JOURNAL OF INTEGRAL EQUATIONS
AND APPLICATIONS
Volume 2, Number 2, Spring 1990

ON THE STABLE AND UNSTABLE SUBSPACES OF A CRITICAL FUNCTIONAL DIFFERENTIAL EQUATION

OLOF J. STAFFANS

ABSTRACT. We study the asymptotic behavior of the linear, infinite delay, autonomous system of functional differential equations

$$(*)\qquad \begin{aligned} x'(t) + \mu * x(t) &= 0, & t &\geq 0,\\ x(t) &= \phi(t), & t &\leq 0. \end{aligned}$$

Here μ is an n-dimensional matrix-valued measure supported on $[0,\infty)$, finite with respect to a weight function, ϕ is a $\mathbf{C}^n$-valued continuous function in a fading memory space, and x is a locally absolutely continuous function for $t \geq 0$, satisfying $(*)$. We find conditions that ensure that the state space of $(*)$ can be written as a direct sum of a stable subspace, characterized by the fact that solutions are small at infinity, a finite dimensional central subspace in which solutions are neither small nor large at infinity, and a finite dimensional exponentially unstable subspace consisting of exponentially growing solutions. This work is heavily based on earlier joint work [**2**] with Jordan and Wheeler, and it extends the main result in [**3**]. The basic difference is that here we do not allow an explicit forcing term on the right-hand side of the first of the two equations in $(*)$, but instead we are able to relax the assumptions on the kernel.

1. Introduction. We study the asymptotic behavior of the solutions of the linear, infinite delay, autonomous system of functional differential equations

$$(1.1)\qquad \begin{aligned} x'(t) + \mu * x(t) &= 0, & t &\in \mathbf{R}^+,\\ x(t) &= \phi(t), & t &\in \mathbf{R}^-. \end{aligned}$$

Here $\mathbf{R}^+ = [0,\infty), \mathbf{R}^- = (-\infty, 0], \mu$ is an n by n matrix-valued measure supported on $\mathbf{R}^+$ which is finite with respect to a weight function, and

Received on March 1, 1989 and, in revised form, on May 22, 1989.
1980 *Mathematics Subject Classification.* Primary 34K25, 45D05, 45E10, 45F05, 45M05.

x and ϕ are $\mathbf{C}^n$-valued functions. The initial function ϕ belongs to a certain fading memory space compatible with the weighted measure space containing μ. As usual, $\mu * x$ denotes the convolution

$$(\mu * x)(t) = \int_{\mathbf{R}^+} d\mu(s)x(t-s).$$

We find conditions that ensure that the solution subspace of (1.1) can be decomposed into a direct sum of a *stable subspace* $\mathcal{S}$, which is characterized by the fact that the solutions in $\mathcal{S}$ are small at infinity; a finite dimensional *central subspace* $\mathcal{C}$ in which solutions do not decay, but also do not grow at exponential rates; and, finally, a finite dimensional *unstable subspace* $\mathcal{U}$ consisting of exponentially growing solutions.

The question which we consider here has been studied earlier in a very similar setting in [**3**]. The basic difference is that the equation in [**3**] was a more general version of (1.1), namely,

$$\text{(1.2)} \qquad \begin{aligned} x'(t) + \mu * x(t) &= f(t), \qquad && t \in \mathbf{R}^+, \\ x(t) &= \phi(t), \qquad && t \in \mathbf{R}^-. \end{aligned}$$

Of course, one gets (1.1) from (1.2) by taking $f \equiv 0$; hence the results of [**3**] can be applied to (1.1) as well as to (1.2). The result given in [**3**] is more or less optimal for (1.2), but, as we shall see below, it is possible to prove a somewhat sharper version for the special case (1.1) of (1.2). A more detailed comparison of our present result with the result in [**3**] is given in §5, but, roughly speaking, if the largest growth rate in the central subspace is of order t^m, for some $m \geq 0$, then here we need μ to have a total of just a little more than m finite moments, whereas the requirement in [**3**] is just a little more than $2m+1$ finite moments. In addition we make a structural assumption on the central critical exponents.

Throughout we expect the reader to have [**2**] and [**3**] at hand and make frequent references to these papers.

2. The general setting. The basic setting that we use is essentially the same as the setting in [**2**], adapted to our equation (1.1). We let η be an influence function on $\mathbf{R}$ dominated by a submultiplicative function

ρ on $\mathbf{R}$. Some additional assumptions on η and ρ will be mentioned later, but let us remark at this point that the values of η on $\mathbf{R}^-$ will determine our state space of fading memory type and that the values of η on $\mathbf{R}^+$ will determine the rate of convergence to zero in the stable subspace. In addition, both the values of η on $\mathbf{R}^-$ and the values of η on $\mathbf{R}^+$ will affect the values of ρ on $\mathbf{R}^+$. For an example, see §5.

We assume that $\rho(t) \equiv 1$ on $\mathbf{R}^-$, and that $\mu \in M(\mathbf{R}^+; \mathbf{C}^{n\times n}; \rho)$. We let $\mathcal{B}$ be one of the spaces $L^p, 1 \le p \le \infty, BUC$ or BC_0 and suppose that our initial function ϕ belongs to $\mathcal{B}^{m+1}(\mathbf{R}^-; \mathbf{C}^n; \eta)$ for some $m \ge 0$ (this extra smoothness assumption is not important; the discussion can be carried out in $\mathcal{B}$ instead). Let $\mathcal{L}$ be the operator

$$\mathcal{L}\phi = \phi' + \mu * \phi \tag{2.1}$$

and define

$$g(t) = \begin{cases} \mathcal{L}\phi(t), & t < 0, \\ 0, & t \ge 0. \end{cases} \tag{2.2}$$

Then the solution of (1.1) belongs locally to $\mathcal{B}^{m+1}$ if and only if the function g in (2.2) belongs to $\mathcal{B}^m(\mathbf{R}; \mathbf{C}^n; \eta)$ (that is, if and only if all those derivatives of the function $\mathcal{L}\phi$ that can be evaluated pointwise vanish at zero). Our state space $\mathcal{D}$ will be the space of all those initial functions $\phi \in \mathcal{B}^{m+1}(\mathbf{R}^-; \mathbf{C}^n; \eta)$ for which the corresponding function g in (2.2) belongs to $B^m(\mathbf{R}; \mathbf{C}^n; \eta)$, i.e., for which the solution of (1.1) belongs locally to $\mathcal{B}^{m+1}$.

Since we assume that $\rho(t) \equiv 1$ for $t \le 0$, we have $\lim_{t\to-\infty} t^{-1} \log \rho(t) = 0$, i.e., the number which was called α in [**3**] is zero. Define $\omega = -\lim_{t\to\infty} t^{-1} \log \rho(t)$. Then $-\infty < \omega \le 0$. If $\omega < 0$ then our present results will not differ in any essential way from those in [**3**], and, therefore, we assume in the sequel that $\omega = 0$. In other words, we have

$$\lim_{t\to-\infty} \frac{\log \rho(t)}{t} = \lim_{t\to\infty} \frac{\log \rho(t)}{t} = 0. \tag{2.3}$$

Thus, the maximal ideal space II of $M(\mathbf{R}; \mathbf{C}^{n\times n}; \rho)$ is the imaginary axis; see [**1**, p. 752].

Let $\widehat{L}$ be the formal Laplace transform of $\mathcal{L}$, i.e.,

$$\widehat{L}(z) = zI + \hat{\mu}(z), \qquad \Re z \ge 0. \tag{2.4}$$

This function is sometimes called the *characteristic function* of (1.1). The equation

$$\det[\widehat{L}(z)] = 0, \qquad \Re z \geq 0, \tag{2.5}$$

is called the *characteristic equation* of (1.1), and its roots are called *characteristic exponents.* Let us call those characteristic exponents that belong to the imaginary axis *central characteristic exponents*, and those that belong to the open half plane $\Re z > 0$ *unstable characteristic exponents.* If there are no central characteristic exponents, then the result presented here becomes essentially the same as in [**3**]. Thus, let us suppose that there is at least one central characteristic exponent. (It follows from (2.6) below that the number of central characteristic exponents must be finite.)

The construction of a central subspace in [**3**] was based on the assumption that, at each central characteristic exponent, the inverse of the characteristic function should have a singular part expansion with respect to a certain submultiplicative function ρ^S (different from the function ρ above; we shall return to this function in §5). We recall from [**3**, Proposition 5.1] that this implies that $\widehat{L}$ has a local Smith factorization at each central characteristic exponent. As in [**3**], we shall suppose that the total number of central exponents is finite. Thus, by [**2**, Theorem 3.2], $\widehat{L}$ has a global Smith factorization with respect to ρ^S. In this paper we do not a priori require $\widehat{L}$ to have a singular part expansion, but instead we ask that

(2.6) $\widehat{L}$ has a global Smith factorization with respect to the submultiplicative function ρ.

Observe, in particular, that the factorization is with respect to ρ, not with respect to ρ^S. Fur a further discussion of this assumption, see §5.

In the sequel it will be important that $\widehat{L}$ has a global Smith factorization with respect to another submultiplicative function ρ_α, too:

LEMMA 2.1. *Define*

$$\rho_\alpha(t) = \begin{cases} \rho(t), & t \geq 0, \\ e^{-\alpha t}, & t < 0, \end{cases}$$

where $\alpha > 0$. Then ρ_α is submultiplicative, and $\widehat{L}$ has a global Smith factorization with respect to ρ_α.

PROOF. That ρ_α is submultiplicative follows from the facts that the product of two submultiplicative functions is submultiplicative , that ρ is submultiplicative , and that the function which is $e^{-\alpha t}$ for negative t and one for positive t is submultiplicative (recall that we assume $\rho(t) \equiv 1$ for $t \leq 0$).

To prove that $\widehat{L}$ has a global Smith factorization we have to show that it has a right global factorization and a left global factorization. The proofs are completely similar, so, below, we only give the argument for the right global factorization .

According to (2.6) and [**2**, §5], the function $\widehat{L}$ has a right global Smith factorization of the type

$$\widehat{L}(z) = (z+1)R(z)D(z)P(z), \qquad \Re z = 0,$$

where R, D, and P are transforms of measures in $M(\mathbf{R}; \mathbf{C}^{n\times n}; \rho)$, P is a unimodular (determinant identically one) quasipolynomial (a polynomial in $(z+1)^{-1}$), D is a diagonal quasipolynomial, $\det[D]$ vanishes only at the central critical exponents, D tends to the identity matrix at infinity, and R^{-1} is a transform of a measure in $M(\mathbf{R}; \mathbf{C}^{n\times n}; \rho)$. A direct inspection of D and P shows that these functions are transforms of measures in $M(\mathbf{R}^+; \mathbf{C}^{n\times n}; \rho)$, i.e., transforms of measures in $M(\mathbf{R}; \mathbf{C}^{n\times n}; \rho)$ that are supported on $\mathbf{R}^+$. We claim that the same statement is true for R (but not in general for R^{-1}). To see this, simply observe that the equation above implies that R has a bounded analytic extension to the half plane $\Re z \geq 0$, satisfying $R(x) \to 0$ as $x \to +\infty$ (along the real axis), and apply the standard argument used in the proof of the Paley-Wiener theorem. Thus, we conclude that, since R is the transform of a measure vanishing on $(-\infty, 0)$, it can be regarded as a transform of a measure in $M(\mathbf{R}; \mathbf{C}^{n\times n}; \rho_\alpha)$ rather than as a transform of a measure in $M(\mathbf{R}; \mathbf{C}^{n\times n}; \rho)$. In other words, the factors in the factorization above are locally analytic with respect to the submultiplicative function ρ_α. When we pass from ρ to ρ_α, the corresponding maximal ideal space II expands from the original line $\Re z = 0$ to the strip $0 \leq \Re z \leq \alpha$, and to get a global right Smith factorization with respect to ρ_α we have to factor out all those unstable characteristic exponents that belong to the strip $0 < \Re z \leq \alpha$ from R, transferring the factors to D and P. However, this can be done in exactly the same way as in the construction of the right global Smith factorization given in the proof of [**2**, Theorem 3.2], since we know that the factors are locally analytic with respect to ρ_α and R is analytic at all the unstable critical exponents; hence R has local Smith factorizations at these points. □

3. The unstable and central subspaces. Our treatment of the unstable subspace is identical to the one in [**3**, §3]. Thus, we have

DEFINITION 3.1. A function $\phi \in \mathcal{D}$ belongs to the unstable subspace $\mathcal{U}$ if $\mathcal{L}\phi(t) = 0$ for $t \in \mathbf{R}^-$, and $\phi(t) = O(e^{-|\varepsilon t|})$ as $t \to -\infty$ for some $\varepsilon > 0$.

The construction of the central subspace $\mathcal{C}$ relies on the assumption (2.6). If we choose α so large that all the characteristic exponents lie in the strip $0 \leq \Re z < \alpha$ and define

$$\eta_\alpha(t) = [\rho_\alpha(-t)]^{-1}, \qquad t \in \mathbf{R},$$

where ρ_α is the function defined in Lemma 2.1, then η_α is dominated by ρ_α and the solution x of (1.1) belongs to $\mathcal{B}^{m+1}(\mathbf{R}; \mathbf{C}^n; \eta_\alpha)$, see, e.g., [**3**, §3]. We can use [**2**, Theorem 5.1] and Lemma 2.1 to conclude that the null-space of $\mathcal{L}$, regarded as an operator from $\mathcal{B}^{m+1}(\mathbf{R}; \mathbf{C}^n; \eta_\alpha)$ into $\mathcal{B}^m(\mathbf{R}; \mathbf{C}^n; \eta_\alpha)$, consists of functions that are certain sums of exponential polynomials, the exponents of which correspond to the characteristic exponents of (1.1) and the coefficients being determined by the right Jordan chains of $\widehat{L}$ at each characteristic exponent. The unstable characteristic exponents generate the unstable subspace, and the central characteristic exponents generate the central subspace.

DEFINITION 3.2. A function $\phi \in \mathcal{D}$ belongs to the central subspace $\mathcal{C}$ if $\mathcal{L}\phi(t) = 0$ for $t \in \mathbf{R}^-$, and the solution $x(\phi)$ of (1.1) satisfies $x(\phi)(t) = \mathrm{O}(|t|^m)$ as $|t| \to \infty$ for some finite m.

It follows from the construction above that the null-space of $\mathcal{L}$ in $\mathcal{B}^{m+1}(\mathbf{R}^-; \mathbf{C}^n; \eta)$ is the sum $\mathcal{C} \oplus \mathcal{U}$.

The maximal growth rate of a function $\phi \in \mathcal{C}$ at $-\infty$ is determined by two things. First of all, it depends on the maximal length of the central

Jordan chains of $\widehat{L}$. As it was seen in [**2**, §5], if the maximal partial multiplicity of $\widehat{L}$ at a central characteristic exponent $i\omega_0$ is k, then the largest of the characteristic solutions generated by this characteristic exponent is proportional to $t^{k-1}e^{i\omega_0 t}$ at plus and minus infinity. Of course, the corresponding initial function $\phi \in \mathcal{C}$ has the same growth rate at $-\infty$. However, we have a second requirement to fulfill, namely the requirement $\phi \in \mathcal{D}$, which puts an additional size restriction on ϕ at $-\infty$. If η is too large at $-\infty$, then this second requirement might reduce the size of $\mathcal{C}$ from its maximal theoretical size. In the sequel we shall assume that this does *not* happen, but that

(3.1) The function $t \mapsto t^{k-1}$ belongs to $\mathcal{B}(\mathbf{R}^-; \mathbf{C}; \eta)$, where k is the largest of the partial multiplicities of $\widehat{L}$ at the central characteristic exponents.

The same assumption occurs in [**3**] in the form of [**3**, formula (6.9)]. Thus, our central subspace coincides with the central subspace in [3].

4. The stable subspace. So far nothing really new has emerged compared to [**3**]. In particular, all that has been said above could easily be adapted to the more general equation (1.2). The situation becomes different when we begin to discuss the stable subspace.

Our definition of a stable subspace is the same as in [**3**]:

DEFINITION 4.1. A function $\phi \in \mathcal{D}$ belongs to the stable subspace $\mathcal{S}$ if the solution $x(\phi)$ of (1.1) satisfies $x(\phi)(t) \to 0$ as $t \to \infty$.

Clearly, the intersection of any two of the spaces $\mathcal{S}, \mathcal{C}$ and $\mathcal{U}$ is the zero function. However, at this point it is far from clear that $\mathcal{D}$ can be written as the sum of $\mathcal{S}, \mathcal{C}$ and $\mathcal{U}$. This is true if and only if every solution x of (1.1) can be decomposed into a sum of an exponential polynomial p, satisfying $\mathcal{L}p = 0$, and a remainder which tends to zero at infinity. The rest of this paper will be devoted to the construction of such a decomposition. It differs substantially from the corresponding construction in [**3**], and it is based on the description of the range of the operator $\mathcal{L}$, described in [**2**, §6].

Let us recall the following two results from [**2**]:

DEFINITION 4.2. [**2**, Definition 6.1] The row vectors $v_0, v_1, \ldots, v_p$, with $v_0 \neq 0$, form a Jordan chain of $f \in \mathcal{B}^m(\mathbf{R}; \mathbf{C}^n; \eta)$ of length $p+1$ at a point $i\omega_0$ if there exist scalar functions $F_1, F_2, \ldots, F_{p+1} \in \mathcal{B}(\mathbf{R}; \mathbf{C}; \eta)$ satisfying

$$\left(\frac{d}{dt} - i\omega_0\right) F_1 = v_0 f, \qquad \left(\frac{d}{dt} - i\omega_0\right) F_{j+1} = F_j + v_j f, \qquad 1 \le j \le p.$$

THEOREM 4.3. [**2**, Theorem 6.1] *A function $f \in \mathcal{B}^m(\mathbf{R}; \mathbf{C}^n; \eta)$ can be written in the form $f = \mathcal{L}y$, for some $y \in \mathcal{B}^{m+1}(\mathbf{R}; \mathbf{C}^n; \eta)$, if and only if, for each central characteristic exponent $i\omega_0$, every left Jordan chain of $\widehat{L}$ is a Jordan chain of f at $i\omega_0$.*

In the formulation above we have made use of (2.3) and (2.6).

In the sequel we shall, in addition, need the concept of a Jordan chain of a function belonging to $\mathcal{B}^m(\mathbf{R}^-; \mathbf{C}^n; \eta)$. The definition is the same as in Definition 4.2, with $\mathbf{R}$ replaced by $\mathbf{R}^-$ throughout.

LEMMA 4.4. *Let $\phi \in \mathcal{B}^{m+1}(\mathbf{R}^-; \mathbf{C}^n; \eta)$. Then, for each central characteristic exponent $i\omega_0$, every left Jordan chain of $\widehat{L}$ is a Jordan chain of $\mathcal{L}\phi$ at $i\omega_0$.*

PROOF. Extend ϕ to a function $\psi \in \mathcal{B}^{m+1}(\mathbf{R}; \mathbf{C}^n; \eta)$ in an arbitrary way. Then, by Theorem 4.3, every left Jordan chain of $\widehat{L}$ at $i\omega_0$ is a Jordan chain of $\mathcal{L}\psi$. Since $\mathcal{L}$ is causal, $(\mathcal{L}\psi)(t) = (\mathcal{L}\phi)(t)$ for $t \in \mathbf{R}^-$; hence, every Jordan chain of ψ is a Jordan chain of ϕ. □

LEMMA 4.5. *Let $\phi \in \mathcal{D}$, and define g by (2.2). Then there is a function $y \in \mathcal{B}^{m+1}(\mathbf{R}; \mathbf{C}^n; \eta)$ satisfying $\mathcal{L}y = g$.*

PROOF. By Theorem 4.3 and Lemma 4.4, in order to prove Lemma 4.5 it suffices to show that every Jordan chain of $\mathcal{L}\phi \in \mathcal{B}^m(\mathbf{R}^-; \mathbf{C}^n; \eta)$ of length $p+1 \le k$ is a Jordan chain of $g \in \mathcal{B}^m(\mathbf{R}; \mathbf{C}^n; \eta)$, where k is the number in (3.1).

At this stage the possible non-uniqueness of the functions $F_1, F_2, \ldots, F_j$ in Definition 4.2 becomes important. Let us take a look at the situation in $\mathcal{B}^m(\mathbf{R}^-;\mathbf{C}^n;\eta)$. Suppose that we have two different series of functions $F_1, F_2, \dots, F_{p+1}$ and $G_1, G_2, \dots, G_{p+1}$ corresponding to the same Jordan chain $v_0, v_1, \dots, v_p$ of $f \in \mathcal{B}^m(\mathbf{R}^-;\mathbf{C}^n;\eta)$ at $i\omega_0$, with length $p+1 \le k$. Then $F_1(t) - G_1(t) = c_1 e^{i\omega_0 t}$, where $c_1 = F_1(0) - G_1(0)$; $F_2(t) - G_2(t) = (c_1 t + c_2)e^{i\omega_0 t}$, where $c_2 = F_2(0) - G_2(0)$; etc. In particular, $F_j(t) - G_j(t), 1 \le j \le p+1$, is of the form $\sum_{q=0}^{j} c_{q,j} t^{q-1} e^{i\omega_0 t}$ for some constants $c_{q,j}$. Thus, recalling our assumption (3.1), we conclude that all these differences belong to $B^{m+1}(\mathbf{R}^-;\mathbf{C}^n;\eta)$. Hence, if one wants to test the existence of the functions $F_1, F_2, \dots, F_{p+1}$ in Definition 4.2, it is permitted to prescribe the initial values $F_1(0), F_2(0), \dots, F_{p+1}(0)$ in an arbitrary manner. In particular, it is possible to require $F_1(0) = F_2(0) = \cdots = F_{p+1}(0) = 0$.

Let us return to the function g in (2.2). Let $v_0, v_1, \dots, v_p$ be a Jordan chain of $\mathcal{L}\phi \in \mathcal{B}^m(\mathbf{R}^-;\mathbf{C}^n;\eta)$ of length $p+1 \le k$. Then there exist functions $F_1, F_2, \dots F_{p+1}$ in $\mathcal{B}^{m+1}(\mathbf{R}^-;\mathbf{C}^n;\eta)$ satisfying the differential equations in Definition 4.2 on $\mathbf{R}^-$, with f replaced by $\mathcal{L}\phi$. As we observed above, we may, without loss of generality, take $F_1(0) = F_2(0) = \cdots = F_{p+1}(0) = 0$. Define $F_1(t) = F_2(t) = \cdots = F_{p+1}(t) = 0$ for $t > 0$. Then the extended functions satisfy the required differential equations on $\mathbf{R}$, with f replaced by g, and we conclude that $v_0, v_1, \dots, v_p$ is a Jordan chain of $g \in \mathcal{B}^m(\mathbf{R}^-;\mathbf{C}^n;\eta)$. □

Before we can state our main theorem we need one more assumption, namely,

$$\mathcal{B}^{m+1}(\mathbf{R}^+;\mathbf{C}^n;\eta) \subset BC_0(\mathbf{R}^+;\mathbf{C}^n;1), \tag{4.1}$$

i.e., we assume that every $y \in \mathcal{B}^{m+1}(\mathbf{R};\mathbf{C}^n;\eta)$ tends to zero at $+\infty$.

THEOREM 4.6. *Assume* (2.6), (3.1), *and* (4.1). *Then* $\mathcal{D} = \mathcal{S} \oplus \mathcal{C} \oplus \mathcal{U}$.

PROOF. We already know that the pairwise intersections of $\mathcal{S}, \mathcal{C}$ and $\mathcal{U}$ contain nothing but the zero function, and that $\mathcal{C} \oplus \mathcal{U}$ is the null-space of $\mathcal{L}$ in $\mathcal{B}^{m+1}(\mathbf{R}^-;\mathbf{C}^n;\eta)$. Thus, to prove the theorem, it suffices to show that every $\phi \in \mathcal{D}$ can be written as the sum of two components, one in $\mathcal{S}$, and the other in the null-space of $\mathcal{L}$.

Let $\phi \in \mathcal{D}$. By Lemma 4.5, there is a function $y \in \mathcal{B}^{m+1}(\mathbf{R};\mathbf{C}^n;\eta)$ satisfying $(\mathcal{L}y)(t) = (\mathcal{L}x)(t)$ for all $t \in \mathbf{R}$. Because of (4.1), $y(t) \to 0$ as

$t \to \infty$. Hence, if we let ψ be the initial function of y (the restriction of y to $\mathbf{R}^-$), then $\psi \in \mathcal{S}$. Clearly the difference $z = y - x$ satisfies $(\mathcal{L}z)(t) = 0$ for all $t \in \mathbf{R}$. □

COROLLARY 4.7. *A function $\phi \in \mathcal{D}$ belongs to $\mathcal{S}$ if and only if the solution $x(\phi)$ of* (1.1) *satisfies $x(\phi) \in \mathcal{B}^{m+1}(\mathbf{R}^+; \mathbf{C}^n; \eta)$.*

To some extent the proof of Theorem 4.6 may be regarded as constructive. In order to perform the decomposition above, it suffices to compute y. This function can be constructed from the sequences of functions F_j found in the proof of Lemma 4.5, as explained in the proof of [**2**, Theorem 6.1].

5. A comparison with [**3**]. For simplicity, let us take $\mathcal{B} = BUC$, and let us take η to be of the form

$$\eta(t) = \begin{cases} (1-t)^{-q_-}, & t \in \mathbf{R}^-, \\ (1+t)^{q_+}, & t \in \mathbf{R}^+, \end{cases} \tag{5.1}$$

for some nonnegative, real numbers q_- and q_+. Clearly, (3.1) is true if and only if $q_- \geq k - 1$, and (4.1) holds if and only if $q_+ > 0$. We may take ρ to be

$$\rho(t) = \begin{cases} 1, & t \in \mathbf{R}^-, \\ (1+t)^p, & t \in \mathbf{R}^+, \end{cases} \tag{5.2}$$

provided we take $p \geq q_- + q_+$; cf. [**3**, Example 6.1]. Thus, our basic size assumption on μ becomes

$$\int_{\mathbf{R}^+} (1+t)^{k+\varepsilon-1} |\mu|(dt) < \infty, \tag{5.3}$$

for some $\varepsilon > 0$.

Our second main assumption on μ is (2.6), which says that $\widehat{L}$ should have a global Smith factorization with respect to ρ. By [**1**, Lemma 4.3] and [**3**, Theorem 5.1], this implies that μ has a singular part expansion with respect to the submultiplicative function

$$\rho^S(t) = \begin{cases} 1, & t \in \mathbf{R}^-, \\ (1+t)^\delta, & t \in \mathbf{R}^+, \end{cases} \tag{5.4}$$

provided $\varepsilon - 1 \geq \delta \geq 0$ (for $0 < \varepsilon < 1$ we do not necessarily have a singular part expansion). Thus, if the constant ε in (5.3) satisfies the additional condition $\varepsilon > 1$, then the assumption made in [**3**] that $\widehat{L}$ has a singular part expansion with respect to ρ^S is satisfied, with ρ^S defined as above and $0 < \delta < \varepsilon - 1$.

In addition to the assumptions on the existence of a singular part expansion at the central critical exponents, it was assumed in [**3**] (see [**3**, (6.8)–(6.10)]) that

$$\rho(t) \geq (1+t)^{k+1}, \qquad t \in \mathbf{R}^+, \tag{5.5}$$

$$BUC(\mathbf{R}^-; \mathbf{C}^n; \eta_S) \subset \mathcal{B}(\mathbf{R}^-; \mathbf{C}^n; \eta), \tag{5.6}$$

$$\mathcal{B}(\mathbf{R}^+; \mathbf{C}^n; \eta) \subset L^1(\mathbf{R}^+; \mathbf{C}^n; \rho_S), \tag{5.7}$$

where (if we let ρ^S be the function in (5.4))

$$\rho_S(t) = \begin{cases} 1, & t \in \mathbf{R}^-, \\ (1+t)^{k+\delta-1}, & t \in \mathbf{R}^+, \end{cases} \tag{5.8}$$

$$\eta_S(t) = \begin{cases} (1-t)^{1-k}, & t \in \mathbf{R}^-, \\ 1, & t \in \mathbf{R}^-. \end{cases} \tag{5.9}$$

Clearly, (5.5) is an immediate consequence of (5.2), with $p > k-1$, and (5.6) is essentially the same as our (3.1). The most significant difference between our present assumptions and those in [3] is that (5.7) puts an additional restriction on ρ, not present here. Clearly, if we still choose η to be of the form (5.1), then, to satisfy (5.7), we have to take

$$q_+ > k + \delta.$$

This means that $q_- + q_+ > 2k + \delta - 1$. In other words, the results of [**3**] need μ to satisfy

$$\int_{\mathbf{R}^+} (1+t)^{2k+\gamma-1} |\mu|(dt) < \infty, \tag{5.10}$$

for some $\gamma > 0$. Compare this to our (5.3).

Earlier we pointed out that our assumption (2.6) nearly implies the assumption made in [**3**] concerning the existence of a singular part expansion at the central characteristic exponents. Conversely, in the example given above, (5.10) implies (2.6). To see this, choose $\varepsilon \le \gamma$, let ρ be given by (5.2) with $p = k + \varepsilon - 1$, and use [**1**, Lemma 4.3(iii)] and [2, Theorem 3.1].

REFERENCES

1. G.S. Jordan, O.J. Staffans and R.L. Wheeler, *Local analyticity in weighted L^1-spaces and applications to stability problems for Volterra equations*, Trans. Amer. Math. Soc. **274** (1982), 749-782.

2. ———, ——— and ———, *Convolution operators in a fading memory space: The critical case*, SIAM J. Math. Anal. **18** (1987), 366-386.

3. ———, ——— and ———, *Subspaces of stable and unstable solutions of a functional differential equation in a fading memory space: The critical case*, SIAM J. Math. Anal. **18** (1987), 1323-1340.

INSTITUTE OF MATHEMATICS, HELSINKI UNIVERSITY OF TECHNOLOGY SF-02150 ESPOO 15, FINLAND