u \in H^1(\Omega)$ with $A_i$ defined similarly as in (2.5), we now obtain for $\gamma u_1 = \gamma u_2$ that

$$\int_0^\tau (a_1(\xi) - a_2(\xi))\,d\xi = c_1 - c_2 \quad \text{for all} \quad \tau \in \mathcal{R}(u_1|_\Gamma)\,.$$

This together with [2, Remark 3.93] yields that $a_1 = a_2$ a.e. on $\mathcal{R}(u_1|_\Gamma)$. The remaining assertions are now trivial. □

Obviously, the result above is only of importance if $\mathcal{R}(u_1|_\Gamma)$ has positive measure which is at least the case if $\gamma u_1$ is continuous on $\Gamma$ and not constant.

It is well known that the problem of identifying $a$ is ill-posed. Therefore, one has to use regularization techniques to solve this problem. Usually, this is done by using the operator $F$ that maps $a$ onto a solution $u$. Even if we choose the unique solution $u \in \mathcal{V}$, which makes the operator $F$ well defined, $F$ is still a nonlinear operator. We will choose a different way that allows to work merely with the measurements of $u$ on the boundary and where we have to find a regularized solution of a linear equation.

Let us assume in the following that $u \in H^1(\Omega)$ is a solution of the forward problem for some $a$ satisfying (2.1) and that $u$ is bounded on $\Gamma$ and not constant, hence,

$$\mathcal{R}(u|_\Gamma) \subset I := [u_{min}, u_{max}] \quad \text{with} \quad u_{max} > u_{min}\,. \tag{2.10}$$

This is not a severe restriction, since in practice temperatures will be bounded. The inverse problem of identifying $a$ in (1.1) is now equivalent to finding $a$ and $c \in \mathbb{R}$ such that

$$Ta(u(x)) := \int_{u_{min}}^{u(x)} a(\xi)\,d\xi = w(x) + c\,, \quad x \in \Gamma\,, \tag{2.11}$$

where $w \in \mathcal{V}$ is the solution of (2.3). Since it is obvious in the notation above, we wrote $u$ and $w$ instead of $\gamma u$ and $\gamma w$, respectively.

**3. Adaptive grid regularization** We mentioned above that the identification problem is ill-posed and has to be solved by regularization techniques. We use Tikhonov regularization combined with adaptive grid regularization. The method is described as follows:

The regularized solutions are computed iteratively. In each iteration a Tikhonov functional is minimized over a finite-dimensional space $a_* + \mathcal{X}_n$, where $a_*$ is an initial guess and $\mathcal{X}_n$ is the space of piecewise linear functions on the interval $I$ corresponding to the grid $\tau_n$. After each iteration not only the grid but also the regularizing norm is adjusted. Obviously, there is no need for a regularization with respect to $c$. The details of the algorithm are described below.

**Algorithm 3.1. (Adaptive grid regularization)** Let $\alpha, \beta > 0$.

(i) Start with a uniform (rather coarse) grid $\tau_1$ in $I = [u_{min}, u_{max}]$. Set $n := 1$ and $a_0 := 0$.

(ii) Compute a minimizer $a_n, c_n$ in $\mathcal{X}_n \times \mathbb{R}$ of

$$g_{\alpha,\beta}(a, c, w_n) := \|Ta(u(x)) - (w(x) + c - Ta_*(u(x))\|^2 + \alpha \int_I |\dot{a}(\xi)|^2 \, w_n^{-1}(\xi) \, d\xi \tag{3.1}$$

$$w_n(\xi) := |I| \sqrt{\beta^2 + |\dot{a}_{n-1}(\xi)|^2} \Big/ \int_I \sqrt{\beta^2 + |\dot{a}_{n-1}(\xi)|^2} \, d\xi.$$

(iii) If a stopping criterion is satisfied, the iteration is finished. The final solution of the identification problem is given by $a_* + a_n$. Otherwise:

(iv) Perform a local grid refinement

$$\tau_{n+1} := G(\tau_n, a_n)\,, \tag{3.2}$$

where $\tau_n$ is the grid corresponding to the finite-dimensional space $\mathcal{X}_n$. Set $n := n + 1$ and go to step (ii).

For the motivation and justification of the regularizing norm in step (ii) see [10]. As grid refinement in (3.2) we have chosen the following procedure:

Let us assume that the grid $\tau_n$ is given by the nodes

$$u_{min} = \xi_0 < \xi_1 < \cdots < \xi_{m(n)} = u_{max}$$

and set $S_i := [\xi_{i-1}, \xi_i]$. A subinterval $S_i$ is refined according to the following rules: a refinement is only performed if the length of the subinterval is larger than a certain threshold, i.e.,

$$|\xi_i - \xi_{i-1}| > h_{\min} \tag{3.3}$$

Under all admissible subintervals only those are refined where the corresponding weight $w_{n,i} := w_n|_{S_i}$, which is constant on each subinterval $S_i$, since we use piecewise linear functions, is not smaller than the $k$-th largest weight and close enough to the largest weight, i.e.,

$$w_{n,i} \geq \bar{w}_k := k\text{-th largest element among all } w_{n,j} \text{ with } |\xi_j - \xi_{j-1}| \geq h_{\min}, \tag{3.4}$$

$$w_{n,i} > \kappa * \max\{w_{n,j} \,:\, |\xi_j - \xi_{j-1}| \geq h_{\min}\}. \tag{3.5}$$

for some $k \in \mathbb{N}$ and $h_{\min}, \kappa > 0$. If a subinterval $S_i$ satisfies all criteria then two equally spaced nodes are added within $S_i$.

Since the operator $T$ defined in (2.11) is continuous from $L^1(I)$ into $L^2(\Gamma)$ and $1 \notin \mathcal{N}(T)$ we may apply the convergence analysis from [10]:

Noting that, due to the refinement condition (3.3), no refinement will occur anymore after some iteration step $\bar{n}$, i.e., $\mathcal{X}_n = \mathcal{X}_{\bar{n}}$ for all $n \geq \bar{n}$, the question arises if $(a_n, c_n)$ is convergent in $\mathcal{X}_{\bar{n}} \times \mathbb{R}$. An answer to this question follows directly from Proposition 2.2 and Theorem 2.3 in [10]:

**Theorem 3.2.** *The regularized solutions $(a_n, c_n)$ in step* (ii) *of Algorithm* 3.1 *exist and are unique. Moreover, it holds that $(a_n, c_n)$ converge to the unique minimizer $(a_{\alpha,\beta,\bar{n}}, c_{\alpha,\beta,\bar{n}})$ of*

$$\begin{aligned} f_{\alpha,\beta}(a,c) := \|Ta(u(x)) - (w(x) + c - Ta_*(u(x))\|^2 \\ +\alpha|I|^{-1} J_\beta(a)^2 \\ \text{in } \mathcal{X}_{\bar{n}} \times \mathbb{R} \text{ as } n \to \infty, \end{aligned} \tag{3.6}$$

*where*

$$J_\beta(a) := \sup\left\{ \int_\Omega \left( a(\xi)\dot{v}(\xi) + \beta\sqrt{1 - |v(\xi)|^2} \right) d\xi \,:\, v \in C_c^1(I), |v| \leq 1 \right\}.$$

Note that in $\mathcal{X}_{\bar{n}}$ the definition of $J_\beta(a)$ above coincides with $\int_I \sqrt{\beta^2 + |\dot{a}(\xi)|^2}\, d\xi$. However, the definition also makes sense for functions of bounded variation.

As the threshold $h_{\min}$ in (3.3) gets smaller and smaller, then $\bar{n}$ obviously becomes larger and larger. Therefore, we are also interested if $a_{\alpha,\beta,\bar{n}} \in \mathcal{X}_{\bar{n}}$ converges to a minimizer in some infinite-dimensional space if $\bar{n}$ tends to infinity. The following result is an immediate consequence of Theorem 2.4 and Example 2.6 in [10]:

**Theorem 3.3.** *The sequence $\{(a_{\alpha,\beta,\bar{n}}, c_{\alpha,\beta,\bar{n}})\}$ has a weakly* convergent subsequence in $BV(I) \times \mathbb{R}$. The limit $(\tilde{a}, \tilde{c})$ of every*

*weakly* convergent subsequence is a minimizer of* $f_{\alpha,\beta}$ *in* $BV(I)$ *and* $\lim_{n\to\infty} f_{\alpha,\beta}(a_{\alpha,\beta,\bar{n}}, c_{\alpha,\beta,\bar{n}}) = f_{\alpha,\beta}(\tilde{a}, \tilde{c})$.

*If the minimizer* $(a_{\alpha,\beta}, c_{\alpha,\beta})$ *of* $f_{\alpha,\beta}$ *in* $BV(I)$ *is unique, then* $\{(a_{\alpha,\beta,\bar{n}}, c_{\alpha,\beta,\bar{n}})\}$ *weakly* converges towards* $(a_{\alpha,\beta}, c_{\alpha,\beta})$ *and hence*

$$a_{\alpha,\beta,\bar{n}} \xrightarrow{L^p} a_{\alpha,\beta}\,, \quad \text{for all } 1 \le p < \infty\,, \quad \text{and} \quad c_{\alpha,\beta,\bar{n}} \to c_{\alpha,\beta,\bar{n}}\,.$$

$BV(I)$ *denotes the space of functions of bounded variation consisting of all functions* $a \in L^1(I)$ *with* $J_0(a) < \infty$.

Of course, in practice the data $u$ are not given exactly but only noisy measurements $u^\delta$ are available with $\|u^\delta - u\|_{L^2(\Gamma)} \le \delta$. Moreover, in general the function $w$ in (2.3) can not be computed exactly which yields additional data noise for the solution of the inverse problem. Covergence results for $(a_{\alpha,\beta}, c_{\alpha,\beta})$ when the noise and the parameters $\alpha$ and $\beta$ go to 0 can be derived similary as in [3, 4].

**4. Numerical results** We will now present numerical results for the problem of identifying $a$ in (1.1), where $\overline{\Omega} = [0,1]^2$ and the functions $f \in L^2(\Omega)$ and $h \in L^2(\Gamma)$ are defined by

$$f(x,y) := \begin{cases} -8 & t(x,y) \le 0\,, \\ -8\left(\frac{157}{70} + \frac{58}{7}\,t(x,y)\right) & 0 < t(x,y) \le \frac{7}{29}\,, \\ -16 & \frac{7}{29} < t(x,y)\,, \end{cases}$$

$$h(x,0) := \begin{cases} 1\,, & t(x,0) \le 0\,, \\ 1 + \frac{29}{7}\,t(x,y)\,, & 0 < t(x,0) \le \frac{7}{29}\,, \\ 2\,, & \frac{7}{29} < t(x,0)\,, \end{cases}$$

$$h(0,y) := \begin{cases} \frac{4}{3}\,, & t(0,y) \le 0\,, \\ \frac{4}{3}\left(1 + \frac{29}{7}\,t(x,y)\right)\,, & 0 < t(0,y) \le \frac{7}{29}\,, \\ \frac{8}{3}\,, & \frac{7}{29} < t(0,y)\,, \end{cases}$$

$$h(x,1) := 6\,,$$

$$h(1,y) := \tfrac{16}{3}\,,$$

with

$$t(x,y) := 2((x-\tfrac{1}{3})^2 + (y-\tfrac{1}{4})^2) - \tfrac{3}{10}\,.$$

For the parameter $a$ defined via

$$a(t) := \begin{cases} 1\,, & t \le 0\,, \\ 1 + \frac{29}{7}\,t\,, & 0 < t \le \frac{7}{29}\,, \\ \frac{13}{10}\,, & \frac{7}{29} < t \le \frac{15}{29} \\ 5\,, & \frac{15}{29} < t \le \frac{9}{10} \\ \frac{5}{2}\,, & \frac{9}{10} < t\,, \end{cases}$$

one can show that the direct problem of (1.1) has the weak solution

$$u(x,y) := \begin{cases} t(x,y)\,, & t(x,y) \le \frac{7}{29}\,, \\ \frac{7}{29} + \frac{20}{13}(t(x,y) - \frac{7}{29})\,, & \frac{7}{29} < t(x,y) \le \frac{61}{145}\,, \\ \frac{15}{29} + \frac{2}{5}(t(x,y) - \frac{61}{145})\,, & \frac{61}{145} < t(x,y) \le \frac{799}{580}\,, \\ \frac{9}{10} + \frac{4}{5}(t(x,y) - \frac{799}{580})\,, & \frac{799}{580} < t(x,y)\,, \end{cases}$$

with $t(x,y)$ as above.

As mentioned in Section 2, the inverse problem now consists in identifying $a$ from a single measurement of $u$ at the boundary. Using Algorithm 3.1 we have to compute Tikhonov regularized solutions (cf. (3.1)). This is achieved by solving the linear system

$$(A_n + \alpha M_n)\eta_n = v_n\,, \qquad \eta_n = (c_n, a_{n,0}, \dots, a_{m(n),0})\,,$$

where

$$A_n := \begin{bmatrix} \langle 1,1\rangle_{L^2(\Gamma)} & -\langle 1, T\varphi_{n,j}(u)_{L^2(\Gamma)} \\ -\langle 1, T\varphi_{n,i}(u)\rangle_{L^2(\Gamma)} & \langle T\varphi_{n,i}(u), T\varphi_{n,j}(u)\rangle_{L^2(\Gamma)} \end{bmatrix}\,,$$

$$M_n := \begin{bmatrix} 0 & 0 \\ 0 & \displaystyle\int_I \dot\varphi_{n,i}(\xi)\,\dot\varphi_{n,j}(\xi)\,w_n^{-1}(\xi)\,d\xi \end{bmatrix}\,,$$

$$v_n := \begin{bmatrix} -\langle w-u, 1\rangle_{L^2(\Gamma)} \\ \langle w-u, T\varphi_{n,j}(u)\rangle_{L^2(\Gamma)} \end{bmatrix}$$

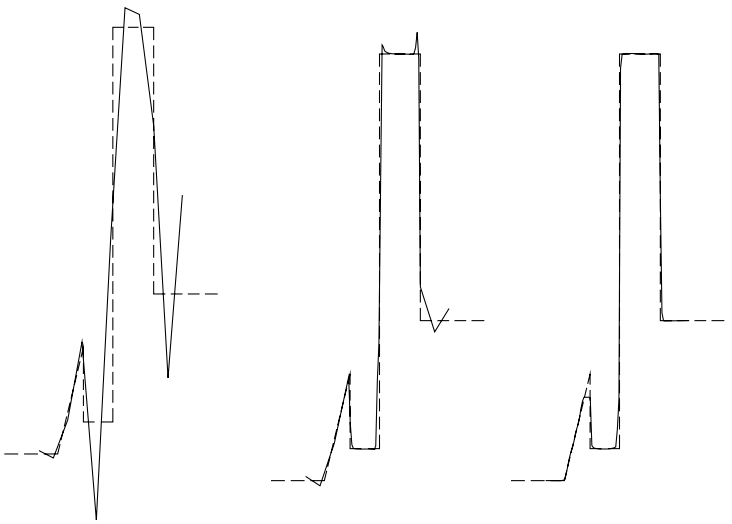

FIGURE 4.1: results for exact $u$: left: $n = 1$; middle: $n = 10$; right: $n = 25$.

with $I = [u_{min}, u_{max}]$, $w_n$ as in (3.1) and $w$ as in (2.3). The functions $\varphi_{n,i}$ are the usual piecewise linear hat functions. As initial guess we have chosen $a_* = 1$.

The parameters in the refinement rules (3.3) - (3.5) and $\beta$ were chosen as: $k = 20$, $h_{\min} = 0.02$, $\kappa = 0.3$, and $\beta^2 = 0.1$. The initial grid $\tau_1$ was a uniform mesh with dimension $m(1) = 10$.

The solution $w$ of the Neumann problem (2.3) was calculated approximately using bilinear finite elements on a uniform mesh of $101 \times 101$ nodes. All integrals were approximated using a Gaussian quadrature rule based on 4 nodes in each subinterval. The number of subintervals for each side of the square $[0, 1]^2$ is denoted by $s_I$.

To further speed up the algorithm in addition to refinement also coarsening steps were performed after 20 iterations according to the following rule: a node is dropped if the difference to the linear approximation without this node is below a certain bound.

Figure 4.1 shows the results for the case where $u$ is assumed to be known exactly everywhere at the boundary:

$u_{min} = -\frac{7}{40} = -0.175$, $u_{max} = \frac{7628}{6525} \approx 1.169$, $s_I = 10$, $\alpha = 10^{-7}$. After 10 iterations (dimension of $\tau_{10}$ : $m(10) = 68$) one can already

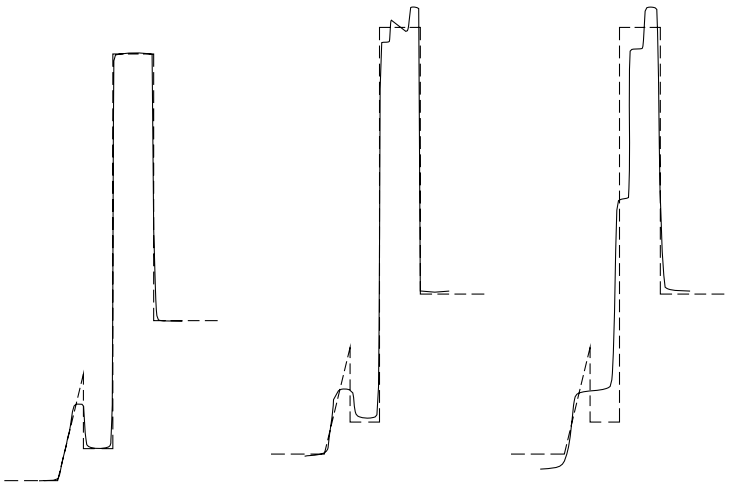

FIGURE 4.2: results for noisy $u$: left: $n = 28$, 0.017% noise; middle: $n = 30$, 1% noise; right: $n = 30$, 5% noise.

clearly see the jumps. Up to the 20th iteration the dimension grows to 148. Then coarsening starts: iterate 24 with dimension $m(24) = 53$ is almost identical to the exact solution. Only the sharp corner can not be identified.

Of course, in practice data are not measured everywhere. Thus, we now assume that the solution $u$ is only measured at the corners of the subintervals, where the Gaussian quadrature is used. Choosing $s_I = 100$ means that $u$ is evaluated at 400 equally spaced nodes at the boundary. Between the nodes, $u$ is calculated via linear interpolation. This already yields to a noise of 0.017 % with resepect to $\|u\|_{L^2(\Gamma)}$. The 28th iterate ($m(28) = 60$, $u_{min}$ and $u_{max}$ as above, $\alpha = 10^{-6}$) is shown in the left part of Figure 4.2. The result is almost as good as for the exact $u$; close to the sharp corner it is slightly worse.

Then we added random noise to the values of $u$ at the nodes. The middle part of Figure 4.2 shows the 30th iterate ($m(30) = 100$, $u_{min} = -0.185$, $u_{max} = 1.171$, $\alpha = 5 \cdot 10^{-6}$) when 1% noise was added. At least the jumps could be identified, however not exactly the values between the jumps. When adding 5% noise, the result were even worse, however still much better than with standard Tikhonov regularization; see the right part showing the 30th iterate ($m(30) = 106$, $u_{min} = -0.226$,

$u_{max} = 1.179$, $\alpha = 5 \cdot 10^{-5}$).

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Institut für Industriemathematik, Johannes-Kepler-Universität, A-4040 Linz, Austria
**Email address: neubauer@indmath.uni-linz.ac.at**