JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 20, Number 1, Spring 2008

ON THE CONNECTION BETWEEN MOLCHAN-GOLOSOV AND MANDELBROT-VAN NESS REPRESENTATIONS OF FRACTIONAL BROWNIAN MOTION

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Communicated by Charles Groetsch

ABSTRACT. We prove analytically a connection between the generalized Molchan-Golosov integral transform, see [**4**, Theorem 5.1], and the generalized Mandelbrot-Van Ness integral transform, see [**8**, Theorem 1.1], of fractional Brownian motion (fBm). The former changes fBm of arbitrary Hurst index K into fBm of index H by integrating over $[0,t]$, whereas the latter requires integration over $(-\infty,t]$ for $t>0$. This completes an argument in [**4**], where the connection is mentioned without full proof.

1. Introduction. The *fractional Brownian motion with Hurst index* $H\in(0,1)$, or H-fBm, is the continuous, centered Gaussian process $(B_t^H)_{t\in\mathbf{R}}$ with $B_0^H=0$, almost surely, and

$$\mathrm{Cov}_{\mathbf{P}}\left(B_s^H,B_t^H\right)=\frac{1}{2}\left(|s|^{2H}+|t|^{2H}-|t-s|^{2H}\right),\quad s,t\in\mathbf{R}.$$

H-fBm is H-self-similar and has stationary increments. For $H=1/2$, fractional Brownian motion is standard Brownian motion and denoted by W. FBm is interesting from a theoretical point of view, since it is fairly simple, but neither a Markov process, nor a semi-martingale. Recently, the process has been studied extensively in connection to various applications, for example in finance and telecommunications. Important tools when working with fBm are its integral representations:

2000 *Mathematics subject classification.* Primary 60G15, 26A33, 60G18.
Keywords and phrases. Fractional Brownian motion, integral transform, fractional calculus.
Received by the editors on August 18, 2006, and in revised form on December 19, 2006.
DOI:10.1216/JIE-2008-20-1-93

for a fixed Hurst index $K \in (0,1)$, on the one hand, there exists a K-fBm $(B^K_t)_{t\in\mathbf{R}}$, such that for all $t \in [0,\infty)$, we have that

$$B^H_t = C(K,H)\int_0^t (t-s)^{H-K} \times F\left(1-K-H, H-K, 1+H-K, \frac{s-t}{s}\right) dB^K_s, \quad \text{a.s.,} \tag{1.1}$$

see [**4**, Theorem 5.1]. Here

$$C(K,H) := \left(\frac{C(H)C(K)^{-1}}{\Gamma(H-K+1)}\right),$$
$$C(H) := \left(\frac{2H\Gamma(H+(1/2))\Gamma((3/2)-H)}{\Gamma(2-2H)}\right)^{1/2},$$

Γ denotes the gamma function, and F is Gauss's hypergeometric function. B^K is unique, up to modification, on $[0,\infty)$. On the other hand, there exists a unique, up to modification, K-fBm $(\widetilde{B}^K_t)_{t\in\mathbf{R}}$, such that for all $t \in \mathbf{R}$, it holds that

$$B^H_t = C(K,H)\int_{\mathbf{R}} \left((t-s)^{H-K} 1_{(-\infty,t)}(s) - (-s)^{H-K} 1_{(-\infty,0)}(s)\right) d\widetilde{B}^K_s, \quad \text{a.s.,} \tag{1.2}$$

see [**8**, Theorem 1.1]. For $K = 1/2$, (1.1) corresponds to the Molchan-Golosov representation and (1.2) is the Mandelbrot-Van Ness representation of H-fBm, see [**6, 5**], respectively. The integrals in (1.1) and (1.2) are fractional Wiener integrals. A priori, representations (1.1) and (1.2) are very different. Indeed, the integrand in (1.1) is a weighted fractional integral over $[0,t]$, whereas the integrand in (1.2) is a simple fractional integral over $\mathbf{R}$. Moreover, the filtrations generated by $(B^H_t)_{t\in[0,\infty)}$ and $(B^K_t)_{t\in[0,\infty)}$ coincide, but this is not the case for the natural filtrations of $(B^H_t)_{t\in\mathbf{R}}$ and $(\widetilde{B}^K_t)_{t\in\mathbf{R}}$.

In this work, we demonstrate how analytical facts of fractional integrals, combined with shifting properties of fBm, are used in order to establish a natural connection between the (generalized) Molchan-Golosov integral transform (1.1) and the (generalized) Mandelbrot-Van Ness integral transform (1.2). More precisely, we show that the latter one emerges as a boundary case of a suitable time-shifted former one: Based on (1.1), we construct a sequence of H-fBms which, for fixed t, converges in $L^2(\mathbf{P})$-sense to (1.2). We will specify the rate of convergence. In particular, the generalized Mandelbrot-Van Ness representation is a consequence of the generalized Molchan-Golosov representation.

The article is organized as follows. In Section 2, we first review the definition and some relevant facts of Gauss's hypergeometric function. Second, we define fractional integrals and derivatives over $\mathbf{R}$ and show the connection to fBm. Third, we recall the definition of the fractional Wiener integral over the real line. In Section 3, we derive the connection between the integral representations.

2. Preliminaries.

2.1. Gauss's hypergeometric function. The *Gauss hypergeometric function* of parameters a, b, c and variable $z \in \mathbf{R}$ is defined by the formal power series

$$F(a,b,c,z) := {}_2F_1(a,b,c,z) := \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{z^k}{k!},$$

where $(a)_0 := 1$ and $(a)_k := a \cdot (a+1) \cdot \dots \cdot (a+k-1)$, $k \in \mathbf{N}$. We assume that $c \in \mathcal{A} := \mathbf{R} \setminus \{\dots, -2, -1, 0\}$ for this to make sense. If $|z| < 1$ or $|z| = 1$ and $c - b - a > 0$, then the series converges absolutely. If, furthermore, $c > b > 0$ for $z \in [-1, 1)$ and $b > 0$ for $z = 1$, then it can be represented by the Euler integral

$$F(a,b,c,z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 v^{b-1}(1-v)^{c-b-1}(1-zv)^{-a}\, dv, \tag{2.1}$$

see [**3**, page 59]. If $c > b > 0$, then the expression on the righthand side of (2.1) is well defined for all $z \in (-\infty, 1)$. For these parameters, we can hence extend the definition of F to all $z \in (-\infty, 1)$ via (2.1). In order to extend F for fixed $z \in (-\infty, 1]$ to more general parameters, we consider Gauss's relations for neighbor functions. Functions of type $F(a \pm m, b, c, z)$, $F(a, b \pm m, c, z)$ and $F(a, b, c \pm m, z)$, $m \in \mathbf{N}$, are called *contiguous* to $F(a, b, c, z)$. If $m = 1$, then they are also called *neighbors*. For any two neighbors $F_1(z)$, $F_2(z)$ of $F(a,b,c,z)$, one has a linear relation of type

$$A(z)F(a,b,c,z) + A_1(z)F_1(z) + A_2(z)F_2(z) = 0, \tag{2.2}$$

where A, A_1 and A_2 are first-degree polynomials with coefficients depending on a, b and c. See [**1**, page 558] for all 15 relations. We use the neighbor relations in order to extend F for $z \in (-\infty, 1)$ to all parameters such that $c \in \mathcal{A}$, and for $z = 1$ to all parameters that satisfy $c, c-b-a \in \mathcal{A}$. Among the most important properties of F are the symmetry

$$F(a,b,c,z) = F(b,a,c,z)$$

and the reduction formula

$$F(0,b,c,z) = F(a,b,c,0) = 1.$$

Also we have the linear transformation formula, see [**1**, page 559],

$$F(a,b,c,z) = (1-z)^{-a} F\left(a, c-b, c, \frac{z}{z-1}\right), \quad z < 1. \tag{2.3}$$

In particular,

$$F(a,b,b,z) = (1-z)^{-a}, \quad z < 1. \tag{2.4}$$

F is smooth in z, and we have (see [**1**, page 557])

$$\frac{d}{dz}F(a,b,c,z) = \frac{ab}{c}F(a+1, b+1, c+1, z) \tag{2.5}$$

and

$$\frac{d}{dz}\big(z^a F(a,b,c,\pm z)\big) = az^{a-1}F(a+1,b,c,\pm z). \tag{2.6}$$

F is left-continuous in $z = 1$ and it holds that, see [**3**, page 9],

$$F(a,b,c,1) = \frac{\Gamma(c)\Gamma(c-b-a)}{\Gamma(c-b)\Gamma(c-a)}. \tag{2.7}$$

Let $a, b > -1$ and $w < x < y$. Substituting $u := (y-x)v + x$ in (2.1) and using (2.3) implies that

$$\begin{aligned}(2.8)\quad \int_x^y (y-u)^b (u-w)^c (u-x)^a du &= \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+2+b)}(y-x)^{1+a+b}(x-w)^c \\ &\quad \times F\left(-c, a+1, a+2+b, \frac{y-x}{w-x}\right) \\ &= \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+2+b)}(y-x)^{1+a+b}(y-w)^c \\ &\quad \times F\left(-c, b+1, a+2+b, \frac{y-x}{y-w}\right).\end{aligned}$$

If $x < y < w$, then we have that

$$\begin{aligned}(2.9)\quad \int_x^y (y-u)^b (w-u)^c (u-x)^a \, du &= \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+2+b)}(y-x)^{1+a+b}(w-x)^c \\ &\quad \times F\left(-c, a+1, a+2+b, \frac{y-x}{w-x}\right) \\ (2.10)\quad &= \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+2+b)}(y-x)^{1+a+b}(w-y)^c \\ &\quad \times F\left(-c, b+1, a+2+b, \frac{y-x}{y-w}\right).\end{aligned}$$

By linearly combining neighbor relations, we obtain relations of type (2.2), where $F_1(z)$ and $F_2(z)$ are contiguous to $F(a, b, c, z)$ and A, A_1 and A_2 are polynomials of higher degree. An example for a contiguity relation is

$$\begin{aligned}(2.11)\quad -cF(a, b-1, c, z) + (c - b + zb - za)F(a, b, c+1, z) \\ + b(1-z)F(a, b+1, c+1, z) = 0.\end{aligned}$$

It can be checked easily by using series.

2.2. *Fractional calculus over the real line.* For more information on fractional calculus in the context of fractional Brownian motion, see [**7**] or [**4**]. See [**10**] for general information on fractional calculus.

Definition 2.1. Let $\alpha > 0$. The (*right-sided*) *Riemann-Liouville fractional integral operator of order* α is defined by

$$\big(\mathcal{I}_-^\alpha f\big)(s) := \frac{1}{\Gamma(\alpha)} \int_s^\infty f(u)(u-s)^{\alpha-1}\,du, \quad s \in \mathbf{R}.$$

Let $\alpha \in (0,1)$. The (*right-sided*) *Riemann-Liouville fractional derivative operator of order* α is defined by

$$\begin{aligned}\big(\mathcal{D}_-^\alpha f\big)(s) &:= \frac{-d}{ds}\big(\mathcal{I}_-^{1-\alpha} f\big)(s) \\ &= \frac{1}{\Gamma(1-\alpha)}\frac{-d}{ds}\int_s^\infty f(u)(u-s)^{-\alpha}\,du, \quad s \in \mathbf{R}.\end{aligned}$$

The (*right-sided*) *Marchaud fractional derivative operator of order* α is defined by

$$\left(\mathbf{D}_-^\alpha f\right)(s) := \lim_{\varepsilon \searrow 0} \left(\mathbf{D}_{-,\varepsilon}^\alpha f\right)(s), \quad \text{a.e. } s \in \mathbf{R},$$

where

$$\left(\mathbf{D}_{-,\varepsilon}^\alpha f\right)(s) := \frac{\alpha}{\Gamma(1-\alpha)} \int_\varepsilon^\infty \big(f(s) - f(u+s)\big) u^{-\alpha-1}\,du.$$

Moreover,

$$\mathcal{D}_-^0 f := \mathbf{D}_-^0 f := f.$$

We set

$$\mathcal{I}_-^{-\alpha} := \mathbf{D}_-^\alpha, \quad \alpha \in [0,1).$$

If $\alpha \in (0,1)$, $p \in [1,(1/\alpha))$ and $f \in L^p(\mathbf{R})$, then $\mathcal{I}_-^\alpha f$ is well defined, see [**10**, page 94]. Clearly, $\mathcal{I}_-^1 f$ exists for $f \in L^1(\mathbf{R})$. We have the composition formula

$$\mathcal{I}_-^\alpha \mathcal{I}_-^\beta f = \mathcal{I}_-^{\alpha+\beta} f, \quad f \in L^1(\mathbf{R}), \qquad \mathcal{I}_-^\beta f \in L^1(\mathbf{R}), \quad \alpha, \beta \in (0,1].$$

Furthermore,

$$\mathcal{D}_-^{\alpha}\mathcal{I}_-^{\beta}f = \mathcal{I}_-^{\beta-\alpha}f, \quad f \in L^1(\mathbf{R}), \quad \mathcal{I}_-^{\beta}f \in L^1(\mathbf{R}), \quad 0 < \alpha \leq \beta \leq 1. \tag{2.12}$$

If f is piecewise differentiable with $\operatorname{supp}(f') \subseteq (-\infty, y)$ for some $y \in \mathbf{R}$ and so that $\mathcal{D}_-^{\alpha}f$ exists, then $\mathbf{D}_-^{\alpha}f = \mathcal{D}_-^{\alpha}f$.

Let $t \in \mathbf{R} \setminus \{0\}$, and set $1_{[0,t)} := -1_{[t,0)}$ for $t < 0$. For all $H \in (0,1)$, we have that

$$\left(\mathcal{I}_-^{H-(1/2)}1_{[0,t)}\right)(s) = \frac{1}{\Gamma(H+(1/2))}\left((t-s)^{H-(1/2)}1_{(-\infty,t)}(s) - (-s)^{H-(1/2)}1_{(-\infty,0)}(s)\right).$$

So for $K = 1/2$, (1.2) can be written as

$$B_t^H = C(H)\int_{\mathbf{R}}\left(\mathcal{I}_-^{H-(1/2)}1_{[0,t)}\right)(s)\,d\widetilde{W}_s, \quad \text{a.s.}, \quad t \in \mathbf{R}. \tag{2.13}$$

2.3. *Fractional Wiener integrals over the real line.* We combine (2.13) with the standard Wiener integral in order to obtain a meaning for the expression $\int_{\mathbf{R}} f(s)\,dB_s^H$ for suitable deterministic integrands f. For details on this topic, see [**9**]. For $H > 1/2$, the space of integrands is given by

$$\Lambda(H) := \left\{f \in L^1(\mathbf{R}) \;\middle|\; \int_{\mathbf{R}}\left(\mathcal{I}_-^{H-(1/2)}f\right)(s)^2\,ds < \infty\right\}.$$

For $H < 1/2$, we have that

$$\Lambda(H) := \left\{f : \mathbf{R} \to \mathbf{R} \;\middle|\; \exists\, \varphi_f \in L^2(\mathbf{R}) \text{ such that } f = \mathcal{I}_-^{(1/2)-H}\varphi_f\right\}.$$

For $f \in \Lambda(H)$, the (*time domain*) *fractional Wiener integral* with respect to B^H is defined by

$$I^H(f) := \int_{\mathbf{R}} f(s)\,dB_s^H := C(H)\int_{\mathbf{R}}\left(\mathcal{I}_-^{H-(1/2)}f\right)(s)\,d\widetilde{W}_s.$$

3. The connection between the integral transforms. Consider transform (1.1). B^K has stationary increments so, for every $s > 0$, the process

$$B_t^{H,s} := C(K,H)\int_0^t (t-u)^{H-K}\widehat{F}\left(\frac{u-t}{u}\right) dB_{u-s}^K, \quad t \in [0,\infty),$$

where

$$\widehat{F}(z) := F(1-K-H, H-K, 1+H-K, z), \tag{3.1}$$

is an H-fBm. The increments of $B^{H,s}$ are stationary, hence the time-shifted process

$$Z_t^{H,s} := B_{t+s}^{H,s} - B_s^{H,s}, \quad t \in [-s,\infty),$$

is an H-fBm. By substituting $v := u - s$, we obtain that

$$\begin{aligned} Z_t^{H,s} = C(K,H)\bigg(&\int_{-s}^t (t-v)^{H-K}\widehat{F}\left(\frac{v-t}{v+s}\right) dB_v^K \\ &- \int_{-s}^0 (-v)^{H-K}\hat{F}\left(\frac{v}{v+s}\right) dB_v^K\bigg), \quad \text{a.s.} \end{aligned}$$

As $s \to \infty$, we formally obtain that

$$\begin{aligned} Z_t^H := Z_t^{H,\infty} := C(K,H)\int_{\mathbf{R}} \big(&(t-v)^{H-K}1_{(-\infty,t)}(v) \\ &- (-v)^{H-K}1_{(-\infty,0)}(v)\big)\, dB_v^K \end{aligned}$$

for $t \in \mathbf{R}$. Note that by definition, the processes $B^{H,s}$, $Z^{H,s}$ and Z^H depend on K. We can state and prove the following:

Theorem 3.1. *For every $K \geq 1/2$ and $t \in \mathbf{R}$, there exist constants $C_1 = C_1(K,H,t)$ and $s_1 = s_1(t) > 0$, such that*

$$E\big[Z_t^{H,s} - Z_t^H\big]^2 \leq C_1 s^{2H-2}, \quad s > s_1.$$

Moreover, for every $K < 1/2$ and $t \in \mathbf{R}$, there exist constants $C_2 = C_2(K,H,t)$, $C_3 = C_3(K,H,t)$ and $s_2 = s_2(t) > 0$ with

$$E\big[Z_t^{H,s} - Z_t^H\big]^2 \leq C_2 s^{2H-2} + C_3 s^{2K-2}, \quad s > s_2.$$

Proof. Clearly, we can assume that $H \neq K$ and $t \neq 0$. Moreover, we assume that $t > 0$. The result is derived similarly for $t < 0$. Recall that $\widehat{F}$ is defined in (3.1), and denote

$$\begin{aligned}
\Delta f_t^s(v) &:= \big((t-v)^{H-K} - (-v)^{H-K}\big)\, 1_{(-\infty,-s)}(v),\\
\Delta g_t^s(v) &:= (t-v)^{H-K}\widehat{F}\left(\frac{v-t}{v+s}\right) 1_{(-s,t)}(v)\\
&\qquad - (-v)^{H-K}\widehat{F}\left(\frac{v}{v+s}\right) 1_{(-s,0)}(v),\\
\Delta h_t^s(v) &:= (t-v)^{H-K} 1_{(-s,t)}(v) - (-v)^{H-K} 1_{(-s,0)}(v)
\end{aligned}$$

and

$$\begin{aligned}
\Delta k_t^s(v) &:= k_t^s(v) - k_0^s(v)\\
&:= (t-v)^{H-K}\left(\widehat{F}\left(\frac{v-t}{v+s}\right) - 1\right) 1_{(-s,t)}(v)\\
&\qquad - (-v)^{H-K}\left(\widehat{F}\left(\frac{v}{v+s}\right) - 1\right) 1_{(-s,0)}(v).
\end{aligned}$$

For continuous G, set ${}^*G := \max_{z\in[-1,0]} |G(z)|$ and $G^* := \max_{z\in[0,1]} |G(z)|$. For $K = 1/2$, we have, by independence of increments of $B^K = W$, that

$$\left(\frac{\Gamma(H+(1/2))}{C(H)}\right)^2 E\big[Z_t^{H,s} - Z_t^H\big]^2 = \int_{\mathbf{R}} \Delta f_t^s(v)^2\, dv + \int_{\mathbf{R}} \Delta k_t^s(v)^2\, dv. \tag{3.2}$$

Note that

$$\left(\sum_{i=1}^n a_i\right)^2 \le n \sum_{i=1}^n a_i^2, \quad n \in \mathbf{N}. \tag{3.3}$$

So for $K \neq 1/2$, we obtain by using (3.3) with $n = 2$, that

$$\begin{aligned} \frac{1}{2C(K,H)^2} &\cdot E\big[Z_t^{H,s} - Z_t^H\big]^2 \\ &\leq E\left[\int_{\mathbf{R}} \Delta f_t^s(v)\, dB_v^K\right]^2 + E\left[\int_{\mathbf{R}} \Delta k_t^s(v)\, dB_v^K\right]^2 \\ &= C(K)^2 \bigg(\int_{\mathbf{R}} \Big(\mathcal{I}_-^{K-(1/2)} \Delta f_t^s\Big)(v)^2\, dv \\ &\quad + \int_{\mathbf{R}} \Big(\mathcal{I}_-^{K-(1/2)} \Delta k_t^s\Big)(v)^2\, dv \bigg). \end{aligned} \tag{3.4}$$

A. For all $K \in (0,1)$, we show that there exists a constant $c_1(K,H,t)$, such that

$$\int_{\mathbf{R}} \Big(\mathcal{I}_-^{K-(1/2)} \Delta f_t^s\Big)(v)^2\, dv \leq c_1(K,H,t) s^{2H-2}, \quad s > t. \tag{3.5}$$

For all $K \in (0,1)$, we have that

$$\begin{aligned} \Big(\mathcal{I}_-^{K-(1/2)} \Delta f_t^s\Big)(v) = \frac{(-s-v)^{H-(1/2)}}{\Gamma(K+(1/2))} &\bigg(G_0\Big(\frac{-s-v}{t-v}\Big) \\ &- G_0\Big(\frac{-s-v}{-v}\Big)\bigg) 1_{(-\infty,-s)}(v), \end{aligned}$$

where

$$G_0(z) := z^{K-H} F\Big(K-H, K-\frac{1}{2}, K+\frac{1}{2}, z\Big).$$

For $K = 1/2$, this is trivial. For $K > 1/2$, this follows from (2.9). For $K < 1/2$, it follows from the fact that $(\mathcal{I}_-^{K-(1/2)} \Delta f_t^s)(v) = (-d/dv)(\mathcal{I}_-^{K+(1/2)} \Delta f_t^s)(v)$ by using (2.9), (2.6) and (2.11). By using (2.6), we have that

$$\begin{aligned} \frac{dG_0}{dz}(z) &= (K-H) z^{K-H-1} F\Big(K-H+1, K-\frac{1}{2}, K+\frac{1}{2}, z\Big) \\ &=: (K-H) z^{K-H-1} G_1(z). \end{aligned} \tag{3.6}$$

By the mean value theorem, there exists $\theta_{v,s} \in ((-s-v)/(t-v), (-s-v)/(-v)) \subset (0,1)$, such that

$$
\begin{aligned}
&\left(\mathcal{I}_{-}^{K-(1/2)} \Delta f_t^s\right)(v)^2 \\
&\qquad = \left(\frac{K-H}{\Gamma(K+(1/2))}\right)^2 (-s-v)^{2H-1} \left(\frac{t(-v-s)}{-v(t-v)}\right)^2 \\
&\qquad\quad \times \theta_{v,s}^{2(K-H-1)} G_1^2(\theta_{v,s}) 1_{(-\infty,-s)}(v) \\
&\qquad \le \left(\frac{K-H}{\Gamma(K+(1/2))}\right)^2 G_1^{*2} t^2 \max\left(1, \left(\frac{t+s}{s}\right)^{2(H-K)}\right) \\
&\qquad\quad \times (-s-v)^{2K-1} (-v)^{2(H-K-1)} \cdot 1_{(-\infty,-s)}(v).
\end{aligned}
$$

Hence, from (2.10) and (2.3), we obtain that

$$
\begin{aligned}
&\int_{\mathbf{R}} (\mathcal{I}_{-}^{K-(1/2)} \Delta f_t^s)(v)^2 \, dv \\
&\qquad \le \left(\frac{K-H}{\Gamma(K+(1/2))}\right)^2 G_1^{*2} t^2 \max\left(1, \left(\frac{t+s}{s}\right)^{2(H-K)}\right) \\
&\qquad\quad \times \lim_{x\to-\infty} \frac{1}{2K} (-x-s)^{2K} s^{2(H-K-1)} \left(\frac{-x}{s}\right)^{-2K} \\
&\qquad\quad \times F\left(2H-1, 2K, 2K+1, \frac{x+s}{x}\right) \\
&\qquad = \left(\frac{K-H}{\Gamma(K+(1/2))}\right)^2 G_1^{*2} \frac{t^2}{2K} F(2H-1, 2K, 2K+1, 1) \\
&\qquad\quad \times \max\left(1, \left(\frac{t+s}{s}\right)^{2(H-K)}\right) s^{2H-2}.
\end{aligned}
$$

By using (2.7), we obtain (3.5) with

$$
c_1(K,H,t) = \left(\frac{K-H}{\Gamma(K+(1/2))}\right)^2 \frac{G_1^{*2} 4\Gamma(2K)\Gamma(2-2H)}{\Gamma(2K-2H+2)} t^2.
$$

B. Fix $d > 0$. For $K \ge 1/2$, we show that there exists a constant $c_2(K,H,t,d)$, such that

$$
\int_{\mathbf{R}} \left(\mathcal{I}_{-}^{K-(1/2)} \Delta k_t^s\right)(v)^2 \, dv \le c_2(K,H,t,d) s^{2H-2}, \qquad s > 2t+4d+1. \tag{3.7}
$$

For $K < 1/2$, we show that there exist constants $c_3(K,H,t,d)$ and $c_4(K,H,t,d)$ with

$$\int_{\mathbf{R}} \left(\mathcal{I}_{-}^{K-(1/2)}\Delta k_t^s\right)(v)^2\,dv \le c_3(K,H,t,d)s^{2H-2} + c_4(K,H,t,d)s^{2K-2},\ s > 2t+4d+1. \tag{3.8}$$

Let $s > 2d$. We have that

$$\Delta k_t^s(v) = k_t^s(v)1_{[-d,t)}(v) - k_0^s(v)1_{[-d,0)}(v) + \Delta k_t^s(v)1_{[(-s/2),-d)}(v) + \Delta g_t^s(v)1_{(-s,(-s/2))}(v) - \Delta h_t^s(v)1_{(-s,(-s/2))}(v).$$

First, let $K = 1/2$. Note that $(-s,(-s/2))\cap[(-s/2),-d)\cap[-d,t) = \varnothing$. By using (3.3) with $n = 2$, we have that

$$\frac{1}{2}\int_{\mathbf{R}} \Delta k_t^s(v)^2\,dv \le \int_{-d}^{t} k_t^s(v)^2\,dv + \int_{-d}^{0} k_0^s(v)^2\,dv + \int_{-s/2}^{-d} \Delta k_t^s(v)^2\,dv + \int_{-s}^{-s/2} \Delta g_t^s(v)^2\,dv + \int_{-s}^{-s/2} \Delta h_t^s(v)^2\,dv. \tag{3.9}$$

Second, let $K > 1/2$. Then

$$
\begin{aligned}
\Gamma(K-(1/2))(\mathcal{I}_-^{K-(1/2)}\Delta k_t^s)(v)
&= \int_{-d}^{t} k_t^s(u)(u-v)^{K-(3/2)}du \cdot 1_{(-\infty,-d)}(v) \\
&\quad + \int_{v}^{t} k_t^s(u)(u-v)^{K-(3/2)}du \cdot 1_{[-d,t)}(v) \\
&\quad - \int_{-d}^{0} k_0^s(u)(u-v)^{K-(3/2)}du \cdot 1_{(-\infty,-d)}(v) \\
&\quad - \int_{v}^{0} k_0^s(u)(u-v)^{K-(3/2)}du \cdot 1_{[-d,0)}(v) \\
&\quad + \int_{-s/2}^{-d} \Delta k_t^s(u)(u-v)^{K-(3/2)}du \cdot 1_{(-\infty,(-s/2))}(v) \\
&\quad + \int_{v}^{-d} \Delta k_t^s(u)(u-v)^{K-(3/2)}du \cdot 1_{[(-s/2),-d)}(v) \\
&\quad + \int_{-s}^{-s/2} \Delta g_t^s(u)(u-v)^{K-(3/2)}du \cdot 1_{(-\infty,-s)}(v) \\
&\quad + \int_{v}^{-s/2} \Delta g_t^s(u)(u-v)^{K-(3/2)}du \cdot 1_{[-s,(-s/2))}(v) \\
&\quad - \int_{-s}^{-s/2} \Delta h_t^s(u)(u-v)^{K-(3/2)}du \cdot 1_{(-\infty,-s)}(v) \\
&\quad - \int_{v}^{-s/2} \Delta h_t^s(u)(u-v)^{K-(3/2)}du \cdot 1_{[-s,(-s/2))}(v) \\
&=: A_1(v) + B_1(v) + A_2(v) + B_2(v) + A_3(v) + B_3(v) \\
&\quad + A_4(v) + B_4(v) + A_5(v) + B_5(v).
\end{aligned}
$$

Hence, by using (3.3) with $n = 5$, we obtain that

$$
\begin{aligned}
\frac{\Gamma(K-(1/2))^2}{5} \int_{\mathbf{R}} \left(\mathcal{I}_-^{K-(1/2)}\Delta k_t^s\right)(v)^2\, dv
\le \sum_{i=1}^{5} \int_{\mathbf{R}} A_i(v)^2\, dv + \sum_{i=1}^{5} \int_{\mathbf{R}} B_i(v)^2\, dv.
\end{aligned} \tag{3.10}
$$

Third, let $K < 1/2$. Let $x \le v < y$ and f be differentiable with $f' \in L^1[x,y]$. From (2.12), it follows that

$$\begin{aligned}
&(\mathcal{I}_-^{K+(1/2)}\mathcal{D}_-^1 f1_{(x,y)})(v)\\
&\qquad = (\mathcal{D}_-^{(1/2)-K}\mathcal{I}_-^1\mathcal{D}_-^1 f1_{(x,y)})(v)\\
&\qquad = (\mathcal{D}_-^{(1/2)-K} f1_{(x,y)} - f(y)1_{(x,y)})(v)\\
&\qquad = (\mathcal{D}_-^{(1/2)-K} f1_{(x,y)})(v) - \frac{f(y)}{\Gamma(K+(1/2))}(y-v)^{K-(1/2)}.
\end{aligned}$$

Since $\lim_{v\nearrow t} k_t^s(v) = \lim_{v\nearrow 0} k_0^s(v) = 0$, we obtain that

$$\begin{aligned}
&\Gamma\left(K-\frac{1}{2}\right)\left(\mathbf{D}_-^{(1/2)-K}\Delta k_t^s\right)(v)\\
&= \Gamma\left(K-\frac{1}{2}\right)\left(\mathbf{D}_-^{(1/2)-K} k_t^s 1_{(-d,t)}\right)(v)\\
&\quad - \Gamma\left(K-\frac{1}{2}\right)\left(\mathcal{D}_-^{(1/2)-K} k_0^s 1_{(-d,0)}\right)(v)\\
&\quad + \Gamma\left(K-\frac{1}{2}\right)\left(\mathcal{D}_-^{(1/2)-K} \Delta k_t^s 1_{((-s/2),-d)}\right)(v)\\
&\quad + \Gamma\left(K-\frac{1}{2}\right)\left(\mathcal{D}_-^{(1/2)-K} \Delta g_t^s 1_{(-s,(-s/2))}\right)(v)\\
&\quad - \Gamma\left(K-\frac{1}{2}\right)\left(\mathcal{D}_-^{(1/2)-K} \Delta h_t^s 1_{(-s,(-s/2))}\right)(v)\\
&= \int_{-d}^{t} k_t^s(u)(u-v)^{K-(3/2)}\,du \cdot 1_{(-\infty,-d)}(v)\\
&\quad + \Gamma\left(K-\frac{1}{2}\right)\left(\mathcal{I}_-^{K+(1/2)}\frac{-d}{du}k_t^s 1_{(-d,t)}\right)(v)\cdot 1_{[-d,t)}(v)\\
&\quad - \int_{-d}^{0} k_0^s(u)(u-v)^{K-(3/2)}\,du \cdot 1_{(-\infty,-d)}(v)\\
&\quad - \Gamma\left(K-\frac{1}{2}\right)\left(\mathcal{I}_-^{K+(1/2)}\frac{-d}{du}k_0^s 1_{(-d,0)}\right)(v)\cdot 1_{[-d,0)}(v)\\
&\quad + \int_{-s/2}^{-d} \Delta k_t^s(u)(u-v)^{K-(3/2)}\,du \cdot 1_{(-\infty,(-s/2))}(v)\\
&\quad + \Gamma\left(K-\frac{1}{2}\right)\left(\mathcal{I}_-^{K+(1/2)}\frac{-d}{du}\Delta k_t^s 1_{((-s/2),-d)}\right)(v)\cdot 1_{[(-s/2),-d)}(v)
\end{aligned}$$

$$
\begin{aligned}
&+\int_{-s}^{-s/2} \Delta g_t^s(u)(u-v)^{K-(3/2)}\,du \cdot 1_{(-\infty,-s)}(v)\\
&+\Gamma\left(K-\frac{1}{2}\right)\left(\mathcal{I}_-^{K+(1/2)}\frac{-d}{du}\Delta g_t^s 1_{(-s,(-s/2))}\right)(v)\cdot 1_{[-s,(-s/2))}(v)\\
&-\int_{-s}^{-s/2} \Delta h_t^s(u)(u-v)^{K-(3/2)}\,du \cdot 1_{(-\infty,-s)}(v)\\
&-\Gamma\left(K-\frac{1}{2}\right)\left(\mathcal{I}_-^{K+(1/2)}\frac{-d}{du}\Delta h_t^s 1_{(-s,(-s/2))}\right)(v)\cdot 1_{[-s,(-s/2))}(v)\\
&+\frac{\Delta k_t^s(-d)}{K-(1/2)}(-d-v)^{K-(1/2)}\cdot 1_{[(-s/2),-d)}(v)\\
&+\frac{\Delta k_t^s(-s/2)}{K-(1/2)}\left(\frac{-s}{2}-v\right)^{K-(1/2)}\cdot 1_{[-s,(-s/2))}(v)\\
=:{}& A_1(v)+C_1(v)+A_2(v)+C_2(v)+A_3(v)+C_3(v)+A_4(v)\\
&+C_4(v)+A_5(v)+C_5(v)+D(v)+E(v).
\end{aligned}
$$

Hence, (3.3) with $n=6$ yields

$$
\begin{aligned}
\text{(3.11)}\quad \frac{\Gamma\left(K-(1/2)\right)^2}{6}\int_{\mathbf{R}}\left(\mathcal{I}_-^{K-(1/2)}\Delta k_t^s\right)(v)^2\,dv &\le \sum_{i=1}^{5}\int_{\mathbf{R}} A_i(v)^2\,dv\\
&+\sum_{i=1}^{5}\int_{\mathbf{R}} C_i(v)^2\,dv+\int_{\mathbf{R}} D(v)^2\,dv+\int_{\mathbf{R}} E(v)^2\,dv.
\end{aligned}
$$

Next we estimate the integrals on the righthand sides of (3.9), (3.10) and (3.11). In what follows, B denotes the beta function.

1. Estimation of $\int_{-d}^{t} k_t^s(v)^2\,dv$, $\int_{\mathbf{R}} A_1(v)^2\,dv$, $\int_{\mathbf{R}} B_1(v)^2\,dv$ and $\int_{\mathbf{R}} C_1(v)^2\,dv$. Let $u\in(-d,t)$. By the mean value theorem, there exists $\theta_{s,u}\in((u-t)/(u+s),0)\subset((-d-t)/(-d+s),0)$, such that

$$
\begin{aligned}
|k_t^s(u)| &= (t-u)^{H-K}\left(\frac{t-u}{u+s}\right)\left|\frac{d\widehat{F}}{dz}(\theta_{s,u})\right|\\
&\le (t-u)^{H-K+1}(s-d)^{-1}\max_{z\in((-d-t)/(-d+s),0)}\left|\frac{d\widehat{F}(z)}{dz}\right|.
\end{aligned}
$$

We assume that $s > 2d + t$. Then, it follows from (2.5) that

$$|k_t^s(u)| \le (t-u)^{H-K+1}(s-d)^{-1}\frac{|1-K-H||H-K|^*G_2}{|1+H-K|},$$

where

$$G_2(z) := F(2-K-H, H-K+1, 2+H-K, z).$$

For $K = 1/2$, we obtain that

$$\int_{-d}^{t} k_t^s(u)^2\,du \le \frac{{}^*G_2^2(t+d)^{2H+2}}{2}s^{-2}.$$

Denote

$$G_3(z) := F\left(\frac{3}{2}-K, H-K+2, H-K+3, z\right).$$

For $K \neq 1/2$, we obtain by using (2.8) that

$$\int_{\mathbf{R}} A_1(v)^2\,dv \le \frac{2{}^*G_2^2G_3^{*2}(t+d)^{2H+2}}{(1+H-K)^2(1-K)}s^{-2}.$$

Also, for $K > 1/2$, we have that

$$\int_{\mathbf{R}} B_1(v)^2\,dv \le \frac{2{}^*G_2^2B\big(H-K+2, K-(1/2)\big)^2}{(1+H-K)^2}(t+d)^{2H+2}s^{-2}.$$

Furthermore, it holds that

$$\begin{aligned}\left|\frac{d}{du}k_t^s(u)\right| &\le |K-H|(t-u)^{H-K-1}\left(\frac{t-u}{u+s}\right)\left|\frac{d\hat{F}}{dz}(\theta_{s,u})\right| \\ &\quad + (t-u)^{H-K}\left|\frac{d\hat{F}}{dz}\left(\frac{u-t}{u+s}\right)\right|\frac{s+t}{(u+s)^2} \\ &\le \left(\frac{|1-K-H||H-K|^*G_2}{|1+H-K|}\right)\left(\frac{|K-H|}{s-d}+\frac{s+t}{(s-d)^2}\right) \\ &\quad \times (t-u)^{H-K}.\end{aligned}$$

Hence, for $K < 1/2$, we obtain by using (3.3) with $n = 2$ that

$$\int_{\mathbf{R}} C_1(v)^2\,dv \le \frac{68 \cdot^* G_2^2 B\left(H-K+1, K+(1/2)\right)^2}{(1+H-K)^2\left(K-(1/2)\right)^2}(t+d)^{2H+2}s^{-2}.$$

2. Estimation of $\int_{-d}^{0} k_0^s(v)^2\,dv$, $\int_{\mathbf{R}} A_2(v)^2 dv$, $\int_{\mathbf{R}} B_2(v)^2\,dv$ and $\int_{\mathbf{R}} C_2(v)^2 dv$. We obtain estimates by replacing t by 0 in the results of 1.

3. Estimation of $\int_{-s/2}^{-d} \Delta k_t^s(v)^2\,dv$, $\int_{\mathbf{R}} A_3(v)^2\,dv$, $\int_{\mathbf{R}} B_3(v)^2\,dv$, $\int_{\mathbf{R}} C_3(v)^2\,dv$, $\int_{\mathbf{R}} D(v)^2\,dv$ and $\int_{\mathbf{R}} E(v)^2\,dv$. Denote

$$G_4(z) := z^{H-K}(\widehat{F}(-z)-1).$$

For $u \in ((-s/2), -d)$, we have that

$$\Delta k_t^s(u) = (u+s)^{H-K}\left(G_4\left(\frac{t-u}{u+s}\right) - G_4\left(\frac{-u}{u+s}\right)\right).$$

From (2.6) and (2.4), it follows that

$$\frac{dG_4}{dz}(z) = (H-K)z^{H-K-1}((1+z)^{H+K-1}-1).$$

Hence, by the mean value theorem, there exist $\theta_{s,u} \in ((-u/u+s), (t-u/u+s)) \subset (0, (t+(s/2)/s/2))$ and $\eta_{s,u} \in (0, \theta_{s,u})$, such that

$$\begin{aligned}\Delta k_t^s(u) &= (u+s)^{H-K}\left(\frac{t}{u+s}\right)(H-K)\theta_{s,u}^{H-K-1}\left((1+\theta_{s,u})^{H+K-1}-1\right)\\ &= (u+s)^{H-K}\left(\frac{t}{u+s}\right)(H-K)\theta_{s,u}^{H-K-1}(H+K-1)\\ &\quad\times \theta_{s,u}(1+\eta_{s,u})^{H+K-2}.\end{aligned}$$

Thus,

$$\begin{aligned}(3.12)\quad |\Delta k_t^s(u)| \le |H-K||H &\\ +K-1|(u+s)^{H-K-1}t\left(\frac{-u}{u+s}\right)^{H-K-1}\frac{t-u}{u+s}.&\end{aligned}$$

In particular, we have that

$$(3.13)\qquad |\Delta k_t^s(u)| \le |H-K||H+K-1|\left(\frac{s}{2}\right)^{-1} t\frac{t+d}{d}(-u)^{H-K}.$$

So, for $K=1/2$, we obtain that

$$\int_{-s/2}^{-d} \Delta k_t^s(u)^2\,du \le \frac{t^2}{8H}\left(\frac{t+d}{d}\right)^2 s^{2H-2}.$$

Also, from (3.12), it follows for $u \in ((-s/2), -d)$ that

$$(3.14)\qquad |\Delta k_t^s(u)| \le |H-K||H+K-1|t\frac{t+(s/2)}{s/2}(-u)^{H-K-1}.$$

Denote

$$G_5(z) := F\left(\frac{3}{2}-K, H-K, H-K+1, z\right).$$

Let $K \neq 1/2$. For $H > K$ and $s > \max(2t, 4d)$, we obtain by using (3.14) and (2.8) that

$$\begin{aligned}\int_{\mathbf{R}} A_3(v)^2\,dv &\le (H-K)^2 t^2\left(\frac{t+(s/2)}{s/2}\right)^2 \\ &\quad\times\int_{-\infty}^{-s/2}\left(\int_{-s/2}^{-d}(-d-u)^{H-K-1}(u-v)^{K-(3/2)}\,du\right)^2 dv \\ &\le\ t^2\left(\frac{t+(s/2)}{s/2}\right)^2 G_5^{*2}\frac{((s/2)-d)^{2H-2}}{2-2K} \le \frac{32G_5^{*2}}{1-K}t^2 s^{2H-2}.\end{aligned}$$

Similarly, for $H < K$, we have by using (3.13) and (2.8) that

$$\int_{\mathbf{R}} A_3(v)^2\,dv \le \frac{2G_6^{*2}t^2}{(H-K+1)^2(1-K)}\left(\frac{t+d}{d}\right)^2 s^{2H-2},$$

where

$$G_6(z) := F\left(\frac{3}{2}-K, H-K+1, H-K+2, z\right).$$

Let

$$G_7(z) := F\big(2(K-H+1), 1, 2K+1, z\big).$$

For $K > 1/2$ and $s > 2t$, we have by using (3.14) and then (2.9) twice that

$$\begin{aligned}\int_{\mathbf{R}} B_3(v)^2\,dv &\le t^2\left(\frac{t+(s/2)}{s/2}\right)^2\\ &\quad\times\int_{-s/2}^{-d}\left(\int_v^{-d}(-u)^{H-K-1}(u-v)^{K-(3/2)}\,du\right)^2 dv\\ &\le t^2\left(\frac{t+(s/2)}{s/2}\right)^2\frac{G_1^{*2}}{(K-(1/2))^2}\frac{1}{2K}\left(\frac{s}{2}-d\right)^{2K}\\ &\quad\times\left(\frac{s}{2}\right)^{2(H-K-1)}G_7^*\le\frac{8G_1^{*2}G_7^*}{(K-(1/2))^2K}t^2s^{2H-2},\end{aligned}$$

where G_1 is defined as in (3.6).

Let $K < 1/2$. It holds that

$$\begin{aligned}\frac{d^2G_4}{d^2z}(z) &= (H-K-1)(H-K)z^{H-K-2}\left((1+z)^{H+K-1}-1\right)\\ &\quad+(H-K)(H+K-1)z^{H-K-1}(1+z)^{H+K-2}.\end{aligned}$$

For $u\in((-s/2),-d)$, there exists $\theta_{s,u}\in((-u/u+s),(t-u/u+s))\subset(0,(t+(s/2)/s/2))$, such that

$$\begin{aligned}\left|\frac{-d}{du}\Delta k_t^s(u)\right| &\le |H-K|(u+s)^{H-K-1}\left|G_4\left(\frac{t-u}{u+s}\right)-G_4\left(\frac{-u}{u+s}\right)\right|\\ &\quad+(u+s)^{H-K-2}t\left|\frac{dG_4}{dz}\left(\frac{t-u}{u+s}\right)\right|\\ &\quad+(u+s)^{H-K-2}s\frac{t}{u+s}\left|\frac{d^2G_4}{d^2z}(\theta_{s,v})\right|\\ &\le (H-K)^2|H+K-1|\frac{t+(s/2)}{s/2}\left(\frac{s}{2}\right)^{-1}t(-u)^{H-K-1}\\ &\quad+|H-K||H+K-1|\frac{t+(s/2)}{s/2}\left(\frac{s}{2}\right)^{-1}t(t-u)^{H-K-1}\\ &\quad+|H-K-1||H-K||H+K-1|\frac{t+(s/2)}{s/2}\\ &\quad\times st\left(\frac{s}{2}\right)^{-1}(-u)^{H-K-2}\end{aligned}$$

$$+ |H-K||H+K-1| s^{H+K-1} \left(\frac{s}{2}\right)^{-H-K}$$
$$\times\, t(-u)^{H-K-1}.$$

Let

$$G_8(z) := F\left(1+K-H, K+\frac{1}{2}, K+\frac{3}{2}, z\right),$$
$$G_9(z) := F\Big(2(K+1-H), 1, 2K+3, z\Big),$$
$$G_{10}(z) := F\left(2+K-H, K+\frac{1}{2}, K+\frac{3}{2}, z\right)$$

and

$$G_{11}(z) := F\big(2(K+2-H), 1, 2K+3, z\big).$$

First, for $\hat{t} \in \{0, t\}$, we have by using (2.9) twice that

$$\int_{-s/2}^{-d} \left(\int_{v}^{-d} (\hat{t}-u)^{H-K-1} (u-v)^{K-(1/2)}\, du \right)^2 dv \leq \left(\frac{G_8^*}{K+(1/2)} \right)^2 \frac{G_9^*}{2K+2} \left(\frac{s}{2}\right)^{2H}.$$

Second, in the same way, we obtain that

$$\int_{-s/2}^{-d} \left(\int_{v}^{-d} (-u)^{H-K-2} (u-v)^{K-(1/2)}\, du \right)^2 dv \leq \left(\frac{G_{10}^*}{K+(1/2)} \right)^2 \frac{G_{11}^*}{2K+2} \left(\frac{s}{2}\right)^{2H-2}.$$

By using (3.3) with $n=4$, we obtain for $s>2t$, that

$$\int_{\mathbf{R}} C_3(v)^2\, dv \leq \frac{4}{(K-(1/2))^2} \left(\frac{t+(s/2)}{s/2}\right)^2 t^2 \left(\frac{s}{2}\right)^{-2}$$
$$\times \left(\frac{G_8^*}{K+(1/2)} \right)^2 \frac{G_9^*}{2K+2} \left(\frac{s}{2}\right)^{2H}$$
$$+ \frac{4}{(K-(1/2))^2} \left(\frac{t+(s/2)}{s/2}\right)^2 t^2 \left(\frac{s}{2}\right)$$

$$
\begin{aligned}
&\times \frac{G_8^{*2}}{(K+(1/2))^2}\frac{G_9^*}{2K+2}\left(\frac{s}{2}\right)^{2H}\\
&+\frac{4}{(K-(1/2))^2}(H-K-1)^2\left(\frac{t+(s/2)}{s/2}\right)^2\\
&\times s^2t^2\left(\frac{s}{2}\right)^{-2}\left(\frac{G_{10}^*}{K+(1/2)}\right)^2\frac{G_{11}^*}{2K+2}\left(\frac{s}{2}\right)^{2H-2}\\
&+\frac{4}{(K-(1/2))^2}s^{2(H+K-1)}t^2\left(\frac{s}{2}\right)^{-2K-2H}\\
&\times\left(\frac{G_8^*}{K+(1/2)}\right)^2\frac{G_9^*}{2K+2}\left(\frac{s}{2}\right)^{2H}\\
\leq{}&\frac{\left(68G_8^{*2}G_9^*+512G_{10}^{*2}G_{11}^*\right)}{(K-(1/2))^2(K+(1/2))^2}t^2s^{2H-2}.
\end{aligned}
$$

By using (3.13), we have that

$$|\Delta k_t^s(-d)|\leq|H-K||H+K-1|(s/2)^{-1}t\frac{t+d}{d}\,d^{H-K}.$$

Hence,

$$\int_{\mathbf{R}}D(v)^2\,dv\leq\frac{2d^{2(H-K)}}{K(K-(1/2))^2}\left(\frac{t+d}{d}\right)^2t^2s^{2K-2}.$$

From (3.14), it follows that

$$|\Delta k_t^s(-s/2)|\leq|H-K||H+K-1|t\frac{t+(s/2)}{s/2}(s/2)^{H-K-1}.$$

Hence for $s>2t$, it holds that

$$\int_{\mathbf{R}}E(v)^2\,dv\leq\frac{8}{K(K-(1/2))^2}t^2s^{2H-2}.$$

4. Estimation of $\int_{-s}^{-s/2}\Delta g_t^s(v)^2\,dv$, $\int_{\mathbf{R}}A_4(v)^2\,dv$, $\int_{\mathbf{R}}B_4(v)^2\,dv$ and $\int_{\mathbf{R}}C_4(v)^2\,dv$. By using (2.3), we have that

$$
\begin{aligned}
G_{12}(z)&:=z^{H-K}F(2H,H-K,H-K+1,z)\\
&=\left(\frac{1-z}{z}\right)^{K-H}\widehat{F}\left(\frac{z}{z-1}\right).
\end{aligned}
$$

From (2.6) and (2.4), it follows that

$$\frac{dG_{12}}{dz}(z) = (H-K)z^{H-K-1}(1-z)^{-2H}.$$

Let $u \in (-s, (-s/2))$. There exists $\theta_{s,u} \in ((-u/s), (t-u/s+t)) \subset ((1/2), 1)$, such that

$$\begin{aligned}\Delta g_t^s(u) &= (u+s)^{H-K}\left(G_{12}\left(\frac{t-u}{s+t}\right) - G_{12}\left(\frac{-u}{s}\right)\right) \\ &= (u+s)^{H-K}\left(\frac{(s+u)t}{(s+t)s}\right)(H-K)\theta_{s,u}^{H-K-1}(1-\theta_{s,u})^{-2H}.\end{aligned}$$

In particular,

$$|\Delta g_t^s(u)| \leq \frac{|H-K|t}{(s+t)^{1-2H}s}\left(\frac{1}{2}\right)^{H-K-1}(u+s)^{1-K-H}.$$

Hence, for $K = 1/2$ and $s > t$, we have that

$$\int_{-s}^{-s/2} \Delta g_t^s(u)^2\, du \leq \frac{1}{1-H} t^2 s^{2H-2}.$$

Denote

$$G_{13}(z) := F\left(\frac{3}{2} - K, 1, 3 - K - H, z\right).$$

For $K \neq 1/2$ and $s > t$, (2.8) yields that

$$\int_{\mathbf{R}} A_4(v)^2\, dv \leq \frac{2G_{13}^{*2}t^2}{(1-K)(2-K-H)^2} s^{2H-2}.$$

Similarly, for $K > 1/2$ and $s > t$, we obtain that

$$\int_{\mathbf{R}} B_4(v)^2\, dv \leq \frac{2G_{14}^{*2}}{(K-(1/2))^2 K} t^2 s^{2H-2},$$

where

$$G_{14}(z) := F\left(H + K - 1, 1, K + \frac{1}{2}, z\right).$$

Furthermore, we have that

$$\frac{d^2G_{12}}{d^2z}(z) = (H-K)\big((H-K-1)z^{H-K-2}(1-z)^{-2H} \\ +2Hz^{H-K-1}(1-z)^{-2H-1}\big).$$

So for $u \in (-s, (-s/2))$, it holds that

$$\begin{aligned}
\left|\frac{d}{du}\Delta g_t^s(u)\right| &\le |H-K|(u+s)^{H-K-1}\left|G_{12}\left(\frac{t-u}{s+t}\right) - G_{12}\left(\frac{-u}{s}\right)\right| \\
&\quad + (u+s)^{H-K} \\
&\qquad \left|\frac{dG_{12}}{dz}\left(\frac{t-u}{s+t}\right)\left(\frac{-1}{s+t}\right) - \frac{dG_{12}}{dz}\left(\frac{-u}{s}\right)\frac{-1}{s}\right| \\
&\le (H-K)^2\frac{t(1/2)^{H-K-1}}{(s+t)^{1-2H}s}(u+s)^{-H-K} \\
&\quad + (u+s)^{H-K}\left|\frac{dG_{12}}{dz}\left(\frac{t-u}{s+t}\right) - \frac{dG_{12}}{dz}\left(\frac{-u}{s}\right)\right|\frac{1}{s+t} \\
&\quad + (u+s)^{H-K}\left|\frac{dG_{12}}{dz}\left(\frac{-u}{s}\right)\right|\frac{t}{(s+t)s} \\
&\le \frac{(H-K)^2t(1/2)^{H-K-1}}{(s+t)^{1-2H}s}(u+s)^{-H-K} \\
&\quad + \frac{|H-K||H-K-1|t(1/2)^{H-K-2}}{s(s+t)^{2-2H}}(u+s)^{1-H-K} \\
&\quad + \frac{|H-K|2Ht(1/2)^{H-K-1}}{s(s+t)^{1-2H}}(u+s)^{-H-K} \\
&\quad + \frac{|H-K|t(1/2)^{H-K-1}}{(s+t)s^{1-2H}}(u+s)^{-H-K} \\
&= \Bigg(\frac{(H-K)^2t(1/2)^{H-K-1}}{(s+t)^{1-2H}s} + \frac{|H-K|2Ht(1/2)^{H-K-1}}{s(s+t)^{1-2H}} \\
&\qquad\qquad + \frac{|H-K|t(1/2)^{H-K-1}}{(s+t)s^{1-2H}}\Bigg)(u+s)^{-H-K} \\
&\quad + \frac{|H-K||H-K-1|t(1/2)^{H-K-2}}{s(s+t)^{2-2H}}(u+s)^{1-H-K}.
\end{aligned}$$

Denote

$$G_{15}(z) := F\left(K+H, 1, K+\frac{3}{2}, z\right)$$

and

$$G_{16}(z) := F\left(K + H - 1, 1, K + \frac{3}{2}, z\right).$$

Then, for $K < 1/2$ and $s > t$, we obtain by using (2.8) and (3.3) with $n = 3$ that

$$\begin{aligned}\int_{\mathbf{R}} C_4(v)^2\,dv \le{}& \left(\frac{1}{K-(1/2)}\right)^2 \\ & \left(\frac{t\,(1/2)^{H-K-1}}{(s+t)^{1-2H}s} + \frac{2t\,(1/2)^{H-K-1}}{s(s+t)^{1-2H}} + \frac{t\,(1/2)^{H-K-1}}{(s+t)s^{1-2H}}\right)^2 \\ & \times \frac{G_{15}^{*2}}{(K+(1/2))^2}\frac{(s/2)^{2-2H}}{2K+2} + \left(\frac{1}{K-(1/2)}\right)^2 \\ & \times \left(\frac{|H-K-1|t\,(1/2)^{H-K-2}}{s(s+t)^{2-2H}}\right)^2 \frac{G_{16}^{*2}}{(K+(1/2))^2}\frac{(s/2)^{4-2H}}{2K+2} \\ \le{}& \frac{\left(63G_{15}^{*2} + 4G_{16}^{*2}\right)}{(K-1/2)^2\,(K+(1/2))^2}t^2 s^{2H-2}.\end{aligned}$$

5. Estimation of $\int_{-s}^{-s/2} \Delta h_t^s(v)^2\,dv$, $\int_{\mathbf{R}} A_5(v)^2\,dv$, $\int_{\mathbf{R}} B_5(v)^2\,dv$ and $\int_{\mathbf{R}} C_5(v)^2\,dv$. For $u \in (-s, (-s/2))$, we have that

$$|\Delta h_t^s(u)| \le |H - K| t (s/2)^{H-K-1}.$$

For $K = 1/2$, it follows that

$$\int_{-s}^{-s/2} \Delta h_t^s(u)^2\,du \le t^2 s^{2H-2}.$$

Let

$$G_{17}(z) := F\left(\frac{3}{2} - K, 1, 2, z\right).$$

For $K \ne 1/2$, we obtain by using (2.8) that

$$\int_{\mathbf{R}} A_5(v)^2\,dv \le \frac{2G_{17}^{*2}}{1-K}t^2 s^{2H-2}.$$

Moreover, for $K > 1/2$, we have that

$$\int_{\mathbf{R}} B_5(v)^2\,dv \le \frac{2t^2}{K\,(K-(1/2))^2} s^{2H-2}.$$

For $u \in (-s, (-s/2))$, we have that

$$\left|\frac{d}{du}\Delta h_t^s(u)\right| \le |H-K||H-K-1|t\left(\frac{s}{2}\right)^{H-K-2}.$$

For $K < 1/2$, it follows that

$$\int_{\mathbf{R}} C_5(v)^2\,dv \le \frac{8\Gamma\,(K-(1/2))^2}{\Gamma\,(K+(3/2))^2} t^2 s^{2H-2}.$$

By combining (3.9) and (3.10), respectively, with these estimates, we obtain (3.7) where

$$c_2\left(\frac{1}{2}, H, t, d\right) = 2\cdot{}^* G_2^2 (t+d)^{2H+2} + \left(\frac{(t+d)^2}{4Hd^2} + \frac{2}{1-H} + 2\right)t^2$$

and

$$\begin{aligned} &c_2(K,H,t,d) \\ &= \left(\frac{20\cdot{}^*G_2^2 G_3^{*2}}{\Gamma(K-(1/2))^2(1+H-K)^2(1-K)} + \frac{20\cdot{}^*G_2^2\Gamma(H-K+1)^2}{\Gamma(H+(3/2))^2}\right)(t+d)^{2H+2} \\ &\quad + \left(\frac{\max\left(160G_5^{*2}, (10G_6^{*2}(t+d)^2)/((H-K+1)^2 d^2)\right) + 10G_{17}^{*2} + (10G_{13}^{*2})/((2-K-H)^2)}{(1-K)\Gamma(K-(1/2))^2}\right. \\ &\qquad\qquad \left. + \frac{10G_{14}^{*2} + 40G_1^{*2}G_7^* + 10}{K\Gamma(K+(1/2))^2}\right)t^2, \ \ K > \frac{1}{2}. \end{aligned}$$

In the same way, by combing (3.11) with the estimates, we obtain (3.8) with

$$\begin{aligned} &c_3(K,H,t,d) \\ &= \left(\frac{24\cdot{}^*G_2^2 G_3^{*2}}{\Gamma(K-(1/2))^2(1+H-K)^2(1-K)} + \frac{816\cdot{}^*G_2^2\Gamma(H-K+1)^2}{(1+H-K)^2\Gamma(H+(3/2))^2}\right)(t+d)^{2H+2} \\ &\quad + \left(\frac{\max\left(192G_5^{*2}, (12G_6^{*2}(t+d)^2)/((H-K+1)^2 d^2)\right) + (12G_{13}^{*2})/((2-H-K)^2) + 12G_{17}^{*2}}{\Gamma(K-(1/2))^2(1-K)}\right)t^2 \\ &\quad + \frac{48}{K\Gamma(K+(1/2))^2}t^2 + \left(\frac{408G_8^{*2}G_9^* + 3072G_{10}^{*2}G_{11}^* + 378G_{15}^{*2} + 24G_{16}^{*2} + 48}{\Gamma(K+(3/2))^2}\right)t^2 \end{aligned}$$

and

$$c_4(K,H,t,d) = \frac{12d^{2(H-K-1)}(t+d)^2}{\Gamma\left(K+(1/2)\right)^2 K} t^2.$$

Eventually, by combining (3.2) and (3.4) with the results in A and B, we obtain the following: For $K = 1/2$, we have that

$$\left(\frac{\Gamma\left(H+(1/2)\right)}{C(H)}\right)^2 E\big[Z_t^{H,s} - Z_t^H\big]^2 \\ \leq \left(c_1\left(\frac{1}{2}, H, t\right) + c_2\left(\frac{1}{2}, H, t, d\right)\right) s^{2H-2}, \quad s > 2t + 4d + 1.$$

For $K > 1/2$, we have that

$$\frac{\Gamma(H-K+1)^2}{2C(H)^2} E\big[Z_t^{H,s} - Z_t^H\big]^2 \\ \leq \big(c_1(K,H,t) + c_2(K,H,t,d)\big) s^{2H-2}, \quad s > 2t + 4d + 1.$$

For $K < 1/2$, it holds that

$$\frac{\Gamma(H-K+1)^2}{2C(H)^2} \cdot E\big[Z_t^{H,s} - Z_t^H\big]^2 \\ \leq \big(c_1(K,H,t) + c_3(K,H,t,d)\big) s^{2H-2} \\ + c_4(K,H,t,d) s^{2K-2}, \quad s > 2t + 4d + 1. \qquad \square$$

The (generalized) Mandelbrot-Van Ness representation is a direct consequence of Theorem 3.1, and hence of the (generalized) Molchan-Golosov representation:

Corollary 3.2. *For every $K \in (0,1)$, the process $\left(Z_t^H\right)_{t \in \mathbf{R}}$ is an H-fBm.*

Proof. It follows from Theorem 3.1 that $\lim_{s\to\infty} E\big[Z_t^{H,s} - Z_t^H\big]^2 = 0$ for all $t \in \mathbf{R}$. Hence, for all $t, t' \in \mathbf{R}$, we have that $E\big[Z_t^H \cdot Z_{t'}^H\big] = \lim_{s\to\infty} E\big[Z_t^{H,s} \cdot Z_{t'}^{H,s}\big] = (1/2)\left(|t|^{2H} + |t'|^{2H} - |t-t'|^{2H}\right)$. $\square$

Acknowledgments. Thanks to my supervisor Esko Valkeila for suggesting the topic studied here and for useful remarks. Thanks are also due to the referee of Stochastic Processes and their Applications for a careful reading in connection with [**4**]. I would like to thank DYNSTOCH Network and the Finnish Graduate School in Stochastics (FGSS) for financial support.

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